H^1 BOUNDEDNESS OF DETERMINANTS OF VECTOR FIELDS

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Abstract. We consider multilinear operators $T(f_1, \dots, f_l)$ given by determinants of matrices of the form $(X_k f_j)_{1\le j,k\le l}$, where the X_k's are C^∞ vector fields on $\mathbb{R}^n$. We give conditions on the X_k's so that the corresponding operator T map products of Lebesgue spaces $L^{p_1}\times\cdots\times L^{p_l}$ into some anisotropic space H^1, when $\frac{1}{p_1}+\cdots+\frac{1}{p_l}=1$.

0. Introduction and statement of results.

A well known Theorem of Coifman, Lions, Meyer, and Semmes [CLMS] states that the Jacobian $J(F)$ of a map $F=(f_1,\dots,f_n)$ from $\mathbb{R}^n$ into itself maps the product of Sobolev spaces $L_1^{p_1}\times\cdots\times L_1^{p_n}$ into the Hardy space H^1, when $\frac{1}{p_1}+\cdots+\frac{1}{p_n}=1$. $J(F)$ is given by the determinant of the matrix $(\frac{\partial}{\partial x_k} f_j)_{1\le j,k\le n}$, where $\{\frac{\partial}{\partial x_k}\}_{1\le k\le n}$ is the usual basis of the tangent space of $\mathbb{R}^n$ at every point. Replacing the standard basis $\{\frac{\partial}{\partial x_k}\}$ by general vector fields $\{X_k\}$, we form the multilinear operator $T(\{f_j\})=\det(X_k f_j)$. We consider the following question: Under what conditions on the X_k's do we have that T maps products of Lebesgue spaces into some Hardy space H^1 as before?

The purpose of this paper is to give a satisfactory answer to the question posed above. If the X_k are taken from the usual basis of the Heisenberg group in $\mathbb{R}^{2n+1}$, Rochberg and the author [GR] prove that the corresponding T maps into the group space H^1.

In this work we show that if the vector fields X_k satisfy Hörmander's condition, then the corresponding T maps suitable products of Lebesgue spaces into the local anisotropic Hardy space H^1 with respect to the metric associated with the vector fields defined by Nagel, Stein, and Wainger [NSW]. Precise statements of results are given in Theorems A and B.

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Suppose that $S = \{Y_1, Y_2, \dots, Y_l\}$ is a set of smooth vector fields defined on a bounded open connected subset Ω of $\mathbb{R}^n$ for some $n \geq 2$. Assume that S is a Hörmander system. This means that there exists an integer s such that the vector fields $Y_1, \dots, Y_l$ together with their commutators of order at most s span the tangent space of Ω at every point x.

[NSW] define a (quasi)metric ρ on Ω by setting $\rho(x,y) = \inf\{t : \exists$ piecewise smooth curve $\gamma : [0,t] \to \mathbb{R}$, $\gamma(0) = x$, $\gamma(t) = y$ and $\gamma'(s) = \sum_{j=1}^{l} \beta_j(s) Y_j(\gamma(s))$ with $\sum_{j=1}^{l} |\beta_j(s)|^2 \leq 1$ for all $s \in [0,t]\}$, for all x, y in Ω. Intuitively, $\rho(x,y)$ is the least time taken to move from x to y along a path pointing in the directions of the Y_j's. For $x \in \Omega$ and $\delta > 0$, let $B(x,\delta) = \{y \in \Omega : \rho(x,y) < \delta\}$ be the ball centered at x with radius δ with respect to the metric ρ. [NSW] prove that these balls satisfy a doubling property for Lebesgue measure. More precisely, they prove that for any compact subset K of Ω, there exist positive constants C_K and $\delta_1(K)$ such that for all $0 < \delta < \delta_1(K)$ and all x in K

$$|B(x, 2\delta)| \leq C_K |B(x,\delta)|, \tag{0.1}$$

where $|\cdot|$ denotes Lebesgue measure. As a corollary of (0.1), the local Hardy-Littlewood maximal function

$$(Mf)(x) = (M_{\delta_1(K)} f)(x) = \sup_{0<\delta<\delta_1(K)} |B(x,\delta)|^{-1} \int_{B(x,\delta)} |f(y)|\, dy$$

maps $L^p(\Omega)$ to $L^p(K)$ for any K compact subset of Ω and $1 < p < \infty$.

We now define the space $H^1_{\mathrm{loc}}(\Omega)$. Fix a smooth bump ϕ in the unit ball of $\mathbb{R}^n$ and let $\phi_\delta(y) = \delta^{-n} \phi_\delta(\delta^{-1} y)$. For any x_0 in Ω and $\delta > 0$ small enough, the push-forward of ϕ_δ by any of the coordinate maps constructed in [NSW] gives a smooth bump $\psi_\delta^{x_0}$ supported in the ball $B(x_0, \delta)$. One can check that for any compact subset K of Ω and for all $j = 1, \dots, l$ and $x \in K$

$$|\psi_\delta^{x_0}(x)| \leq C_K |B(x_0,\delta)|^{-1} \quad \text{and} \qquad |Y_j(\psi_\delta^{x_0})(x)| \leq C_K \delta^{-1} |B(x_0,\delta)|^{-1} \tag{0.2}$$

when $0 < \delta < \delta_2(K)$, where $\delta_2(K)$ is a small constant depending on K.

For a function f on Ω and $\delta > 0$, let

$$(\mathcal{M}_\delta f)(x_0) = \sup_{0<\sigma<\delta} \left| \int f(z) \psi_\sigma^{x_0}(z)\, dz \right|. \tag{0.3}$$

We call $\mathcal{M}_\delta$ the "smooth" maximal function of f. We say that f lies in $H^1_{\mathrm{loc}}(\Omega)$ if for all compact subsets K of Ω, there exists a $\delta_0(K) > 0$ such that $\mathcal{M}_{\delta_0(K)} f$ is in $L^1(K)$. We define the *Hardy-1 space norm of f on K* by setting

$$\|f\|_{H^1(K)} = \|\mathcal{M}_{\delta_0(K)} f\|_{L^1(K)}. \tag{0.4}$$

For a C^1 map $F = (f_1, \dots, f_l) : \Omega \to \mathbb{R}^l$, define

$$\text{Jac}(F) = \text{Jac}_{\{Y_1,\dots,Y_l\}}(F) = \det \begin{pmatrix} Y_1 f_1 & Y_1 f_2 & \dots & Y_1 f_l \\ Y_2 f_1 & Y_2 f_2 & \dots & Y_2 f_l \\ \vdots & \vdots & & \vdots \\ Y_l f_1 & Y_l f_2 & \dots & Y_l f_l \end{pmatrix}. \tag{0.5}$$

Our results are:

Theorem A. *For $1 \le j \le l$, let $1 < p_j < \infty$ be given with $\frac{1}{p_1} + \dots + \frac{1}{p_l} = 1$. Suppose that $Jac_{\{Y_1,\dots,Y_l\}}(F)$ has integral zero for all C^∞ compactly supported functions F on Ω. Then for any compact subset K of Ω, there exists a constant $C_K > 0$ such that for all C^1 functions $F = (f_1, \dots, f_l) : \Omega \to \mathbb{R}^l$, we have:*

$$\|Jac_{\{Y_1,\dots,Y_l\}}(F)\|_{H^1(K)} \le C_K \prod_{j=1}^{l} \Big[\sum_{k=1}^{l} \|Y_k f_j\|_{L^{p_j}(\Omega)} \Big]. \tag{0.6}$$

The integral zero condition is trivially satisfied by the usual basis $\frac{\partial}{\partial x_j}$ of the tangent bundle of Ω. For other non trivial examples see section 3.

We now turn to the situation where $Jac_{\{Y_1,\dots,Y_l\}}$ does not satisfy the integral zero condition as in Theorem A. A C^∞ vector field $X = \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j}$ in the tangent space of Ω at x_0 is called divergence-free if

$$\int_\Omega (Xf) g \, dx = - \int_\Omega f(Xg) \, dx \tag{0.7}$$

for all f, g smooth compactly supported functions on Ω. This happens exactly when $\text{div}((a_j(x)) = \sum_{j=1}^n \frac{\partial a_j}{\partial x_j}(x) = 0$ for all x in Ω.

Assuming divergence-free, we can do away with the integral zero condition of Theorem A. We have the following:

Theorem B. *For $1 \le j \le l$, let $1 < p_j < \infty$ be given with $\frac{1}{p_1} + \dots + \frac{1}{p_l} = 1$. Suppose all the Y_j are divergence free. Then for any compact subset K of Ω, there exists a constant $C_K > 0$, such that for any C^1 map $F = (f_1, \dots, f_l) : \Omega \to \mathbb{R}^l$, we have*

$$\|Jac_{\{Y_1,\dots,Y_l\}}(F)\|_{H^1(K)} \le C_K \prod_{j=1}^{l} \Big[\sum_{k=1}^{l} \|Y_k f_j\|_{L^{p_j}(\Omega)} + \sum_{k_1,k_2=1}^{l} \|[Y_{k_1}, Y_{k_2}] f_j\|_{L^{p_j}(\Omega)} \Big]. \tag{0.8}$$

Above, $[X, Y]$ denotes the commutator of the vector fields X and Y.

As a corollary of our results, we obtain improved integrability for positive Jacobians formed by vector fields satisfying Hörmander's condition. The corollary below generalizes the classical result of Müller [M] for Euclidean Jacobians.

Corollary. *Let p_j and F be as in Theorem B. Suppose that, $Jac(F) \geq 0$, and that the right hand-side of (0.8) is finite. Then $Jac(F)$ lies in $(L^1 \log L)_{\text{loc}}(\Omega)$, i.e.*

$$\int_K Jac(F) \log(2 + Jac(F))\, dx < +\infty \tag{3.3}$$

for all compact subsets of K of Ω.

As the reader has observed our results are only local. This is not due to insufficiency of the methods, but to the lack of global dilation structure associated with the metric constructed by [NSW]. As a result of this, two key ingredients of the proof, the Poincaré inequality for vector fields satisfying Hörmander's condition, and the L^p boundedness of the (metric ball) maximal function, are only local in this setting.

1. Proof of Theorem A.

Denote by f_A the average of a function f over the set A. We will need the following version of the Poincaré inequality for vector fields satisfying Hörmander's condition.

Theorem. *Let Q be the homogeneous dimension of the graded nilpotent group generated by the left invariant vector fields corresponding to the lifted vector fields $\{\tilde{Y}_j\}$ of the $\{Y_j\}$ as in [RS]. Let q and r satisfy $\frac{1}{Q} < \frac{1}{r} < 1$ and $\frac{1}{r} - \frac{1}{Q} < \frac{1}{q} \leq 1$. Then for any compact subset K of Ω there exist positive constants C_K and $\delta_3(K)$ such that for every $x \in K$ and $\delta > 0$ with $B(x,\delta) \subset \Omega$ and $0 < \delta < \delta_3(K)$ and for all C^∞ functions f on the closure of $B(x,\delta)$, we have*

$$\left(\int_{B(x,\delta)} |f(x) - f_{B(x,\delta)}|^q dx\right)^{\frac{1}{q}} \leq C\delta |B(x,\delta)|^{\frac{1}{q}-\frac{1}{r}} \sum_{j=1}^{l} \left(\int_{B(x,\delta)} |(Y_j f)(x)|^r dx\right)^{\frac{1}{r}}. \tag{1.0}$$

For a proof of the above and a precise definition of Q see the article of Lu [L1] (Theorem C). Observe that $Q \geq l$. This Theorem was first proved by Jerison [J] when $q = r$.

Fix a compact subset K of Ω and a point x_0 in K. Define $\psi_\delta^{x_0}$ as in the previous section.

Let $\delta_0(K) = \min\{\delta_1(K), \delta_2(K), \delta_3(K), \frac{1}{2}\text{dist}(K, \Omega^c)\}$, where $\delta_3(K)$ are as in the Theorem above, and $\delta_1(K)$, $\delta_2(K)$ are as in (0.1) and (0.2). We will estimate the L^1 norm of the smooth maximal function $\mathcal{M}_{\delta_0(K)}(\text{Jac}_{\{Y_1,\dots,Y_l\}})(F)$ on K, where $F = (f_1, f_2, \dots, f_l)$ is a C^∞ compactly supported map : $\Omega \to \mathbb{R}^l$. Once we prove (0.6) for such F, a simple density argument will give (0.6) for all C^1 $F : \Omega \to \mathbb{R}^l$. Throughout, C_K will be a constant depending on K.

Since $\sum_{j=1}^{l} \frac{1}{p_j} = 1$, it follows that $p_j \leq l$ for some j. Relabeling indices, we may assume that $p_1 \leq l$. Since $l \leq Q$ as observed, we have that $p_1 \leq Q$. For any $0 < \delta < \delta_0(K)$, we let $c_1 = \frac{1}{|B_\delta|}\int_{B_\delta} f_1\ dx$, where $B_\delta = B(x_0,\delta)$, and we replace f_1 by $f_1 - c_1$ in (0.5). Then $\mathrm{Jac}(F) = \mathrm{Jac}_{\{Y_1,\dots,Y_l\}}(F)$ remains unchanged. By the linearity of the Jacobian (as a function of f_1), we have the identity

$$\mathrm{Jac}(F)(x)\,\psi_\delta^{x_0}(x) = J_1(x,x_0,\delta) + J_2(x,x_0,\delta), \tag{1.1}$$

where

$$J_1(\cdot,x_0,\delta) = -\ \det\begin{pmatrix} (f_1-c_1)(Y_1\psi_\delta^{x_0}) & Y_1 f_2 & \dots & Y_1 f_l \\ (f_1-c_1)(Y_2\psi_\delta^{x_0}) & Y_2 f_2 & \dots & Y_2 f_l \\ \vdots & \vdots & & \vdots \\ (f_1-c_1)(Y_l\psi_\delta^{x_0}) & Y_l f_2 & \dots & Y_l f_l \end{pmatrix}, \qquad \text{and}$$

$$J_2(\cdot,x_0,\delta) = +\ \det\begin{pmatrix} Y_1((f_1-c_1)\psi_\delta^{x_0}) & Y_1 f_2 & \dots & Y_1 f_l \\ Y_2((f_1-c_1)\psi_\delta^{x_0}) & Y_2 f_2 & \dots & Y_2 f_l \\ \vdots & \vdots & & \vdots \\ Y_l((f_1-c_1)\psi_\delta^{x_0}) & Y_l f_2 & \dots & Y_l f_l \end{pmatrix}.$$

We build on the ideas of [CLMS]. We begin by estimating

$$\sup_{0<\delta<\delta_0(K)} \left| \int_\Omega J_1(x,x_0,\delta)\,dx \right|. \tag{1.2}$$

Expand the determinant defining J_1 along its first column. We obtain

$$\begin{aligned} \left| \int_\Omega J_1(x,x_0,\delta)\ dx \right| &= \left| \int_\Omega \sum_{j=1}^{l} (-1)^{j+1}\,(f_1-c_1)(Y_j\psi_\delta^{x_0}) M_j(f_2,\dots,f_l)\ dx \right| \\ &\le \sum_{j=1}^{l} \left| \int_\Omega (f_1-c_1)(Y_j\psi_\delta^{x_0}) M_j(f_2,\dots,f_l)\ dx \right|, \end{aligned} \tag{1.3}$$

where the M_j's are the minors. Let us only estimate the first term of the sum in (1.3), since the remaining terms are similar. The minor M_1 is a sum of terms of the form $\pm\prod_{j=2}^{l} Y_{r_j} f_j$ where $\{r_2,\dots,r_l\}$ is a permutation of the set $\{2,\dots,l\}$. For every such permutation, by (0.2), we have the pointwise bound for (1.3):

$$\int_\Omega |f_1-c_1|\ |Y_1\psi_\delta^{x_0}| \prod_{j=2}^{l} |Y_{r_j} f_j|\ dx \le C\delta^{-1}|B_\delta|^{-1} \int_{B_\delta} |f_1-c_1| \prod_{j=2}^{l} |Y_{r_j} f_j|\ dx. \tag{1.4}$$

For any $1 \leq j \leq l$ select $1 < s_j < p_j$ such that $1 < \sum_{j=1}^{l} \frac{1}{s_j} < 1 + \frac{1}{Q}$ and define q by $\frac{1}{q} = 1 - \sum_{j=2}^{l} \frac{1}{s_j}$. One can check that $0 < \frac{1}{s_1} - \frac{1}{Q} < \frac{1}{q} < 1$ and that $\frac{1}{Q} < \frac{1}{s_1} < 1$. The latter follows from the choice of $p_1 \leq Q$. Therefore the hypotheses of the Poincaré inequality (1.0) are satisfied. We now apply Hölder's inequality with exponents $\frac{1}{q} + \frac{1}{s_2} + \ldots + \frac{1}{s_l} = 1$ to control (1.4) by

$$C_K \delta^{-1} \ |B_\delta|^{-1} \|f_1 - c_1\|_{L^q(B_\delta)} \prod_{j=2}^{l} \ \|Y_{r_j} f_j\|_{L^{s_j}(B_\delta)} \tag{1.5}$$

for all $x_0 \in K$. Applying the Poincaré inequality (1.0) we bound (1.5) by

$$\begin{aligned} & C_K \delta^{-1} \ |B_\delta|^{-1} \ \delta \ |B_\delta|^{\frac{1}{q} - \frac{1}{s_1}} \Big[\sum_{k=1}^{l} \|Y_k f_1\|_{L^{s_1}(B_\delta)} \Big] \prod_{j=1}^{l} \|Y_{r_j} f_j\|_{L^{s_j}(B_\delta)} \\ \leq & C_K |B_\delta|^{-\sum_{j=1}^{l} \frac{1}{s_j}} \prod_{j=1}^{l} \Big[\sum_{k=1}^{l} \|Y_k f_j\|_{L^{s_j}(B_\delta)} \Big] \\ \leq & C_K \prod_{j=1}^{l} \Big[\sum_{k=1}^{l} \big[\big(M(|Y_k f_j|^{s_j}) \big)^{\frac{1}{s_j}} (x_0) \big] \Big], \end{aligned} \tag{1.6}$$

where the $M = M_{\delta_1(K)}$ is the (local) Hardy-Littlewood maximal function with the respect to the family of the metric balls. (1.6) now controls (1.4). Summing (1.4) over all possible permutations $\{r_2, \ldots, r_l\} = \{2, \ldots, l\}$ we get the estimate below for the first term of the sum in (1.3):

$$C_K \prod_{j=1}^{l} \Big[\sum_{k=1}^{l} \big[\big(M(|Y_k f_j|^{s_j}) \big)^{\frac{1}{s_j}} (x_0) \big] \Big]. \tag{1.7}$$

Similar estimates hold for the other terms of the sum in (1.3). Therefore (1.7) majorizes (1.3), and since it is independent of δ, it also majorizes (1.2). We now estimate the L^1 norm of (1.2) in x_0 over K by the L^1 norm of (1.7) over K. We apply Hölder's inequality with exponents $\frac{1}{p_1} + \ldots + \frac{1}{p_l} = 1$ to (1.7) and since $s_j < p_j$ and the maximal function $M = M_{\delta_1(K)}$ maps $L^{\frac{p_j}{s_j}}(\Omega)$ to $L^{\frac{p_j}{s_j}}(K)$, we obtain that the L^1 norm of (1.2) on K is bounded above by

$$C_K \prod_{j=1}^{l} \big[\sum_{k=1}^{l} \|Y_k f_j\|_{L^{p_j}(\Omega)} \big]. \tag{1.8}$$

This concludes the estimate for the term J_1. To finish the proof of Theorem A, note that the zero integrability assumption for the Jacobian gives that $J_2(\cdot, x_0, \delta)$ has integral zero. By (1.1) we obtain that (1.8) majorizes the L^1 norm of $\sup_{0<\delta<\delta_0(K)} \left| \int_\Omega \mathrm{Jac}(F)(x)\psi_\delta^{x_0}\, dx \right|$ on K.

2. Estimates involving commutators and the proof of Theorem B.

In this section we take up the estimates for term J_2. We don't assume that $\mathrm{Jac}(F)$ has integral zero but we assume that the Y_j's are divergence-free. We need to control

$$\sup_{0<\delta<\delta_0(K)} \left| \int_\Omega J_2(x, x_0, \delta) dx \right|. \tag{2.1}$$

Expanding the determinant defining J_2 along its first column and using (0.7) we obtain

$$\left| \int_\Omega J_2(x, x_0, \delta)\; dx \right| = \left| - \int_\Omega (f_1 - c_1)\; \psi_\delta^{x_0} \sum_{j=1}^{l} (-1)^{j+1} Y_j [M_j(f_2, \dots, f_l)]\; dx \right|, \tag{2.2}$$

where M_j is a suitable minor. It can be checked that $\sum_{j=1}^{l}(-1)^{j+1}Y_j[M_j(f_2,\dots,f_l)] = \Sigma_l(f_2,\dots,f_l)$ is a sum of terms of the form

$$\Pi_r(f_2, \dots, f_l) = \pm \Bigg(\prod_{\substack{2 \le j \le l \\ j \ne m_1, m_2}} Y_j f_{r(j)} \Bigg) [Y_{m_1}, Y_{m_2}] f_{r(j)}, \tag{2.3}$$

where $r(\cdot)$ is a permutation of the set $\{2, \dots, l\}$. In other words, exactly one commutator of order 2 appears in the product and the alternation of the signs in the expansion of the determinant produces all possible commutators of order 2. For instance, when $l = 3$, $\Sigma_3(f_2, f_3) = \{(Y_1 f_3)([Y_2, Y_3] f_2) - (Y_1 f_2)([Y_2, Y_3] f_3)\} + \{(Y_2 f_3)([Y_3, Y_1] f_2) - (Y_2 f_2)([Y_3, Y_1] f_3)\} + \{(Y_3 f_3)([Y_1, Y_2] f_2) - (Y_3 f_2)([Y_1, Y_2] f_2)\}$.

Let us now estimate the integral over Ω of a typical term $(f_1 - c_1)\psi_\delta^{x_0}\Pi_r(f_2,\dots,f_l)$ by

$$\begin{aligned} &\int_\Omega |f_1 - c_1|\; |\psi_\delta^{x_0}|\; |[Y_{m_1}, Y_{m_2}] f_{r(j)}| \prod_{\substack{1 \le j \le l \\ j \ne m_1, m_2}} |Y_j f_{r(j)}|\; dx \\ \le & C_K |B_\delta|^{-1} \int_{B_\delta} |f_1 - c_1|\; |[Y_{m_1}, Y_{m_2}] f_{r(j)}| \prod_{\substack{1 \le j \le l \\ j \ne m_1, m_2}} |Y_j f_{r(j)}|\; dx. \end{aligned} \tag{2.4}$$

For any $1 \le j \le l$ select $1 < s_j < p_j$ and q as before. We apply Hölder's inequality with exponents $\frac{1}{q} + \frac{1}{s_2} + \ldots + \frac{1}{s_l} = 1$ to control (2.4) by

$$(2.5) \qquad C_K |B_\delta|^{-1} \, \|f_1 - c_1\|_{L^q(B_\delta)} \prod_{\substack{1 \le j \le l \\ j \ne m_1, m_2}} \|Y_j f_{r(j)}\|_{L^{s_{r(j)}}(B_\delta)} \|[Y_{m_1}, Y_{m_2}] f_{r(j)}\|_{L^{s_{r(j)}}(B_\delta)}.$$

Let us denote by $\{Z_k\}$ all commutators of order 2 of the Y_k's. Poincaré's inequality (1.0) allows us to deduce that (2.5) is bounded by

$$\begin{aligned}
& C_K \delta |B_\delta|^{-1+\frac{1}{q}-\frac{1}{s_1}} \left[\sum_{k=1}^{l} \|Y_k f_1\|_{L^{s_1}(B_\delta)} \right] \\
& \qquad \left[\prod_{\substack{1 \le j \le l \\ j \ne m_1, m_2}} \|Y_j f_{r(j)}\|_{L^{s_{r(j)}}(B_\delta)} \right] \|[Y_{m_1}, Y_{m_2}] f_{r(j)}\|_{L^{s_{r(j)}}(B_\delta)} \\
& \le C_K \delta |B_\delta|^{-\sum_{j=1}^{l} \frac{1}{s_j}} \prod_{j=1}^{l} \left[\sum_{k=1}^{l} \left(\|Y_k f_j\|_{L^{s_j}(B_\delta)} \right) + \sum_{k=1}^{\frac{l(l-1)}{2}} \left(\|Z_k f_j\|_{L^{s_j}(B_\delta)} \right) \right] \\
(2.6) \qquad & \\
& \le C_K \delta_0(K) \prod_{j=1}^{l} \left[\sum_{k=1}^{l} \left[\left(M(|Y_k f_j|^{s_j}) \right)^{\frac{1}{s_j}} (x_0) \right] + \sum_{k=1}^{\frac{l(l-1)}{2}} \left[\left(M(|Z_k f_j|^{s_j}) \right)^{\frac{1}{s_j}} (x_0) \right] \right],
\end{aligned}$$

where we used $\delta < \delta_0(K)$ since we are only interested in a local estimate. Summing over all possible functions $r(\cdot)$, we obtain the same estimate (with a larger constant) for (2.2). Since (2.6) controls (2.2), and is independent of $\delta < \delta_0(K)$, it controls (2.1). We have therefore proved that

$$\begin{aligned}
& \sup_{0<\delta<\delta_0(K)} \left| \int_\Omega J_2(x, x_0, \delta) \, dx \right| \\
& \le C_K \prod_{j=1}^{l} \left[\sum_{k=1}^{l} \left[\left(M(|Y_k f_j|^{s_j}) \right)^{\frac{1}{s_j}} (x_0) \right] + \sum_{k=1}^{\frac{l(l-1)}{2}} \left[\left(M(|Z_k f_j|^{s_j}) \right)^{\frac{1}{s_j}} (x_0) \right] \right].
\end{aligned}$$

As before, Hölder's inequality with exponents $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_l} = 1$ gives a bound analogous to (1.8). This completes the estimate for J_2. This estimate together with the estimate for J_1 given in the previous section give (0.8).

3. Remarks, Examples, and Consequences.

A few remarks are in order. First note that in view of the doubling property (0.1), it would be sufficient for our purposes to use the weaker form of the Poincaré inequality (1.0) where the ball $B(x,\delta)$ on the right hand side of the inequality is replaced by its double $2B(x,\delta)$. The final estimates (1.7) and (2.6) will be the same (only the constant C_K will be slightly larger).

We have proved our results for C^1 functions F on Ω. By density the results are true for all F such that the right hand side of the inequalities (0.6) and (0.8) are finite.

We now explain how are the hypotheses of Theorems A and B related.

Proposition. *If all the vector fields Y_j are divergence-free and all the commutators of the Y_j's vanish identically on Ω, then $Jac_{\{Y_1,\dots,Y_l\}}(F)$ has integral zero for all F compactly supported smooth functions on Ω.*

Indeed, expand the Jacobian (0.5) along the column $(Y_1f_1, Y_2f_1, \dots, Y_lf_1)$, integrate over Ω, and use (0.7). The result is the integral over Ω of a sum of terms of the form $\pm f_1\Pi_r(f_2,\dots,f_l)$, where $\Pi_r(f_2,\dots,f_l)$ is as in (2.3). By assumption each of these terms is zero. Thus the Jacobian has integral zero.

The converse of this proposition need not be true. There exist divergence-free smooth vector fields with nonvanishing commutators such that the associated Jacobian has integral zero. For instance, let x_1, x_2, x_3 be coordinates in $\Omega = \mathbb{R}^3 - \{\text{the coordinate axes}\}$. Define

$$Y_1 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}, \qquad Y_2 = x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}, \qquad Y_3 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}. \tag{3.1}$$

Note that the Y_j's are divergence-free, they satisfy $[Y_1,Y_2] = Y_3$, $[Y_2,Y_3] = Y_1$, $[Y_3,Y_1] = Y_2$, and the associated Jacobian $\mathrm{Jac}_{\{Y_1,Y_2,Y_3\}}(F)$ is identically equal to zero for all F.

Below are examples of sets of vector fields that can be treated by our Theorems:

1. Let Y_1, Y_2, Y_3 be the vector fields in (3.1), and $\Omega = \mathbb{R}^3 - \{\text{the coordinate axes}\}$. If $\frac{1}{p} + \frac{1}{q} = 1$ for some $1 < p,q < \infty$ and $\|Y_1f\|_{L^p(\Omega)} + \|Y_2f\|_{L^p(\Omega)} + \|Y_3f\|_{L^p(\Omega)} < \infty$ and $\|Y_1g\|_{L^q(\Omega)} + \|Y_2g\|_{L^q(\Omega)} + \|Y_3g\|_{L^q(\Omega)} < \infty$, it follows from Theorem B that all 2×2 determinants $\mathrm{Jac}_{\{Y_j,Y_k\}}(f,g)$ lie in the Hardy space $H^1_{\mathrm{loc}}(\Omega)$. Theorem A doesn't apply in this case since the 2×2 determinants $\mathrm{Jac}_{\{Y_j,Y_k\}}(f,g)$ may not have integral zero.

2. On $\mathbb{R}^2 - \{\text{the coordinate axes}\}$, let

$$Y_1 = 2x_1\frac{\partial}{\partial x_1} + \frac{1}{x_2}\frac{\partial}{\partial x_2}, \qquad Y_2 = x_2\frac{\partial}{\partial x_1} + \frac{1}{x_1}\frac{\partial}{\partial x_2}. \tag{3.2}$$

These vector fields are not divergence-free but the associated Jacobian $\mathrm{Jac}_{\{Y_1,Y_2\}}(F)$ has integral zero for all smooth compactly supported F. This example only falls under the scope of Theorem A.

3. On $\mathbb{R}^3$, let Y_1 and Y_2 be the Heisenberg group vector fields

$$Y_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}, \qquad Y_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}. \tag{3.3}$$

The 2×2 Jacobian $\mathrm{Jac}_{\{Y_1,Y_2\}}(F)$ doesn't satisfy the integral zero condition and only falls under the scope of Theorem B. In this case $[Y_1, Y_2] = -4\frac{\partial}{\partial x_3}$.

In the all examples above, the vector fields satisfy Hörmander's condition.

We now sketch the proof of the Corollary stated in the introduction, which extends the classical result of [M].

To prove te corollary, note that if $g \geq 0$, the (local) Hardy-Littlewood maximal function $Mg = M_{\delta_0(K)}g$ and $\mathcal{M}_{\delta_0(K)}g$ are comparable on any compact $K \subset \Omega$. (For this to be true one needs to select $\psi_\delta^{x_0} \geq 0$.) Then $M(\mathrm{Jac}(F))$ is in $H^1_{\mathrm{loc}}(\Omega)$ and Stein's Theorem [S1] extended to spaces of homogeneous type, gives the required result. The details are omitted.

We end by observing that, like the usual Jacobian on $\mathbb{R}^n$, the Jacobians considered above map into the local H^p spaces for some p below 1. More precisely, let $(0.6)'$ and $(0.8)'$ be (0.6) and (0.8) respectively where $H^1(K)$ is replaced by $H^p(K)$. Let $1 < p_j < \infty$ be such that $\frac{1}{p_1} + \cdots + \frac{1}{p_l} = \frac{1}{p}$, where $1 \geq p > \frac{Q}{Q+1}$. Then under the hypothesis of Theorem A $(0.6)'$ holds and under the hypothesis of Theorem B $(0.8)'$ holds.

The proof is the same. Note that since $\frac{1}{p_1} + \cdots + \frac{1}{p_l} < \frac{Q+1}{Q}$, we can still select $1 < s_j < p_j$ such that $\frac{1}{s_1} + \cdots + \frac{1}{s_l} < \frac{Q+1}{Q}$.

Finally observe that when when the vector fields Y_j are free and $p = \frac{Q}{Q+1}$, then $(0.6)'$ and $(0.8)'$ hold if $H^p(K)$ is replaced by the *weak*-$H^p(K)$, that is, the space of all functions on Ω whose smooth maximal function $\mathcal{M}_{\delta(K)}$ lies in *weak*-$L^p(K)$. Here one uses the Poincaré inequality for free vector fields that satisfy Hörmander's condition in the endpoint case $\frac{1}{q} = \frac{1}{r} - \frac{1}{Q}$ due to Lu [L2]. Then one adapts the argument in [G] pages 77-78. This seems to be a new observation even for the usual Jacobian $J(F) = \det(\frac{\partial f_j}{\partial x_k})$ on $\mathbb{R}^n$. It is sharp in the sense that J doesn't map into $H^p(\mathbb{R}^n)$ for $p < \frac{n}{n+1}$.

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