UNIFORM BOUNDS FOR THE BILINEAR HILBERT TRANSFORMS, I

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ABSTRACT. It is shown that the bilinear Hilbert transforms

$$H_{\alpha,\beta}(f,g)(x) = \text{p.v.} \int_{\mathbf{R}} f(x-\alpha t) g(x-\beta t) \, \frac{dt}{t}$$

map $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R}) \to L^p(\mathbf{R})$ uniformly in the real parameters α, β when $2 < p_1, p_2 < \infty$ and $1 < p = \frac{p_1 p_2}{p_1+p_2} < 2$. Combining this result with the main result in [9], it follows that the operators $H_{1,\alpha}$ map $L^2(\mathbf{R}) \times L^\infty(\mathbf{R}) \to L^2(\mathbf{R})$ uniformly in the real parameter $\alpha \in [0,1]$, as conjectured by A. Calderón.

1. INTRODUCTION

The study of the Cauchy integral along Lipschitz curves during the period 1965–1995 has provided a formidable impetus and a powerful driving force for significant developments in euclidean harmonic analysis during that period and later. The Cauchy integral along a Lipschitz curve Γ is given by

$$C_\Gamma(h)(z) = \frac{1}{2\pi i} \, \text{p.v.} \int_\Gamma \frac{h(\zeta)}{\zeta - z} \, d\zeta \,,$$

where h is a function on Γ, which is taken to be the graph of a Lipschitz function $A: \mathbf{R} \to \mathbf{R}$. Calderón [2] wrote $C_\Gamma(h)(z)$ as the infinite sum

$$\frac{1}{2\pi i} \sum_{m=0}^{\infty} (-i)^m \mathcal{C}_m(f;A)(x) \,,$$

where $z = x + iA(x)$, $f(y) = h(y + iA(y))(1 + iA'(y))$, and

$$\mathcal{C}_m(f;A)(x) = \text{p.v.} \int_{\mathbf{R}} \left(\frac{A(x)-A(y)}{x-y} \right)^m \frac{f(y)}{x-y} \, dy \,,$$

reducing the boundedness of $C_\Gamma(h)$ to that of the operators $\mathcal{C}_m(f;A)$ with constants with suitable growth in m. The operators $\mathcal{C}_m(f;A)$ are called the commutators of f with A and they are archetypes of nonconvolution singular integrals whose action on the function 1 has inspired the fundamental work on the $T1$ theorem [5] and its subsequent ramifications. The family of bilinear Hilbert transforms

$$H_{\alpha_1,\alpha_2}(f_1,f_2)(x) = \text{p.v.} \int_{\mathbf{R}} f_1(x-\alpha_1 t) f_2(x-\alpha_2 t) \, \frac{dt}{t}, \quad \alpha_1, \alpha_2, x \in \mathbf{R},$$

was also introduced by Calderón in one of his attempts to show that the commutator $\mathcal{C}_1(f;A)$ is bounded on $L^2(\mathbf{R})$ when $A(t)$ is a function on the line with derivative A' in L^∞.

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In fact, in the mid 1960's Calderón observed that the linear operator $f \to \mathcal{C}_1(f;A)$ can be written as the average

$$\mathcal{C}_1(f;A)(x) = \int_0^1 H_{1,\alpha}(f,A')(x)\,d\alpha\,,$$

and the boundedness of $C_1(f;A)$ can be therefore reduced to the uniform (in α) boundedness of $H_{1,\alpha}$. Although the boundedness of $C_1(f;A)$ was settled in [1] via a different approach, the issue of the uniform boundedness of the operators $H_{1,\alpha}$ from $L^2(\mathbf{R})\times L^\infty(\mathbf{R})$ into $L^2(\mathbf{R})$ remained open up to now. The purpose of this article and its subsequent, part II, is to obtain exactly this, i.e. the uniform boundedness (in α) of the operators $H_{1,\alpha}$ for a range of exponents that completes in particular the above program initiated by A. Calderón about 40 years ago. This is achieved in two steps. In this article we obtain bounds for $H_{1,\alpha}$ from $L^{p_1}(\mathbf{R})\times L^{p_2}(\mathbf{R})$ into $L^p(\mathbf{R})$ uniformly in the real parameter α when $2 < p_1, p_2 < \infty$ and $1 < p = \frac{p_1 p_2}{p_1+p_2} < 2$. In part II of this work, the second author obtains bounds for $H_{1,\alpha}$ from $L^{p_1}(\mathbf{R})\times L^{p_2}(\mathbf{R})$ into $L^p(\mathbf{R})$, uniformly in α satisfying $|\alpha-1|\ge c>0$ when $1<p_1,p_2<2$ and $\frac{2}{3}<p=\frac{p_1p_2}{p_1+p_2}<1$. Interpolating between these two results yields the uniform boundedness of $H_{1,\alpha}$ from $L^p(\mathbf{R})\times L^\infty(\mathbf{R})$ into $L^p(\mathbf{R})$ for $\frac{4}{3}<p<4$ when α lies in a compact subset of $\mathbf{R}$. This in particular implies the boundedness of the commutator $\mathcal{C}_1(\,\cdot\,;A)$ on $L^p(\mathbf{R})$ for $\frac{4}{3}<p<4$ via the Calderón method described above but also has other applications. See [9] for details. We note that the restriction to compact subsets of $\mathbf{R}$ is necessary, as uniform $L^p\times L^\infty\to L^p$ bounds for $H_{1,\alpha}$ cannot hold as $\alpha\to\pm\infty$.

Boundedness for the operators $H_{1,\alpha}$ was first obtained by M. Lacey and C. Thiele in [7] and [8]. Their proof, though extraordinary and pioneering, gives bounds that depend on the parameter α, in particular that blow up polynomially as α tends to 0, 1 and $\pm\infty$. The approach taken in this work is based on powerful ideas of C. Thiele [10], [11] who obtained that the $H_{1,\alpha}$'s map $L^2(\mathbf{R})\times L^2(\mathbf{R})\to L^{1,\infty}(\mathbf{R})$ uniformly in α satisfying $|\alpha-1|\ge\delta>0$.

The theorem below is the main result of this article.

Theorem. *Let $2 < p_1, p_2 < \infty$ and $1 < p = \frac{p_1p_2}{p_1+p_2} < 2$. Then there is a constant $C = C(p_1,p_2)$ such that for all f_1, f_2 Schwartz functions on $\mathbf{R}$ we have*

$$\sup_{\alpha_1,\alpha_2\in\mathbf{R}} \|H_{\alpha_1,\alpha_2}(f_1,f_2)\|_p \le C\,\|f_1\|_{p_1}\|f_2\|_{p_2}.$$

By dilations we may take $\alpha_1 = 1$. It is easy to see that the boundedness of the operator $H_{1,-\alpha}$ on any product of Lebesgue spaces is equivalent to that of the operator

$$(f_1,f_2)\to\int_{\mathbf{R}}\int_{\mathbf{R}}\widehat{f}_1(\xi)\widehat{f}_2(\eta)e^{2\pi i(\xi+\eta)x}1_{\{\eta<\alpha^{-1}\xi\}}(\xi,\eta)\,d\xi\,d\eta\,,$$

where 1_A denotes the characteristic (indicator) function of the set A. Moreover in the range $2<p_1,p_2<\infty$ and $1<p=\frac{p_1p_2}{p_1+p_2}<2$, in view of duality considerations, it suffices to obtain uniform bounds near only one of the three 'bad' directions $\alpha=-1,0,\infty$ of $H_{1,-\alpha}$. In this article we choose to work with the 'bad' direction 0. This direction corresponds to bilinear multipliers whose symbols are characteristic functions of planes of the form $\eta<\frac{1}{\alpha}\xi$. For simplicity we will only consider the case where $\frac{1}{\alpha}=2^m$, $m\in\mathbf{Z}^+$. The arguments here can be suitably adjusted to cover the more general situation where $2^m\le\frac{1}{\alpha}<2^{m+1}$ as well.

For a positive integer m, we consider the following pseudodifferential operator

$$T_m(f_1,f_2)(x)=\int_{\mathbf{R}}\int_{\mathbf{R}}\widehat{f}_1(\xi)\widehat{f}_2(\eta)e^{2\pi i(\xi+\eta)x}1_{\{\eta<2^m\xi\}}(\xi,\eta)\,d\xi\,d\eta, \tag{1.1}$$

and we prove that it satisfies

$$\|T_m(f_1,f_2)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2} \tag{1.2}$$

uniformly in $m \ge 2^{200}$ where p_1, p_2, p are as in the statement of the theorem.

The rest of the paper is devoted to the proof of (1.2). In the following sections, $L = 2^{100}$ will be a fixed large integer. We will use the notation $|S|$ for the Lebesgue measure of set S and S^c for its complement. By $c(J)$ we denote the center of an interval J and by AJ the interval with length $A|J|$ $(A > 0)$ and center $c(J)$. For J, J' sets we will use the notation

$$J < J' \iff \sup_{x\in J} x \le \inf_{x\in J'} x.$$

The Hardy-Littlewood maximal operator of g is denoted by Mg and $M_p g$ will be $(M|g|^p)^{1/p}$. The derivative of order α of a function f will be denoted by $D^\alpha f$. When L^p norms or limits of integration are not specified, they are to be taken as the whole real line. Also C will be used for any constant that depends only on the exponents p_1, p_2 and is independent of any other parameter, in particular of the parameter m. Finally N will denote a large (but fixed) integer whose value may be chosen appropriately at different times.

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2. The decomposition of the bilinear operator T_m

We begin with a decomposition of the half plane $\eta < 2^m\xi$ on the ξ-η plane. We can write the characteristic function of the half plane $\eta < 2^m\xi$ as a union of rectangles of size $2^{-k} \times 2^{-k+m}$ as in Figure 1. Precisely, for $k, l \in \mathbf{Z}$ we set

$$\begin{aligned}
J_1^{(1)}(k,l) &= [2^{-k}(2l), 2^{-k}(2l+1)] & J_2^{(1)}(k,l) &= [2^{-k+m}(2l-2), 2^{-k+m}(2l-1)]\\
J_1^{(2)}(k,l) &= [2^{-k}(2l+1), 2^{-k}(2l+2)] & J_2^{(2)}(k,l) &= [2^{-k+m}(2l-2), 2^{-k+m}(2l-1)]\\
J_1^{(3)}(k,l) &= [2^{-k}(2l+1), 2^{-k}(2l+2)] & J_2^{(3)}(k,l) &= [2^{-k+m}(2l-1), 2^{-k+m}(2l)].
\end{aligned}$$

We call the rectangles $J_1^{(r)}(k,l) \times J_2^{(r)}(k,l)$ of type r, $r \in \{1,2,3\}$. It is easy to see that

$$1_{\eta<2^m\xi} = \sum_{k\in\mathbf{Z}}\sum_{l\in\mathbf{Z}} \big(1_{J_1^{(1)}(k,l)}(\xi)1_{J_2^{(1)}(k,l)}(\eta) + 1_{J_1^{(2)}(k,l)}(\xi)1_{J_2^{(2)}(k,l)}(\eta) + 1_{J_1^{(3)}(k,l)}(\xi)1_{J_2^{(3)}(k,l)}(\eta)\big),$$

which provides a (nonsmooth) partition of unity of the the half-plane $\eta < 2^m\xi$. Next we pick a smooth partition of unity $\{\Psi_{k,l}^{(r)}(\xi,\eta)\}_{k,l,r}$ of the half-plane $\eta < 2^m\xi$ with each $\Psi_{k,l}^{(r)}$ supported only in a small enlargement of the rectangle $J_1^{(r)}(k,l) \times J_2^{(r)}(k,l)$ and satisfying standard derivative estimates. Since the functions $\Psi_{k,l}^{(r)}(\xi,\eta)$ are not of tensor type, (i.e. products of functions of ξ and functions of η) we apply the Fourier series method of Coifman and Meyer [4] (pp 55–57) to write

$$\Psi_{k,l}^{(r)}(\xi,\eta) = \sum_{n\in\mathbf{Z}^2} C(n)(\Phi_{1,k,l,n}^{(r)})\widehat{\ }(\xi)(\Phi_{2,k,l,n}^{(r)})\widehat{\ }(\eta)$$

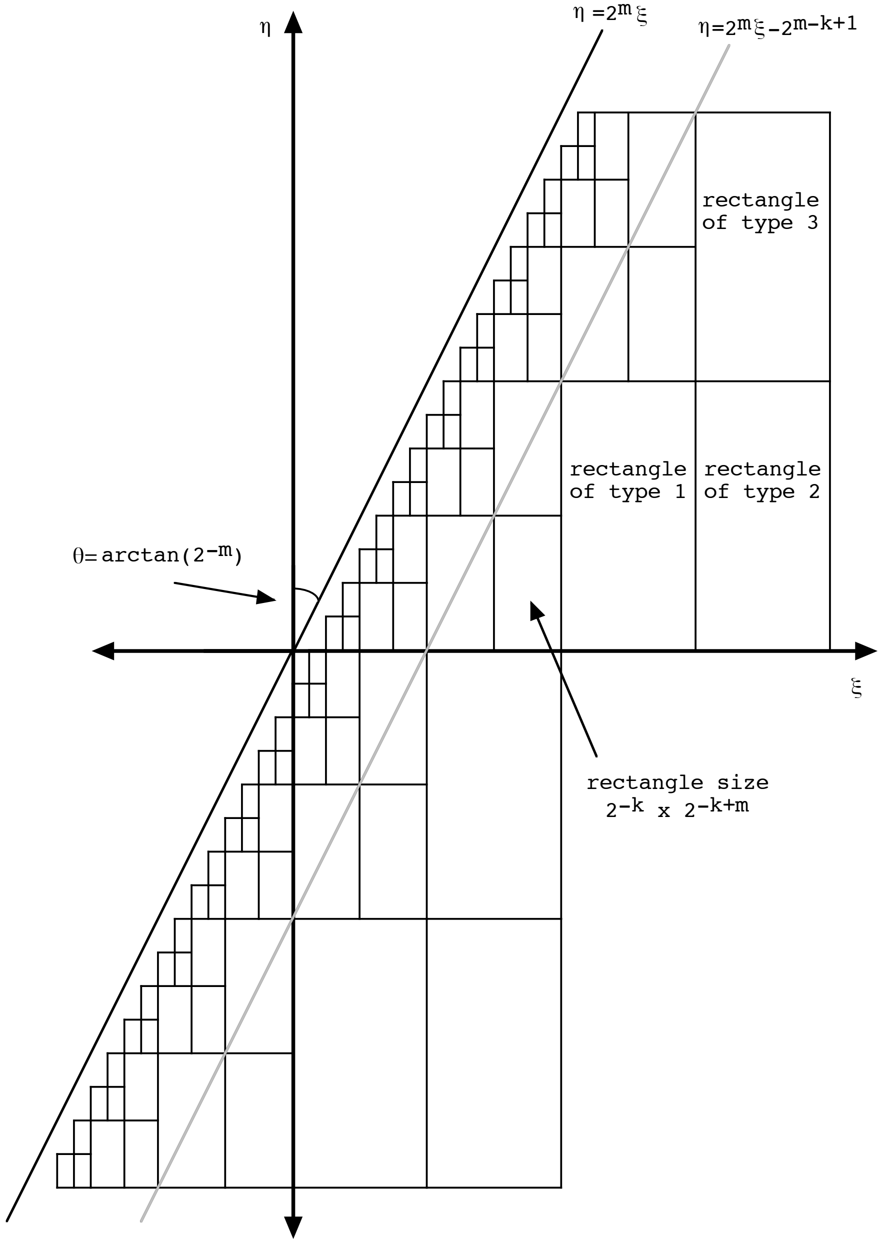

FIGURE 1. The decomposition of the plane $\eta < 2^m\xi$.

where $|C(n)| \le C_M(1+|n|^2)^{-M}$ for all $M > 0$ ($n = (n_1, n_2)$, $|n|^2 = n_1^2 + n_2^2$) and the functions $\Phi^{(r)}_{1,k,l,n}$ and $\Phi^{(r)}_{2,k,l,n}$ are Schwartz and satisfy:

$$\begin{aligned} &|D^\alpha((\Phi^{(r)}_{1,k,l,n})\widehat{\ })| \le C_\alpha(1+|n|)^\alpha\, 2^{\alpha k}, \ \operatorname{supp}(\Phi^{(r)}_{1,k,l,n})\widehat{\ } \subset (1+2^{-2L})J_1^{(r)}(k,l), \\ &(\Phi^{(r)}_{1,k,l,n})\widehat{\ }(\xi) = e^{2\pi i n_1 2^k(\xi - c(J_1^{(r)}(k,l)))} \ \text{ on } \ (1-2^{-2L})J_1^{(r)}(k,l), \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&|D^\alpha((\Phi^{(r)}_{2,k,l,n})\widehat{\ })| \le C_\alpha (1+|n|)^\alpha\, 2^{\alpha(k-m)},\ \text{supp}\,(\Phi^{(r)}_{2,k,l,n})\widehat{\ } \subset (1+2^{-2L})J_2^{(r)}(k,l),\\
&(\Phi^{(r)}_{2,k,l,n})\widehat{\ }(\eta) = e^{2\pi i n_2 2^{k-m}(\eta - c(J_2^{(r)}(k,l)))} \ \text{ on } \ (1-2^{-2L})J_2^{(r)}(k,l),
\end{aligned}
\tag{2.2}
$$

for all nonnegative integers α and all $r \in \{1,2,3\}$. In the sequel, for notational convenience, we will drop the dependence of these functions on r and we will concentrate on the case $n = (n_1, n_2) = (0,0)$. In the cases $n \neq 0$, the polynomial appearance of $|n|$ in the estimates will be controlled by the rapid decay of $C(n)$, while the exponential functions in (2.1) and (2.2) can be thought as almost "constant" locally (such as when $n_1 = n_2 = 0$), and thus a small adjustment of the case $n = (0,0)$ will yield the case for general n in $\mathbf{Z}^2$.

Based on these remarks, we may set $\Phi_{j,k,l} = \Phi_{j,k,l,0}$ and it will be sufficient to prove the uniform (in m) boundedness of the operator T_m^0 defined by

$$
T_m^0(f_1,f_2)(x) = \sum_{k\in\mathbf{Z}}\sum_{l\in\mathbf{Z}} \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta) e^{2\pi i(\xi+\eta)x}\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,l}}(\eta)\,d\xi d\eta\,. \tag{2.3}
$$

The representation of T_m^0 into a sum of products of functions of ξ and η will be crucial in its study. If follows from (2.1) and (2.2) that we have the following size estimates for the functions $\Phi_{1,k,l}$ and $\Phi_{2,k,l}$.

$$
|\Phi_{1,k,l}(x)| \le C_N 2^{-k}(1+2^{-k}|x|)^{-N}, \tag{2.4}
$$

$$
|\Phi_{2,k,l}(x)| \le C_N 2^{-k+m}(1+2^{-k+m}|x|)^{-N} \tag{2.5}
$$

for any $N \in \mathbf{Z}^+$. The following lemma is also a consequence of (2.1) and (2.2).

Lemma 1. *For all $N \in \mathbf{Z}^+$, there exists $C_N > 0$ such that for all $f \in \mathcal{S}(\mathbf{R})$, we have*

$$
\sum_{l\in\mathbf{Z}} |(f*\Phi_{1,k,l})(x)|^2 \le C_N \int |f(y)|^2 \frac{2^{-k}}{(1+2^{-k}|x-y|)^N}\,dy, \tag{2.6}
$$

$$
\sum_{l\in\mathbf{Z}} |(f*\Phi_{2,k,l})(x)|^2 \le C_N \int |f(y)|^2 \frac{2^{-k+m}}{(1+2^{-k+m}|x-y|)^N}\,dy, \tag{2.7}
$$

where C_N is independent of m.

Proof. To prove the lemma we first observe that whenever $\Phi_l \in \mathcal{S}$ has Fourier transform supported in the interval $[2l-3, 2l+3]$ and satisfies $\sup_l \|D^\alpha \widehat{\Phi_l}\|_\infty \le C_\alpha$ for all sufficiently large integers α, then we have

$$
\sum_{l\in\mathbf{Z}} |(f*\Phi_l)(x)|^2 \le C_N \int_{\mathbf{R}} \frac{|f(y)|^2}{(1+|x-y|)^N}\,dy. \tag{2.8}
$$

Once (2.8) is established, we apply it to the function $\Phi_l(x) = 2^k\Phi_{1,k,l}(2^k x)$, which by (2.1) satisfies $|D^\alpha\widehat{\Phi_l}(\xi)| \le C_\alpha$, to obtain (2.6). Similarly, applying (2.8) to the function $\Phi_l(x) = 2^{k-m}\Phi_{2,k,l}(2^{k-m}x)$, which by (2.2) also satisfies $|D^\alpha\widehat{\Phi_l}(\xi)| \le C_\alpha$, we obtain (2.7).

By a simple translation, it will suffice to prove (2.8) when $x = 0$. Then we have

$$
\begin{aligned}
\sum_{l\in\mathbf{Z}} |(f*\Phi_l)(0)|^2 &= \sum_{l\in\mathbf{Z}} \left| \int_{[2l-3,2l+3]} \left(\frac{f(-\,\cdot\,)}{(1+4\pi^2|\cdot|^2)^N} \right)^{\wedge}\!(y) \overline{\left((I-\Delta)^N \widehat{\Phi_l}\right)}(y)\,dy \right|^2 \\
&\le \sum_{l\in\mathbf{Z}} \int_{[2l-3,2l+3]} \left| \left(\frac{f(-\,\cdot\,)}{(1+4\pi^2|\cdot|^2)^N} \right)^{\wedge}\!(y) \right|^2 dy \int |(I-\Delta)^N \widehat{\Phi_l}(y)|^2 dy
\end{aligned}
$$

$$\leq \quad C_N \int_{\mathbf{R}} \frac{|f(-y)|^2}{(1+4\pi^2|y|^2)^N}\, dy \leq C_N \int_{\mathbf{R}} \frac{|f(y)|^2}{(1+|y|)^N}\, dy.$$

$\square$

3. The truncated trilinear form

Let ψ be a nonnegative Schwartz function such that $\widehat{\psi}$ is supported in $[-1,1]$ and satisfies $\widehat{\psi}(0)=1$. Let $\psi_k(x)=2^{-k}\psi(2^{-k}x)$. For $E\subset\mathbf{R}$ and $k\in\mathbf{Z}$ define

$$E_k = \{x\in E : \operatorname{dist}(x,E^c)\geq 2^k\}, \tag{3.1}$$

$$\psi_{1,k}(x) = (1_{(E_k)^c} * \psi_k)(x), \qquad \text{and} \qquad \psi_{2,k}(x)=\psi_{3,k}(x)=\psi_{1,k-m}(x). \tag{3.2}$$

Note that $\psi_{1,k}$, $\psi_{2,k}$, and $\psi_{3,k}$ depend on the set E but we will suppress this dependence for notational convenience, since we will be working with a fixed set E. Also note that the functions $\psi_{2,k}$ and $\psi_{3,k}$ depend on m, but this dependence will also be suppressed in our notation. The crucial thing is that all of our estimates will be independent of m. Define

$$\Lambda_E(f_1,f_2,f_3) = \sum_{k\in\mathbf{Z}}\sum_{l\in\mathbf{Z}} \int \prod_{j=1}^{3} \psi_{j,k}(x)(f_j * \Phi_{j,k,l})(x)dx \tag{3.3}$$

where for any $\alpha\geq 0$, $\Phi_{3,k,l}$ depends on $\Phi_{1,k,l}$ and $\Phi_{2,k,l}$ and is chosen so that it satisfies

$$\begin{aligned} &|D^\alpha \widehat{\Phi_{3,k,l}}| \leq C2^{\alpha(k-m)}, \ \ \operatorname{supp}\widehat{\Phi_{3,k,l}} \subset (1+2^{-2L})J_3^{(r)}(k,l), \ \text{ and} \\ &\widehat{\Phi_{3,k,l}} = 1 \ \text{ on } \ J_3^{(r)}(k,l) = -(1+2^{-2L})J_1^{(r)}(k,l) - (1+2^{-2L})J_2^{(r)}(k,l), \end{aligned} \tag{3.4}$$

for all nonnegative integers α. (The number r in (3.4) is the type of the rectangle in which the Fourier transforms of $\Phi_{1,k,l}$ and $\Phi_{2,k,l}$ are supported.) One easily obtains the size estimate

$$|\Phi_{3,k,l}(x)| \leq C2^{-k+m}(1+2^{-k+m}|x|)^{-N}\,. \tag{3.5}$$

Because of the assumption on the indices p_1,p_2, there exists a $2<p_3<\infty$ such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$. Fix such a p_3 throughout the rest of the paper. The following two lemmata reduce matters to the truncated trilinear form (3.3).

Lemma 2. *Let $2<p_1,p_2,p_3<\infty$, $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$, and $\|f_j\|_{p_j}=1$ for $f_j\in\mathcal{S}$ and $j\in\{1,2,3\}$. Define*

$$E=\bigcup_{j=1}^{3}\{x\in\mathbf{R} : M_{p_j}(Mf_j)(x)>2\}.$$

Then for some constant C independent of m and f_1,f_2,f_3 we have

$$|\Lambda_E(f_1,f_2,f_3)|\leq C\,.$$

Lemma 2 will be proved in the next sections. Next we have

Lemma 3. *Lemma 2 implies (1.2).*

Proof. To prove (1.2), it will be sufficient to prove that for all $\lambda>0$ we have

$$|\{x : |T_m^0(f_1,f_2)(x)|>\lambda\}| \leq C\lambda^{-\frac{p_1p_2}{p_1+p_2}}$$

whenever $\|f_1\|_{p_1}=\|f_2\|_{p_2}=1$. By linearity and scaling invariance, it suffices to show that

$$|\{x : |T_m^0(f_1,f_2)(x)|>2\}| \leq C. \tag{3.6}$$

Let $E=\bigcup_{j=1}^{2}\{x\in\mathbf{R}: M_{p_j}(Mf_j)(x)>1\}$. Since $|E|\le C$, it will be enough to show that

$$|\{x\in E^c: |T_m^0(f_1,f_2)(x)|>2\}|\le C. \tag{3.7}$$

Let $G=E^c\bigcap\{|T_m^0(f_1,f_2)|>2\}$, and assuming $|G|\ge 1$ (otherwise there is nothing to prove) choose $f_3\in\mathcal{S}$ with $\|f_3\|_{L^\infty(E^c)}\le 1$, $\operatorname{supp} f_3\subset E^c$, and

$$\left\|f_3-\frac{1_G}{|G|^{1/p_3}}\frac{T_m^0(f_1,f_2)}{|T_m^0(f_1,f_2)|}\right\|_{p_3}\le \min\{1,\|T_m^0(f_1,f_2)\|_{p_3'}^{-1}\}.$$

Note that for the f_3 chosen we have $\|f_3\|_{p_3}\le 2$ and thus the set $\{x\in\mathbf{R}: M_{p_3}(Mf_3)(x)>2\}$ is empty. Now define

$$\Lambda(f_1,f_2,f_3)=\sum_{k\in\mathbf{Z}}\sum_{l\in\mathbf{Z}}\int\prod_{j=1}^{3}(f_j*\Phi_{j,k,l})(x)dx. \tag{3.8}$$

Then by Lemma 2 it follows that

$$|G|^{1/p_3'}\le\left\langle T_m^0(f_1,f_2),\frac{1_G}{|G|^{1/p_3}}\frac{T_m^0(f_1,f_2)}{|T_m^0(f_1,f_2)|}\right\rangle\le|\Lambda(f_1,f_2,f_3)-\Lambda_E(f_1,f_2,f_3)|+C\,.$$

Therefore, to prove (3.7), we only need to show that

$$|\Lambda(f_1,f_2,f_3)-\Lambda_E(f_1,f_2,f_3)|\le C \tag{3.9}$$

whenever $\|f_3\|_{L^\infty(E^c)}\le 1$ and $\operatorname{supp} f_3\subset E^c$. To prove (3.9) note that

$$|\Lambda(f_1,f_2,f_3)-\Lambda_E(f_1,f_2,f_3)|\le\left|\sum_{k\in\mathbf{Z}}\sum_{l\in\mathbf{Z}}\int(1-\prod_{j=1}^{3}\psi_{j,k}(x))\prod_{j=1}^{3}(f_j*\Phi_{j,k,l})(x)dx\right|. \tag{3.10}$$

But recall that $\psi_{2,k}=\psi_{3,k}$, hence

$$|1-\prod_{j=1}^{3}\psi_{j,k}(x)|\le|1-\psi_{1,k}(x)|+2|1-\psi_{2,k}(x)|.$$

Thus the expression on the right in (3.10) is at most equal to the sum of the following two quantities

$$\sum_{k\in\mathbf{Z}}\int|1-\psi_{1,k}(x)|\prod_{j=1}^{2}\left(\sum_{l}|f_j*\Phi_{j,k,l}(x)|^2\right)^{\frac12}\sup_l|f_3*\Phi_{3,k,l}(x)|dx, \tag{3.11}$$

$$2\sum_{k\in\mathbf{Z}}\int|1-\psi_{2,k}(x)|\prod_{j=1}^{2}\left(\sum_{l}|f_j*\Phi_{j,k,l}(x)|^2\right)^{\frac12}\sup_l|f_3*\Phi_{3,k,l}(x)|dx. \tag{3.12}$$

Using (2.6) and the fact that $p_1>2$, for any point $z_0\in E^c$, we obtain

$$(\sum_{l\in\mathbf{Z}}|f_1*\Phi_{1,k,l}(x)|^2)^{\frac12}\le C\left(\int|f_1(y)|^{p_1}\frac{2^{-k}}{(1+2^{-k}|x-y|)^N}dy\right)^{\frac{1}{p_1}}\le C\left(1+2^{-k}\operatorname{dist}(x,E^c)\right).$$

Similarly, using (2.7) and the fact that $p_2>2$ we obtain

$$(\sum_{l\in\mathbf{Z}}|f_2*\Phi_{2,k,l}(x)|^2)^{\frac12}\le C\left(1+2^{-k+m}\operatorname{dist}(x,E^c)\right).$$

Using (3.5) and the facts that $\|f_3\|_{L^\infty(E^c)} \le 1$ and $\operatorname{supp} f_3 \subset E^c$, we also obtain

$$|f_3 * \Phi_{3,k,l}(x)| \le C_N \left(1 + 2^{-k+m}\operatorname{dist}(x, E^c)\right)^{-N} \tag{3.13}$$

for all $N > 0$. Therefore, (3.11) can be estimated by

$$C \sum_k \int \int_{E_k} \frac{2^{-k}}{(1 + 2^{-k}|x-y|)^N} dy \frac{1}{\left(1 + 2^{-k+m}\operatorname{dist}(x, E^c)\right)^{N-2}} dx$$
$$\le C \int_E \sum_{\substack{k \in \mathbf{Z} \\ 2^k \le \operatorname{dist}(y, E^c)}} \frac{1}{\left(1 + 2^{-k}\operatorname{dist}(y, E^c)\right)^{N-2}} dy \le C|E| \le C.$$

A similar works for (3.12). This completes the proof of (3.9) and therefore of Lemma 3. $\square$

We now set up some notation. For $k, n \in \mathbf{Z}$, define $I_{k,n} = [2^k n, 2^k(n+1)]$ and let

$$\begin{aligned} \phi_{1,k,n}(x) &= (1_{I_{k,n}} * \psi_k)(x), \\ \phi_{j,k,n}(x) &= (1_{I_{k,n}} * \psi_{k-m})(x), \quad \text{when } j \in \{2, 3\}. \end{aligned} \tag{3.14}$$

Therefore we can write

$$\Lambda_E(f_1, f_2, f_3) = \sum_{k \in \mathbf{Z}} \sum_{l \in \mathbf{Z}} \int \prod_{j=1}^3 \left(\sum_{n \in \mathbf{Z}} \phi_{j,k,n}(x)\psi_{j,k}(x)(f_j * \Phi_{j,k,l})(x) \right) dx. \tag{3.15}$$

For an integer r with $0 \le r < L$, let $\mathbf{Z}_r = \{\ell \in \mathbf{Z} : \ell = \kappa L + r \text{ for some } \kappa \in \mathbf{Z}\}$. Also for $S \subset \mathbf{Z}_r \times \mathbf{Z} \times \mathbf{Z}_r$ we let $S_{k,l} = \{n \in \mathbf{Z} : (k, n, l) \in S\}$ and we define

$$\Lambda_{E,S}(f_1, f_2, f_3) = \sum_{k \in \mathbf{Z}_r} \sum_{l \in \mathbf{Z}_r} \int \prod_{j=1}^3 \left(\sum_{n \in S_{k,l}} \phi_{j,k,n}(x)\psi_{j,k}(x)(f_j * \Phi_{j,k,l})(x) \right) dx. \tag{3.16}$$

For simplicity we will only consider the case where $m \in \mathbf{Z}_0$. The argument below can be suitably adjusted to the case where m has a different remainder when divided by L. We will therefore concentrate in proving Lemma 2 for the expression $\Lambda_{E,S}(f_1, f_2, f_3)$ when $m \in \mathbf{Z}_0$. To achieve this goal, we introduce the grid structure.

Definition 1. *A set of intervals $\mathcal{G}$ is called a grid if the condition below holds:*

$$\textit{for} \quad J, J' \in \mathcal{G}, \quad \textit{if} \quad J \cap J' \ne \emptyset, \quad \textit{then} \quad J \subset J' \quad \textit{or} \quad J' \subset J. \tag{3.17}$$

If a grid $\mathcal{G}$ satisfies the additional condition:

$$\textit{for} \quad J, J' \in \mathcal{G}, \quad \textit{if} \quad J \subsetneqq J', \quad \textit{then} \quad 5J \subset J', \tag{3.18}$$

then it will be called a central grid.

Given $S \subset \mathbf{Z}_r \times \mathbf{Z} \times \mathbf{Z}_r$ and $s = (k, n, l) \in S$ we set $I_s = I_{k,n}$. For each function $\Phi_{j,k,l}$ and each $n \in \mathbf{Z}$ we define a family of intervals $\omega_{j,s}$, $s = (k, n, l) \in S$ so that conditions (3.19)-(3.25) below hold: Say that $\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,l}}(\eta)$ is supported in a small neighborhood of a rectangle of type $r = 1$. Then we define $\omega_{j,s}$ such that

$$|c(\omega_{1,s}) - 2^{-k}(2l + \tfrac{1}{2})| \le 5 \cdot 2^{-L} 2^{-k}, \tag{3.19}$$

$$\left|c(\omega_{2,s}) - 2^{-k+m}(2l - \tfrac{3}{2})\right| \le 5 \cdot 2^{-L} 2^{-k+m} \quad \text{and} \quad \omega_{2,s} = \omega_{3,s}, \tag{3.20}$$

$$\operatorname{supp} \widehat{\Phi_{j,k,l}} \subset \omega_{j,s} \quad \text{for} \quad j \in \{1, 2\}, \tag{3.21}$$

$$\operatorname{supp}\widehat{\Phi_{3,k,l}} \subset [-(1+2^{-m})a, -(1+2^{-m})b), \text{ where } [a,b) = \omega_{3,s}, \tag{3.22}$$

$$(1+2^{-2L})2^{-k} \le |\omega_{1,s}| \le (1+10\cdot 2^{-L})2^{-k}, \tag{3.23}$$

$$(1+2^{-2L})2^{-k+m} \le |\omega_{j,s}| \le (1+2\cdot 2^{-2L})(1+5\cdot 2^{-L})2^{-k+m} \quad \text{for } j\in\{2,3\}, \tag{3.24}$$

$$\{\omega_{j,s}\}_{s\in S} \text{ is a central grid, for } j\in\{1,2,3\}. \tag{3.25}$$

These properties are trivially adjusted when $\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,l}}(\eta)$ is supported in a small neighborhood a rectangle of type $r=2$ or $r=3$.

As in [7], we prove the existence of $\omega_{j,s}$ by induction. If S is nonempty, pick $s_0 = (k,n,l)\in S$ such that k is minimal and define $S' = S\backslash\{s_0\}$. By induction, we may assume that for any $s'\in S'$ there exists a $\omega_{j,s'}$ so that the collection of all such intervals satisfies (3.19)-(3.25). Now we try to define ω_{j,s_0} so that (3.19)-(3.25) still hold. Let $[a_1,b_1)$ be an interval with length $(1+2^{-2L})2^{-k}$ which contains $\operatorname{supp}\widehat{\Phi_{1,k,l}}$. And for $j=2,3$, let $[a_j,b_j)$ be an interval with length $(1+2\cdot 2^{-2L})2^{-k+m}$ which contains $\operatorname{supp}\widehat{\Phi_{j,k,l}}$. By (3.22), we know that $\operatorname{supp}\widehat{\Phi_{3,k,l}} \subset [-(1+2^{-m})a_3, -(1+2^{-m})b_3)$. Define $\omega_{j,s_0,1}$ as the union of $[a_j,b_j)$ and all intervals $5\omega_{j,s'}$ with $s'\in S'$ which satisfy $\operatorname{dist}(\omega_{j,s'},[a_j,b_j)) \le 2|\omega_{j,s'}|$ and $\omega_{j,s'}$ be the next smaller interval in S. Inductively we define $\omega_{j,s_0,l}$ for $l\ge 1$. Let $\omega_{j,s_0} = \bigcup_{l\ge 1}\omega_{j,s_0,l}$. It is easy to verify conditions (3.19)-(3.25) for ω_{j,s_0}. This completes the proof of the existence of a grid structure.

Furthermore, we have the following geometric picture for $\omega_{j,s}$.

Lemma 4. *For $s,s'\in S$ and $\omega_{j,s}\neq\omega_{j,s'}$, the following properties hold*
(1) If $\omega_{1,s}\subset\omega_{1,s'}$, then $\omega_{j,s'}<\omega_{j,s}$ and $\frac{1}{2}|\omega_{j,s'}| < \operatorname{dist}(\omega_{j,s},\omega_{j,s'}) < 2|\omega_{2,s'}|$ for $j=2,3$.
(2) If $\omega_{j,s}\subset\omega_{j,s'}$ for $j=2,3$, then $\omega_{1,s}<\omega_{1,s'}$ and $\frac{1}{8}|\omega_{1,s'}| < \operatorname{dist}(\omega_{1,s},\omega_{1,s'}) < 2|\omega_{1,s'}|$.

Proof. For simplicity, let us assume that the $\omega_{j,s}$ are associated with rectangles of type 1.

$$\omega_{1,s} = [2^{-k}(2l-L^{-1}), 2^{-k}(2l+1+L^{-1})],$$

$$\omega_{1,s'} = [2^{-k'}(2l'-L^{-1}), 2^{-k'}(2l'+1+L^{-1})],$$

$$\omega_{2,s} = [2^{-k+m}(2l-2-2L^{-1}), 2^{-k+m}(2l-1+2L^{-1})],$$

$$\text{and}\quad \omega_{2,s'} = [2^{-k'+m}(2l'-2-2L^{-1}), 2^{-k'+m}(2l'-1+2L^{-1})].$$

For (1), note if $\omega_{1,s}\subsetneqq\omega_{1,s'}$, we have $2^{-k'}(2l'-L^{-1}) < 2^{-k}(2l-L^{-1}) < 2^{-k}(2l+L^{-1}+1) < 2^{-k'}(2l'+L^{-1}+1)$. Thus we have $2l' < 2^{k'-k}(2l-L^{-1})+L^{-1}$ and $2l+L^{-1}+1 < 2^{k-k'}(2l'+L^{-1}+1)$. By this, we have $2^{-k'+m-1} < 2^{-k+m}(2l-2L^{-1}-2) - 2^{-k'+m}(2l'-1+2L^{-1}) < 2^{-k'+m+1}$, which proves (1). We omit the proof of (2) since it is similar. □

As in [10] we give the following definition.

Definition 2. *A subset S of $\mathbf{Z}_r\times\mathbf{Z}\times\mathbf{Z}_r$ is called convex if for all $s,s''\in S$, $s'\in\mathbf{Z}_r\times\mathbf{Z}\times\mathbf{Z}_r$, $j\in\{1,2\}$ with $I_s\subset I_{s'}\subset I_{s''}$ and $\omega_{j,s''}\subset\omega_{j,s'}\subset\omega_{j,s}$, we have $s'\in S$.*

It is sufficient to prove bounds on $\Lambda_{E,S}$ for all finite convex sets S of triples of integers, provided the bound is independent of S and of course m.

4. The selection of the trees

Definition 3. *Fix $T \subset S$ and $t \in T$. If for any $s \in T$, we have $I_s \subset I_t$ and $\omega_{j,s} \supset \omega_{j,t}$, then we call T a tree of type j with top t. T is called a maximal tree of type $j \in \{1,2\}$ with top t in S if there does not exist a larger tree of type j with the same top strictly containing T. Let T be a maximal tree of type $j \in \{1,2\}$ with top t in S, and $i \in \{1,2\}$, $i \neq j$. Denote the maximal tree of type i with top t in S by $\widetilde{T}$.*

Lemma 5. *Let $S \subset \mathbf{Z}_r \times \mathbf{Z} \times \mathbf{Z}_r$ be a convex set and $T \subset S$ be a maximal tree of type $j \in \{1,2\}$ with top t in S. Then T is a convex set.*

Proof. Let $s, s'' \in T$, $s' \in \mathbf{Z}_r \times \mathbf{Z} \times \mathbf{Z}_r$, $i \in \{1,2\}$ with $I_s \subset I_{s'} \subset I_{s''}$ and $\omega_{i,s''} \subset \omega_{i,s'} \subset \omega_{i,s}$. Then $s' \in S$ by the convexity of S. Since $s \neq s''$, it follows from Lemma 4 that $i = j$. Using that $I_{s'} \subset I_{s''} \subset I_t$, $\omega_{j,t} \subset \omega_{j,s''} \subset \omega_{j,s'}$, and the maximality of T, we obtain that $s' \in T$, hence the convexity of T follows. □

Lemma 6. *Let $S \subset \mathbf{Z}_r \times \mathbf{Z} \times \mathbf{Z}_r$ be a convex set and T be a maximal tree of type $j \in \{1,2\}$ with top t in S. Then $S \backslash (T \bigcup \widetilde{T})$ is convex.*

Proof. Assume that $S \backslash (T \bigcup \widetilde{T})$ is not convex. Then there exist $s, s'' \in S \backslash (T \bigcup \widetilde{T})$, $s' \in T \bigcup \widetilde{T}$, $i \in \{1,2\}$ with $I_s \subset I_{s'} \subset I_{s''}$ and $\omega_{i,s''} \subset \omega_{i,s'} \subset \omega_{i,s}$. If $s' \in T$, then $I_{s'} \subset I_t$ and $\omega_{j,t} \subset \omega_{j,s'}$. Since s is not in T, we have $i \neq j$. By Lemma 4, we have $\operatorname{dist}(\omega_{i,t}, \omega_{i,s'}) < 2|\omega_{j,s'}|$. Since $5\omega_{i,s'} \subset \omega_{i,s}$ we have $\omega_{i,t} \subset \omega_{i,s}$. Thus $s \in \widetilde{T}$, which is a contradiction. □

For a given subset T of S we define $T_{k,l}$ to be the set $\{n \in \mathbf{Z} : (k,n,l) \in T\}$. If T is a tree of type j for $j \in \{1,2,3\}$ and $k \in \mathbf{Z}_r$, then there is at most one $l \in \mathbf{Z}_r$ such that $T_{k,l} \neq \emptyset$. If such an l exists, then let $T_k = T_{k,l}$ and

$\Phi_{j,k,T} = \Phi_{j,k,l}$. Otherwise, let $T_k = \emptyset$ and $\Phi_{j,k,T} = 0$. For brevity, we write $(k,n) \in T$ if and only if there exists an $l \in \mathbf{Z}_r$ with $(k,n,l) \in T$. Thus we can identify trees with sets of pairs of integers and we will use this identification throughout.

Therefore, if $(k,n,l) \in T$, we can write $\omega_{j,k,n,l} = \omega_{j,k,l} = \omega_{j,k,T}$, and

$$\Lambda_{E,T}(f_1,f_2,f_3) = \sum_{k \in \mathbf{Z}_r} \int \prod_{j=1}^{3} \left(\sum_{n \in T_k} \phi_{j,k,n}(x) \psi_{j,k}(x) (f_j * \Phi_{j,k,T})(x) \right) dx. \tag{4.1}$$

Let $t = (k_T, n_T, l_T)$ be the top of T. We write $I_T = I_{k_T,n_T}$ and $\omega_{j,T} = \omega_{j,k_T,T}$.

For a tree T of type 2 (or 3) with top t and $k \in \mathbf{Z}_r$, define $\theta^+_{j,k,T}$ and $\theta^-_{j,k,T}$ by

$$\widehat{\theta^+_{j,k,T}}(\xi) = (\Phi_{j,k-L,T} - \Phi_{j,k,T})^\wedge(\xi) 1_{\xi \geq \alpha_j c(w_{j,t})}(\xi),$$

$$\widehat{\theta^-_{j,k,T}}(\xi) = (\Phi_{j,k-L,T} - \Phi_{j,k,T})^\wedge(\xi) 1_{\xi \leq \alpha_j c(w_{j,t})}(\xi),$$

where $\alpha_j = 1$ if $j = 2$ and $\alpha_j = 1 + 2^{-m}$, if $j = 3$. Let $\psi^*(x) = (1+x^2)^{-N}$. In accordance with the definitions of $\phi_{j,k,n}$ and $\psi_{j,k}$ we define the functions

$$\psi^*_{1,k}(x) = (1_{(E_k)^c} * \psi^*_k)(x), \qquad \psi^*_{j,k}(x) = \psi^*_{1,k-m}(x), \quad \text{when } j \in \{2,3\}. \tag{4.2}$$

$$\phi^*_{1,k,n}(x) = (1_{I_{k,n}} * \psi^*_k)(x), \qquad \phi^*_{j,k,n}(x) = (1_{I_{k,n}} * \psi^*_{k-m})(x), \quad \text{when } j \in \{2,3\}. \tag{4.3}$$

Let Δ_k be the set of all connected components of $E_k \backslash E_{k+L}$. Obviously Δ_k is a set of intervals. Observe that if $J \in \Delta_k$, then $2^k \leq |J| < 2^{k+L}$, and $\bigcup_k \Delta_k$ is a set of pairwise disjoint intervals. Define

$$\Delta_{k,T} = \{J \in \Delta_k : J \subset I_{k+m+L,n}, \ \text{for some } (k+m+L,n) \in T\},$$

and for $J \in \Delta_{k,T}$ define

$$\rho_{k,J}(x) = 1_J * \psi_k^*(x), \text{ where } \psi_k^*(x) = 2^{-k}\psi^*(2^{-k}x). \tag{4.4}$$

Throughout this paper fix $0 < \eta \le L^{-1}\left(\sum_{j=1}^3 \frac{1}{p_j} - 1\right) \min_{j\in\{1,2,3\}}\{\frac{1}{p_j}\}$ and let

$$H = \bigcup_{j=1}^{3}\{(1,j,1),(2,1,1),(3,1,1)\} \bigcup \left(\bigcup_{\nu=2}^{5}\{(2,2,\nu),(2,3,\nu),(3,2,\nu),(3,3,\nu)\}\right).$$

We now describe a procedure for selecting a collection of trees $T^\nu_{\mu,i,j,l}$ and $\widetilde{T}^\nu_{\mu,i,j,l}$ by induction on μ and l. Let $S_{-1} = S$, and for $\mu \ge 0$ let

$$S_\mu = S_{\mu-1} \backslash \bigcup_{(i,j,\nu)\in H} \bigcup_{l\ge 0} (T^\nu_{\mu,i,j,l} \bigcup \widetilde{T}^\nu_{\mu,i,j,l})$$

where $T^\nu_{\mu,i,j,l}, \widetilde{T}^\nu_{\mu,i,j,l}$ are defined as follows:

Let $l \ge 0$ be an integer and assume that we have already defined $T^\nu_{\mu,i,j,\lambda}$, $\widetilde{T}^\nu_{\mu,i,j,\lambda}$ for $\lambda < l$. If one of the sets $T^\nu_{\mu,i,j,\lambda}, \widetilde{T}^\nu_{\mu,i,j,\lambda}$ with $\lambda < l$ is empty, then let $T^\nu_{\mu,i,j,l} = \widetilde{T}^\nu_{\mu,i,j,l} = \emptyset$. Otherwise, let $\mathcal{F}$ denote the set of all trees T of type i which satisfy conditions **(1)**-**(8)** below:

(1) For $(i,j,\nu) \in H$,

$$T \subset S_{\mu-1} \backslash \bigcup_{\lambda<l} (T^\nu_{\mu,i,j,\lambda} \bigcup \widetilde{T}^\nu_{\mu,i,j,\lambda}) \tag{4.5}$$

and T is a maximal tree of type i in $S_{\mu-1} \backslash \bigcup_{\lambda<l} (T^\nu_{\mu,i,j,\lambda} \bigcup \widetilde{T}^\nu_{\mu,i,j,\lambda})$.

(2) If $(i,j,\nu) = (1,1,1)$, then for $(k,n) \in T$, one of the following inequalities holds:

$$\left\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1 * \Phi_{1,k,l})\right\|_2 \ge 2^{-\eta\mu}2^{-\frac{\mu}{p_1}}|I_{k,n}|^{\frac12}, \tag{4.6}$$

$$\left\|\phi^*_{1,k,n}\psi^*_{1,k}\left(e^{-2\pi i c(\omega_{1,k,l})(\cdot)}(f_1 * \Phi_{1,k,l})(\cdot)\right)'\right\|_2 \ge 2^{-\eta\mu}2^{-\frac{\mu}{p_1}}|I_{k,n}|^{-\frac12}. \tag{4.7}$$

(3) If $(i,j,\nu) = (1,2,1)$ or $(1,3,1)$, then

$$\left(\sum_{(k,n)\in T}\left\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j * \Phi_{j,k,T})\right\|_2^2\right)^{\frac12} \ge 2^4 2^{-\frac{\mu}{p_j}}|I_{k_T,n_T}|^{\frac12}. \tag{4.8}$$

(4) If $(i,j,\nu) = (2,1,1)$ or $(3,1,1)$, then one of the following inequalities holds:

$$\left(\sum_{(k,n)\in T}\left\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1 * \Phi_{1,k,T})\right\|_2^2\right)^{\frac12} \ge 2^4 2^{-\frac{\mu}{p_1}}|I_{k_T,n_T}|^{\frac12}. \tag{4.9}$$

(5) If $i = 2$ or 3, $j = 2$ or 3, $\nu = 2$, then there exists $\tilde{k} \in \{-L, 0, L, 2L, 3L, 4L\}$ such that, for $(k,n) \in T$, one of the following inequalities holds:

$$\left\|\phi^*_{j,k+\tilde{k},n}\psi^*_{j,k+\tilde{k}}(f_j * \Phi_{j,k+\tilde{k},l})\right\|_2 \ge 2^{-\eta\mu}2^{-\frac{\mu}{p_j}}|I_{k,n}|^{\frac12}, \tag{4.10}$$

$$\left\|\phi^*_{1,k,n}\psi^*_{j,k+m+\tilde{k}}(f_j * \Phi_{j,k+m+\tilde{k},l})\right\|_2 \ge 2^{-\eta\mu}2^{-\frac{\mu}{p_j}}|I_{k,n}|^{\frac12}, \tag{4.11}$$

$$\left\|\phi^*_{1,k,n}\psi^*_{j,k+m+\tilde{k}}\left(e^{-2\pi i c(\omega_{j,k+m+\tilde{k},T})(\cdot)}(f_j*\Phi_{j,k+m+\tilde{k},l})(\cdot)\right)'\right\|_2 \geq 2^{-\eta\mu}2^{-\frac{\mu}{p_j}}|I_{k,n}|^{-\frac{1}{2}}. \tag{4.12}$$

(6) If $i=2$ or 3, $j=2$ or 3, $\nu=3$, then

$$\left(\sum_{(k,n)\in T}\left\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\theta^+_{j,k,T})\right\|_2^2\right)^{\frac{1}{2}} \geq 2^4 2^{-\frac{\mu}{p_j}}|I_{k_T,n_T}|^{\frac{1}{2}}. \tag{4.13}$$

(7) If $i=2$ or 3, $j=2$ or 3, $\nu=4$, then

$$\left(\sum_{(k,n)\in T}\left\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\theta^-_{j,k,T})\right\|_2^2\right)^{\frac{1}{2}} \geq 2^4 2^{-\frac{\mu}{p_j}}|I_{k_T,n_T}|^{\frac{1}{2}}. \tag{4.14}$$

(8) If $i=2$ or 3, $j=2$ or 3, $\nu=5$, then there exists $\tilde{k}\in\{-L,0,L,2L,3L,4L\}$ such that

$$\left(\sum_{k}\sum_{J\in\Delta_{k-m,T}}\left\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde{k},T})\right\|_2^2\right)^{\frac{1}{2}} \geq 2^4 2^{-\frac{\mu}{p_j}}|I_{k_T,n_T}|^{\frac{1}{2}}. \tag{4.15}$$

If no such trees exist, in other words if $\mathcal{F}=\emptyset$, then we set $T^\nu_{\mu,i,j,l}=\widetilde{T}^\nu_{\mu,i,j,l}=\emptyset$. Otherwise, we select $T^\nu_{\mu,i,j,l}$ and $\widetilde{T}^\nu_{\mu,i,j,l}$ as follows

(9) If $(i,j,\nu)\in\{(1,2,1),(1,3,1),(2,2,4),(2,3,4),(3,2,4),(3,3,4)\}$, then select $T^\nu_{\mu,i,j,l}\in\mathcal{F}$ such that for any $T\in\mathcal{F}$ we have

$$\omega_{j,T^\nu_{\mu,i,j,l}} \not> \omega_{j,T} \tag{4.16}$$

Let $\widetilde{T}^\nu_{\mu,i,j,l}$ be the maximal tree of type i' with top t in $S_{\mu-1}\setminus\bigcup_{\lambda<l}(T^\nu_{\mu,i,j,\lambda}\bigcup\widetilde{T}^\nu_{\mu,i,j,\lambda})$, where $i'=2$ if $i=1$, $i'=1$ if $i\in\{2,3\}$, and t is the top of $T^\nu_{\mu,i,j,l}$.

(10) If $(i,j,\nu)\in\{(2,1,1),(3,1,1),(2,2,3),(2,3,3),(3,2,3),(3,3,3)\}$, then select $T^\nu_{\mu,i,j,l}\in\mathcal{F}$ such that for any $T\in\mathcal{F}$ we have

$$\omega_{j,T^\nu_{\mu,i,j,l}} \not< \omega_{j,T} \tag{4.17}$$

Let $\widetilde{T}^\nu_{\mu,i,j,l}$ be the maximal tree of type i' with top t in $S_{\mu-1}\setminus\bigcup_{\lambda<l}(T^\nu_{\mu,i,j,\lambda}\bigcup\widetilde{T}^\nu_{\mu,i,j,\lambda})$, where $i'=2$ if $i=1$, $i'=1$ if $i\in\{2,3\}$, and t is the top of $T^\nu_{\mu,i,j,l}$.

This completes the selection of trees. Observe that as a consequence of Lemma 5 and Lemma 6, we have that S_μ, $T^\nu_{\mu,i,j,l}$ and $\widetilde{T}^\nu_{\mu,i,j,l}$ are convex.

Lemma 7. *For $\mu\geq 0$, $(i,j,\nu)\in H$,*

$$\sum_{l}|I_{T^\nu_{\mu,i,j,l}}| \leq C2^{10\eta p_j\mu}2^{\mu} \tag{4.18}$$

where C is independent of m.

Let $2<q_1,q_2,q_3<\infty$ with $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$. Then we have

Lemma 8. *Let $\mu\geq 0$, $j\in\{1,2,3\}$, T be a tree of type j and $T\subset S_\mu$, then*

$$|\Lambda_{E,T}(f_1,f_2,f_3)| \leq C2^{-\eta\mu}2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}|I_T| \quad \text{if} \quad j=1, \tag{4.19}$$

And if T is a convex set, then

$$|\Lambda_{E,T}(f_1,f_2,f_3)| \le C_{q_1} 2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})\mu}|I_T| \text{ if } j=2,3, \tag{4.20}$$

where C, C_{q_1} are independent of m.

The core of the proof consists of the proofs of these lemmata. These will be given in the next section. We now state and prove one more lemma which will allow us to conclude the proof of (1.2), assuming the validity of Lemmata 7 and 8.

Lemma 9. *Let $\mu \ge 0$, $T \subset S_{\mu-1}$ be a tree of type $j \in \{1,2,3\}$, $P \subset S_{\mu-1}$, and $T \bigcap P = \emptyset$. Suppose T is a maximal tree in $T \bigcup P$. Then*

$$|\Lambda_{E,T\bigcup P}(f_1,f_2,f_3) - \Lambda_{E,P}(f_1,f_2,f_3)| \le |\Lambda_{E,T}(f_1,f_2,f_3)| + C2^{-\eta\mu}2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}|I_T|,$$

where C is independent of μ, P and T.

Proof. Notice there exists at most one l such that $T_{k,l} \ne \emptyset$ and T is a maximal tree in $T\bigcup P$, we have

$$|\Lambda_{E,T\bigcup P}(f_1,f_2,f_3) - \Lambda_{E,P}(f_1,f_2,f_3)| \le |\Lambda_{E,T}(f_1,f_2,f_3)| + \sum_{k\le k_T}\int |A_k(x)|dx,$$

where A_k satisfies

$$|A_k(x)| \le \frac{C}{(1+2^{-k}\mathrm{dist}(x,\partial I_T))^N}\prod_{j=1}^{3}\Bigg(\sum_{n\in(P\bigcup T)_k}\phi^*_{j,k,n}(x)\psi^*_{j,k}(x)|f_j*\Phi_{j,k,T}(x)|\Bigg), \tag{4.21}$$

and $(P\bigcup T)_k = (P\bigcup T)_{l,k}$ if there exists an l such that $T_{l,k}\ne\emptyset$, and $(P\bigcup T)_k=\emptyset$ if such an l does not exist. Thus we have

$$\int |A_k(x)|dx \le \sum_{n'\in\mathbf{Z}}\frac{C}{(1+2^{-k}\mathrm{dist}(I_{k,n'},\partial I_T))^N}\Bigg\|\sum_{n\in(P\bigcup T)_k}\phi^*_{1,k,n}\psi^*_{1,k}(f_1*\Phi_{1,k,T})\Bigg\|_{L^\infty(I_{k,n'})}$$

$$\cdot\prod_{j=2}^{3}\Bigg\|\sum_{n\in(P\bigcup T)_k}\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\Bigg\|_{L^2(I_{k,n'})}.$$

Note that, since $P\bigcup T \in S_{\mu-1}$,

$$\Bigg\|\sum_{n\in(P\bigcup T)_k}\phi^*_{1,k,n}\psi^*_{1,k}(f_1*\Phi_{1,k,T})\Bigg\|_{L^\infty(I_{k,n'})}$$

$$\le C\big\|\phi^*_{1,k,n''}\psi^*_{1,k}f_1*\Phi_{1,k,T}\big\|_2^{\frac12}\big\|(\phi^*_{1,k,n''}\psi^*_{1,k}e^{-2\pi ic(\omega_{1,k,T})(\cdot)}(f_1*\Phi_{1,k,T})(\cdot))'\big\|_2^{\frac12} \le C2^{-\eta\mu}2^{-\frac{\mu}{p_1}},$$

where $n''\in(P\bigcup T)_k$ is so that it minimizes the distance to n'. And

$$\Bigg\|\sum_{n\in(P\bigcup T)_k}\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\Bigg\|_{L^2(I_{k,n'})} \le C2^{-\eta\mu}2^{-\frac{\mu}{p_j}}|I_{k,n}|^{\frac12}.$$

Hence we obtain

$$\begin{aligned}&|\Lambda_{E,T\bigcup P}(f_1,f_2,f_3)-\Lambda_{E,P}(f_1,f_2,f_3)|\\ \le\ & |\Lambda_{E,T}(f_1,f_2,f_3)| + \sum_{k\le k_T}\sum_{n'\in\mathbf{Z}}\frac{C2^{-\eta\mu}2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}2^k}{(1+2^{-k}\mathrm{dist}(I_{k,n'},\partial I_T))^N}\\ \le\ & |\Lambda_{E,T}(f_1,f_2,f_3)| + C2^{-\eta\mu}2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}|I_T|.\end{aligned}$$

This conclues the proof of Lemma 9. □

We now deduce the proof of (1.2) using assumed Lemmata 7 and 8 and Lemma 9.

$$
\begin{aligned}
|\Lambda_{E,S}(f_1,f_2,f_3)| \leq\ & C_{q_1}\sum_{(i,j,\nu)\in H}\sum_{\mu\geq 0}2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})\mu}\sum_{l}|I_{T^{\nu}_{\mu,i,j,l}}| \\
& + C\sum_{(i,j,\nu)\in H}\sum_{\mu\geq 0}2^{-\eta\mu}2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}\sum_{l}|I_{T^{\nu}_{\mu,i,j,l}}| \\
\leq\ & C\sum_{(i,j,\nu)\in H}\sum_{\mu\geq 0}2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}2^{10\eta p_j\mu}2^{\mu} \\
& + C_{q_1}\sum_{(i,j,\nu)\in H}\sum_{\mu\geq 0}2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})\mu}2^{10\eta p_j\mu}2^{\mu} \\
\leq\ & C_{p_1,p_2,p_3}<\infty,
\end{aligned}
$$

It remains to prove Lemmata 7 and 8. This will be achieved in the following sections.

5. Some technical material

In this section we prove a variety of technical facts that will be used in the proofs of Lemmata 7 and 8 presented in the next sections.

Lemma 10. *For any $(k,n,l)\in S$ we have the following:*

$$\left\|\phi^*_{1,k,n}(f_1*\Phi_{1,k,l})\right\|_2\leq C\inf_{x\in I_{k,n}}Mf_1(x)|I_{k,n}|^{\frac{1}{2}}, \tag{5.1}$$

$$\left\|\phi^*_{1,k,n}\left(e^{-2\pi ic(\omega_{1,k,l})(\cdot)}(f_1*\Phi_{1,k,l})(\cdot)\right)'\right\|_2\leq C\inf_{x\in I_{k,n}}Mf_1(x)|I_{k,n}|^{-\frac{1}{2}}, \tag{5.2}$$

$$\left\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1*\Phi_{1,k,l})\right\|_2\leq C|I_{k,n}|^{\frac{1}{2}}, \tag{5.3}$$

$$\left\|\phi^*_{1,k,n}\psi^*_{1,k}\left(e^{-2\pi ic(\omega_{1,k,l})(\cdot)}(f_1*\Phi_{1,k,l})(\cdot)\right)'\right\|_2\leq C|I_{k,n}|^{-\frac{1}{2}}. \tag{5.4}$$

Proof. Since $\phi^*_{1,k,n}(x)\leq C\left(1+2^{-k}\text{dist}(x,I_{k,n})\right)^{-N}$ we obtain

$$\left\|\phi^*_{1,k,n}(f_1*\Phi_{1,k,l})\right\|_2^2\leq C\left(\inf_{x\in I_{k,n}}Mf_1(x)\right)^2|I_{k,n}|.$$

This proves (5.1). Now observe that

$$\left(e^{-2\pi ic(\omega_{1,k,l})(\cdot)}(f_1*\Phi_{1,k,l})(\cdot)\right)'(x)=\int f_1(y)e^{-2\pi ic(\omega_{1,k,l})y}\left(\Phi_{1,k,l}(\cdot)e^{-2\pi ic(\omega_{1,k,l})(\cdot)}\right)'(x-y)dy,$$

and

$$\left|\left(\Phi_{1,k,l}(\cdot)e^{-2\pi ic(\omega_{1,k,l})(\cdot)}\right)'(x)\right|\leq C\,2^{-2k}(1+2^{-k}|x|)^{-N}.$$

Using this estimate and a similar argument as before we obtain (5.2).

We now prove (5.3). We may assume that $I_{k,n}\subset E$, otherwise (5.3) follows immediately from (5.1) and from the fact that $Mf_1(x)\leq M_{p_1}f_1(x)$. Pick a number $A\geq 1$ such that $AI_{k,n}\subset E$ and $2AI_{k,n}\bigcap E^c\neq\emptyset$. Then by $\psi^*_{1,k}(x)\leq\left(1+2^{-k}\text{dist}(x,E^c)\right)^{-2N}$, we have

$$\left\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1*\Phi_{1,k,l})\right\|_2^2\leq CA^{-N}A\left(\inf_{x\in 2AI_{k,n}}M_{p_1}f_1(x)\right)^2|I_{k,n}|\leq C|I_{k,n}|.$$

This completes the proof of (5.3). The proof of (5.4) is similar. □

Applying the same idea and the fact that the Littlewood-Paley square function is bounded from L^2 to L^2, we can prove the following.

Lemma 11. *For any tree T of type 1 and any $j \in \{2,3\}$ we have*

$$\Big(\sum_{(k,n)\in T} \big\|\phi^*_{j,k,n}(f_j * \Phi_{j,k,T})\big\|_2^2\Big)^{\frac12} \le C \inf_{x\in I_T} M_2 f_j(x) |I_{k_T,n_T}|^{\frac12}, \tag{5.5}$$

$$\Big(\sum_{(k,n)\in T} \big\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j * \Phi_{j,k,T})\big\|_2^2\Big)^{\frac12} \le C |I_{k_T,n_T}|^{\frac12}. \tag{5.6}$$

Similarly we obtain the following lemmata whose proofs we omit.

Lemma 12. *For any tree T of type j, $j \in \{2,3\}$ we have*

$$\Big(\sum_{(k,n)\in T} \big\|\phi^*_{1,k,n}(f_1 * \Phi_{1,k,T})\big\|_2^2\Big)^{\frac12} \le C \inf_{x\in I_T} M_2 f_1(x) |I_{k_T,n_T}|^{\frac12}, \tag{5.7}$$

$$\Bigg(\sum_{(k,n)\in T} |I_{k,n}|^2 \bigg\|\phi^*_{1,k,n}\Big(e^{-2\pi i c(\omega_{1,k,T})(\cdot)}(f_1 * \Phi_{1,k,T})(\cdot)\Big)'\bigg\|_2^2\Bigg)^{\frac12} \le C \inf_{x\in I_T} M_2 f_1(x) |I_T|^{\frac12}, \tag{5.8}$$

$$\Bigg(\sum_{(k,n)\in T} |I_{k,n}|^2 \bigg\|\phi^*_{1,k,n}\psi^*_{1,k}\Big(e^{-2\pi i c(\omega_{1,k,T})(\cdot)}(f_1 * \Phi_{1,k,T})(\cdot)\Big)'\bigg\|_2^2\Bigg)^{\frac12} \le C |I_{k_T,n_T}|^{\frac12}, \tag{5.9}$$

$$\Big(\sum_{(k,n)\in T} \big\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1 * \Phi_{1,k,T})\big\|_2^2\Big)^{\frac12} \le C |I_{k_T,n_T}|^{\frac12}. \tag{5.10}$$

Lemma 13. *For $(k,n,l) \in S$, $\tilde{k} \in \{-L, 0, L, 2L, 3L, 4L\}$, and $j \in \{2,3\}$ we have*

$$\big\|\phi^*_{j,k+\tilde{k},n}(f_j * \Phi_{j,k+\tilde{k},l})\big\|_2 \le C \inf_{x\in I_{k,n}} M_2 f_j(x) |I_{k,n}|^{\frac12}, \tag{5.11}$$

$$\big\|\phi^*_{1,k,n}(f_j * \Phi_{j,k+m+\tilde{k},l})\big\|_2 \le C \inf_{x\in I_{k,n}} M f_j(x) |I_{k,n}|^{\frac12}, \tag{5.12}$$

$$\bigg\|\phi^*_{1,k,n}\Big(e^{-2\pi i c(\omega_{j,k+m+\tilde{k},l})(\cdot)}(f_j * \Phi_{j,k+m+\tilde{k},l})(\cdot)\Big)'\bigg\|_2 \le C \inf_{x\in I_{k,n}} M f_j(x) |I_{k,n}|^{-\frac12}, \tag{5.13}$$

$$\big\|\phi^*_{j,k+\tilde{k},n}\psi^*_{j,k+\tilde{k}}(f_j * \Phi_{j,k+\tilde{k},l})\big\|_2 \le C |I_{k,n}|^{\frac12}, \tag{5.14}$$

$$\big\|\phi^*_{1,k,n}\psi^*_{2,k+m+\tilde{k}}(f_j * \Phi_{j,k+m+\tilde{k},l})\big\|_2 \le C |I_{k,n}|^{\frac12}, \tag{5.15}$$

$$\bigg\|\phi^*_{1,k,n}\psi^*_{2,k+m+\tilde{k}}\Big(e^{-2\pi i c(\omega_{j,k+m+\tilde{k},l})(\cdot)}(f_j * \Phi_{j,k+m+\tilde{k},l})(\cdot)\Big)'\bigg\|_2 \le C |I_{k,n}|^{-\frac12}. \tag{5.16}$$

Lemma 14. *For a convex tree T of type j, $j\in\{2,3\}$ we have*

$$\Big(\sum_{(k,n)\in T}\big\|\phi^*_{j,k,n}(f_j*\theta^+_{j,k,T})\big\|_2^2\Big)^{\frac12}\le C\inf_{x\in I_T}M_2f_j(x)|I_{k_T,n_T}|^{\frac12},\tag{5.17}$$

$$\Big(\sum_{(k,n)\in T}\big\|\phi^*_{j,k,n}(f_j*\theta^-_{j,k,T})\big\|_2^2\Big)^{\frac12}\le C\inf_{x\in I_T}M_2f_j(x)|I_{k_T,n_T}|^{\frac12},\tag{5.18}$$

$$\Big(\sum_{(k,n)\in T}\big\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\theta^+_{j,k,T})\big\|_2^2\Big)^{\frac12}\le C|I_{k_T,n_T}|^{\frac12},\tag{5.19}$$

$$\Big(\sum_{(k,n)\in T}\big\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\theta^-_{j,k,T})\big\|_2^2\Big)^{\frac12}\le C|I_{k_T,n_T}|^{\frac12}.\tag{5.20}$$

Lemma 15. *For $\tilde k\in\{-L,0,L,2L,3L,4L\}$, let T be a tree of type j, $j\in\{2,3\}$,*

$$\Big(\sum_k\sum_{J\in\Delta_{k-m,T}}\big\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde k,T})\big\|_2^2\Big)^{\frac12}\le C\inf_{x\in I_T}M_2f_j(x)|I_{k_T,n_T}|^{\frac12},\tag{5.21}$$

$$\Big(\sum_k\sum_{J\in\Delta_{k-m,T}}\big\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde k,T})\big\|_2^2\Big)^{\frac12}\le C|I_{k_T,n_T}|^{\frac12}.\tag{5.22}$$

Proof. We prove (5.22) first. Since

$$\begin{aligned}\big\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde k,T})\big\|_2^2 &\le \int\frac{1}{(1+2^{-k+m}\mathrm{dist}(x,J))^N}\int\frac{|f_j(y)|^2 2^{-k+m}dy}{(1+2^{-k+m}|x-y|)^N}dx\\ &\le C|J|\Big(\inf_{x\in 8J}M_2f_j(x)\Big)^2\ \le C|J|,\end{aligned}$$

we have

$$\sum_k\sum_{J\in\Delta_{k-m,T}}\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde k,T})\|_2^2\le C\sum_k\sum_{J\in\Delta_{k-m,T}}|J|\le C|I_T|,$$

because the union of Δ_{k-m} is a set of pairwise disjoint intervals. We now prove (5.21). We have

$$\sum_k\sum_{J\in\Delta_{k-m,T}}\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde k,T})\|_2^2\le D_1+D_2,$$

where

$$\begin{aligned}D_1&=\sum_k\sum_{J\in\Delta_{k-m,T}}\big\|\rho_{k-m,J}((f_j1_{2I_T})*\Phi_{j,k+\tilde k,T})\big\|_2^2,\\ D_2&=\sum_k\sum_{J\in\Delta_{k-m,T}}\big\|\rho_{k-m,J}((f_j1_{(2I_T)^c})*\Phi_{j,k+\tilde k,T})\big\|_2^2.\end{aligned}$$

It is easy to see that

$$D_1\ \le\ C\|f_j1_{2I_T}\|_2^2\ \le\ C|I_T|\Big(\inf_{x\in I_T}M_2f_j(x)\Big)^2.$$

To control D_2 we have to work a bit harder. For any $z_0 \in I_T$ we have

$$\sum_k \sum_{J\in\Delta_{k-m,T}} \left\|\rho_{k-m,J}((f_j 1_{(2I_T)^c}) * \Phi_{j,k+\tilde{k},T})\right\|_2^2$$

$$\leq \sum_k \sum_{J\in\Delta_{k-m,T}} \frac{CM_2 f_j(z_0)}{(1+2^{-k+m}\mathrm{dist}((2I_T)^c, J))^N} \int \frac{(1+2^{-k+m}|z_0-x|)^2}{(1+2^{-k+m}\mathrm{dist}(x,J))^{N+2}}dx$$

$$\leq \sum_k \sum_{J\in\Delta_{k-m,T}} \frac{C|J|}{(1+2^{-k+m}\mathrm{dist}((2I_T)^c, J))^N} \left(\inf_{x\in I_T} M_2 f_j(x)\right)^2$$

$$\leq C\sum_k \sum_{J\in\Delta_{k-m,T}} |J|\left(\inf_{x\in I_T} M_2 f_j(x)\right)^2 \quad \leq C|I_T|\left(\inf_{x\in I_T} M_2 f_j(x)\right)^2,$$

which proves (5.21) and thus completes the proof of Lemma 15. □

The following lemma is just a version of the boundedness of the Littlewood-Paley square function from L^∞ to BMO. Its proof follows standard arguments and is also omitted.

Lemma 16. *Let $j \in \{2,3\}$ and $T \subset S$ be a convex tree of type j. Then we have*

$$\left\|\psi^*_{j,k}(f_j * \Phi_{j,k,l})\right\|_\infty \leq C, \tag{5.23}$$

$$\left\|\left(\sum_k \left|\sum_{n\in T_k} \phi^*_{j,k,n}\psi^*_{j,k}\big(f_j * (\Phi_{j,k-L,T} - \Phi_{j,k,T})\big)\right|^2\right)^{\frac12}\right\|_{BMO} \leq C, \tag{5.24}$$

where C is independent of m and BMO denotes dyadic BMO.

6. The size estimate for the trees

Having proved all these preliminary lemmata we now concentrate on the proof of Lemma 8. This section is entirely devoted to its proof.

We begin by showing (4.19). For a tree T of type 1 and $T \subset S_\mu$, we have

$$|\Lambda_{E,T}(f_1,f_2,f_3)|$$

$$\leq \sum_k \int \prod_{j=1}^3 \left|\left(\sum_{n\in T_k} \phi_{j,k,n}(x)\psi_{j,k}(x)(f_j * \Phi_{j,k,T})\right)\right|dx$$

$$\leq \left\|\sup_k \left|\sum_{n\in T_k} \phi_{1,k,n}\psi_{1,k}(f_1 * \Phi_{1,k,T})\right|\right\|_\infty \prod_{j=2}^3 \left(\sum_k \left\|\sum_{n\in T_k} \phi_{j,k,n}\psi_{j,k}(f_j * \Phi_{j,k,T})\right\|_2^2\right)^{\frac12}$$

$$\leq C \sup_{(k,n)\in T} \left\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1 * \Phi_{1,k,T})\right\|_\infty 2^{-\frac{\mu}{p_2}} 2^{-\frac{\mu}{p_3}} |I_T|.$$

Observe that

$$\left\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1 * \Phi_{1,k,T})\right\|_\infty$$

$$\leq \left\|\phi^*_{1,k,n}\psi^*_{1,k}(f_1 * \Phi_{1,k,T})\right\|_2^{\frac12} \left\|\left(\phi^*_{1,k,n}\psi^*_{1,k} e^{-2\pi i c(\omega_{1,k,T})(\cdot)}(f_1 * \Phi_{1,k,T})(\cdot)\right)'\right\|_2^{\frac12}$$

$$\leq C2^{-\eta\mu}2^{-\frac{\mu}{p_1}}.$$

Thus we have

$$|\Lambda_{E,T}(f_1,f_2,f_3)| \leq C2^{-\eta\mu}2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}.$$

This completes the proof of (4.19) for trees of type 1. We now turn our attention to the proof of (4.20). Let

$$f_{i,k}(x) = \sum_{n\in T_k} \phi_{i,k,n}(x)\psi_{i,k}(x)(f_i * \Phi_{i,k,T})(x),$$

for $i = 1, 2, 3$. Then we write the sum $\sum_{k\in\mathbf{Z}_r} f_{1,k}f_{2,k}f_{3,k}$ as

$$\sum_{k\in\mathbf{Z}_r} f_{1,k}f_{2,k+m+L}f_{3,k+m+L} + \sum_{k\in\mathbf{Z}_r}\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k}(f_{2,k+\tilde{k}}f_{3,k+\tilde{k}} - f_{2,k+\tilde{k}+L}f_{3,k+\tilde{k}+L}).$$

Note $\operatorname{supp}\widehat{f}_{1,k} \subset \omega_{1,k,T}$, $\operatorname{supp}\widehat{f}_{2,k+m+L} \subset \omega_{2,k+m+L,T}$, and $-\operatorname{supp}\widehat{f}_{3,k+m+L} \subset \omega_{1,k+m+L,T} + \omega_{2,k+m+L,T}$. Since T is a tree of type 2 or 3, by Lemma 4, we have $\omega_{1,k+m+L,T} < \omega_{1,k,T}$ and $\operatorname{dist}(\omega_{1,k+m+L,T}, \omega_{1,k,T}) > |\omega_{1,k,L}|/8$. Therefore, we have $(\omega_{1,k+m+L,T} + \omega_{2,k+m+L,T}) < (\omega_{1,k,T} + \omega_{2,k+m+L,T})$, which implies $-\operatorname{supp}\widehat{f}_{3,k+m+L} < \operatorname{supp}\widehat{f}_{1,k} + \operatorname{supp}\widehat{f}_{2,k+m+L}$. Thus, we have

$$\int \sum_k f_{1,k}(x)f_{2,k+m+L}(x)f_{3,k+m+L}(x)dx = 0.$$

Therefore, it is sufficient to consider

$$\sum_{k\in\mathbf{Z}_r}\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k}\big(f_{2,k+\tilde{k}}f_{3,k+\tilde{k}} - f_{2,k+\tilde{k}+L}f_{3,k+\tilde{k}+L}\big).$$

We write this term as

$$\sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k-\tilde{k}}\Bigg)(f_{2,k}f_{3,k} - f_{2,k+L}f_{3,k+L}) = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k-\tilde{k}}\Bigg)(f_{2,k} - f_{2,k+L})(f_{3,k} - f_{3,k+L}),$$

$$I_2 = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k-\tilde{k}}\Bigg)(f_{2,k+L} - f_{2,k+2L})(f_{3,k} - f_{3,k+L}),$$

$$I_3 = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k-\tilde{k}}\Bigg)(f_{2,k} - f_{2,k+L})(f_{3,k+L} - f_{3,k+2L}),$$

$$I_4 = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k-\tilde{k}}\Bigg)f_{2,k+2L}(f_{3,k} - f_{3,k+L}),$$

$$I_5 = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m}} f_{1,k-\tilde{k}}\Bigg)(f_{2,k} - f_{2,k+L})f_{3,k+2L}.$$

Therefore,

$$|I_1| \le \sup_k \Big| \sum_{\substack{\tilde{k}\in \mathbf{Z}_0 \\ 0\le \tilde{k}\le m}} f_{1,k-\tilde{k}} \Big| \Big(\sum_k |f_{2,k} - f_{2,k+L}|^2 \Big)^{\frac{1}{2}} \Big(\sum_k |f_{3,k} - f_{3,k+L}|^2 \Big)^{\frac{1}{2}}$$

and thus for $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$ and $2 < q_1, q_2, q_3 < \infty$, we have

$$\begin{aligned} \int |I_1(x)|\, dx \ &\le \Big\| \sup_k \Big| \sum_{\tilde{k}\in \mathbf{Z}_0, 0\le \tilde{k}\le m} f_{1,k-\tilde{k}} \Big| \Big\|_{q_1} \prod_{j=2}^{3} \Big\| \Big(\sum_k |f_{j,k} - f_{j,k+L}|^2 \Big)^{\frac{1}{2}} \Big\|_{q_j} \\ &\le C_{q_1} \Big\| \sum_k f_{1,k} \Big\|_{q_1} \prod_{j=2}^{3} \Big\| \Big(\sum_k |f_{j,k} - f_{j,k+L}|^2 \Big)^{\frac{1}{2}} \Big\|_{q_j}, \end{aligned} \tag{6.1}$$

where the L^{q_1} norm estimate above is a consequence of the Carleson-Hunt theorem [3] and [6], since the Fourier transforms of $f_{1,k}$'s have disjoint supports.

To control the product of the last three terms in (6.1) we will need the following lemma.

Lemma 17. *Let $\mu \ge 0$, $j \in \{2,3\}$, T be a tree of type j and $T \subset S_\mu$, then*

$$\| \sum_k f_{1,k} \|_2 \le C 2^{-\frac{\mu}{p_1}} |I_T|^{\frac{1}{2}}, \tag{6.2}$$

$$\| e^{-2\pi i c(\omega_{1,T})(\cdot)} \sum_k f_{1,k}(\cdot) \|_{BMO} \le C 2^{-\frac{\mu}{p_1}}. \tag{6.3}$$

Proof. The proof of (6.2) follows from the selection of trees, (in particular (4.9) which fails for $\mu - 1$), since

$$\begin{aligned} \| \sum_k f_{1,k} \|_2^2 \ &= \| \sum_k \sum_{n\in T_k} \phi_{1,k,n} \psi_{1,k} (f_1 * \Phi_{1,k,T}) \|_2^2 \\ &\le \sum_k \sum_{n\in T_k} \| \phi_{1,k,n} \psi_{1,k} (f_1 * \Phi_{1,k,T}) \|_2^2 \le C 2^{-\frac{2\mu}{p_1}} |I_T|. \end{aligned}$$

We now prove (6.3). Let $J = [2^{k_J} n_J, 2^{k_J}(n_J+1)]$ for some $k_J \in \mathbf{Z}$ and define $T_J := \{(k,n) \in T : I_{k,n} \subset J\}$. Then

$$|J|^{-1} \inf_c \int_J \Big| \sum_{(k,n)\in T} \phi_{1,k,n}(x) \psi_{1,k}(x) (f_1 * \Phi_{1,k,T})(x) e^{-2\pi i c(\omega_{1,T})x} - c \Big| dx \le J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 \ &= |J|^{-1} \int_J \Big| \sum_{(k,n)\in T_J} \phi_{1,k,n}(x) \psi_{1,k}(x) (f_1 * \Phi_{1,k,T})(x) \Big| dx \\ J_2 \ &= |J|^{-1} \int_J \Big| \sum_{(k,n)\in T\setminus T_J, k\le k_J} \phi_{1,k,n}(x) \psi_{1,k}(x) (f_1 * \Phi_{1,k,T})(x) \Big| dx \\ J_3 \ &= |J|^{-1} \inf_c \int_J \Big| \sum_{(k,n)\in T, k> k_J} \phi_{1,k,n}(x) \psi_{1,k}(x) (f_1 * \Phi_{1,k,T})(x) e^{-2\pi i c(\omega_{1,T})x} - c \Big| dx \end{aligned}$$

Since T_J is a union of trees of type 2 or 3, we have

$$J_1 \le |J|^{-\frac{1}{2}} \Big(\sum_{(k,n)\in T_J} \big\| \phi^*_{1,k,n} \psi^*_{1,k} (f_1 * \Phi_{1,k,T}) \big\|_2^2 \Big)^{\frac{1}{2}} \le C 2^{-\frac{\mu}{p_1}},$$

which proves the required estimate for J_1.

For J_2, we use (4.6) (which fails for $\mu - 1$) to obtain

$$J_2 \le C 2^{-\frac{\mu}{p_1}} |J|^{-\frac{1}{2}} \sum_{\substack{(k,n)\in T\setminus T_J \\ k\le k_J}} \frac{|I_{k,n}|^{\frac{1}{2}}}{(1+2^{-k}\mathrm{dist}(J, I_{k,n}))^N} \le C 2^{-\frac{\mu}{p_1}}.$$

Finally we can control J_3 by

$$\int_J \Big| \sum_{\substack{(k,n)\in T \\ k>k_J}} \Big(\phi_{1,k,n}(\cdot) \psi_{1,k}(\cdot) (f_1 * \Phi_{1,k,T})(\cdot) e^{-2\pi i c(\omega_{1,T})(\cdot)} \Big)'(x) \Big| \, dx$$

which is equal to

$$\int_J \Big| \sum_{\substack{(k,n)\in T \\ k>k_J}} \Big(\phi_{1,k,n}(\cdot) \psi_{1,k}(\cdot) (f_1 * \Phi_{1,k,T})(\cdot) e^{-2\pi i c(\omega_{1,k,T})(\cdot)} e^{-2\pi i (c(\omega_{1,T}) - c(\omega_{1,k,T}))(\cdot)} \Big)'(x) \Big| \, dx.$$

Thus we obtain the estimate $J_3 \le J_{31} + J_{32}$, where

$$J_{31} = \int_J \sum_{(k,n)\in T, k>k_J} \Big| \Big(\phi_{1,k,n}(\cdot) \psi_{1,k}(\cdot) (f_1 * \Phi_{1,k,T})(\cdot) e^{-2\pi i c(\omega_{1,k,T})(\cdot)} \Big)'(x) \Big| \, dx,$$

$$J_{32} = C \int_J \Big| \sum_{(k,n)\in T, k>k_J} \phi_{1,k,n}(x) \psi_{1,k}(x) (f_1 * \Phi_{1,k,T})(x) e^{-2\pi i c(\omega_{1,k,T})x} (c(\omega_{1,T}) - c(\omega_{1,k,T})) \Big| \, dx.$$

Since T is a tree of type 2 or 3 it follows from Lemma 4 that $|c(\omega_{1,T}) - c(\omega_{1,k,T})| \le 2|\omega_{1,k,T}|$. Thus we have

$$\begin{aligned} J_{32} &\le C|J|^{\frac{1}{2}} \sum_{(k,n)\in T, k>k_J} \frac{2^{-k}}{(1+2^{-k}\mathrm{dist}(J, I_{k,n}))^N} \big\| \phi^*_{1,k,n} \psi^*_{1,k} (f_1 * \Phi_{1,k,T}) \big\|_2 \\ &\le C 2^{-\frac{\mu}{p_1}} |J|^{\frac{1}{2}} \sum_{k>k_J} \sum_{n\in T_k} \frac{2^{-k}|I_{k,n}|^{\frac{1}{2}}}{(1+2^{-k}\mathrm{dist}(J, I_{k,n}))^N} \le C 2^{-\frac{\mu}{p_1}}. \end{aligned}$$

Similarly, we prove $J_{31} \le C 2^{-\frac{\mu}{p_1}}$, by using (4.6) and (4.7) (which failed at the step $\mu - 1$). This completes the proof of (6.3). □

Now interpolate between (6.2) and (6.3) to obtain

$$\Big\| \sum_k f_{1,k} \Big\|_{q_1} \le C 2^{-\frac{\mu}{p_1}} |I_T|^{\frac{1}{q_1}}, \tag{6.4}$$

where C is independent of q_1. Next we write $\big(\sum_k |f_{j,k} - f_{j,k+L}|^2\big)^{\frac{1}{2}}$ as

$$\Big(\sum_k \Big| \sum_{n\in T_k} \phi_{j,k,n} \psi_{j,k} (f_j * \Phi_{j,k,T}) - \sum_{n\in T_{k+L}} \phi_{j,k+L,n} \psi_{j,k+L} (f_j * \Phi_{j,k+L,T}) \Big|^2 \Big)^{\frac{1}{2}}$$

which we control by $I_{11}+I_{12}+I_{13}$, where

$$\begin{aligned} I_{11} &= \Big(\sum_k \Big|\sum_{n\in T_{k+L}} \phi_{j,k+L,n}\psi_{j,k+L}\big(f_j*(\Phi_{j,k,T}-\Phi_{j,k+L,T})\big)\Big|^2\Big)^{\frac12} \\ I_{12} &= \Big(\sum_k \Big|\sum_{n\in T_{k+L}} \phi_{j,k+L,n}(\psi_{j,k}-\psi_{j,k+L})(f_j*\Phi_{j,k,T})\Big|^2\Big)^{\frac12} \\ I_{13} &= \Big(\sum_k \Big|\Big(\sum_{n\in T_k}\phi_{j,k,n} - \sum_{n\in T_{k+L}}\phi_{j,k+L,n}\Big)\psi_{j,k}(f_j*\Phi_{j,k,T})\Big|^2\Big)^{\frac12}. \end{aligned}$$

By (4.13) and (4.14) we obtain

$$\|I_{11}\|_2 \le \Big(\sum_{(k+L,n)\in T}\Big\|\phi_{j,k+L,n}\psi_{j,k+L}\big(f_j*(\Phi_{j,k,T}-\Phi_{j,k+L,T})\big)\Big\|_2^2\Big)^{\frac12} \le C2^{-\frac{\mu}{p_j}}|I_T|^{\frac12}.$$

Thus by Lemma 16 and interpolation, we have $\|I_{11}\|_{q_j} \le C2^{-\frac{\mu}{p_j}\frac{2}{q_j}}|I_T|^{\frac{1}{q_j}}$, where C is independent of q_j. As in [10] we observe that

$$\begin{aligned} |\psi_{j,k}-\psi_{j,k+L}| &\le |1_{E_{k-m}\setminus E_{k-m+L}}*\psi_{k-m}| + |1_{(E_{k-m+L})^c}*(\psi_{k-m}-\psi_{k-m+L})| \\ &\le C\sum_{J\in\Delta_{k-m}}\rho^2_{k-m,J}. \end{aligned}$$

Introduce sets $V_k^+=\{n\in T_k:\ n+1\notin T_k\}$ and $V_k^-=\{n\in T_k:\ n-1\notin T_k\}$. Then we have

$$\begin{aligned} &\Big|\sum_{n\in T_{k+L}}\phi_{j,k+L,n}(\psi_{j,k}-\psi_{j,k+L})\Big| \\ &\le \sum_{n\in T_{k+L}}\phi^*_{j,k+L,n}\sum_{J\in\Delta_{k-m,T}}\rho^2_{k-m,J} + \sum_{n\in V^+_{k+L}\bigcup V^-_{k+L}}\phi^*_{j,k+L,n}\psi^*_{j,k+L} \\ &\quad + \sum_{n\in T_{k+L}}\phi^*_{j,k+L,n}\psi^*_{j,k+L}\frac{C}{(1+2^{-k+L}\mathrm{dist}(I_{k+L,n},(2I_T)^c))^N}. \end{aligned}$$

Using this, (4.15), and (4.10) we obtain $\|I_{12}\|_2\le C2^{-\frac{\mu}{p_j}}|I_T|^{\frac12}$. which provides an L^2 estimate for I_{12}. We now obtain a BMO estimate for I_{12}. For any $I=[2^{k_I}n_I, 2^{k_I}(n_I+1)]$, we have

$$\inf_c\int_I|I_{12}-c|^2dx \le \inf_c\int_I\Big|\sum_k\Big|\sum_{n\in T_{k+L}}\phi_{j,k+L,n}(\psi_{j,k}-\psi_{j,k+L})(f_j*\Phi_{j,k,T})\Big|^2 - c\Big|dx$$

and we control the last expression above by the sum $I'_{12}+I''_{12}$ where

$$\begin{aligned} I'_{12} &= \inf_c\int_I\Big|\sum_{k>k_I+m}\Big|\sum_{n\in T_{k+L}}\phi_{j,k+L,n}(x)(\psi_{j,k}-\psi_{j,k+L})(x)(f_j*\Phi_{j,k,T})(x)\Big|^2-c\Big|dx, \\ I''_{12} &= \inf_c\int_I\Big|\sum_{k\le k_I+m}\Big|\sum_{n\in T_{k+L}}\phi_{j,k+L,n}(x)(\psi_{j,k}-\psi_{j,k+L})(x)(f_j*\Phi_{j,k,T})(x)\Big|^2-c\Big|dx. \end{aligned}$$

Using (5.23) we can estimate I'_{12} by

$$
\begin{aligned}
&C|I|^2 \sum_{k>k_I+m} \Big\| \Big(\Big(\sum_{n\in T_{k+L}} \phi_{j,k+L,n}(\psi_{j,k}-\psi_{j,k+L})(f_j*\Phi_{j,k,T}) e^{-2\pi i c(\omega_{j,k,T})(\cdot)} \Big)^2 \Big)' \Big\|_\infty \\
&\le C|I|^2 \sum_{k>k_I+m} 2^{-k+m} \quad \le \quad C|I|.
\end{aligned}
$$

Also using (5.23) we obtain the sequence of estimates

$$
\begin{aligned}
I''_{12} \ &\le \int_I \sum_{k\le k_I+m} \Big| \sum_{n\in T_{k+L}} \phi_{j,k+L,n}(x)(\psi_{j,k}-\psi_{j,k+L})(x)(f_j*\Phi_{j,k,T})(x) \Big|^2 dx \\
&\le C \sum_{k\le k_I+m} \sum_{J\in\Delta_{k-m}} \int_I \rho_{k-m,J}(x)dx \\
&\le C \sum_{k\le k_I+m} \sum_{J\in\Delta_{k-m}, J\bigcap 2I\neq\emptyset} \int_I\int_J \frac{2^{-k+m}}{(1+2^{-k+m}|x-y|)^N} dydx \\
&\qquad + C \sum_{k\le k_I+m} \sum_{J\in\Delta_{k-m}, J\bigcap 2I=\emptyset} \int_I\int_J \frac{2^{-k+m}}{(1+2^{-k+m}|x-y|)^N} dydx \\
&\le C \sum_{k\le k_I+m} \sum_{J\in\Delta_{k-m}, J\subset 5^L I} |J| + C \sum_{k\le k_I+m} \frac{|I|}{(1+2^{-k+m}|I|)^N} \\
&\le C|I| + C|I| \sum_{k\le k_I+m} \frac{2^{(k-m)N}}{|I|^N} \ \le C|I|.
\end{aligned}
$$

Thus we have proved that $\|I_{12}\|_{BMO} \le C$. Then by interpolation, we obtain

$$
\|I_{12}\|_{q_j} \le C 2^{-\frac{\mu}{p_j}\frac{2}{q_j}} |I_T|^{\frac{1}{q_j}}, \tag{6.5}
$$

where C is independent of q_j. For I_{13}, we have

$$
\begin{aligned}
\|I_{13}\|_2 \ &\le \Big(\sum_k \sum_{n'} \Big\| \Big(\sum_{n\in T_k} \phi_{j,k,n} - \sum_{n\in T_{k+L}} \phi_{j,k+L,n} \Big) \psi_{j,k}(f_j*\Phi_{j,k,T}) \Big\|^2_{L^2(I_{k,n'})} \Big)^{\frac12} \\
&\le I^{(1)}_{13} + \big\| \phi^*_{j,k_T,n_T} \psi^*_{j,k_T}(f_j*\Phi_{j,k_T,T}) \big\|_2 \ \le \ I^{(1)}_{13} + C2^{-\eta\mu} 2^{-\frac{\mu}{p_j}} |I_T|^{\frac12},
\end{aligned}
$$

in view of (4.10), where we set $I^{(1)}_{13}$ to be the expression

$$
\Big\{ \sum_{k\neq k_T} \sum_{n'} \Big\| \Big| \sum_{n\in T_k} \phi_{j,k,n} - \sum_{n\in T_{k+L}} \phi_{j,k+L,n} \Big|^{\frac12} \Big\|_{L^\infty(I_{k,n'})} \Big\| \sum_{n\in T_k} \phi^*_{j,k,n} \psi^*_{j,k}(f_j*\Phi_{j,k,T}) \Big\|^2_{L^2(I_{k,n'})} \Big\}^{\frac12}.
$$

Note that by (4.10), we have

$$
\Big\| \sum_{n\in T_k} \phi^*_{j,k,n} \psi^*_{j,k}(f_j*\Phi_{j,k,T}) \Big\|^2_{L^2(I_{k,n'})} \le C 2^{-2\eta\mu} 2^{-\frac{2\mu}{p_j}} |I_{k,n}|.
$$

Thus, we have

$$
I^{(1)}_{13} \le C2^{-\eta\mu} 2^{-\frac{\mu}{p_j}} \Big(\sum_{k<k_T} \sum_{n'} \Big\| \Big| \sum_{n\in T_k} \phi_{j,k,n} - \sum_{n\in T_{k+L}} \phi_{j,k+L,n} \Big|^{\frac12} \Big\|_{L^\infty(I_{k,n'})} 2^k \Big)^{\frac12}.
$$

We next observe that

$$\Big|\sum_{n\in T_k}\phi_{j,k,n}-\sum_{n\in T_{k+L}}\phi_{j,k+L,n}\Big|\leq \sum_{n\in W_{k+L}}\phi^*_{j,k+L,n}+\sum_{n\in V^+_{k+L}\bigcup V^-_{k+L}}\phi^*_{j,k+L,n},$$

where

$$W_{k+L}=\{n\in\mathbf{Z}:\ (k,n)\notin T\quad \text{but there exists } (k+L,n')\in T \text{ such that } I_{k,n}\subset I_{k+L,n'}\}.$$

Note that by the convexity of T the set $\bigcup_k\bigcup_{n\in W_{k+L}}\{I_{k,n}\}$ is a set of pairwise disjoint intervals. Hence, we have

$$\begin{aligned} I^{(1)}_{13} &\leq C2^{-\eta\mu}2^{-\frac{\mu}{p_j}}\Big(\sum_{k<k_T}\sum_{n'}\sum_{n\in W_{k+L}}2^k\big\|\phi^*_{j,k+L,n}\big\|_{L^\infty(I_{k,n'})}\Big)^{\frac12}\\ &\qquad + C2^{-\eta\mu}2^{-\frac{\mu}{p_j}}\Big(\sum_{k<k_T}\sum_{n'}\sum_{n\in V^+_{k+L}\bigcup V^-_{k+L}}2^k\big\|\phi^*_{j,k+L,n}\big\|_{L^\infty(I_{k,n'})}\Big)^{\frac12}\\ &\leq C2^{-\eta\mu}2^{-\frac{\mu}{p_j}}\Big(\sum_{k<k_T}\sum_{n\in W_{k+L}}|I_{k,n}|+\sum_{k<k_T}\sum_{n\in V^+_{k+L}\bigcup V^-_{k+L}}|I_{k,n}|\Big)^{\frac12}\leq\ C2^{-\eta\mu}2^{-\frac{\mu}{p_j}}|I_T|^{\frac12}. \end{aligned}$$

Therefore, we obtain

$$\|I_{13}\|_2\leq C2^{-\eta\mu}2^{-\frac{\mu}{p_j}}|I_T|^{\frac12}.$$

By an argument similar to that used for the BMO estimate of I_{12}, we obtain that $\|I_{13}\|_{BMO}$ is at most a constant. Therefore interpolation yields

$$\|I_{13}\|_{q_j}\leq C2^{-\frac{\mu}{p_j}\frac{2}{q_j}}|I_T|^{\frac{1}{q_j}}, \tag{6.6}$$

where C is independent of q_j. We conclude that (6.4)-(6.6) imply

$$\|I_1\|_1\leq C_{q_1}2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})}|I_T|.$$

Similarly for $j=2$ and $j=3$ we get

$$\|I_j\|_1\leq C_{q_1}2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})}|I_T|.$$

Now we write

$$I_4=I_{41}+I_{42}+I_{43}+I_{44},$$

where

$$I_{41} = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m-3L}} f_{1,k-\tilde{k}}\Bigg) f_{2,k+2L}\Bigg(\sum_{n\in T_{k+L}}\phi_{3,k+L,n}\psi_{3,k+L}(f_3*(\Phi_{3,k,T}-\Phi_{3,k+L,T}))\Bigg),$$

$$I_{42} = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m-3L}} f_{1,k-\tilde{k}}\Bigg) f_{2,k+2L}\Bigg(\sum_{n\in T_{k+L}}\phi_{3,k+L,n}(\psi_{3,k}-\psi_{3,k+L})(f_3*\Phi_{3,k,T})\Bigg),$$

$$I_{43} = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m-3L}} f_{1,k-\tilde{k}}\Bigg) f_{2,k+2L}\Bigg(\sum_{n\in T_k}\phi_{3,k,n}-\sum_{n\in T_{k+L}}\phi_{3,k+L,n}\Bigg)\psi_{3,k}(f_3*\Phi_{3,k,T}),$$

$$I_{44} = \sum_{k\in\mathbf{Z}_r}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ m-3L\le\tilde{k}\le m}} f_{1,k-\tilde{k}}\Bigg) f_{2,k+2L}(f_{3,k}-f_{3,k+L}).$$

Next we observe the following three set-theoretic inclusions: $\operatorname{supp}\widehat{f}_{2,k+2L}\subset\omega_{2,k+2L}$,

$$\operatorname{supp}\Bigg(\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m-3L}}\widehat{f}_{1,k-\tilde{k}}\Bigg)\subset\bigcup_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m-3L}}\omega_{1,k-\tilde{k}}\,,\qquad\text{and}$$

$$-\operatorname{supp}\Bigg(\sum_{n\in T_{k+L}}\phi_{3,k+L,n}\psi_{3,k+L}(f_3*(\Phi_{3,k,T}-\Phi_{3,k+L,T}))\Bigg)^{\wedge}\subset(\omega_{1,k}+\omega_{2,k})\backslash(\omega_{1,k+L}+\omega_{2,k+L}).$$

Since by Lemma 4 we have

$$\bigcup_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m-3L}}\omega_{1,k-\tilde{k}}+\omega_{2,k+2L}\subset\omega_{1,k+L}+\omega_{2,k+L},$$

we obtain that the function I_{41} has vanishing integral. For I_{42}, we have

$$\begin{aligned}|I_{42}| \;&\le\; \sup_{k\in\mathbf{Z}_r}\Bigg|\sum_{\substack{\tilde{k}\in\mathbf{Z}_0\\ 0\le\tilde{k}\le m-3L}} f_{1,k-\tilde{k}}\Bigg|\Bigg(\sum_{k\in\mathbf{Z}_r}\Bigg|\sum_{n\in T_{k+L}}\phi_{2,k+L,n}(\psi_{2,k}-\psi_{2,k+L})\Bigg||f_2*\Phi_{2,k+2L,T}|^2\Bigg)^{\frac12}\\ &\quad\cdot\Bigg(\sum_{k\in\mathbf{Z}_r}\Bigg|\sum_{n\in T_{k+L}}\phi_{3,k+L,n}(\psi_{3,k}-\psi_{3,k+L})\Bigg||f_3*\Phi_{3,k,T}|^2\Bigg)^{\frac12}.\end{aligned}$$

As for the estimates regarding I_1 and I_{12}, we have

$$\begin{aligned}\|I_{42}\|_1 \;&\le C_{q_1}\Bigg\|\sum_{k\in\mathbf{Z}_r} f_{1,k}\Bigg\|_{q_1}\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}_r}\Bigg|\sum_{n\in T_{k+L}}\phi_{2,k+L,n}(\psi_{2,k}-\psi_{2,k+L})\Bigg||f_2*\Phi_{2,k+2L,T}|^2\Bigg)^{\frac12}\Bigg\|_{q_2}\\ &\qquad\cdot\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}_r}\Bigg|\sum_{n\in T_{k+L}}\phi_{3,k+L,n}(\psi_{3,k}-\psi_{3,k+L})\Bigg||f_3*\Phi_{3,k,T}|^2\Bigg)^{\frac12}\Bigg\|_{q_3}\\ &\le C_{q_1}2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})}|I_T|.\end{aligned}$$

As in the estimates for I_1 and I_{13}, it is easy to obtain

$$\|I_{43}\|_1 \le C_{q_1} 2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})}|I_T|.$$

In I_{44}, the index $\tilde{k}$ runs through three values. We estimate each of the three summands separately. For $\tilde{k} \in \{0, L, 2L\}$ we have

$$\sum_{k\in\mathbf{Z}_r} |f_{1,k-m+\tilde{k}} f_{2,k+2L}(f_{3,k} - f_{3,k+L})| \le \left\{\sum_{k\in\mathbf{Z}_r} |f_{1,k-m+\tilde{k}} f_{2,k+2L}|^2\right\}^{\frac{1}{2}} \left\{\sum_{k\in\mathbf{Z}_r} |f_{3,k} - f_{3,k+L}|^2\right\}^{\frac{1}{2}}$$

from which it follows

$$\begin{aligned}\|I_{44}\|_1 &\le C\left(\sum_{k\in\mathbf{Z}_r} \|f_{1,k-m+\tilde{k}} f_{2,k+2L}\|_2^2\right)^{\frac{1}{2}} \left\|\left(\sum_{k\in\mathbf{Z}_r} |f_{3,k} - f_{3,k+L}|^2\right)^{\frac{1}{2}}\right\|_2 \\ &\le C 2^{-(\frac{\mu}{p_1}+\frac{\mu}{p_3})}|I_T| \sup_{(k,n)\in T} \left\|\phi^*_{1,k,n}\psi^*_{2,k+m-\tilde{k}+2L}(f_2 * \Phi_{2,k+m-\tilde{k}+2L})\right\|_\infty.\end{aligned}$$

Note that

$$\begin{aligned}&\sup_{(k,n)\in T} \left\|\phi^*_{1,k,n}\psi^*_{2,k+m-\tilde{k}+2L}(f_2 * \Phi_{2,k+m-\tilde{k}+2L})\right\|_\infty \\ &\le C \sup_{(k,n)\in T} \left\|\phi^*_{1,k,n}\psi^*_{2,k+m-\tilde{k}+2L}(f_2 * \Phi_{2,k+m-\tilde{k}+2L})\right\|_2^{\frac{1}{2}} \\ &\qquad \cdot \left\|\left(\phi^*_{1,k,n}\psi^*_{2,k+m-\tilde{k}+2L}\left(e^{-2\pi i c(\omega_{2,k+m-\tilde{k}+2L,T})(\cdot)}(f_2 * \Phi_{2,k+m-\tilde{k}+2L})(\cdot)\right)\right)'\right\|_2^{\frac{1}{2}} \\ &\le C 2^{-\eta\mu} 2^{-\frac{\mu}{p_2}}.\end{aligned}$$

Thus, we obtain $\|I_{44}\|_1 \le C 2^{-\eta\mu} 2^{-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})\mu}|I_T|$. Hence, we have

$$\|I_4\|_1 \le C_{q_1} 2^{-(\frac{1}{p_1}+\frac{1}{p_2}\frac{2}{q_2}+\frac{1}{p_3}\frac{2}{q_3})}|I_T|$$

and a similar estimate is valid for $\|I_5\|_1$. This completes the proof of (4.20).

7. Counting the trees, part I

Having established the proof of Lemma 8, we now turn our attention to Lemma 7. The proof of this lemma will be presented in this and in the next two sections. In this section we prove (4.18) for $(i, j, \nu) \in \bigcup_{i,j\in\{2,3\}}\{(i, j, 2)\} \bigcup \{(1, 1, 1)\}$. We will need the following separation lemma from [7] and [8].

Lemma 18. *Let $S \subset \mathbf{Z}^3$, $s \in S$ and $I_s, \omega_s \in \mathcal{J}$ such that*
1) $\frac{1}{2}|I_s||\omega_s| \le |I_{s'}||\omega_{s'}| \le 2|I_s||\omega_s|$, for $s, s' \in S$.
2) $\{I_s\}_{s\in S}$, $\{\omega_s\}_{s\in S}$ are grids.
3) If $s \ne s'$, then $I_s \bigcap I_{s'} = \emptyset$ or $\omega_s \bigcap \omega_{s'} = \emptyset$.
Then for any $A \ge 1$, we can write $S = \left(\bigcup_{l=1}^{A^{10}} S_l\right) \bigcup S'$ where
4) For $1 \le l \le A^{10}$, if $s, s' \in S_l$ and $s \ne s'$, then $(AI_s \times \omega_s) \bigcap (AI_{s'} \times \omega_{s'}) = \emptyset$.
5) $\sum_{s\in S'} |I_s| \le Ce^{-A} \sum_{s\in S_1} |I_s|$.

We give a couple more definitions. For $s \in S \subset \mathbf{Z}^3$ we set

$$\phi_s(x) = \int_{I_s} \frac{2^m}{|I_s|} \psi^* \left(\frac{(x-y)2^m}{|I_s|} \right) dy, \tag{7.1}$$

and

$$\phi_s^*(x) = \int_{I_s} \frac{2^m}{|I_s|} \left(1 + \frac{|x-y|2^m}{|I_s|} \right)^{-N} dy. \tag{7.2}$$

Since $\psi^*(x) = (1+x^2)^{-N}$, we clearly have $|\phi_s(x)| \leq 2^N |\phi_s^*(x)| \big(1 + \frac{2^m \operatorname{dist}(x, I_s)}{|I_s|}\big)^{-N}$. Then we have the following almost orthogonality lemma.

Lemma 19. *Let $S \subset \mathbf{Z}^3$, $s \in S$ and $I_s, \omega_s \in \mathcal{J}$. Define the counting function of S by*

$$\mathcal{N}_S(x) = \sum_{s \in S} 1_{I_s}(x),$$

and for $m \geq 0$, let $\Phi_s \in \mathcal{S}(\mathbf{R})$ be such that $\operatorname{supp} \widehat{\Phi_s} \subset \omega_s$ and

$$|\Phi_s(x)| \leq C 2^m |I_s|^{-1} (1 + 2^m |I_s|^{-1} |x|)^{-N}.$$

Let S, I_s, ω_s satisfy 1), 2), and 3) below:
1) $2^{m-1} \leq |I_s||\omega_s| \leq 2^{m+1}$.
2) $\{I_s\}_{s \in S}$, $\{\omega_s\}_{s \in S}$ are grids.
3) $(AI_s \times \omega_s) \bigcap (AI_{s'} \times \omega_{s'}) = \emptyset$ for $s, s' \in S$, $s' \neq s$ for some $A \geq 1$.
Then

$$\sum_{s \in S} \|\phi_s (f * \Phi_s)\|_2^2 \leq C_N (1 + \|\mathcal{N}_S\|_\infty A^{-N}) \|f\|_2^2,$$

and for any $J \in \{I_s\}_{s \in S}$, we have

$$\sum_{\substack{s \in S \\ I_s \subset J}} \|\phi_s (f * \Phi_s)\|_2^2 \leq C_N |J| \|\mathcal{N}_S\|_\infty^{\frac{1}{N}} (1 + \|\mathcal{N}_S\|_\infty A^{-N}) \left(\inf_{x \in J} M_2(Mf)(x) \right)^2,$$

where C_N is independent of m. And if $m = 0$, we have

$$\sum_{s \in S} |I_s|^2 \left\| \phi_s \left(e^{-2\pi i c(\omega_s)(\cdot)} (f * \Phi_s)(\cdot) \right)' \right\|_2^2 \leq C_N (1 + \|\mathcal{N}_S\|_\infty A^{-N}) \|f\|_2^2,$$

and for any $J \in \{I_s\}_{s \in S}$, we have

$$\sum_{\substack{s \in S \\ I_s \subset J}} |I_s|^2 \left\| \phi_s \left(e^{-2\pi i c(\omega_s)(\cdot)} (f * \Phi_s)(\cdot) \right)' \right\|_2^2 \leq C_N |J| \|\mathcal{N}_S\|_\infty^{\frac{1}{N}} \Big\{ 1 + \frac{\|\mathcal{N}_S\|_\infty}{A^N} \Big\} \Big\{ \inf_J M_2(Mf) \Big\}^2.$$

Proof. We may assume that S is a finite set. Let $A_s f(x) = \phi_s (f * \Phi_s)(x)$, then $A_s^* f(x) = (f \overline{\phi}_s) * \tilde{\Phi}_s(x)$, where $\tilde{\Phi}_s(x) = \overline{\Phi}_s(-x)$. Thus, we have

$$\sum_{s \in S} \|A_s f\|_2^2 = \sum_{s \in S} \langle A_s^* A_s f, f \rangle \leq \| \sum_{s \in S} A_s^* A_s f \|_2 \|f\|_2 := B \|f\|_2.$$

Assume $\|f\|_2 \leq B$, otherwise there is nothing need to prove. Notice that

$$\begin{aligned} B^2 &= \sum_{s\in S}\sum_{s'\in S}\langle A_s^*A_sf, A_{s'}^*A_{s'}f\rangle \\ &\leq C\sum_{s\in S}\|A_sf\|_2^2 + \sum_{s\in S}\|A_sf\|_2 \sum_{s'\in S: s'\neq s}\|A_{s'}A_s^*\|\|A_{s'}f\|_2 \\ &\leq CB\|f\|_2 + \sum_{s\in S}\|A_sf\|_2 \sum_{s'\in S:s'\neq s}\|A_{s'}A_s^*\|\|A_{s'}f\|_2 := CB\|f\|_2 + \mathcal{O}. \end{aligned}$$

Note that $A_{s'}A_s^*f(x) = \int K(x,y)f(y)dy$, where $K(x,y) = \phi_{s'}(x)(\tilde{\Phi}_s * \Phi_{s'})(x-y)\overline{\phi}_s(y)$. It is easy to see that $K(x,y) = 0$ if $\omega_s \bigcap \omega_{s'} = \emptyset$. We claim that

$$\|A_{s'}A_s^*\| \leq C_N \frac{2^m|I_{s'}|^{\frac12}}{|I_s|^{\frac12}}\left(1+\frac{2^m\text{dist}(I_s,I_{s'})}{|I_s|}\right)^{-N}, \text{ if } |I_s'| \leq 2|I_s|. \tag{7.3}$$

Now we prove (7.3). Notice that

$$\int |K(x,y)|\,dx \leq C_N \frac{2^m|I_{s'}|}{|I_s|}\left(1+\frac{2^m\text{dist}(I_s,I_{s'})}{|I_s|}\right)^{-N}.$$

Similarly, we have

$$\int |K(x,y)|\,dy \leq C_N 2^m\left(1+\frac{2^m\text{dist}(I_s,I_{s'})}{|I_s|}\right)^{-N}.$$

Thus by Schur's lemma, we obtain (7.3). Hence

$$\begin{aligned} \mathcal{O} &\leq 2\sum_{s\in S}\|A_sf\|_2 \sum_{\substack{s'\in S\\ s'\neq s\\ \omega_s\subset\omega_{s'}}}\|A_{s'}A_s^*\|\|A_{s'}f\|_2 \\ &\leq C\sum_{s\in S}\|A_sf\|_2 \sum_{\substack{s'\in S\\ s'\neq s\\ \omega_s\subset\omega_{s'}}}\frac{2^m|I_{s'}|^{\frac12}}{|I_s|^{\frac12}}\frac{\|A_{s'}f\|_2}{\left(1+\frac{2^m\text{dist}(I_s,I_{s'})}{|I_s|}\right)^{N+2}}. \end{aligned}$$

Notice $\|A_sf\|_2 \leq C2^m \inf_{x\in I_s} Mf(x)|I_s|^{\frac12}$, where C is independent of m. The proof of this fact is the same as the proof of (5.1). Therefore, we have the estimate

$$\begin{aligned} \mathcal{O} &\leq C\sum_{s\in S}\inf_{x\in I_s} Mf(x) \sum_{\substack{s'\in S\\ s'\neq s\\ \omega_s\subset\omega_{s'}}}\frac{2^{3m}|I_{s'}|}{\left(1+\frac{2^m\text{dist}(I_s,I_{s'})}{|I_s|}\right)^{N+2}}\inf_{x\in I_{s'}} Mf(x) \\ &\leq CA^{-N}\sum_{s\in S}\inf_{x\in I_s} Mf(x) \sum_{\substack{s'\in S\\ s'\neq s\\ \omega_s\subset\omega_{s'}}}\int_{I_{s'}}\frac{Mf(x)}{\left(1+\frac{\text{dist}(x,I_s)}{|I_s|}\right)^2}dx \\ &\leq CA^{-N}\sum_{s\in S}\int_{I_s}\left(M(Mf)(x)\right)^2dx \\ &\leq CA^{-N}\int\sum_{s\in S}1_{I_s}(x)\left(M(Mf)(x)\right)^2dx \,\leq\, CA^{-N}\|\mathcal{N}_S\|_\infty B\|f\|_2. \end{aligned}$$

Thus, we have obtained $B \leq C_N(1+A^{-N}\|\mathcal{N}_S\|_\infty)\|f\|_2$ and

$$\sum_{s\in S}\|A_s f\|_2^2 \leq C_N(1+A^{-N}\|\mathcal{N}_S\|_\infty)\|f\|_2^2. \tag{7.4}$$

If we set $J_N = 2\|\mathcal{N}_S\|_\infty^{\frac{1}{N}} J$, then we have

$$\frac{1}{2}\sum_{\substack{s\in S\\ I_s\subset J}}\|A_s f\|_2^2 \leq \sum_{\substack{s\in S\\ I_s\subset J}}\|A_s(f1_{J_N})\|_2^2 + \sum_{\substack{s\in S\\ I_s\subset J}}\|A_s(f1_{(J_N)^c})\|_2^2 := G_1+G_2.$$

First we use (7.4) to obtain

$$G_1 \leq C_N(1+A^{-N}\|\mathcal{N}_S\|_\infty)\|f1_{J_N}\|_2^2 \leq C_N|J|\|\mathcal{N}_S\|_\infty^{\frac{1}{N}}(1+A^{-N}\|\mathcal{N}_S\|_\infty)\big\{\inf_J M_2 f\big\}^2.$$

Notice $\operatorname{dist}(y,I_s)\geq \|\mathcal{N}_S\|_\infty^{\frac{1}{N}}|J|$ if $y\in(J_N)^c$. Thus we have

$$\begin{aligned}
G_2 &\leq \sum_{s\in S: I_s\subset J}\int|\phi_s(x)|^2\left|\int_{(J_N)^c} f(y)\Phi_s(x-y)dy\right|^2 dx\\
&\leq C\sum_{s\in S: I_s\subset J}\|\mathcal{N}_S\|_\infty^{-1}\big(\inf_{I_s} Mf(x)\big)^2\int|\phi_s^*(x)|dx\\
&\leq C\sum_{s\in S: I_s\subset J}\|\mathcal{N}_S\|_\infty^{-1}\big(\inf_{I_s} Mf\big)^2|I_s| \;\leq\; C\|Mf\|_{L^2(J)}^2 \;\leq\; C|J|\big(\inf_J M_2(Mf)\big)^2.
\end{aligned}$$

Therefore, we obtain

$$\sum_{s\in S: I_s\subset J}\|A_s f\|_2^2 \leq C_N|J|\|\mathcal{N}_S\|_\infty^{\frac{1}{N}}(1+A^{-N}\|\mathcal{N}_S\|_\infty)\left(\inf_{x\in J} M_2(Mf)(x)\right)^2.$$

The proofs of the last two statements of this lemma ($m=0$) is entirely similar. This completes the proof of Lemma 19. □

We now turn to the proof of (4.18) for $\{(i,j,\nu)\}\in\bigcup_{i,j\in\{2,3\}}(i,j,2)\bigcup\{(1,1,1)\}$. We only prove the case $(i,j,2)$ if $i,j\in\{2,3\}$. The proof for the case $(1,1,1)$ is similar. Let $\mathcal{F}_{i,j,2}=\bigcup_l\{T^2_{\mu,i,j,l}\}$, $\mathcal{N}_{\mathcal{F}_{i,j,2}}(x)=\sum_{T\in\mathcal{F}_{i,j,2}}1_{I_T}(x)$, and

$$\mathcal{N}^\sharp_{\mathcal{F}_{i,j,2}}(x)=\sup_{\substack{J\in\{I_T:T\in\mathcal{F}_{i,j,2}\}\\ J\ni x}}\frac{1}{|J|}\sum_{\substack{T\in\mathcal{F}_{i,j,2}\\ I_T\subset J}}|I_T|.$$

It is sufficient to prove

$$\|\mathcal{N}_{\mathcal{F}_{i,j,2}}\|_1 \leq C2^{10\eta p_j\mu}2^\mu. \tag{7.5}$$

Since $\mathcal{N}_{\mathcal{F}_{i,j,2}}$ is integer-valued, to prove (7.5), it suffices to show that there exists $0<\varepsilon<\eta$ such that, for any $\lambda\geq 1$,

$$|\{x\in\mathbf{R}:\mathcal{N}_{\mathcal{F}_{i,j,2}}(x)\geq\lambda\}| \leq C_\varepsilon 2^{10\eta p_j\mu}2^\mu\lambda^{12p_j\varepsilon-\frac{p_j}{2}}, \tag{7.6}$$

since $\frac{p_j}{2}-12p_j\varepsilon>1$. Take $\mathcal{F}'\subset\mathcal{F}_{i,j,2}$ such that $\mathcal{N}_{\mathcal{F}'}(x)=\min\{\mathcal{N}_{\mathcal{F}_{i,j,2}}(x),\lambda\}$. See [7]. Then we have

$$|\{x\in\mathbf{R}:\mathcal{N}_{\mathcal{F}'}(x)\geq\lambda\}| = |\{x\in\mathbf{R}:\mathcal{N}_{\mathcal{F}_{i,j,2}}(x)\geq\lambda\}|.$$

Let $A=\lambda^\varepsilon$. By Lemma 18, we have $\mathcal{F}'=\bigcup_{l=1}^{A^{10}}\mathcal{F}_l\bigcup\mathcal{F}''$, where:

1) for $T, T' \in \mathcal{F}_l$ and $T' \neq T$, $(AI_T \times \omega_{i,T}) \bigcap (AI_{T'} \times \omega_{i,T'}) = \emptyset$, and
2) $\sum_{T\in\mathcal{F}''} |I_T| \leq Ce^{-A} \sum_{T\in\mathcal{F}_1} |I_T|$.

Applying Lemma 19 with $S := \{(k_T, n_T, l_T)\}_{T\in\mathcal{F}_l}$, $\mathcal{N}_S := \mathcal{N}_{\mathcal{F}_l}$, $I_s := I_T$, $\omega_s := \omega_{j,T}$, $\phi_s := \phi^*_{j,k_T+\tilde{k},n_T}$, $\Phi_s := \Phi_{j,k_T+\tilde{k},T}$, $N := \frac{1}{\varepsilon}$, we obtain for $J \in \{I_T : T \in \mathcal{F}_l\}$,

$$\begin{aligned}\sum_{\substack{T\in\mathcal{F}_l\\ I_T\subset J}} \|\phi^*_{j,k_T+\tilde{k},n_T}(f_j * \Phi_{j,k_T+\tilde{k},T})\|_2^2 &\leq C|J|\|\mathcal{N}_{\mathcal{F}_l}\|_\infty^\varepsilon (1 + \|\mathcal{N}_{\mathcal{F}_l}\|_\infty A^{-\frac{1}{\varepsilon}}) \big\{ \inf_J M_2(Mf_j) \big\}^2 \\ &\leq C|J|\lambda^\varepsilon \big\{ \inf_J M_2(Mf_j) \big\}^2,\end{aligned}$$

since $\mathcal{N}_{\mathcal{F}_l} \leq \mathcal{N}_{\mathcal{F}'} \leq \lambda$.

Assume that $T \in \mathcal{F}_l$ satisfies (4.10). Then we have

$$\begin{aligned}\sum_{\substack{T\in\mathcal{F}_l\\ I_T\subset J}} |I_T| &\leq C2^{2\eta\mu} 2^{\frac{2\mu}{p_j}} \sum_{\substack{T\in\mathcal{F}_l\\ I_T\subset J}} \|\phi^*_{j,k_T+\tilde{k},n_T} \psi^*_{j,k_T+\tilde{k}}(f_j * \Phi_{j,k_T+\tilde{k},T})\|_2^2 \\ &\leq C2^{2\eta\mu} 2^{\frac{2\mu}{p_j}} |J| \lambda^\varepsilon \left(\inf_{x\in J} M_2(Mf_j)(x) \right)^2.\end{aligned}$$

Hence, we obtain

$$\mathcal{N}^\sharp_{\mathcal{F}_l}(x) \leq C2^{2\eta\mu} 2^{\frac{2\mu}{p_j}} \lambda^\varepsilon \left(M_2(Mf_j)(x) \right)^2.$$

Since $p_j > 2$, using [7] we obtain

$$\|\mathcal{N}_{\mathcal{F}_l}\|_{\frac{p_j}{2}} \leq C\|\mathcal{N}^\sharp_{\mathcal{F}_l}\|_{\frac{p_j}{2}} \leq C2^{2\eta\mu} 2^{\frac{2\mu}{p_j}} \lambda^\varepsilon \|M((Mf_j)^2)\|_{\frac{p_j}{2}} \leq C2^{2\eta\mu} 2^{\frac{2\mu}{p_j}} \lambda^\varepsilon \|f_j\|_{p_j}^2 \leq C2^{2\eta\mu} 2^{\frac{2\mu}{p_j}} \lambda^\varepsilon.$$

Thus, we have

$$\left| \left\{ x \in \mathbf{R} : \mathcal{N}_{\mathcal{F}_l}(x) \geq \frac{\lambda}{A^{10}+1} \right\} \right| \leq C2^{4\eta p_j \mu} 2^\mu \lambda^{6p_j\varepsilon - \frac{p_j}{2}}.$$

Therefore, we have

$$\begin{aligned}|\{\mathcal{N}_{\mathcal{F}'} \geq \lambda\}| &\leq \sum_{l=1}^{A^{10}} \left| \left\{ \mathcal{N}_{\mathcal{F}_l} \geq \frac{\lambda}{A^{10}+1} \right\} \right| + \left| \left\{ \mathcal{N}_{\mathcal{F}''} \geq \frac{\lambda}{A^{10}+1} \right\} \right| \\ &\leq C2^{4\eta p_j \mu} 2^\mu \lambda^{12p_j\varepsilon - \frac{p_j}{2}} + C\lambda^{10\varepsilon - 1} \|\mathcal{N}_{\mathcal{F}''}\|_1 \;\leq\; C2^{4\eta p_j \mu} 2^\mu \lambda^{12p_j\varepsilon - \frac{p_j}{2}}.\end{aligned}$$

Here we used the fact that $\mathcal{N}_{\mathcal{F}_1}$ is integer-valued in the estimate $\|\mathcal{N}_{\mathcal{F}_1}\|_1 \leq \|\mathcal{N}_{\mathcal{F}_1}\|_{p_j/2}^{p_j/2}$. This completes the proof of (7.6).

8. Counting the trees, part II

In this section, we prove (4.18) for $(i, j, \nu) \in \bigcup_{i,j\in\{2,3\}} \left(\{(1, j, 1), (i, 1, 1)\} \bigcup_{\nu=3}^{4} \{(i, j, \nu)\} \right)$.
First, we need the following almost orthogonality lemma.

The proof of this lemma is similar to that of Lemma 18 and is therefore omitted.

Lemma 20. *Let $l \in \mathbf{Z}$, $T_l \subset \mathbf{Z}^3$, $S = \bigcup_{l=1}^\infty T_l$, $s \in S$ and $I_s, \omega_s \in \mathcal{J}$. Let $A \geq 1$, and*

$$\mathcal{N}_S(x) = \sum_{l=1}^\infty 1_{I_{T_l}}(x).$$

Also let Φ_s *be in* $\mathcal{S}$ *with* $\operatorname{supp}\widehat{\Phi_s} \subset \omega_s$, $|\Phi_s(x)| \le C2^m|I_s|^{-1}(1+2^m|I_s|^{-1}|x|)^{-N}$. *Suppose that* S, I_s, ω_s *satisfy the following*
1) $2^{m-1} \le |I_s||\omega_s| \le 2^{m+1}$,
2) $\{\omega_s\}_{s\in S}$ *is a grid and* $\{I_s\}_{s\in S}$ *is a dyadic grid,*
3) for $s, s' \in T_l$, *if* $I_s = I_{s'}$, *then* $s = s'$; *and (8.1), (8.2), and (8.3) below hold*

$$AI_s \subset I_{T_l} \quad \text{for any } \; s \in T_l, \tag{8.1}$$

$$\text{if } \; s \in T_l, s' \in T_{l'}, l \neq l', \; \text{ and } \; \omega_s \subset \omega_{s'}, \; \text{ then } \; \omega_s = \omega_{s'} \; \text{ or } \; I_{T_l} \cap I_{s'} = \emptyset, \tag{8.2}$$

and

$$\text{for } \; s, s' \in T_l, \; \text{ we have either } \; \omega_s = \omega_{s'} \; \text{ or } \; \omega_s \cap \omega_{s'} = \emptyset. \tag{8.3}$$

Then we have the inequality

$$\sum_{s\in S} \|\phi_s(f*\Phi_s)\|_2^2 \le C_N(1+\|\mathcal{N}_S\|_\infty A^{-N})\|f\|_2^2, \tag{8.4}$$

and for any $J \in \{I_{T_l}\}_{l=1}^\infty$, *we have*

$$\sum_{s\in \bigcup\limits_{l:I_{T_l}\subset J} T_l} \|\phi_s(f*\Phi_s)\|_2^2 \le C_N|J|\|\mathcal{N}_S\|_\infty^{\frac{1}{N}}(1+\|\mathcal{N}_S\|_\infty A^{-N})\left(\inf_{x\in J} M_2(Mf)(x)\right)^2, \tag{8.5}$$

where C_N *is independent of* m. *And if* $m = 0$, *we have*

$$\sum_{s\in S} |I_s|^2 \left\|\phi_s\left(e^{-2\pi i c(\omega_s)(\cdot)}(f*\Phi_s)(\cdot)\right)'\right\|_2^2 \le C_N(1+\|\mathcal{N}_S\|_\infty A^{-N})\|f\|_2^2,$$

and for any $J \in \{I_{T_l}\}_{l=1}^\infty$, *we also have*

$$\sum_{s\in \bigcup\limits_{l:I_{T_l}\subset J} T_l} |I_s|^2 \left\|\phi_s\big(e^{-2\pi i c(\omega_s)(\cdot)}(f*\Phi_s)(\cdot)\big)'\right\|_2^2 \le C_N|J|\|\mathcal{N}_S\|_\infty^{\frac{1}{N}}\Big\{1+\frac{\|\mathcal{N}_S\|_\infty}{A^N}\Big\}\big(\inf_J M_2(Mf)\big)^2.$$

We now return to the proof of (4.18) for $(i,j,\nu) \in \bigcup\limits_{i,j\in\{2,3\}} \Big(\{(1,j,1),(i,1,1)\} \bigcup\limits_{\nu=3}^{4} \{(i,j,\nu)\}\Big)$.

We only prove the case $(1,j,1)$ for $j \in \{2,3\}$. The proof for the case $(i,1,1)$ for $i \in \{2,3\}$ is similar. Let $\mathcal{F}_{1,j,1} = \bigcup_l T^1_{\mu,1,j,l}$, $\mathcal{N}_{\mathcal{F}_{1,j,1}}(x) = \sum_{T\in\mathcal{F}_{1,j,1}} 1_{I_T}(x)$, and

$$\mathcal{N}^\sharp_{\mathcal{F}_{1,j,1}}(x) = \sup_{\substack{J\in\{I_T:T\in\mathcal{F}_{1,j,1}\}\\ J\ni x}} \frac{1}{|J|} \sum_{\substack{T\in\mathcal{F}_{1,j,1}\\ I_T\subset J}} |I_T|.$$

It is enough to prove that

$$\|\mathcal{N}_{\mathcal{F}_{1,j,1}}\|_1 \le C2^{10\eta p_j\mu}2^\mu.$$

But since $\mathcal{N}_{\mathcal{F}_{1,j,1}}$ is integer-valued, it is sufficient to show that there exists $0<\varepsilon<\eta$ such that, for any $\lambda \ge 1$,

$$|\{x\in\mathbf{R} : \mathcal{N}_{\mathcal{F}_{1,j,1}}(x) \ge \lambda\}| \le C_\varepsilon 2^{10\eta p_j\mu}2^\mu\lambda^{12p_j\varepsilon-\frac{p_j}{2}}, \tag{8.6}$$

since $\frac{p_j}{2} - 12p_j\varepsilon > 1$. Take $\mathcal{F}' \subset \mathcal{F}_{1,j,1}$ such that $\mathcal{N}_{\mathcal{F}'}(x) = \min\{\mathcal{N}_{\mathcal{F}_{1,j,1}}(x), \lambda\}$. Then we have

$$|\{x\in\mathbf{R} : \mathcal{N}_{\mathcal{F}'}(x) \ge \lambda\}| = |\{x\in\mathbf{R} : \mathcal{N}_{\mathcal{F}_{1,j,1}}(x) \ge \lambda\}|.$$

Define a partial order on the set $\{I_s\}_{s\in S}\times\{\omega_s\}_{s\in S}$ by setting

$$I_s\times\omega_s < I_{s'}\times\omega_{s'}$$

only when $I_s\subset I_{s'}$ and $\omega_{s'}\subset\omega_s$. For $T\in\mathcal{F}'$, define

$$T^{\min}=\{s\in T: I_s\times\omega_{1,s} \text{ is minimal w.r.t. } <\},$$

$$T^{\partial}=\{s\in T: I_s\cap(1-2^{-4})I_T=\emptyset\},$$

$$T^{\partial\max}=\{s\in T^{\partial}: I_s\times\omega_{1,s} \text{ is maximal in } T^{\partial} \text{ w.r.t. } <\},$$

$$T^{\lambda}=\{s\in T\backslash T^{\min}: 2^5\lambda^{\varepsilon}|I_s|\geq|I_T|\},$$

$$T^{\text{nice}}=T\backslash(T^{\min}\bigcup T^{\partial}\bigcup T^{\lambda}).$$

We aim to apply Lemma 20 with $S=\bigcup_{T\in\mathcal{F}'}T^{\text{nice}}$, $A=\lambda^{\varepsilon}$, $\mathcal{N}_S=\mathcal{N}_{\mathcal{F}'}$, $I_s=I_{k,n}$, $\omega_s=\omega_{j,k,T}$, $\phi_s=\phi^*_{j,k,n}$, $\Phi_s=\Phi_{j,k,T}$, and $N=\frac{1}{\varepsilon}$. We check (8.1)-(8.3). (8.3) holds because $T\in\mathcal{F}'$ is a tree of type 1. And the subtraction of T^{λ} and T^{∂} makes all $s\in T^{\text{nice}}$ satisfy $AI_s\subset I_T$, which is (8.1).

Now we check (8.2). Assume $T,T'\in\mathcal{F}'$, and $s\in T$, $s'\in T'$ with $\omega_{j,s}\subset\omega_{j,s'}$ and $\omega_{j,s}\neq\omega_{j,s'}$. Since T is a tree of type 1, we have $\omega_{1,T}\subset\omega_{1,s}$. By Lemma 4, we have $\omega_{j,s}<\omega_{j,T}$, $\text{dist}(\omega_{j,s},\omega_{j,T})<2|\omega_{2,s}|$. Notice $5\omega_{j,s}\subset\omega_{j,s'}$, we have $\omega_{j,T}\bigcap\omega_{j,s'}\neq\emptyset$. Thus $\omega_{j,T}\subset\omega_{j,s'}$. Again by Lemma 4, we have $\omega_{1,T}<\omega_{1,s'}$ and $\text{dist}(\omega_{1,T},\omega_{1,s'})<2|\omega_{1,s'}|$. Since there exists $s''\in T'$ such that $s''\in(T')^{\min}$ and $5\omega_{1,s'}\subset\omega_{1,s''}$, we have $\omega_{1,s''}\bigcap\omega_{1,T}\neq\emptyset$. By the maximality of T and (4.16), we know $I_T\bigcap I_{s'}=\emptyset$, which is (8.2).

For $J\in\{I_T: T\in\mathcal{F}'\}$, Lemma 20 gives

$$\sum_{\substack{T\in\mathcal{F}'\\ I_T\subset J}}\sum_{(k,n)\in T^{\text{nice}}}\|\phi^*_{j,k,n}(f_j*\Phi_{j,k,T})\|_2^2\leq C|J|\lambda^{\varepsilon}\left(\inf_{x\in J}M_2(Mf_j)(x)\right)^2. \tag{8.7}$$

By (4.8), we have, for any $T\in\mathcal{F}'$,

$$\begin{aligned}|I_T| &\leq 2^{-8}2^{\frac{2\mu}{p_j}}\sum_{(k,n)\in T}\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\|_2^2\\
&\leq 2^{-8}2^{\frac{2\mu}{p_j}}\sum_{(k,n)\in T^{\text{nice}}}\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\|_2^2+2^{-8}2^{\frac{2\mu}{p_j}}\sum_{(k,n)\in T^{\min}}\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\|_2^2\\
&\quad+2^{-8}2^{\frac{2\mu}{p_j}}\sum_{(k,n)\in T^{\partial}}\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\|_2^2+2^{-8}2^{\frac{2\mu}{p_j}}\sum_{(k,n)\in T^{\lambda}}\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\|_2^2.\end{aligned}$$

Note that by (4.10) we have

$$\begin{aligned}\sum_{(k,n)\in T^{\min}}\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\|_2^2 &\leq \sum_{(k,n)\in T^{\min}}2^{-2\eta(\mu-1)}2^{-\frac{2(\mu-1)}{p_j}}|I_{k,n}|\\
&\leq 2^{-2\eta(\mu-1)}2^{-\frac{2(\mu-1)}{p_j}}|I_T| \leq 2^32^{-\frac{2\mu}{p_j}}|I_T|.\end{aligned}$$

By (4.8) we also have

$$\sum_{(k,n)\in T^{\partial}}\|\phi^*_{j,k,n}\psi^*_{j,k}(f_j*\Phi_{j,k,T})\|_2^2\leq\sum_{(k,n)\in T^{\partial\max}}2^42^{-\frac{2(\mu-1)}{p_j}}|I_{k,n}|\leq 2^22^{-\frac{2\mu}{p_j}}|I_T|.$$

Therefore, we conclude

$$|I_T| \le 2^{\frac{2\mu}{p_j}} \sum_{(k,n)\in T^{\text{nice}}} \|\phi^*_{j,k,n}\psi^*_{j,k}(f_j * \Phi_{j,k,T})\|_2^2 + 2^{-7} 2^{\frac{2\mu}{p_j}} \sum_{(k,n)\in T^{\lambda}} \|\phi^*_{j,k,n}\psi^*_{j,k}(f_j * \Phi_{j,k,T})\|_2^2. \tag{8.8}$$

Now observe that in view of (4.10) we have

$$\begin{aligned} \sum_{(k,n)\in T^{\lambda}} \|\phi^*_{j,k,n}\psi^*_{j,k}(f_j * \Phi_{j,k,T})\|_2^2 &\le \sum_{(k,n)\in T^{\lambda}} 2^{-2\eta(\mu-1)} 2^{-\frac{2(\mu-1)}{p_j}} |I_{k,n}| \\ &\le 2^3 2^{-2\eta\mu} 2^{-\frac{2\mu}{p_j}} (\log_2 \lambda^{\varepsilon} + 5)|I_T|. \end{aligned}$$

Hence, if $\log_2 \lambda^{\varepsilon} \le 2^{2\eta\mu}$, we obtain

$$|I_T| \le C 2^{\frac{2\mu}{p_j}} \sum_{(k,n)\in T^{\text{nice}}} \|\phi^*_{j,k,n}\psi^*_{j,k}(f_j * \Phi_{j,k,T})\|_2^2. \tag{8.9}$$

From (8.7) we deduce

$$\sum_{\substack{T\in\mathcal{F}' \\ I_T\subset J}} |I_T| \le C 2^{\frac{2\mu}{p_j}} |J| \lambda^{\varepsilon} \left(\inf_{x\in J} M_2(Mf_j)(x) \right)^2 . \tag{8.10}$$

This gives $\mathcal{N}^{\sharp}_{\mathcal{F}'}(x) \le C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \lambda^{\varepsilon} \left(M_2(Mf_j)(x)\right)^2$. Since $p_j > 2$, when $\log_2 \lambda^{\varepsilon} \le 2^{2\eta\mu}$ we obtain

$$\|\mathcal{N}_{\mathcal{F}'}\|_{\frac{p_j}{2}} \le C \|\mathcal{N}^{\sharp}_{\mathcal{F}'}\|_{\frac{p_j}{2}} \le C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \lambda^{\varepsilon} . \tag{8.11}$$

The situation where $\log_2 \lambda^{\varepsilon} > 2^{2\eta\mu}$ is similar. By using a similar method, we obtain

$$\|\mathcal{N}_{\mathcal{F}'}\|_{\frac{p_j}{2}} \le C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \lambda^{12\varepsilon} + C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \frac{\lambda^{1+2\varepsilon}}{e^{\lambda^{\varepsilon}}} \le C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \lambda^{12\varepsilon} \tag{8.12}$$

when $\log_2 \lambda^{\varepsilon} > 2^{2\eta\mu}$. Using (8.11) and (8.12) we deduce

$$\|\mathcal{N}_{\mathcal{F}'}\|_{\frac{p_j}{2}} \le C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \lambda^{12\varepsilon} + C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \frac{\lambda^{1+2\varepsilon}}{e^{\lambda^{\varepsilon}}} \le C_{\varepsilon} 2^{\frac{2\mu}{p_j}} \lambda^{12\varepsilon} \tag{8.13}$$

for any $\lambda \ge 1$. Hence, $|\{x \in \mathbf{R} : \mathcal{N}_{\mathcal{F}'}(x) \ge \lambda\}| \le C_{\varepsilon} 2^{\mu} \lambda^{6p_j\varepsilon - \frac{p_j}{2}}$, which implies (8.6).

The proof for $(i,j,\nu) \in \bigcup_{i,j\in\{2,3\}} \bigcup_{\nu=3}^{4} \{(i,j,\nu)\}$ is similar. We only check the following separation condition for trees:
Let $\mathcal{F}_{i,j,\nu} = \bigcup_l T^{\nu}_{\mu,i,j,l}$ and $(\omega_{j,k-L,T} - \omega_{j,k,T})^+ = (\omega_{j,k-L,T} - \omega_{j,k,T}) \bigcap [c(\omega_{j,T}), \infty)$, $(\omega_{j,k-L,T} - \omega_{j,k,T})^- = (\omega_{j,k-L,T} - \omega_{j,k,T}) \bigcap (-\infty, c(\omega_{j,T})]$.

Lemma 21. *Let $(i,j,\nu) \in \bigcup_{i,j\in\{2,3\}} \bigcup_{\nu=3}^{4} \{(i,j,\nu)\}$, $T, T' \in \mathcal{F}_{i,j,\nu}$, $(k,n) \in T^{\text{nice}}$, $(k',n') \in (T')^{\text{nice}}$. Suppose that $T \ne T'$, $k'+L < k < k_T$, $k'+2L \le k_{T'}$, and that either $\nu = 3$ and $(\omega_{j,k-L,T} - \omega_{j,k,T})^+ \bigcap (\omega_{j,k'-L,T'} - \omega_{j,k',T'})^+ \ne \emptyset$, or $\nu = 4$ and $(\omega_{j,k-L,T} - \omega_{j,k,T})^- \bigcap (\omega_{j,k'-L,T'} - \omega_{j,k',T'})^- \ne \emptyset$. Then we have $I_{k',n'} \bigcap I_T = \emptyset$.*

Proof. For simplicity, we only prove the case $(i,j,\nu) = (2,2,3)$. The other cases are similar. Assume $I_{k',n'} \bigcap I_T \ne \emptyset$. Since $(\omega_{2,k-L,T} - \omega_{2,k,T})^+ \bigcap (\omega_{2,k'-L,T'} - \omega_{2,k',T'})^+ \ne \emptyset$, there exists $(k'',n'') \in (T')^{\min}$ such that $\omega_{2,k,T} \subset \omega_{2,k'',T'}$. By assumption, we have $I_{k'',n''} \times \omega_{2,k'',T'} < I_T \times \omega_{2,k_T,T}$. It follows from the maximality of T, that we chose T' before T.

By (4.17) and $\omega_{2,k_T,T} \bigcap \omega_{2,k_{T'},T'} = \emptyset$, we have $\omega_{2,k_T,T} < \omega_{2,k_{T'},T'}$. Note that

$$\text{dist}(\omega_{2,T'}, (\omega_{2,k'-L,T'} - \omega_{2,k',T'})^+) \ge 2 \cdot 2^{-k'-L} \ge 2 \cdot 2^L |\omega_{2,k,T}|,$$

by the convexity of T', the central grid structure of $\{\omega_{2,k,T'}\}$, and the known fact that $\mathrm{dist}(\omega_{2,T}, (\omega_{2,k'-L,T'} - \omega_{2,k',T'})^+) \le 2^L|\omega_{2,k,T}|$. This contradicts that $\omega_{2,k_T,T} < \omega_{2,k_{T'},T'}$. Therefore we have $I_{k',n'} \bigcap I_T = \emptyset$. This completes the proof of Lemma 21. □

9. Counting the trees, part III

In this section, we prove (4.18) for $i, j \in \{2, 3\}$ and $\nu = 5$. Here we will need the following almost orthogonality lemma whose proof we omit since it is similar to that of Lemma 18.

Lemma 22. *Let $\mathcal{F}$ be a collection of trees of type i and*

$$Q := \{(k, J, T) : k \in \mathbf{Z}, T \in \mathcal{F}, J \in \Delta_{k-m,T}\}.$$

For $q \in Q$, define $A_q f(x) := \rho_q(x)(f * \Phi_q)(x)$, where ρ_q and Φ_q are functions satisfying

$$|\rho_q(x)| \le C_N\Big(1 + |J|^{-1}\mathrm{dist}(x, J)\Big)^{-N}, \tag{9.1}$$

$$|\Phi_q(x)| \le C_N|J|^{-1}\Big(1 + |J|^{-1}|x|\Big)^{-N} \text{ and } \mathrm{supp}\,\widehat{\Phi_q} \subset \omega_q, \tag{9.2}$$

where $\omega_q = \omega_{j,k,T}$. Assume furthermore that

$$(AJ \times \omega_q)\bigcap(AJ' \times \omega_{q'}) = \emptyset, \tag{9.3}$$

whenever $q, q' \in Q$ with $q \ne q'$. Let $\mathcal{N}_{\mathcal{F}}(x) = \sum_{T\in\mathcal{F}} 1_{I_T}(x)$. Then we have

$$\sum_{q\in Q} \|A_q f\|_2^2 \le C(1 + \|\mathcal{N}_{\mathcal{F}}\|_\infty A^{-N})\|f\|_2^2. \tag{9.4}$$

And for $I \in \{I_T : T \in \mathcal{F}\}$, we have

$$\sum_{\substack{q\in Q\\ q=(k,J,T)\\ J\subset I}} \|A_q f\|_2^2 \le C|I|\|\mathcal{N}_{\mathcal{F}}\|_\infty^{\frac{1}{N}}(1 + \|\mathcal{N}_{\mathcal{F}}\|_\infty A^{-N})\left(\inf_{x\in I} M_2(Mf)(x)\right)^2. \tag{9.5}$$

We now prove (4.18) for $i, j \in \{2, 3\}$ and $\nu = 5$. Let $\mathcal{F}_{i,j,5} = \bigcup_l T^5_{\mu,i,j,l}$, $\mathcal{N}_{\mathcal{F}_{i,j,5}}(x) = \sum_{T\in\mathcal{F}_{i,j,5}} 1_{I_T}(x)$, and

$$\mathcal{N}^\sharp_{\mathcal{F}_{i,j,5}}(x) = \sup_{\substack{J\in\{I_T:T\in\mathcal{F}_{i,j,5}\}\\ J\ni x}} \frac{1}{|J|}\sum_{\substack{T\in\mathcal{F}_{i,j,5}\\ I_T\subset J}} |I_T|.$$

To establish prove (4.18), it will be enough to prove $\|\mathcal{N}_{\mathcal{F}_{i,j,5}}\|_1 \le C2^{10\eta p_j\mu}2^\mu$ and for this it will suffice to show that there exists $0 < \varepsilon < \eta$ such that, for any $\lambda \ge 1$,

$$|\{x \in \mathbf{R} : \mathcal{N}_{\mathcal{F}_{i,j,5}}(x) \ge \lambda\}| \le C_\varepsilon 2^{10\eta p_j\mu}2^\mu\lambda^{12p_j\varepsilon - \frac{p_j}{2}}, \tag{9.6}$$

since $\mathcal{N}_{\mathcal{F}_{i,j,5}}$ is integer-valued and since $\frac{p_j}{2} - 12p_j\varepsilon > 1$. Take $\mathcal{F}' \subset \mathcal{F}_{i,j,5}$ such that $\mathcal{N}_{\mathcal{F}'}(x) = \min\{\mathcal{N}_{\mathcal{F}_{i,j,5}}(x), \lambda\}$. Then we have

$$|\{x \in \mathbf{R} : \mathcal{N}_{\mathcal{F}'}(x) \ge \lambda\}| = |\{x \in \mathbf{R} : \mathcal{N}_{\mathcal{F}_{i,j,5}}(x) \ge \lambda\}|.$$

Assume $\lambda \ge 2^{\eta\mu}$ first. Let $A = \lambda^\varepsilon$, $Q' := \{(k, J, T) : k \in \mathbf{Z}_r, J \in \Delta_{k-m,T}, T \in \mathcal{F}'\}$ and $q = (k, J, T) \in Q'$, $q' = (k', J', T') \in Q'$. But if $q \ne q'$, then $(J \times \omega_{j,k+L,T})\bigcap(J' \times \omega_{j,k'+L,T'}) = \emptyset$. In fact, otherwise we would have $J = J'$, $k = k'$ and $\omega_{j,k+L,T} = \omega_{j,k'+L,T'}$ by the grid structure. By $J \in \Delta_{k-m,T}\bigcap\Delta_{k-m,T'}$, we know that there exist $(k + L, n) \in T$

and $(k+L,n')\in T'$ such that $J\subset I_{k+L,n}\bigcap I_{k+L,n'}$ which implies $n=n'$. Therefore, T and T' contain a common element $(k+L,n,l)$. This contradicts the maximality of T and T'.

By Lemma 18, we have $Q'=\bigcup_{l=1}^{A^{10}} Q'_l\bigcup Q''$, such that $(AJ\times\omega_{j,k,T})\bigcap(AJ'\times\omega_{j,k',T'})=\emptyset$, for $1\le l\le A^{10}$, $q\ne q'$, $q,q'\in Q'_l$, and

$$\sum_{\substack{q\in Q''\\ q=(k,J,T)}}|J|\le Ce^{-A}\sum_{\substack{q\in Q'_1\\ q=(k,J,T)}}|J|. \tag{9.7}$$

Applying Lemma 22, we obtain, for $I\in\{I_T:T\in\mathcal{F}'\}$,

$$\sum_{\substack{q\in Q'_l\\ q=(k,J,T)\\ J\subset I}}\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde{k},T})\|_2^2\le C_\varepsilon|I|\lambda^\varepsilon\left(\inf_{x\in I}M_2(Mf_j)(x)\right)^2, \tag{9.8}$$

where $\tilde{k}\in\{-L,0,L,2L,3L,4L\}$. Thus, using (4.15), we obtain

$$\begin{aligned}
\sum_{\substack{T\in\mathcal{F}'\\ I_T\subset I}}|I_T| &\le C2^{\frac{2\mu}{p_j}}\sum_{\substack{T\in\mathcal{F}'\\ I_T\subset I}}\sum_k\sum_{J\in\Delta_{k-m,T}}\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde{k},T})\|_2^2\\
&\le C2^{\frac{2\mu}{p_j}}\sum_{\substack{q\in Q'\\ q=(k,J,T)\\ J\subset I}}\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde{k},T})\|_2^2\\
&\le C2^{\frac{2\mu}{p_j}}\sum_{l=1}^{A^{10}}\sum_{\substack{q\in Q'_l\\ q=(k,J,T)\\ J\subset I}}\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde{k},T})\|_2^2+C2^{\frac{2\mu}{p_j}}\sum_{\substack{q\in Q''\\ q=(k,J,T)\\ J\subset I}}\|\rho_{k-m,J}(f_j*\Phi_{j,k+\tilde{k},T})\|_2^2\\
&\le C_\varepsilon 2^{\frac{2\mu}{p_j}}|I|\lambda^{11\varepsilon}\left(\inf_{x\in I}M_2(Mf_j)(x)\right)^2+C2^{\frac{2\mu}{p_j}}\sum_{\substack{q\in Q''\\ q=(k,J,T)\\ J\subset I}}|J|\\
&\le C_\varepsilon 2^{\frac{2\mu}{p_j}}|I|\lambda^{11\varepsilon}\left(\inf_{x\in I}M_2(Mf_j)(x)\right)^2+C2^{\frac{2\mu}{p_j}}\frac{\lambda}{e^{\lambda^\varepsilon}}|I|.
\end{aligned}$$

Therefore we have

$$\mathcal{N}^\sharp_{\mathcal{F}'}(x)\le C_\varepsilon\lambda^{11\varepsilon}\left(M_2(Mf_j)(x)\right)^2+C2^{\frac{2\mu}{p_j}}\frac{\lambda}{e^{\lambda^\varepsilon}}. \tag{9.9}$$

Since $p_j>2$ and $\operatorname{supp}\mathcal{N}^\sharp_{\mathcal{F}'}\subset\bigcup_{T\in\mathcal{F}'}I_T$, we have

$$\|\mathcal{N}_{\mathcal{F}'}\|_{\frac{p_j}{2}}\le C\|\mathcal{N}^\sharp_{\mathcal{F}'}\|_{\frac{p_j}{2}}\le C_\varepsilon 2^{\frac{2\mu}{p_j}}\lambda^{12\varepsilon}+C2^{\frac{2\mu}{p_j}}\frac{\lambda}{e^{\lambda^\varepsilon}}\Big|\bigcup_{T\in\mathcal{F}'}I_T\Big|^{\frac{2}{p_j}}.$$

By (4.15) and (5.21), we have for $T\in\mathcal{F}'$ $\inf_{I_T}M_2f_j\ge C2^{-\frac{\mu}{p_j}}$, which gives

$$\bigcup_{T\in\mathcal{F}'}I_T\subset\{x\in\mathbf{R}:M_2f_j(x)\ge C2^{-\frac{\mu}{p_j}}\}. \tag{9.10}$$

This in turn implies $\left|\bigcup_{T\in\mathcal{F}'} I_T\right| \le C2^{\mu} \le C\lambda^{\frac{1}{\eta}}$. Therefore, we have for $\lambda \ge 2^{\eta\mu}$

$$\|\mathcal{N}_{\mathcal{F}'}\|_{\frac{p_j}{2}} \le C_\varepsilon 2^{\frac{2\mu}{p_j}}\lambda^{12\varepsilon} + C2^{\frac{2\mu}{p_j}}\frac{\lambda^{1+\frac{1}{\eta}}}{e^{\lambda^\varepsilon}} \le C_\varepsilon 2^{\frac{2\mu}{p_j}}\lambda^{12\varepsilon}. \tag{9.11}$$

Finally we obtain for $\lambda \ge 2^{\eta\mu}$

$$|\{x\in\mathbf{R} : \mathcal{N}_{\mathcal{F}'}(x) \ge \lambda\}| \le C_\varepsilon 2^{\mu}\lambda^{6p_j\varepsilon - \frac{p_j}{2}}. \tag{9.12}$$

If $\lambda \le 2^{\eta\mu}$, applying (9.5) with $A := 1$, $Q := Q'$, and $N = 1/\varepsilon$ we obtain

$$\sum_{\substack{q\in Q'\\ q=(k,J,T)\\ J\subset I}} \|\rho_{k-m,J}(f_j * \Phi_{j,k+\tilde{k},T})\|_2^2 \le C|I|\lambda^{1+\varepsilon}\left(\inf_{x\in I} M_2(Mf_j)(x)\right)^2. \tag{9.13}$$

By (4.15) we have for $\lambda \le 2^{\eta\mu}$

$$\sum_{\substack{T\in\mathcal{F}'\\ I_T\subset I}} |I_T| \le C2^{\frac{2\mu}{p_j}}|I|\lambda^{1+\varepsilon}\left(\inf_{x\in I} M_2(Mf_j)(x)\right)^2. \tag{9.14}$$

Thus we have $\mathcal{N}^{\sharp}_{\mathcal{F}'}(x) \le C2^{\frac{2\mu}{p_j}}\lambda^{1+\varepsilon}\left(M_2(Mf_j)(x)\right)^2$ for $\lambda \le 2^{\eta\mu}$ from which we conclude

$$\|\mathcal{N}_{\mathcal{F}'}\|_{\frac{p_j}{2}} \le \|\mathcal{N}^{\sharp}_{\mathcal{F}'}\|_{\frac{p_j}{2}} \le C2^{\frac{2\mu}{p_j}}\lambda^{1+\varepsilon} \le C2^{\eta\mu}2^{\frac{2\mu}{p_j}}\lambda^{\varepsilon}. \tag{9.15}$$

Hence, for $\lambda \le 2^{\eta\mu}$, we have

$$|\{x\in\mathbf{R} : \mathcal{N}_{\mathcal{F}'}(x) \ge \lambda\}| \le C2^{2\eta p_j\mu}2^{\mu}\lambda^{\frac{p_j\varepsilon}{2} - \frac{p_j}{2}}. \tag{9.16}$$

Combining (9.12) and (9.16) we obtain for any $\lambda \ge 1$,

$$|\{x\in\mathbf{R} : \mathcal{N}_{\mathcal{F}'}(x) \ge \lambda\}| \le C2^{2\eta p_j\mu}2^{\mu}\lambda^{12p_j\varepsilon - \frac{p_j}{2}}, \tag{9.17}$$

which implies (9.6). This completes the proof of Lemma 7 and thus of estimate (1.2).

We refer to [9] for an application of the results of this paper.

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