# CHARACTERIZATIONS OF HARDY SPACES ON TUBE DOMAINS OVER POLYHEDRAL CONES

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Abstract. This paper is devoted to the equivalence of various characterizations of holomorphic $H^1$ Hardy spaces on tube domains over polyhedral cones. We establish a new iterated Poisson integral formula which reproduces holomorphic functions on such domains. However, this formula shows that holomorphic $H^1$ functions have boundary values in a new type of Hardy space of real variables on their Shilov boundaries $\mathbb{R}^n$, which cannot be treated by standard classical multi-parameter harmonic analysis. We overcome this difficulty by developing techniques suitably adapted in this setting. Using the iterated Poisson integral as our approximation to the identity, and employing a lifting technique, we introduce various notions of multi-parameter analysis adapted to tube domains, such as twisted rectangles, new non-tangential approach regions, non-tangential maximal functions and Littlewood-Paley type functions. All these notions exhibit new geometric features associated with polyhedral cones and involve hidden parameters, as in the flag setting. We develop the necessary multi-parameter tools to investigate these new Hardy spaces. In particular, we apply these tools to obtain equivalent characterizations of the holomorphic $H^1$ Hardy spaces on tube domains in terms of non-tangential maximal, Lusin-Littlewood-Paley area and Littlewood-Paley $g$-functions.

## Contents



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The first author is supported by the National Natural Science Foundation (NNSF) (No.12431006). The second author is supported by Simons Foundation Grant (No. 00012681). The third author is supported by by NNSF (No.12371082). The fourth author is supported by NNSF (No.12171221) and Taishan Scholars Program for Young Experts of Shandong Province (tsqn202507265).

## 1. Introduction

It is well known that holomorphic functions in the Hardy space $H^p(\mathbb{R}^2_+)$ on the upper half-plane $\mathbb{R}^2_+$ can be characterized by their non-tangential maximal functions or Lusin-Littlewood-Paley area functions, and that the real parts of the boundary values of such functions constitute the Hardy space $H^p(\mathbb{R})$ of real-variable functions on $\mathbb{R}$. In order to define the Hardy space $H^p(\mathbb{R}^n)$ of real variables on $\mathbb{R}^n$, Stein and Weiss [39] introduced the notion of a generalized analytic function $u = (u_0, \dots, u_n) : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$, which satisfies

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \qquad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \quad \text{for } j \neq k,$$

and proved that the Hardy space $H^p(\mathbb{R}^{n+1}_+)$ of generalized analytic functions for $p \geq \frac{n-1}{n}$ can also be characterized by non-tangential maximal functions or area functions. The components $u_0$ of the boundary values of such functions constitute the Hardy space $H^p(\mathbb{R}^n)$ of real variables on $\mathbb{R}^n$.

There is a long history of extending the classical characterizations of analytic Hardy spaces to more general pseudoconvex domains in several complex variables and of establishing a correspondence between the boundary values of analytic Hardy functions and suitable Hardy spaces of real variables on their Shilov boundaries. For the Siegel upper half-space, Geller [13] proved that these characterizations are equivalent and that the boundary values of holomorphic $H^p$ functions belong to the Hardy space $H^p$ on the Heisenberg group. Such characterizations can be extended to a large class of smooth pseudoconvex domains with smooth boundary (cf. e.g. [25, 35]).

However, for pseudoconvex domains with non-smooth boundary, new phenomena appear. Even for the bidisc, there are fundamental difficulties caused by the higher-dimensional geometric complexity. In the 1970s, Malliavin and Malliavin [28] proved that the maximal function characterization implies the characterization in terms of the Lusin-Littlewood-Paley area function, by introducing a technique based on an auxiliary function. This result was generalized to the $H^p$ theory for the bidisc by Gundy and Stein [16]. Biharmonic or holomorphic $H^p$ functions on the bidisc have boundary values in the bi-parameter Hardy space on its Shilov boundary, namely a torus. These results stimulated the development of multi-parameter harmonic analysis (cf. e.g. [4, 9, 23, 26, 32]), which remains an active field of research. Later, Sato [33] extended the theory to the equivalence of characterizations of Hardy spaces for generalized analytic functions on the product $\mathbb{R}^{n_1+1}_+ \times \mathbb{R}^{n_2+1}_+$.

Since general Siegel domains or symmetric spaces have non-smooth boundaries, admissible convergence of Poisson integrals becomes very complicated (cf. e.g. [24, 34, 37]), and the study of holomorphic $H^p$ Hardy spaces is usually restricted to homogeneous Siegel domains and to $p > 1$ (cf. e.g. [1, 2] and the references therein). However, in [41], the second and third named authors proved that any holomorphic $H^1$ function on the product of two Siegel upper half-spaces has a boundary value in a bi-parameter Hardy space $H^1$ on its Shilov boundary, which is the product of two Heisenberg groups. By using bi-parameter harmonic analysis (cf. e.g. [6]), the Cauchy-Szegő projection is used to decompose a holomorphic $H^1$ function into a sum of holomorphic atoms. In [42], the authors study a new class of flag-like singular integral kernels, that includes the Cauchy-Szegő kernels on certain Siegel domains. These kernels are neither of product type nor of flag type on their Shilov boundaries, which have the structure of nilpotent Lie groups of step two, say $\mathscr{N}$. In [42] the authors also introduce new Hardy spaces defined by iterated heat kernel integrals on these groups $\mathscr{N}$.

In general, holomorphic $H^p$ functions on different domains may have boundary values in different types of Hardy spaces of real variables on their Shilov boundaries, and one needs to use different kinds of multi-parameter harmonic analysis to deal with them. It was attested by Stein [38] that "a suitable version of multi-parameter analysis will provide the missing theory of singular integrals needed for a variety of questions in several complex variables. This is indeed an exciting prospect." Under this philosophy, Nagel, Ricci and Stein [31] introduced and studied an important class of Siegel domains: tube domains over polyhedral cones. A *polyhedral cone* $\Omega$ in $\mathbb{R}^n$ is the interior of the convex hull of a finite number of rays meeting at the origin, among which there exist at least $n$ that are linearly independent. The *tube domain* over $\Omega$ is the domain in $\mathbb{C}^n$ defined by

$$T_\Omega := \mathbb{R}^n + \mathbf{i}\Omega = \{x + \mathbf{i}y \in \mathbb{C}^n; x \in \mathbb{R}^n,\ y \in \Omega\},$$

whose Shilov boundary is the Euclidean space $\mathbb{R}^n + \mathbf{i}0$. The holomorphic Hardy space $H^p(T_\Omega)$ consists of all holomorphic functions $F$ on $T_\Omega$ such that

$$\|F\|_{H^p(T_\Omega)} := \left(\sup_{y\in\Omega}\int_{\mathbb{R}^n} |F(x+\mathbf{i}y)|^p\, dx\right)^{1/p} < \infty. \tag{1.1}$$

See [40, Chapter 3] for basic facts about $H^p$ Hardy spaces on tube domains. The Cauchy-Szegő projection $L^2(\mathbb{R}^n) \to H^p(T_\Omega)$ has a kernel, called the *Cauchy-Szegő kernel*, given by

$$C(x+\mathbf{i}y) = \int_{\Omega^*} e^{2\pi\mathbf{i}(x+\mathbf{i}y)\cdot\xi}\, d\xi,$$

where $\Omega^* := \{\xi \in \mathbb{R}^n : y\cdot\xi \geq 0,\ y \in \Omega\}$ is the dual cone of $\Omega$. Nagel, Ricci and Stein [31] proved that this kernel is a sum of flag kernels when restricted to the Shilov boundary. However, the study of the Hardy spaces associated to sums of flag kernels is still in its early stage (cf. e.g. [22] and the references therein). This paper is devoted to developing the necessary tools of a "suitable version of multi-parameter analysis" to investigate new types of Hardy spaces of real variables on the Shilov boundary $\mathbb{R}^n$, to applying them to the holomorphic $H^1$ Hardy spaces on $T_\Omega$, and to proving the equivalence of various of its characterizations.

It is well known [11] that Hardy spaces of real variables are defined in terms of approximations to the identity. Thus our first step is to find an integral representation formula for holomorphic functions on $T_\Omega$, which can also be viewed as an approximation to the identity on the Shilov boundary $\mathbb{R}^n$. We use this representation to define new Hardy spaces corresponding to holomorphic $H^1$ Hardy spaces on $T_\Omega$, and study the latter. For simplicity, we restrict ourselves to $H^p(T_\Omega)$ with $p = 1$ in this paper, and assume that the polyhedral cone is spanned by unit vectors $e_j$ in $\mathbb{R}^n$, $j = 1, \ldots, m$, i.e.

$$\Omega = \{\lambda_1 e_1 + \cdots + \lambda_m e_m; \lambda_1, \ldots, \lambda_m > 0\},$$

and that any $n$ of $e_1, \ldots, e_m$ are linearly independent, in order to avoid degeneracy (see Figure 1). We must have $m \geq n$. By definition, the polyhedral cone $\Omega$ has the form $\{\lambda y; \lambda > 0,\ y \in \mathcal{C}\}$, where $\mathcal{C}$ is the convex hull of the vectors $\{e_1, \ldots, e_m\}$.

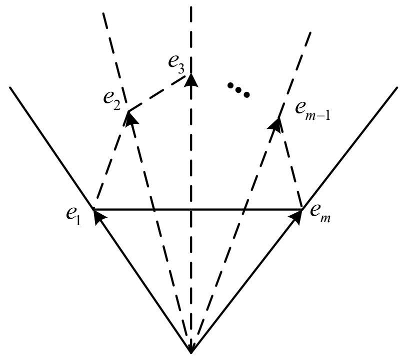


FIGURE 1. Polyhedral cone

Let $P_t(s) = \frac{1}{\pi}\frac{t}{t^2+s^2}$ be the standard Poisson kernel on $\mathbb{R}$. Define the *Poisson integral along the line* $e_\mu$ by

$$f *_\mu P_t(x) := \int_{\mathbb{R}} f(x - se_\mu)P_t(s)\, ds \tag{1.2}$$

for $x \in \mathbb{R}^n$, and the *iterated Poisson integral* by

$$P_{\mathbf{t}}(f)(x) := f *_1 P_{t_1} \cdots *_m P_{t_m}(x), \tag{1.3}$$

which is a function of the variables $(x, \mathbf{t})$ in the space $\mathbb{R}^n \times (\mathbb{R}_+)^m$. We often write $P_{\mathbf{t}}(f)(x)$ as $f * P_{\mathbf{t}}(x)$. As in the flag or flag-like cases (cf. e.g. [30, 42]), we consider the lifting space $\mathbb{R}^m$ and the projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$ given by

$$(\lambda_1, \dots, \lambda_m) \mapsto \lambda_1 e_1 + \cdots + \lambda_m e_m. \tag{1.4}$$

**Theorem 1.1.** *Suppose that $T_\Omega$ is a tube domain over a polyhedral cone as above. For $F \in H^p(T_\Omega)$ with $p > 1$, there exists $F^b \in L^p(\mathbb{R}^n)$ such that*

$$F(x + \mathbf{i}\pi(\mathbf{t})) = F^b * P_{\mathbf{t}}(x), \tag{1.5}$$

*for all $(x, \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$. Moreover, if $F$ is continuous on $\overline{T_\Omega}$, then $F^b = \lim_{\mathbf{t}\to 0} F(x + \mathbf{i}\pi(\mathbf{t}))$.*

**Remark 1.1.** (1) *This reproducing formula is unusual in the sense that we have to use the lifting cone $(\mathbb{R}_+)^m$, where a point $y$ in the cone $\Omega$ is represented by $\pi(\mathbf{t})$. The iterated Poisson integral has an $m$-parameter structure, although $\mathbb{R}^n$ is only $n$-dimensional. Namely, there exist $m - n$ hidden parameters.*

(2) *This formula identifies the new type of Hardy spaces on $\mathbb{R}^n$ to which the boundary values of holomorphic Hardy functions belong, and indicates what kind of multi-parameter analysis is needed to study them. We will develop the necessary tools of the corresponding multi-parameter harmonic analysis on the Shilov boundary $\mathbb{R}^n$ to solve problems concerning holomorphic Hardy functions on the tube domain.*

The iterated Poisson integral is our approximation to the identity; it allows us to introduce natural notions of multi-parameter analysis adapted to tube domains $T_\Omega$, such as twisted rectangles, new non-tangential regions, non-tangential maximal functions, Lusin-Littlewood-Paley area functions and new types of Hardy spaces. All of these differ from the standard ones and have new geometric features associated with polyhedral cones.

For $\mathbf{t} := (t_1, \ldots, t_m) \in (\mathbb{R}_+)^m$, the standard rectangle in $\mathbb{R}^m$ is $\widetilde{R}(0,\mathbf{t}) = (-t_1, t_1) \times \cdots \times (-t_m, t_m)$. Its image $\pi(\widetilde{R}(0,\mathbf{t}))$ under the projection $\pi$ in (1.4) is denoted by $R(0,\mathbf{t})$, and is called a *twisted rectangle* in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, let $R(x,\mathbf{t}) := x + R(0,\mathbf{t})$. Then

$$R(x,\mathbf{t}) := \{x + \lambda_1 e_1 + \cdots + \lambda_m e_m; |\lambda_1| < t_1, \ldots, |\lambda_m| < t_m\}. \tag{1.6}$$

For $x \in \mathbb{R}^n$ and any $\beta > 0$, we define the *non-tangential region in* $\mathbb{R}^n \times (\mathbb{R}_+)^m$ by

$$\Gamma_\beta(x) := \{(x', \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m; x' \in R(x, \beta\mathbf{t})\}. \tag{1.7}$$

We also denote $\Gamma_1(x)$ by $\Gamma(x)$ for short. Define the *non-tangential maximal function* on $\mathbb{R}^n$ by

$$N^\beta(f)(x) := \sup_{(x',\mathbf{t})\in\Gamma_\beta(x)} \left|(f * P_\mathbf{t})(x')\right|. \tag{1.8}$$

The Hardy space $H^1_{\max;\Omega}(\mathbb{R}^n)$ consists of $f \in L^1(\mathbb{R}^n)$ such that $N^\beta(f) \in L^1(\mathbb{R}^n)$. For two polyhedral cones $\Omega, \Omega'$ with $\overline{\Omega} \subset \Omega'$, we have $H^1(T_{\Omega'}) \subsetneqq H^1(T_\Omega)$ and $H^1_{\max;\Omega'}(\mathbb{R}^n) \subsetneq H^1_{\max;\Omega}(\mathbb{R}^n)$ (cf. Proposition 3.5). If there is no ambiguity, we will omit the dependence of $H^1_{\max;\Omega}$ on $\Omega$.

Define the *Lusin-Littlewood-Paley area function* of $f$ by

$$S(f)(x) := \left(\int_{\Gamma(x)} |\nabla_1 \cdots \nabla_m (f * P_\mathbf{t})(x')|^2 \frac{\mathbf{t}\, d\mathbf{t}\, dx'}{|R(x,\mathbf{t})|}\right)^{1/2}, \tag{1.9}$$

for all $x \in \mathbb{R}^n$, where $\nabla_j$ is the gradient in the plane spanned by $e_j$ and $t_j$, and $\mathbf{t}\, d\mathbf{t} = t_1 \cdots t_m\, dt_1 \cdots dt_m$. Define the *S-function Hardy space* $H^1_S(\mathbb{R}^n)$ by setting

$$H^1_S(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n); S(f) \in L^1(\mathbb{R}^n)\}$$

and equipped with the norm $\|f\|_{H^1_S(\mathbb{R}^n)} := \|S(f)\|_{L^1(\mathbb{R}^n)}$. The *Littlewood-Paley g-function* of $f$ is defined by

$$g(f)(x) := \left(\int_{(\mathbb{R}_+)^m} |\nabla_1 \cdots \nabla_m (f * P_\mathbf{t})|^2 (x)\, \mathbf{t}\, d\mathbf{t}\right)^{1/2},$$

for all $x \in \mathbb{R}^n$. We define the *g-function Hardy space*

$$H^1_g(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n); g(f) \in L^1(\mathbb{R}^n)\}$$

and this space is naturally equipped with the norm $\|f\|_{H^1_g(\mathbb{R}^n)} := \|g(f)\|_{L^1(\mathbb{R}^n)}$.

Note that the tube domain $T_\Omega = \mathbb{R}^n + \mathbf{i}\Omega$ is different from $\mathbb{R}^n \times (\mathbb{R}_+)^m$, but $\Omega = \pi((\mathbb{R}_+)^m)$. If we lift the function $F$ on $T_\Omega$ to a function $u(x,\mathbf{t}) = F(x + \mathbf{i}\pi(\mathbf{t}))$ on $\mathbb{R}^n \times (\mathbb{R}_+)^m$, then by Theorem 1.1, $u(x,\mathbf{t}) = F^b * P_\mathbf{t}(x)$ if $F \in H^p(T_\Omega)$ with $p > 1$. Motivated by nontangential maximal function, area function and $g$-function for iterated Poisson integrals above, we define the *nontangential maximal function* of a holomorphic function $F$ on $T_\Omega$ by

$$\mathbb{N}^\beta(F)(x) := \sup_{(x',\mathbf{t})\in\Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t}))|, \tag{1.10}$$

for all $x \in \mathbb{R}^n$. We also define the *Lusin-Littlewood-Paley area function* by

$$\mathbb{S}(F)(x) := \left(\int_{\Gamma(x)} |\nabla_1 \cdots \nabla_m (F(x' + \mathbf{i}\pi(\mathbf{t}))|^2 \frac{\mathbf{t} d\mathbf{t} dx'}{|R(x,\mathbf{t})|}\right)^{\frac{1}{2}},$$

and the *Littlewood-Paley g-function* by

$$\mathbb{G}(F)(x) := \left( \int_{(\mathbb{R}_+)^m} |\nabla_1 \cdots \nabla_m (F(x + \mathbf{i}\pi(\mathbf{t}))|^2 \, \mathbf{t} d\mathbf{t} \right)^{\frac{1}{2}}.$$

Having established this notation, we are now able to state our main theorem.

**Theorem 1.2.** *Suppose that $T_\Omega$ is a tube domain over a polyhedral cone as above. Then, for a holomorphic function $F$ on $T_\Omega$, the following properties are equivalent:*
(1) $F \in H^1(T_\Omega)$;
(2) *For any* $\beta > 0$, $\mathbb{N}^\beta(F) \in L^1(\mathbb{R}^n)$;
(3) $\lim_{\substack{|y| \to +\infty \\ y \in y_0 + \Omega}} F(x + \mathbf{i}y) = 0$ *for any fixed* $y_0 \in \Omega$, *and* $\mathbb{S}(F) \in L^1(\mathbb{R}^n)$.
(4) $\lim_{\substack{|y| \to +\infty \\ y \in y_0 + \Omega}} F(x + \mathbf{i}y) = 0$ *for any fixed* $y_0 \in \Omega$, *and* $\mathbb{G}(F) \in L^1(\mathbb{R}^n)$.
*Moreover,*

$$\|F\|_{H^1(T_\Omega)} \approx \|\mathbb{N}^\beta(F)\|_{L^1(\mathbb{R}^n)} \approx \|\mathbb{S}(F)\|_{L^1(\mathbb{R}^n)} \approx \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)}.$$

**Remark 1.2.** *A key step in our proof is to approximate holomorphic $H^1$ functions by holomorphic $H^p$ functions with $p > 1$. For the latter class, we can apply Theorem 1.1 to establish a correspondence with real-variable $H^p$ functions defined via the iterated Poisson integral on the Shilov boundary. Then, by employing the tools of multi-parameter analysis we have outlined and using the holomorphicity of $F$, we obtain a complete characterization of holomorphic $H^1$ Hardy spaces on such tube domains. This approach also provides a partial characterization of a new type of Hardy space on the Shilov boundary $\mathbb{R}^n$.*

For the equivalence of characterizations of Hardy spaces of real variables in various settings, see, for example, [3, 5, 6, 8, 14, 19, 21, 43].

The proof of Theorem 1.2 proceeds as follows. For holomorphic functions $F$, we can use the subharmonicity of $|F|^p$ for $p > 0$ to prove (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) can be reduced to a Fefferman–Stein type good-$\lambda$ inequality: there exist $C > 0$ and $\beta > 1$ such that for $f \in L^1(\mathbb{R}^n)$ and all $\lambda > 0$,

$$|\{x \in \mathbb{R}^n; S(f)(x) > \lambda\}| \leq C \left|\{x \in \mathbb{R}^n; N^\beta(f)(x) > \lambda\}\right| + \frac{C}{\lambda^2} \int_{\{x \in \mathbb{R}^n; N^\beta(f)(x) \leq \lambda\}} N^\beta(f)(x)^2 \, dx, \tag{1.11}$$

where $C$ is a constant independent of $f$ and $\lambda$. In the one-parameter case, this inequality was proved by Fefferman and Stein [10, 36] via smooth surface approximation and an application of Green's formula in smooth domains. The main difficulty in the bi-parameter case is the lack of such an effective approximation, owing to the higher-dimensional geometric complexity. To overcome this difficulty for biharmonic functions, Malliavin and Malliavin [28] introduced an auxiliary function. So far, maximal function characterizations have been established only in a few cases by proving analogues of (1.11): the $\mathbb{R}^n \times \mathbb{R}^n$ case by Merryfield [29], the Muckenhoupt–Stein Bessel operator setting by Doung, Li, Wick, Yang [8], the multi-parameter flag setting by Han, Lee, Li, Wick [19, 20], product stratified Lie groups by Cowling, Fan, Li, Yan [7], the flag setting on the Heisenberg group by Chen, Cowling, Lee, Li, Ottazzi [5] and the Shilov boundary of products of domains of finite type by Li [27], etc. In our setting, the difficulty lies in the fact that the number of parameters is arbitrary, unlike the aforementioned situations which involve only two. Thus we have to find general inductive formulae to overcome this difficulty.

In the product case $\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_2+1}$, the implication (3) $\Rightarrow$ (1) was proved by use of the estimate $\|\mathfrak{f}\|_1 \lesssim \|S(\mathfrak{f})\|_1$ for Hilbert space-valued functions $\mathfrak{f}$ of one variable [33]. Because our non-tangential region $\Gamma_\beta(x)$ in (1.7) cannot be written as a product, we cannot employ this approach. However, since the domain $(\mathbb{R}_+)^m$ of integration for the $g$-function has a product structure, we add condition (4), which usually does not appear in characterizations of (generalized) analytic Hardy spaces (cf. e.g. [13, 33, 39]). The implication (3) $\Rightarrow$ (4) follows from a pointwise control of the $\mathbb{G}$-function by the $\mathbb{S}$-function via the mean value formula. The real-variable version of (4) $\Rightarrow$ (1) is

$$\sup_{\mathbf{t}\in(\mathbb{R}_+)^m} \|f * P_{\mathbf{t}}\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}. \tag{1.12}$$

We first establish a one-dimensional version of (1.12) for Hilbert space-valued functions by applying the corresponding one-dimensional Plancherel-Pólya type inequality. Then we use this estimate iteratively to prove (1.12).

The paper is organized as follows. In Section 2, we discuss partial convolutions with the Poisson kernel along lines, which satisfy partial Laplace equations. In Section 3, we characterize the geometric shape of twisted rectangles and prove the equivalence of iterated and twisted maximal functions. We also obtain boundary growth estimates for iterated Poisson integrals and for holomorphic $H^p$-functions, respectively, and then show that the iterated Poisson integral reproduces holomorphic functions by using the maximum principle. In Section 4, the Fefferman-Stein type good-$\lambda$ inequality (1.11) is reduced to the estimate of an integral over $\mathbb{R}^n \times (\mathbb{R}_+)^m$ by means of an auxiliary function. We use differential identities to estimate the terms appearing in this integral, the new terms arising after these estimates, and in general all terms appearing inductively. In Section 5, we prove a one-dimensional version of (1.12) for Hilbert space-valued functions, from which we deduce the real-variable version (1.12) of (4) $\Rightarrow$ (1). In Section 6, we prove the equivalence of the various characterizations in the main Theorem 1.2. In the appendix, we give the proof of the Plancherel-Pólya type inequality for Hilbert space-valued functions for the convenience of the readers.

## 2. Preliminaries

2.1. **Partial convolutions along lines and partial Laplace equations.** Consider the *complex line* in $\mathbb{C}^n$ that contains $e_\mu$:

$$L_\mu := \{(s + \mathbf{i}t)e_\mu; s, t \in \mathbb{R}\}, \tag{2.1}$$

and its upper half plane

$$L_\mu^+ = \{(s + \mathbf{i}t)e_\mu; s \in \mathbb{R}, t > 0\}, \tag{2.2}$$

with boundary $\mathbb{R}e_\mu$ contained in the Shilov bounday $\mathbb{R}^n + \mathbf{i}0$ of $T_\Omega$ . Then $x + L_\mu^+$ may be contained in the boundary of the tube domain for any $x \in \mathbb{R}^n$, because $e_\mu$ may be contained in $\partial\Omega$. But we always have $x + iy + L_\mu^+ \subset T_\Omega$ for $y \in \Omega$.

Denote by $*_\mu$ the *partial convolution on* $\mathbb{R}^n$ *along the line* $e_\mu$, i.e. for $f \in L^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R})$,

$$f *_\mu g(x) := \int_{\mathbb{R}} f\left(x - se_\mu\right) g(s) ds, \tag{2.3}$$

for almost all $x \in \mathbb{R}^n$. It is obvious that partial convolutions along two lines $e_\mu$ and $e_\nu$ are commutative:

$$f *_\mu g *_\nu h(x) = f *_\nu h *_\mu g(x), \tag{2.4}$$

for any $g, h \in L^1(\mathbb{R})$. If write

$$e_\mu = (e_{\mu 1}, \ldots, e_{\mu n}) \in \mathbb{R}^n, \tag{2.5}$$

$\mu = 1, \ldots, m$, then the vector field

$$X_\mu := \sum_{k=1}^{n} e_{\mu k} \frac{\partial}{\partial x_k}, \tag{2.6}$$

is the derivative along the direction $e_\mu$. Define the *$\mu$-th Laplacian* and the gradient of a function $u$ along the upper half plane $L_\mu^+$ in (2.2) as

$$\triangle_\mu := X_\mu^2 + \frac{\partial^2}{\partial t_\mu^2}, \qquad \nabla_\mu u := \left( X_\mu u, \frac{\partial u}{\partial t_\mu} \right),$$

respectively, $\mu = 1, \ldots, m$.

**Proposition 2.1.** *(1) If $F$ is holomorphic on the tube domain $T_\Omega$, then for fixed $x + \mathbf{i}y \in T_\Omega$, the restriction of $F$ to $x + \mathbf{i}y + L_\mu^+$ is holomorphic, i.e. $F(x + \mathbf{i}y + ze_\mu)$ is holomorphic in $z \in \mathbb{C}$ for $\operatorname{Im} z > 0$.*
*(2) For $f \in \mathscr{C}^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, we have*

$$\triangle_\mu[f *_\mu P_{t_\mu}(x)] = 0, \qquad \text{and} \qquad \nabla_\mu \left[ f *_\mu P_{t_\mu}(x) \right] = f *_\mu \widetilde{\nabla}_\mu P_{t_\mu}(x), \tag{2.7}$$

*where $\widetilde{\nabla}_\mu$ is the gradient in the plane with variables $s_\mu, t_\mu$: $\widetilde{\nabla}_\mu v(s_\mu, t_\mu) = \left( \frac{\partial v}{\partial s_\mu}(s_\mu, t_\mu), \frac{\partial v}{\partial t_\mu}(s_\mu, t_\mu) \right)$.*

*Proof.* (1) If we write the coordinates $z_1, \ldots, z_n$ of the tube domain as $z_j = x_j + \mathbf{i}y_j$ and $z = s + \mathbf{i}t$, then, for fixed $\mu$ we have

$$\begin{aligned}
2\frac{\partial F}{\partial \bar{z}} &= \left( \frac{\partial}{\partial s} + \mathbf{i}\frac{\partial}{\partial t} \right) [F(x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu)] \\
&= \sum_{k=1}^{n} e_{\mu k} \frac{\partial F}{\partial x_k}(x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu) + \mathbf{i}\sum_{k=1}^{n} e_{\mu k} \frac{\partial F}{\partial y_k}(x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu) \\
&= \sum_{k=1}^{n} e_{\mu k} \left( \frac{\partial F}{\partial x_k} + \mathbf{i}\frac{\partial F}{\partial y_k} \right) (x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu) = 0,
\end{aligned}$$

by using (2.5) and the Cauchy-Riemann equation satisfied by $F$.

(2) It is obvious that $f *_\mu P_{t_\mu}(x)$ lies in $\mathscr{C}^2(\mathbb{R}^n \times \mathbb{R}_+)$. Since

$$\frac{\partial}{\partial s}[f(x - se_\mu)] = -X_\mu f(x - se_\mu), \tag{2.8}$$

by definition (2.6), we get

$$\begin{aligned}
\triangle_\mu[f *_\mu P_{t_\mu}(x)] &= \int_{\mathbb{R}} X_\mu^2 f(x - se_\mu) P_{t_\mu}(s) ds + \int_{\mathbb{R}} f(x - se_\mu) \frac{\partial^2}{\partial t_\mu^2} P_{t_\mu}(s) ds \\
&= \int_{\mathbb{R}} \frac{\partial^2}{\partial s^2}[f(x - se_\mu)] P_{t_\mu}(s) ds + \int_{\mathbb{R}} f(x - se_\mu) \frac{\partial^2}{\partial t_\mu^2} P_{t_\mu}(s) ds \\
&= \int_{\mathbb{R}} f(x - se_\mu) \left( \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t_\mu^2} \right) P_{t_\mu}(s) ds = 0,
\end{aligned}$$

by integration by parts, and

$$\begin{aligned}\nabla_\mu[f *_\mu P_{t_\mu}(x)] &= \left(\int_{\mathbb{R}} X_\mu f(x-se_\mu)P_{t_\mu}(s)ds, \int_{\mathbb{R}} f(x-se_\mu)\frac{\partial}{\partial t_\mu}P_{t_\mu}(s)ds\right)\\ &= \left(\int_{\mathbb{R}} -\frac{\partial}{\partial s}[f(x-se_\mu)]P_{t_\mu}(s)ds, \int_{\mathbb{R}} f(x-se_\mu)\frac{\partial}{\partial t_\mu}P_{t_\mu}(s)ds\right)\\ &= \int_{\mathbb{R}} f(x-se_\mu)\widetilde{\nabla}_\mu P_{t_\mu}(s)ds.\end{aligned}$$

The proposition is proved. □

Denote $\mathfrak{m} := \{1,\ldots,m\}$. For a subset $\mathfrak{j} = \{j_1,\ldots,j_a\}$ of $\mathfrak{m}$, we introduce the notation

$$\mathbf{t}_{\mathfrak{j}} := (t_{j_1},\ldots,t_{j_a}), \qquad \nabla_{\mathfrak{j}} v := \nabla_{j_1}\cdots\nabla_{j_a} v, \qquad \mathbf{t}_{\mathfrak{j}} d\mathbf{t}_{\mathfrak{j}} := t_{j_1}\cdots t_{j_a} dt_{j_1}\cdots dt_{j_a}, \tag{2.9}$$

and for $f \in L^1(\mathbb{R}^n)$ and $\phi \in L^1(\mathbb{R})$, we set

$$f *_{\mathfrak{j}} \phi_{\mathbf{t}_{\mathfrak{j}}} := f *_{j_1} \phi_{t_{j_1}} \cdots *_{j_a} \phi_{t_{j_a}}(x),$$

where $\phi_t(s) = \frac{1}{t}\phi(\frac{s}{t})$. Let

$$\mathfrak{m}_k = \{k,\ldots,m\}.$$

In view of this notation, we have $\mathbf{t}_{\mathfrak{m}} = \mathbf{t}$. We will write $f * \phi_{\mathbf{t}}$ instead of $f *_{\mathfrak{j}} \phi_{\mathbf{t}_{\mathfrak{j}}}$ when $\mathfrak{j} = \mathfrak{m}$.

For a subset $\mathfrak{j}$ of $\mathfrak{m}$ and $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, define the *partial Littlewood-Paley g-function* of $f$ by

$$g_{\mathfrak{j}}(f)(x) := \left(\int_{(\mathbb{R}_+)^{|\mathfrak{j}|}} |\nabla_{\mathfrak{j}}(f *_{\mathfrak{j}} P_{\mathbf{t}_{\mathfrak{j}}})(x)|^2\, \mathbf{t}_{\mathfrak{j}} d\mathbf{t}_{\mathfrak{j}}\right)^{\frac{1}{2}}.$$

**Proposition 2.2.** *For $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, we have $\|g_{\mathfrak{j}}(f)\|_p \lesssim \|f\|_p$.*

The proof is standard (cf. e.g. [21, 42]) by repeatedly using the $L^p$ boundedness of $g$-function for vector-valued functions. See also the proof of Proposition 5.5 for the converse part when $p = 1$. We omit the details.

2.2. **A formula of integration by parts.** It will be repeatedly used in the proof of good-$\lambda$ inequality. For fixed $\mu$, we will use coordinates $x = x^\perp + se_\mu$, where $x^\perp$ is the part of $x$ perpendicular to $e_\mu$.

**Proposition 2.3.** *Suppose that $U \in \mathscr{C}^2(\mathbb{R}^{n+1}_+)$ satisfies the following three properties:*
*(1) For fixed $t_\mu > 0$, we have $\nabla^a U(\cdot,t_\mu) \in L^1(\mathbb{R}^n)$, $a = 0,1,2$, and $X_\mu U(x^\perp + se_\mu, t_\mu) \to 0$ as $s \to \infty$ for almost all $x^\perp \in e_\mu^\perp$;*
*(2) As $t_\mu \to +\infty$, we have $U(\cdot,t_\mu), t_\mu\partial_{t_\mu}U(\cdot,t_\mu) \to 0$ in $L^1(\mathbb{R}^n)$;*
*(3) As $t_\mu \to 0$, we have $U(\cdot,t_\mu) \to U(\cdot,0)$ and $t_\mu\partial_{t_\mu}U(\cdot,t_\mu) \to 0$ in $L^1(\mathbb{R}^n)$.*
*Then*

$$\int_0^\infty \int_{\mathbb{R}^n} \triangle_\mu U(x,t_\mu)t_\mu dt_\mu dx = \int_{\mathbb{R}^n} U(x,0)\,dx. \tag{2.10}$$

*Proof.* Write

$$\int_\varepsilon^T \int_{\mathbb{R}^n} \triangle_\mu U(x,t_\mu)t_\mu dt_\mu dx = \int_\varepsilon^T \int_{\mathbb{R}^n} X_\mu^2 U(x,t_\mu)t_\mu dt_\mu dx + \int_\varepsilon^T \int_{\mathbb{R}^n} \partial_{t_\mu}^2 U(x,t_\mu)t_\mu dt_\mu dx =: \mathrm{I}_1 + \mathrm{I}_2.$$

Note that $x \to (x^\perp, s)$ preserves the volume form since we assume $e_\mu$ to be a unit vector. Hence,

$$\begin{aligned}\int_{\mathbb{R}^n} X_\mu^2 U(x,t_\mu)dx &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \frac{d^2}{ds^2}[U(x^\perp + se_\mu, t_\mu)]\, ds\, dx^\perp \\ &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \left(\lim_{s\to+\infty} X_\mu U(x^\perp + se_\mu, t_\mu) - \lim_{s\to-\infty} X_\mu U(x^\perp + se_\mu, t_\mu)\right) ds\, dx^\perp = 0\end{aligned}$$

by using (2.8) and the assumption (1). Thus, $\mathrm{I}_1 = 0$. For the term $\mathrm{I}_2$, by integration by parts, we get

$$\begin{aligned}\mathrm{I}_2 &= -\int_{\mathbb{R}^n}\int_\varepsilon^T \partial_{t_\mu} U(x,t_\mu)dt_\mu dx + \int_{\mathbb{R}^n} T\partial_{t_\mu} U(x,T)dx - \int_{\mathbb{R}^n} \varepsilon\partial_{t_\mu} U(x,\varepsilon)dx \\ &= -\int_{\mathbb{R}^n} U(x,T)dx + \int_{\mathbb{R}^n} U(x,\varepsilon)dx + \int_{\mathbb{R}^n} T\partial_{t_\mu} U(x,T)dx - \int_{\mathbb{R}^n} \varepsilon\partial_{t_\mu} U(x,\varepsilon)dx.\end{aligned}$$

Then taking limit $T \to +\infty$ and $\varepsilon \to 0$, we deduce the required conclusions in view of assumptions (2) and (3). □

## 3. New maximal functions associated to twisted rectangles and iterated Poisson integral formula

3.1. **Twisted rectangles, iterated and twisted maximal functions.** Write $\nabla_\mu = (X_\mu^{(1)}, X_\mu^{(2)})$, i.e. $X_\mu^{(1)} = X_\mu$ and $X_\mu^{(2)} = \partial_{t_\mu}$. Then $\nabla_{\mathfrak{j}} v$ in (2.9) is a vector valued function with $\mathbb{R}^{2^a}$ entries, and their inner product is defined as

$$\nabla_{\mathfrak{j}} u \cdot \nabla_{\mathfrak{j}} v := \sum_{\alpha_1,\dots,\alpha_a=1,2} X_{j_1}^{(\alpha_1)}\cdots X_{j_a}^{(\alpha_a)} u \cdot X_{j_1}^{(\alpha_1)}\cdots X_{j_a}^{(\alpha_a)} v.$$

Define *maximal function along the line* $e_\mu$ as

$$M_\mu(f)(x) := \sup_{r_\mu\in\mathbb{R}_+} \frac{1}{2r_\mu}\int_{|s|<r_\mu} |f(x+se_\mu)|ds,$$

for $x \in \mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n)$. It is convenient to work with the *iterated maximal function* $M_{it} := M_m \circ \cdots \circ M_1$. Then for $f \in L^1_{loc}(\mathbb{R}^n)$, there exists an absolute constant $C_0 > 0$ such that

$$|P_{\mathbf{t}}(f)(x')| \le C_0 M_{it}(f)(x) \tag{3.1}$$

for any $(x', \mathbf{t}) \in \Gamma(x)$, by the property of the Poisson kernel [36]. Define the *twisted maximal function* as

$$M_t(f)(x) := \sup_{\mathbf{r}\in(\mathbb{R}_+)^m} \frac{1}{|R(x,\mathbf{r})|}\int_{R(x,\mathbf{r})} |f(x')|dx'.$$

Note that the projection of the ball in the lifting space is usually called tube in the flag or flag-like cases (cf. e.g. [5, 42]). But in this paper, the projection $R(x,\mathbf{r})$ in (1.6) is called twisted rectangle for avoiding confusion with the tube domain. Given $\mathbf{r} = (r_1,\dots,r_m) \in (\mathbb{R}_+)^m$, let $r_{l_1},\dots,r_{l_n}$ be the largest $n$ numbers among $\{r_1,\dots,r_m\}$, i.e. $r_j \ge r_\mu$ for any $j \in \mathfrak{l} := \{l_1,\dots,l_n\}$ and any $\mu \notin \mathfrak{l}$. Set

$$R_{\mathfrak{l}}(x,\mathbf{r}) := \{x + \lambda_{l_1}e_{l_1} + \cdots + \lambda_{l_n}e_{l_n}; |\lambda_{l_1}| < r_{l_1},\dots,|\lambda_{l_n}| < r_{l_n}\}, \tag{3.2}$$

which is a parallelohedron. Note that given $\mathbf{r}$, the choice of $\mathfrak{l}$ may not be unique.

On the other hand, given any subset $\mathfrak{l} := \{l_1,\dots,l_n\}$ of $\mathfrak{m}$, $e_{l_1},\dots,e_{l_n}$ are linearly independent by the assumption. So there exists numbers $A^{\mathfrak{l}}_{\mu j}$ such that $e_\mu = \sum_{j=1}^n A^{\mathfrak{l}}_{\mu j} e_{l_j}$, for $\mu \notin \mathfrak{l}$. As the number of choices of such subsets $\mathfrak{l}$ is finite, we can define

$$\mathcal{A} := \sum_{\mathfrak{l}} \sum_{\mu \notin \mathfrak{l}} \sum_{j=1}^{n} |A_{\mu j}^{\mathfrak{l}}| < +\infty. \tag{3.3}$$

**Proposition 3.1.** *Given $\mathbf{r} \in (\mathbb{R}_+)^m$, let $r_{l_1}, \ldots, r_{l_n}$ be the largest $n$ numbers among $\{r_1, \ldots, r_m\}$. Then*

$$R_{\mathfrak{l}}(x, \mathbf{r}) \subset R(x, \mathbf{r}) \subset R_{\mathfrak{l}}(x, \widetilde{\gamma}_0^{-1}\mathbf{r}), \tag{3.4}$$

*for any $x \in \mathbb{R}^n$, where $\widetilde{\gamma}_0 = (1 + \mathcal{A})^{-1} < 1$.*

*Proof.* The first inclusion is obvious by comparing definitions (1.6) and (3.2). For the second one, note that for any $x' \in R(x, \mathbf{r})$, we can write

$$x' = x + \lambda_1 e_1 + \cdots + \lambda_m e_m = x + \sum_{j=1}^{n} \left( \lambda_{l_j} + \sum_{\mu \notin \mathfrak{l}} A_{\mu j}^{\mathfrak{l}} \lambda_\mu \right) e_{l_j},$$

for some $|\lambda_1| < r_\mu \leq r_1, \ldots, |\lambda_m| < r_m$. Then

$$\left| \lambda_{l_j} + \sum_{\mu=n+1}^{m} A_{\mu j}^{\mathfrak{l}} \lambda_\mu \right| \leq (1 + \mathcal{A}) r_{l_j} < \widetilde{\gamma}_0^{-1} r_{l_j},$$

since $|\lambda_\mu| \leq r_{l_j}$ for $\mu \notin \mathfrak{l}$ by definition. □

As a corollary, we get

$$|R(x, \mathbf{r})| \approx |R_{\mathfrak{l}}(x, \mathbf{r})| = 2^n r_{l_1} \cdots r_{l_n} \det(e_{l_1}, \ldots, e_{l_n}). \tag{3.5}$$

Let $\chi$ be the characteristic function of the interval $[-1, 1]$ and let $\chi_t(s) := \frac{1}{t}\chi(\frac{s}{r})$ for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}$. Recall that $|f| * \chi_{\mathbf{t}} := |f| *_{j_1} \chi_{t_{j_1}} \cdots *_{j_m} \chi_{t_{j_m}}$ for $\mathbf{t} = (t_{j_1}, \ldots, t_{j_m}) \in (\mathbb{R}_+)^m$. The iterated maximal function and twisted maximal function are equivalent by the following proposition.

**Proposition 3.2.** *Suppose that $f \in L^1_{loc}(\mathbb{R}^n)$. Then for any $n < m$, $x \in \mathbb{R}^n$, and $\mathbf{t} \in (\mathbb{R}_+)^m$,*

$$|f| * \chi_{\gamma_0 \mathbf{t}}(x) \lesssim \frac{1}{|R(x, \mathbf{t})|} \int_{R(x, \mathbf{t})} |f(x')| dx' \lesssim |f| * \chi_{\gamma_0^{-1}\mathbf{t}}(x),$$

*where $\gamma_0 = (1 + \mathcal{A})^{-2}$. Consequently, $M_t(f) \approx M_{it}(f)$.*

*Proof.* Without loss of generality, we may assume that $t_1, \ldots, t_n$ are the largest $n$ numbers among $\{t_1, \ldots, t_m\}$. The linear independence of $e_1, \ldots, e_n$ implies that there exists numbers $A_{\mu j}$ such that

$$e_\mu = \sum_{j=1}^{n} A_{\mu j} e_j, \tag{3.6}$$

for any $\mu = n + 1, \ldots, m$. Then

$$\begin{aligned} |f| * \chi_{\mathbf{t}}(x) &= \int_{\mathbb{R}^m} \left| f\left( x - \sum_{j=1}^{m} s_j e_j \right) \right| \chi_{t_1}(s_1) \cdots \chi_{t_m}(s_m) ds_1 \cdots ds_m \\ &= \frac{1}{t_1 \cdots t_m} \int_{[-t_1, t_1] \times \cdots \times [-t_m, t_m]} \left| f\left( x - \sum_{j=1}^{n} \left( s_j + \sum_{\mu=n+1}^{m} A_{\mu j} s_\mu \right) e_j \right) \right| ds_1 \cdots ds_m. \end{aligned} \tag{3.7}$$

Take the coordinate transformation $\Theta : \mathbb{R}^m \to \mathbb{R}^m$, $\mathbf{s} \mapsto \mathbf{s}'$ given by

$$s'_j = \begin{cases} s_j + \sum_{\mu=n+1}^m A_{\mu j} s_\mu, & j = 1, \ldots, n, \\ s_j, & j = n+1, \ldots, m. \end{cases} \tag{3.8}$$

It is obvious that $\Theta$ is one to one and its Jacobian identically equals to 1. We claim that

$$\prod_{a=1}^m [-\widetilde{\gamma}_0 t_a, \widetilde{\gamma}_0 t_a] \subset \Theta\left(\prod_{a=1}^m [-t_a, t_a]\right) \subset \prod_{a=1}^m \left[-\widetilde{\gamma}_0^{-1} t_a, \widetilde{\gamma}_0^{-1} t_a\right], \tag{3.9}$$

with $\widetilde{\gamma}_0 = (1+\mathcal{A})^{-1}$. For the first inclusion, note that for each $\mathbf{s}' \in \prod_{a=1}^m [-\widetilde{\gamma}_0 t_a, \widetilde{\gamma}_0 t_a]$, there exists a unique preimage $\mathbf{s}$ in $\prod_{a=1}^m [-t_a, t_a]$, i.e. $\Theta(\mathbf{s}) = \mathbf{s}'$. This is because $|s_j| \leq t_j$ obviously for $j = n+1, \ldots, m$ by definition (3.8), while for $j = 1, \ldots, n$, we have

$$|s_j| = \left| s'_j - \sum_{\mu=n+1}^m A_{\mu j} s'_\mu \right| \leq \widetilde{\gamma}_0 t_j + \mathcal{A}\widetilde{\gamma}_0 t_j = t_j,$$

by $|s'_\mu| \leq \widetilde{\gamma}_0 t_\mu \leq \widetilde{\gamma}_0 t_j$ for any $\mu > n \geq j$ by the assumption. It is similar to show the second inclusion.

Now it follows from (3.7) and (3.9) that

$$\begin{aligned} |f| * \chi_{\mathbf{t}}(x) &\geq \frac{1}{t_1 \cdots t_m} \int_{[-\widetilde{\gamma}_0 t_1, \widetilde{\gamma}_0 t_1] \times \cdots \times [-\widetilde{\gamma}_0 t_m, \widetilde{\gamma}_0 t_m]} \left| f\left(x - \sum_{j=1}^n s'_j e_j\right)\right| ds'_1 \cdots ds'_m \\ &= \frac{(2\widetilde{\gamma}_0)^{m-n}}{t_1 \cdots t_n} \int_{[-\widetilde{\gamma}_0 t_1, \widetilde{\gamma}_0 t_1] \times \cdots \times [-\widetilde{\gamma}_0 t_n, \widetilde{\gamma}_0 t_n]} \left| f\left(x - \sum_{j=1}^n s'_j e_j\right)\right| ds'_1 \cdots ds'_n \\ &= \frac{\widetilde{\gamma}_0^{m-n} 2^m}{|R_{\mathfrak{l}}(0, \mathbf{t})|} \int_{R_{\mathfrak{l}}(0, \widetilde{\gamma}_0 \mathbf{t})} |f(x - x')| dx' \\ &\geq \frac{\widetilde{\gamma}_0^{m-n} 2^m}{|R_{\mathfrak{l}}(0, \mathbf{t})|} \int_{R(0, \widetilde{\gamma}_0^2 \mathbf{t})} |f(x - x')| dx', \end{aligned}$$

by taking coordinates transformation (3.8) and using the transformation $\mathbb{R}^n \to \mathbb{R}^n$ given by

$$(s'_1, \ldots, s'_n) \mapsto x' = \sum_{j=1}^n s'_j e_j \tag{3.10}$$

and Proposition 3.1. The Jacobian of this transformation is $\det(e_1, \ldots, e_n) \neq 0$ since $e_1, \ldots, e_n$ are linearly independent. Similarly,

$$\begin{aligned} |f * \chi_{\mathbf{t}}(x)| &\leq \frac{1}{t_1 \cdots t_m} \int_{[-\widetilde{\gamma}_0^{-1} t_1, \widetilde{\gamma}_0^{-1} t_1] \times \cdots \times [-\widetilde{\gamma}_0^{-1} t_m, \widetilde{\gamma}_0^{-1} t_m]} \left| f\left(x - \sum_{j=1}^n s'_j e_j\right)\right| ds'_1 \cdots ds'_m \\ &\approx \frac{1}{|R_{\mathfrak{l}}(0, \mathbf{t})|} \int_{R_{\mathfrak{l}}(0, \widetilde{\gamma}_0^{-1} \mathbf{t})} f(x - x') \, dx'. \end{aligned}$$

The required conclusion follows by applying Proposition 3.1 . □

**Remark 3.1.** *Such equivalence of maximal functions is also proved in the flag case* [5] *and the flag-like case* [42, Proposition 3.2]. *But the proof given here is simpler than that in* [42].

3.2. **Boundary growth estimates for iterated Poisson integrals and holomorphic $H^p$-functions.**

**Proposition 3.3.** *For $f \in L^1(\mathbb{R}^n)$, let $u(x,\mathbf{t}) := f * P_{\mathbf{t}}$. Then, for $b_1, \cdots, b_m \in \mathbb{N}_0$,*

$$\begin{aligned} t_1^{b_1}\cdots t_m^{b_m}\left\|\nabla_1^{b_1}\cdots\nabla_m^{b_m}u(\cdot,\mathbf{t})\right\|_{L^1(\mathbb{R}^n)} &\lesssim \|f\|_{L^1(\mathbb{R}^n)}, \\ t_1^{b_1}\cdots t_m^{b_m}\left|\nabla_1^{b_1}\cdots\nabla_m^{b_m}u(x,\mathbf{t})\right| &\lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}}{|R(0,\mathbf{t})|}, \end{aligned} \tag{3.11}$$

*for $(x,\boldsymbol{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$.*

*Proof.* In the sequel we use notation $\boldsymbol{\varepsilon} := (\varepsilon,\ldots,\varepsilon) \in (\mathbb{R}_+)^m$ for $\varepsilon > 0$. If we let $f_\varepsilon := f * P_{\boldsymbol{\varepsilon}}$, then we can write $u(x,\mathbf{t}) := f_\varepsilon * P_{\mathbf{t}-\boldsymbol{\varepsilon}}$ for small $\varepsilon > 0$ and $f_\varepsilon$ is smooth. So we may assume that $f$ is smooth. For fixed $\mu$, we use coordinates $x = x^\perp + se_\mu$ as above. Then,

$$\begin{aligned} \left\|f *_\mu P_{t_\mu}\right\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\left|\int_{\mathbb{R}} f(x^\perp + (s-s')e_\mu)P_{t_\mu}(s')ds'\right| ds\, dx^\perp \\ &= \int_{\mathbb{R}^{n-1}} \|f(x^\perp + \cdot e_\mu) * P_{t_\mu}\|_{L^1(\mathbb{R})} dx^\perp \\ &\le \int_{\mathbb{R}^{n-1}} \|f(x^\perp + \cdot e_\mu)\|_{L^1(\mathbb{R})}\|P_{t_\mu}\|_{L^1(\mathbb{R})} dx^\perp \\ &= \int_{\mathbb{R}^{n-1}} \|f(x^\perp + \cdot e_\mu)\|_{L^1(\mathbb{R})} dx^\perp = \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{3.12}$$

Applying this equality repeatedly, we see that $\|u(\cdot,\mathbf{t})\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$. By replacing $P_{t_\mu}$ by $t_\mu^{b_\mu}\widetilde{\nabla}_\mu^{b_\mu}P_{t_\mu}$, the above argument gives us the first estimate in (3.11), since we also have $\|t_\mu^{b_\mu}\widetilde{\nabla}_\mu^{b_\mu}P_{t_\mu}\|_{L^1(\mathbb{R})} \lesssim 1$.

By Proposition 2.1 (2), $u(x+s'_\mu e_\mu, \mathbf{t}')$ is harmonic on $\{(s'_\mu, t'_\mu); t'_\mu > 0\}$, which contains the disc $|s'_\mu + \mathbf{i}(t_\mu - t'_\mu)| < t_\mu$. So we can apply the mean value formula to obtain

$$u(x,\mathbf{t}) = \frac{1}{\pi t_\mu^2}\int_{|s'_\mu + \mathbf{i}t'_\mu| < t_\mu} u\left(x + s'_\mu e_\mu, t_1, \ldots, t_\mu + t'_\mu, \ldots\right) ds'_\mu dt'_\mu.$$

Repeating this procedure, we deduce

$$\begin{aligned} |u(x,\mathbf{t})| &\le \frac{1}{\pi^m t_1^2\cdots t_m^2}\int_{|s'_1+\mathbf{i}t'_1|<t_1}\cdots\int_{|s'_m+\mathbf{i}t'_m|<t_m}\left|u\left(x+\sum_{\mu=1}^m s'_\mu e_\mu, \mathbf{t}+\mathbf{t}'\right)\right| ds'_m dt'_m\cdots ds'_1 dt'_1 \\ &\le \frac{1}{\pi^m t_1^2\cdots t_m^2}\int_{-t_1}^{t_1}\cdots\int_{-t_m}^{t_m}\int_{-t_1}^{t_1}\cdots\int_{-t_m}^{t_m}\left|u\left(x+\sum_{\mu=1}^m s'_\mu e_\mu, \mathbf{t}+\mathbf{t}'\right)\right| dt'_m\cdots dt'_1 ds'_m\cdots ds'_1. \end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned} \frac{1}{t_1\cdots t_m}\int_{-t_1}^{t_1}\cdots\int_{-t_m}^{t_m}\left|u\left(x+\sum_{\mu=1}^m s'_\mu e_\mu, \mathbf{t}+\mathbf{t}'\right)\right| ds'_m\cdots ds'_1 &= \left|u\left(\cdot,\mathbf{t}+\mathbf{t}'\right)\right| * \chi_{\mathbf{t}}(x) \\ &\lesssim \frac{1}{|R(x,\gamma_0^{-1}\mathbf{t})|}\int_{R(x,\gamma_0^{-1}\mathbf{t})}\left|u\left(x',\mathbf{t}+\mathbf{t}'\right)\right| dx' \\ &\lesssim \frac{1}{|R(x,\gamma_0^{-1}\mathbf{t})|}\int_{\mathbb{R}^n}\left|u\left(x',\mathbf{t}+\mathbf{t}'\right)\right| dx' \\ &\lesssim \frac{1}{|R(x,\mathbf{t})|}\|f\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

by applying Proposition 3.2 and the first estimate in (3.11). Substitute this estimate into (3.13) to obtain the second inequality in (3.11) for $b_1 = \cdots = b_m = 0$.

For the general case, note that each component of $\nabla_1^{b_1} \cdots \nabla_m^{b_m} u(x + s'_\mu e_\mu, \mathbf{t}')$ is still harmonic on $\{(s'_\mu, t'_\mu); t'_\mu > 0\}$. The result follows from the same argument by using the first estimate in (3.11). □

**Corollary 3.1.** *Fix $\varepsilon > 0$ and let $f_\varepsilon := f * P_{\boldsymbol{\varepsilon}}$ for $f \in L^1(\mathbb{R}^n)$. Then,*
(a) *$U(x, t_\mu) = (f_\varepsilon *_\mu P_{t_\mu}(x))^2$ satisfies conditions in Proposition 2.3.*
(b) *Let $\mathfrak{j}$ be a subset of $\mathfrak{m}$ and $\mu \notin \mathfrak{j}$. Then for fixed $\mathbf{t}_\mathfrak{j}$, $U(x, t_\mu) = f_\varepsilon *_\mathfrak{j} \widetilde{\nabla}_\mathfrak{j} P_{\mathbf{t}_\mathfrak{j}} *_\mu \widetilde{\nabla}_\mu P_{t_\mu}(x)$ also satisfies conditions in Proposition 2.3 except for $U(\cdot, t_\mu) \to U(\cdot, 0)$ replaced by $U(\cdot, t_\mu) \to 0$.*

*Proof.* (a) For condition (1), write $U(x, t_\mu) = u^2(x, t_\mu)$ with $u(x, t_\mu) := f_\varepsilon *_\mu P_{t_\mu}(x)$. It is obviously smooth on $\mathbb{R}_+^{n+1}$ since $f_\varepsilon$ is. Because $u(\cdot, t_\mu), \nabla_\mu u(\cdot, t_\mu) \in L^1 \cap L^\infty(\mathbb{R}^n)$ with norms only depending on $\varepsilon$ and $\|f\|_1$ by Proposition 3.3, we see that $U(\cdot, t_\mu), \nabla_\mu U(\cdot, t_\mu) = 2u(\cdot, t_\mu)\nabla_\mu u(\cdot, t_\mu) \in L^1 \cap L^\infty(\mathbb{R}^n)$. It is similar to see $\nabla_\mu^2 U(\cdot, t_\mu) \in L^1 \cap L^\infty(\mathbb{R}^n)$. It is standard to see that for $x = x^\perp + s e_\mu$,

$$X_\mu U(x, t_\mu) = 2u(x, t_\mu) \cdot \int_{\mathbb{R}} f_\varepsilon(x^\perp + (s - s')e_\mu) \partial_{s'} P_{t_\mu}(s') ds' \to 0,$$

for almost all $x^\perp \in e_\mu^\perp$ as $s \to \infty$, since $f_\varepsilon(x^\perp + \cdot e_\mu) \in L^1(\mathbb{R})$ for almost all $x^\perp \in e_\mu^\perp$.

For condition (2), apply estimates (3.11) to $u(x, t_\mu) = f * P_{\boldsymbol{\varepsilon}} *_\mu P_{t_\mu}(x)$ to get

$$\|t_\mu \partial_{t_\mu} U(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \lesssim \|u(\cdot, t_\mu)\|_{L^\infty(\mathbb{R}^n)} \|t_\mu \partial_{t_\mu} u(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}^2}{\varepsilon^{n-1}(\varepsilon + t_\mu)} \to 0$$

as $t_\mu \to +\infty$. Similarly, $\|U(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \to 0$ as $t_\mu \to +\infty$.

For condition (3), note that $u(x, 0) = f_\varepsilon$. Write $\triangle(x) := \|f_\varepsilon(\cdot - x) - f_\varepsilon(\cdot)\|_{L^1(\mathbb{R}^n)}$. It is well known that $\triangle(x) \to 0$ as $|x| \to 0$. Hence,

$$\begin{aligned} \|u(\cdot, t_\mu) - u(\cdot, 0)\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}} \left[ f_\varepsilon(x^\perp + (s - s')e_\mu) - f_\varepsilon(x^\perp + s e_\mu) \right] P_{t_\mu}(s') ds' \right| dx^\perp ds \\ &\leq \int_{\mathbb{R}} \triangle(s' e_\mu) P_{t_\mu}(s') ds' = \int_{\mathbb{R}} \triangle(t_\mu s' e_\mu) P_1(s') ds' \to 0, \end{aligned}$$

as $t_\mu \to 0$, by rescaling and using Lebegues' dominated convergence theorem (cf. [36, P. 63]). We see that $\|t_\mu \partial_{t_\mu} u(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \to 0$ as $t_\mu \to 0$ follows by the same argument since

$$t_\mu \partial_{t_\mu} u(x, t_\mu) = \int_{\mathbb{R}} \left[ f_\varepsilon(x - s' e_\mu) - f_\varepsilon(x) \right] t_\mu \partial_{t_\mu} P_{t_\mu}(s') ds'$$

by $\int_{\mathbb{R}} t_\mu \partial_{t_\mu} P_{t_\mu}(s') ds' = 0$. The result for $U(x, t_\mu) = u^2(x, t_\mu)$ follows from that $|u(x, t_\mu)|$ has an upper bound only depending on $\varepsilon$.

(b) We apply the proof in part (a) with $P_{t_\mu}$ replaced by $\widetilde{\nabla}_\mu P_{t_\mu}$ and $f_\varepsilon$ replaced by $f_\varepsilon *_\mathfrak{j} \widetilde{\nabla}_\mathfrak{j} P_{\mathbf{t}_\mathfrak{j}}(x)$, which is in $L^1 \cap L^\infty(\mathbb{R}^n)$ with a norm only depending on $\varepsilon$. □

**Proposition 3.4.** *Let $F \in H^p(T_\Omega)$ with $p \geq 1$. Then for $(x, \mathbf{r}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$, we have*

$$|F(x + \mathbf{i}\pi(\mathbf{r}))| \lesssim \frac{\|F\|_{H^p(T_\Omega)}}{|R(0, \mathbf{r})|^{\frac{1}{p}}}. \tag{3.14}$$

*Proof.* Given $y \in \Omega$, write $y = \pi(\mathbf{r}) = \sum_{\mu=1}^m r_\mu e_\mu \in \Omega$ for some $\mathbf{r} \in (\mathbb{R}_+)^m$. Note that the disc $\{x + \mathbf{i}y + (s_\mu + \mathbf{i}t_\mu)e_\mu; |s_\mu + \mathbf{i}t_\mu| < r_\mu\}$ is contained in the tube domain $T_\Omega$. This is because $r_\mu + t_\mu >$

$r_\mu - r_\mu = 0$. So we can apply the mean value formula for this disc in the complex line $\mathbb{C}e_\mu$ to get

$$F(x+\mathbf{i}y) = \frac{1}{\pi\gamma_0^2 r_\mu^2}\int_{|s_\mu+\mathbf{i}t_\mu|<\gamma_0 r_\mu} F\left(x+\mathbf{i}y+(s_\mu+\mathbf{i}t_\mu)e_\mu\right) ds_\mu dt_\mu.$$

Repeating this procedure, we get

$$\begin{aligned}|F(x+\mathbf{i}y)| &\le \int_{|s_1+\mathbf{i}t_1|<\gamma_0 r_1}\cdots\int_{|s_m+\mathbf{i}t_m|<\gamma_0 r_m} \frac{\left|F\left(x+\mathbf{i}y+\sum_{\mu=1}^m (s_\mu+\mathbf{i}t_\mu)e_\mu\right)\right|}{\pi^m\gamma_0^{2m} r_1^2\cdots r_m^2} ds_m dt_m\cdots ds_1 dt_1\\ &\le \int_{-\gamma_0 r_1}^{\gamma_0 r_1}\cdots\int_{-\gamma_0 r_m}^{\gamma_0 r_m}\int_{-\gamma_0 r_1}^{\gamma_0 r_1}\cdots\int_{-\gamma_0 r_m}^{\gamma_0 r_m} \frac{\left|F\left(x+\mathbf{i}y+\sum_{\mu=1}^m (s_\mu+\mathbf{i}t_\mu)e_\mu\right)\right|}{\pi^m\gamma_0^{2m} r_1^2\cdots r_m^2} dt_m\cdots dt_1 ds_m\cdots ds_1.\end{aligned}$$

Note that

$$\begin{aligned}\int_{-\gamma_0 r_1}^{\gamma_0 r_1}\cdots\int_{-\gamma_0 r_m}^{\gamma_0 r_m} \frac{\left|F\left(x+\mathbf{i}y+\sum_{\mu=1}^m s_\mu e_\mu+\mathbf{i}\sum_{\mu=1}^m t_\mu e_\mu\right)\right|}{\gamma_0^m r_1\cdots r_m} dt_m\cdots dt_1 &= \left|F\left(x+\sum_{\mu=1}^m s_\mu e_\mu+\mathbf{i}\cdot\right)\right| * \chi_{\gamma_0\mathbf{r}}(y)\\ &\lesssim \frac{1}{|R(y,\mathbf{r})|}\int_{R(y,\mathbf{r})}\left|F\left(x+\sum_{\mu=1}^m s_\mu e_\mu+\mathbf{i}y'\right)\right| dy',\end{aligned}$$

by applying Proposition 3.2, where $R(y,\mathbf{r})\subset\Omega$. Apply the same procedure for the integral over $s_1,\ldots,s_m$ to get

$$\begin{aligned}|F(x+\mathbf{i}y)| &\lesssim \frac{1}{|R(0,\mathbf{r})|^2}\int_{R(x,\mathbf{r})}\int_{R(y,\mathbf{r})}|F(x'+\mathbf{i}y')|\,dy'\,dx'\\ &\le \frac{1}{|R(0,\mathbf{r})|^{1+\frac{1}{p}}}\int_{R(y,\mathbf{r})}\left(\int_{\mathbb{R}^n}|F(x'+\mathbf{i}y'|^p\,dx'\right)^{\frac{1}{p}} dy' \qquad (3.15)\\ &\lesssim \frac{1}{|R(0,\mathbf{r})|^{\frac{1}{p}}}\|F\|_{H^p(T_\Omega)}.\end{aligned}$$

This concludes the proof of (3.14). □

**Proposition 3.5.** *For two polyhedral cones $\Omega$ and $\Omega'$ in $\mathbb{R}^n$ such that $\overline{\Omega}\subset\Omega'$, then $H^1(T_{\Omega'})\subsetneqq H^1(T_\Omega)$ and $H^1_{max;\Omega'}(\mathbb{R}^n)\subsetneq H^1_{max;\Omega}(\mathbb{R}^n)$.*

*Proof.* Since $T_\Omega\subset T_{\Omega'}$, $H^1(T_{\Omega'})\subset H^1(T_\Omega)$ in view of definition (1.1). For an $H^1(T_{\Omega'})$-function $F$, we have $F(\cdot+\mathbf{i}y)\in H^2(T_{\Omega'})$ for fixed $y\in\Omega$, by the boundary growth estimate (3.14). So the Fourier transformation $\widehat{F(\cdot+\mathbf{i}y)}$ with respect to variable $x$ must be supported in $(\Omega')^*$ by Theorem 3.1 in [40].

Note that $\overline{(\Omega')^*}\subsetneq\Omega^*$ by definition. Let $\psi$ be a smooth function supported in a bounded subset of $\Omega^*$, but is not contained in $(\Omega')^*$. Consider a holomorphic function on $T_\Omega$

$$F_0(x+\mathbf{i}y) = \int_{\Omega^*} e^{2\pi\mathbf{i}(x+\mathbf{i}y)t}\psi(t)dt.$$

We see that $F_0\in H^1(T_\Omega)$ by the estimates: for $|x|<1$, $|F_0(x+\mathbf{i}y)|\lesssim 1$, since $y\cdot t\ge 0$ for $y\in\Omega, t\in\Omega^*$; while for $|x|>1$,

$$\begin{aligned}|F_0(x+\mathbf{i}y)| &= \left|\int_{\mathbb{R}^n}\frac{\triangle^n e^{2\pi\mathbf{i}(x+\mathbf{i}y)t}}{(2\pi\mathbf{i}(x+\mathbf{i}y))^{2n}}\psi(t)dt\right|\\ &= \left|\int_{\mathbb{R}^n}\frac{1}{(2\pi\mathbf{i}(x+\mathbf{i}y))^{2n}}e^{2\pi\mathbf{i}(x+\mathbf{i}y)t}\triangle^n\psi(t)dt\right| \lesssim \frac{1}{|x|^{2n}},\end{aligned}$$

where $\triangle$ is the Laplacian on $\mathbb{R}^n_t$, and implicit constants are independent of $y$. Since the support of $\widehat{F_0(\cdot+\mathbf{i}y)}(t)=e^{-2\pi yt}\psi(t)$ is not contained in $(\Omega')^*$, we have $F_0\notin H^1(T_{\Omega'})$. Hence, $H^1(T_{\Omega'})\neq H^1(T_\Omega)$.

Since $F_0(\cdot+\mathbf{i}y_0)\in H^2(T_\Omega)$ for fixed $y_0\in\Omega$, by the integral representation in Theorem 1.1, we have $F_0(x+\mathbf{i}\pi(\mathbf{t})+\mathbf{i}y_0)=F_0(\cdot+\mathbf{i}y_0)*P_{\mathbf{t}}(x)$. So by Theorem 1.2, $F_0(\cdot+\mathbf{i}y_0)|_{\mathbb{R}^n}$ belongs to $H^1_{max;\Omega}(\mathbb{R}^n)$, but not to $H^1_{max;\Omega'}(\mathbb{R}^n)$. Otherwise, $F_0(\cdot+\mathbf{i}y_0)\in H^1(T_{\Omega'})$ by Theorem 1.2, which contradicts the fact that $F_0\notin H^1(T_{\Omega'})$. □

3.3. **Proof of iterated Poisson integral formula and harmonic control.** In the integral representation formula (1.5), for $\mathbf{t}\neq\mathbf{t}'$ with $\pi(\mathbf{t})=\pi(\mathbf{t}')$, we must have $F^b*P_{\mathbf{t}}=F^b*P_{\mathbf{t}'}$. This formula follows from the uniqueness of functions satisfying partial Laplace equations $\triangle_\mu u=0$ with the same boundary condition, by using the maximum principle for harmonic functions.

*Proof of Theorem 1.1.* For $F\in H^p(T_\Omega)$, we have $f_\varepsilon:=F(\cdot+\mathbf{i}\pi(\boldsymbol{\varepsilon}))\in L^p(\mathbb{R}^n)$ for $\varepsilon>0$ by definition, and it is obviously smooth on $\mathbb{R}^n$. Then

$$f_{\varepsilon,x^\perp}(\cdot):=F\left(x^\perp+(\cdot)e_\mu+\mathbf{i}\pi(\boldsymbol{\varepsilon})\right)\in L^p(\mathbb{R}) \tag{3.16}$$

for almost all $x^\perp\in e_\mu^\perp\subset\mathbb{R}^n$. Now fix such a point $x^\perp\in e_\mu^\perp$ and let

$$v_{\varepsilon,x^\perp}(s_\mu,t_\mu):=(f_{\varepsilon,x^\perp}*P_{t_\mu})(s_\mu)=(f_\varepsilon *_\mu P_{t_\mu})(x^\perp+s_\mu e_\mu), \tag{3.17}$$

which is smooth on $\overline{\mathbb{R}^2_+}$. Note that $v_{\varepsilon,x^\perp}(\cdot,t_\mu)\to f_{\varepsilon,x^\perp}$ as $t_\mu\to 0$, by the standard property of Poisson integrals [36, P. 62]. Namely,

$$v_{\varepsilon,x^\perp}(s_\mu,0)=f_{\varepsilon,x^\perp}(s_\mu)=F\left(x^\perp+s_\mu e_\mu+\mathbf{i}\pi(\boldsymbol{\varepsilon})\right). \tag{3.18}$$

We claim that for any $\delta>0$, there exists $R,b>0$ sufficiently large such that for $(s_\mu,t_\mu)$ outside the rectangle $\mathcal{R}_{2R,b}:=\{(s_\mu,t_\mu)\in\mathbb{R}^2_+;|s_\mu|\leq 2R,0<t_\mu<b\}$, we have

$$|v_{\varepsilon,x^\perp}(s_\mu,t_\mu)|<\delta,\qquad \left|F\left(x^\perp+(s_\mu+\mathbf{i}t_\mu)e_\mu+\mathbf{i}\pi(\boldsymbol{\varepsilon})\right)\right|<\delta. \tag{3.19}$$

Then by the maximal principle [36] for harmonic functions on the rectangle $\mathcal{R}_{2R,b}$, we obtain

$$|v_{\varepsilon,x^\perp}(s_\mu,t_\mu)-F\left(x^\perp+(s_\mu+\mathbf{i}t_\mu)e_\mu+\mathbf{i}\pi(\boldsymbol{\varepsilon})\right)|\leq 2\delta.$$

Since $\delta>0$ was arbitrarily chosen, we get

$$(f_\varepsilon *_\mu P_{t_\mu})(x^\perp+s_\mu e_\mu)=F\left(x^\perp+(s_\mu+\mathbf{i}t_\mu)e_\mu+\mathbf{i}\pi(\boldsymbol{\varepsilon})\right),$$

for almost all $x^\perp\in e_\mu^\perp$, by (3.17). Consequently, we deduce $(f_\varepsilon *_\mu P_{t_\mu})(x)=F(x+\mathbf{i}t_\mu e_\mu+\mathbf{i}\pi(\boldsymbol{\varepsilon}))$ for any $x\in\mathbb{R}^n$, $t_\mu\in\mathbb{R}$, by continuity. Repeating this procedure, we get

$$(f_\varepsilon *_1 P_{t_1}\cdots *_m P_{t_m})(x)=F(x+\mathbf{i}\pi(\mathbf{t})+\mathbf{i}\pi(\boldsymbol{\varepsilon})) \tag{3.20}$$

for any $x\in\mathbb{R}^n$, $\mathbf{t}\in(\mathbb{R}_+)^m$.

To prove the claim (3.19) of decay, note that by definition, we have

$$\int_{\mathbb{R}^n}|F(x+\mathbf{i}y)|^p\,dx\leq\|F\|^p_{H^p(T_\Omega)}$$

for any $y\in\Omega$, and so for fixed $b>0$,

$$\int_{\mathbb{R}^n}\int_{\{y\in\Omega;|y|<2b\}}|F(x+\mathbf{i}y)|^p\,dxdy<\infty.$$

Hence, for any $\delta' > 0$, there exists $R > 0$ such that

$$\int_{|x|>R}\int_{\{y\in\Omega;|y|<2b\}} |F(x+\mathbf{i}y)|^p\,dxdy < \delta'$$

if $R$ is sufficiently large.

On the other hand, $R(x,\boldsymbol{\varepsilon}) + \mathbf{i}R(y+\pi(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}) \subset T_\Omega$ for any $y \in \Omega$, since

$$R(y+\pi(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}) = \left\{y + \sum_{j=1}^{m}(\varepsilon+\lambda_j)e_j; |\lambda_j| < \varepsilon, j = 1,\ldots,m\right\} \subset \Omega.$$

So by using the estimate similar to (3.15) in Proposition 3.4, for $|x| > 2R, |y| < b$ we obtain

$$\begin{aligned} |F\left(x+\mathbf{i}y+\mathbf{i}\pi(\boldsymbol{\varepsilon})\right)| &\lesssim \frac{1}{|R(0,\boldsymbol{\varepsilon})|^2}\int_{R(x,\boldsymbol{\varepsilon})}\int_{R(y,\boldsymbol{\varepsilon})} |F(x'+\mathbf{i}\pi(\boldsymbol{\varepsilon})+\mathbf{i}y')|\,dy'\,dx' \\ &\lesssim \frac{1}{|R(0,\boldsymbol{\varepsilon})|^{2-\frac{2}{q}}}\left(\int_{R(x,\boldsymbol{\varepsilon})}\int_{R(y,\boldsymbol{\varepsilon})} |F(x'+\mathbf{i}\pi(\boldsymbol{\varepsilon})+\mathbf{i}\hat{y})|^p\,dy'\,dx'\right)^{\frac{1}{p}} \\ &\lesssim \frac{1}{|R(0,\boldsymbol{\varepsilon})|^{2-\frac{2}{q}}}\left(\int_{|\hat{x}|>R}\int_{\{\hat{y}\in\Omega;|y'|<2b\}} |F(x'+\mathbf{i}y')|^p\,dy'\right)^{\frac{1}{p}} dx' < \delta. \end{aligned} \tag{3.21}$$

The claim (3.19) for $F(x+\mathbf{i}y+\mathbf{i}\pi(\boldsymbol{\varepsilon}))$ is proved, while the claim (3.19) for $v_{\varepsilon,x^\perp}$ holds by the decay of the Poisson integral of an $L^p$ function for $p \geq 1$.

For $t_\mu > b$ with $b$ large, the claim (3.19) directly follows from the estimate in Proposition 3.3 and 3.4.

Since $L^p(\mathbb{R}^n)$ for $p > 1$ is reflexive and $\|f_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq \|F\|_{H^p(T_\Omega)}$, there exists a subsequence $\varepsilon_k \to 0$ such that $\{f_{\varepsilon_k}\}$ is weakly convergent to some $F^b \in L^p(\mathbb{R}^n)$ by the Banach-Alaoglu theorem. For fixed $\mathbf{t}$, we can assume $t_1,\ldots,t_n$ are the largest $n$ numbers among $\{t_1,\ldots,t_m\}$ and (3.6) holds. Then, one has

$$\begin{aligned} f_{\varepsilon_k} * P_{\mathbf{t}}(x) &= \int_{\mathbb{R}^m} f_{\varepsilon_k}\left(x - \sum_{j=1}^{m} s_je_j\right) P_{t_1}(s_1)\cdots P_{t_m}(s_m)\,d\mathbf{s} \\ &= \int_{\mathbb{R}^m} f_{\varepsilon_k}\left(x - \sum_{j=1}^{n}\left(s_j + \sum_{\mu=n+1}^{m} A_{\mu j}s_\mu\right)e_j\right) P_{t_1}(s_1)\cdots P_{t_m}(s_m)\,d\mathbf{s} \\ &= \int_{\mathbb{R}^m} f_{\varepsilon_k}\left(x - \sum_{j=1}^{n} s_j'e_j\right)\prod_{j=1}^{n} P_{t_j}\left(s_j' - \sum_{\mu=n+1}^{m} A_{\mu j}s_\mu'\right)\cdot\prod_{j=n+1}^{m} P_{t_j}(s_j')\,d\mathbf{s}' \\ &= \frac{1}{\det(e_1,\ldots,e_n)}\int_{\mathbb{R}^n} f_{\varepsilon_k}(x-x')\,\pi(P_{t_1}\cdots P_{t_m})(x')\,dx', \end{aligned} \tag{3.22}$$

using the coordinates transformations (3.8), where

$$\pi(P_{t_1}\cdots P_{t_m})(x') := \int_{\mathbb{R}^{m-n}}\prod_{j=1}^{n} P_{t_j}\left(s_j' - \sum_{\mu=n+1}^{m} A_{\mu j}s_\mu'\right)\cdot\prod_{j=n+1}^{m} P_{t_j}(s_j')\,ds_{n+1}'\cdots ds_m'$$

for $x' = \sum_{j=1}^n s_j' e_j$. This function lies in $L^q(\mathbb{R}^n)$ for any $q > 1$, since

$$\begin{aligned}\|\pi(P_{t_1}\cdots)\|_{L^q} &\approx \left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^{m-n}}\prod_{j=1}^n P_{t_j}\left(s_j' - \sum_{\mu=n+1}^m A_{\mu j}s_\mu'\right)\prod_{j=n+1}^m P_{t_j}(s_j')ds_{n+1}'\cdots ds_m'\right|^q ds_1'\cdots ds_n'\right)^{\frac{1}{q}}\\ &\le \int_{\mathbb{R}^{m-n}}\left(\int_{\mathbb{R}^n}\left|\prod_{j=1}^n P_{t_j}\left(s_j' - \sum_{\mu=n+1}^m A_{\mu j}s_\mu'\right)\prod_{j=n+1}^m P_{t_j}(s_j')\right|^q ds_1'\cdots ds_n'\right)^{\frac{1}{q}} ds_{n+1}'\cdots ds_m'\\ &= (t_1\cdots t_n)^{\frac{1}{q}-1}\|P_1\|^n_{L^q(\mathbb{R})} < \infty,\end{aligned}$$

in view of the coordinates transformation (3.10) and Minkowski's inequality.

Now by the weak convergence of $\{f_{\varepsilon_k}\}$ to $F^b \in L^p(\mathbb{R}^n)$, we get

$$f_{\varepsilon_k} * \pi(P_{t_1}\cdots P_{t_m})(x) \to F^b * \pi(P_{t_1}\cdots P_{t_m})(x),$$

for any $x \in \mathbb{R}^n$. Then, $f_{\varepsilon_k} * P_{\mathbf{t}}(x) \to F^b * P_{\mathbf{t}}(x)$ as $k \to +\infty$ by (3.22). On the other hand,

$$f_{\varepsilon_k} * P_{\mathbf{t}}(x) = F(x + \mathbf{i}\pi(\mathbf{t}) + \mathbf{i}\pi(\boldsymbol{\varepsilon}_k)) \to F(x + \mathbf{i}\pi(\mathbf{t}))$$

by (3.20), since $F$ is smooth on $T_\Omega$. The integral representation formula (1.5) follows. □

**Proposition 3.6.** *Suppose $F \in H^1(T_\Omega)$. Then for $0 < q < +\infty$, we have*

$$|F(x + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\varepsilon}))|^q \le |f_\varepsilon|^q * P_{\mathbf{t}}(x), \tag{3.23}$$

*for any $(x, \boldsymbol{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$, where $f_\varepsilon(x) := F(x + \mathbf{i}\pi(\boldsymbol{\varepsilon}))$.*

*Proof.* It is well known that if $h$ is holomorphic on a domain $D$ in $\mathbb{C}$, then $|h|^q$ for $q > 0$ is subharmonic on $D$. We see that $|F(x^\perp + (s_\mu + \mathbf{i}t_\mu)e_\mu + \mathbf{i}\pi(\boldsymbol{\varepsilon}))|^q$ is subharmonic on $\mathbb{R}^2_+$, since $F(x^\perp + ze_\mu + \mathbf{i}\pi(\boldsymbol{\varepsilon}))$ is holomorphic in $z \in \mathbb{C}$ for $\operatorname{Im} z > 0$ by Proposition 2.1. As in the proof of Theorem 1.1, let

$$v_{x^\perp,\varepsilon}(s_\mu, t_\mu) := (|f_\varepsilon|^q *_\mu P_{t_\mu})(x^\perp + s_\mu e_\mu + \mathbf{i}\pi(\varepsilon)), \tag{3.24}$$

on the upper half plane $\mathbb{R}^2_+$, which is harmonic by Proposition 2.1. So

$$|F(x^\perp + (s_\mu + \mathbf{i}t_\mu)e_\mu + \mathbf{i}\pi(\boldsymbol{\varepsilon}))|^q - v_{x^\perp,\varepsilon}(s_\mu, t_\mu)$$

is also subharmonic on $\mathbb{R}^2_+$. Moreover, it vanishes at $t_\mu = 0$, i.e.

$$v_{x^\perp,\varepsilon}(s_\mu, 0) = |f_\varepsilon(x^\perp + s_\mu e_\mu)|^q = |F(x^\perp + s_\mu e_\mu + \mathbf{i}\pi(\boldsymbol{\varepsilon}))|^q,$$

by definition (3.24). The claim (3.19) holds similarly by (3.21) for $p = 1$ and $|f_\varepsilon|^q \in L^{\frac{1}{q}}(\mathbb{R}^n)$ with $\frac{1}{q} > 1$. Then, we obtain

$$|F(x^\perp + (s_\mu + \mathbf{i}t_\mu)e_\mu + \mathbf{i}\pi(\boldsymbol{\varepsilon}))|^q - v_{x^\perp,\varepsilon}(s_\mu, t_\mu) \le 0$$

by applying the maximal principle for subharmonic functions. The result follows by repeating the procedure as in the proof of Theorem 1.1. However, we do not need the last step of taking limit $\varepsilon_k \to 0$ here. □

# 4. The maximal function characterization

## 4.1. Reduction of the Fefferman-Stein type good-$\lambda$ inequality to an integral estimate.

**Theorem 4.1.** *For $f \in L^1(\mathbb{R}^n)$ and all $\lambda > 0$, if we choose $\beta$ sufficient large, then the Fefferman-Stein type good-$\lambda$ inequality* (1.11) *holds.*

The following is a direct corollary of this theorem.

**Corollary 4.1.** *If we choose $\beta$ sufficient large, then $\|S(f)\|_{L^1(\mathbb{R}^n)} \lesssim \|N^\beta(f)\|_{L^1(\mathbb{R}^n)}$.*

To prove the Fefferman-Stein type good-$\lambda$ inequality in Theorem 4.1, let

$$E_\beta(\lambda) := \left\{x \in \mathbb{R}^n; N^\beta(f)(x) \le \lambda\right\},$$

for $f \in L^1(\mathbb{R}^n)$ with $N^\beta(f) \in L^1(\mathbb{R}^n)$ and $\lambda \ge 0$, and let

$$A_\beta(\lambda) := \left\{x \in \mathbb{R}^n; M_{it}(\chi_{E_\beta(\lambda)^c})(x) \le \frac{1}{10C_0}\right\},$$

where $C_0$ is given by (3.1). By definition, $E_\beta(\lambda)^c \subset A_\beta(\lambda)^c$, and so $A_\beta(\lambda) \subset E_\beta(\lambda)$, up to a set of measure zero. Moreover,

$$|A_\beta(\lambda)^c| \lesssim \|M_{it}(\chi_{E_\beta(\lambda)^c})\|_{L^2}^2 \le C\,|E_\beta(\lambda)^c|,$$

by the $L^2$-boundedness of iterated maximal function, where $C$ is independent of $\beta$. Therefore, we have

$$\begin{aligned}|\{x \in \mathbb{R}^n; S(f) > \lambda\}| &\le |\{x \in A_\beta(\lambda)^c; S(f)(x) > \lambda\}| + |\{x \in A_\beta(\lambda); S(f)(x) > \lambda\}| \\ &\le C\,|E_\beta(\lambda)^c| + \frac{1}{\lambda^2}\int_{A_\beta(\lambda)} S(f)^2(x)dx.\end{aligned} \tag{4.1}$$

Consider the domains (see Figure 2)

$$W_\beta := \bigcup_{x \in A_\beta(\lambda)} \Gamma(x), \qquad \text{and} \qquad \widetilde{W}_\beta := \bigcup_{x \in E_\beta(\lambda)} \Gamma_\beta(x). \tag{4.2}$$

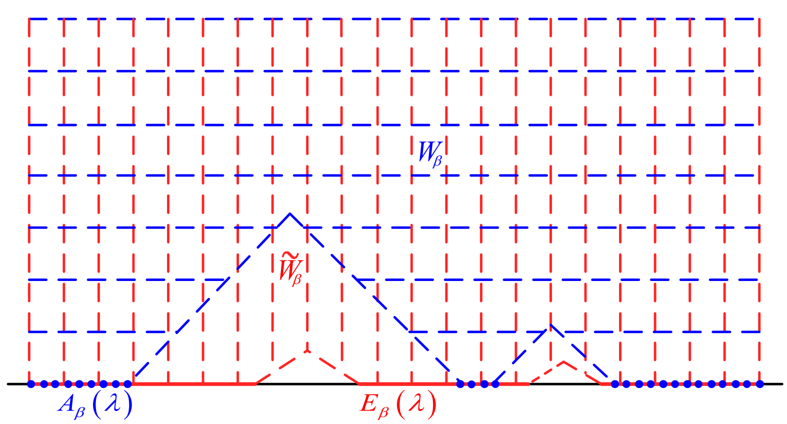


Figure 2. Domains $W_\beta$ and $\widetilde{W}_\beta$

**Proposition 4.1.** (1) *If* $(x,\mathbf{t})\in W_\beta$, *then* $P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)>\frac{9}{10}$.
(2) *If we choose* $\beta$ *sufficient large, then there is a constant* $C_1\in(0,9/10)$ *such that for any* $(x,\mathbf{t})\in\widetilde{W}_\beta^c=(\mathbb{R}^n\times(\mathbb{R}_+)^m)\setminus\widetilde{W}_\beta$, *we have* $P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)\le C_1$.

*Proof.* (1) For $(x,\mathbf{t})\in W_\beta$, there exists $x'\in A_\beta(\lambda)$ such that $(x,\mathbf{t})\in\Gamma(x')$. Thus

$$P_{\mathbf{t}}(\chi_{E_\beta(\lambda)^c})(x)\le C_0M_{it}(\chi_{E_\beta(\lambda)^c})(x')\le C_0\frac{1}{10C_0}=\frac{1}{10},$$

by (3.1), and so $P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)=P_{\mathbf{t}}(1-\chi_{E_\beta(\lambda)^c})(x)=1-P_{\mathbf{t}}(\chi_{E_\beta(\lambda)^c})(x)>\frac{9}{10}$.

(2) Let $(x,\mathbf{t})\in\widetilde{W}_\beta^c$. Given $\mathbf{s}\in(\mathbb{R}_+)^m$, if $x'=x-\pi(\mathbf{s})\in E_\beta(\lambda)$, we must have $x\notin R(x',\beta\mathbf{t})$ by definition, otherwise, we have $(x,\mathbf{t})\in\widetilde{W}_\beta$. So $\pi(\mathbf{s})\notin R(0,\beta\mathbf{t})$, i.e. $\mathbf{s}\notin\widetilde{R}(0,\beta\mathbf{t})$. Consequently, $|s_\mu|\ge\beta t_\mu$ holds at least for one $\mu$. Thus,

$$\begin{aligned}0\le P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)&=\int\chi_{E_\beta(\lambda)}\left(x-\sum_{j=1}^m s_je_j\right)P_{t_1}(s_1)\cdots P_{t_m}(s_m)\,d\mathbf{s}\\&\le\sum_{\mu=1}^m\int_{|s_\mu|\ge\beta t_\mu}P_{t_\mu}(s_\mu)\,ds_\mu\prod_{\nu\ne\mu}\int_{\mathbb{R}}P_{t_\nu}(s_\nu)\,ds_\nu\\&\le\sum_{\mu=1}^m\int_{|s_\mu|\ge\beta t_\mu}P_{t_\mu}(s_\mu)\,ds_\mu\le\frac{C}{\beta}\to0,\end{aligned}$$

as $\beta\to+\infty$. □

Note that the Littlewood-Paley area function (1.9) of $f$ can be written as

$$S(f)(x):=\left(\int_{\Gamma(x)}\left|f*_1\widetilde{\nabla}_1P_{t_1}*_2\cdots*_m\widetilde{\nabla}_mP_{t_m}(x')\right|^2\frac{\mathbf{t}d\mathbf{t}dx'}{|R(0,\mathbf{t})|}\right)^{\frac12},\tag{4.3}$$

for all $x\in\mathbb{R}^n$, by Proposition 2.1 (2). For $\mathbf{t}\in(\mathbb{R}_+)^m$ and $x\in\mathbb{R}^n$, define a function on $\mathbb{R}^n\times(\mathbb{R}_+)^m$:

$$U_1(x,\mathbf{t}):=f*_1P_{t_1}*_2\widetilde{\nabla}_2P_{t_2}*\cdots*_m\widetilde{\nabla}_mP_{t_m}(x).\tag{4.4}$$

Let $\phi$ be a non-negative $\mathscr{C}^\infty(\mathbb{R})$ functions such that $\phi(r)=1$ if $r\ge\frac{9}{10}$, $\phi(r)=0$ if $r\le C_1$ (cf. [28, 33]). But by Proposition 4.1 (1), for $(x,\mathbf{t})\in W_\beta$, we have $P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)>\frac{9}{10}$. Therefore, $\phi(P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x))=1$ and so

$$\begin{aligned}\int_{A_\beta(\lambda)}S(f)^2(x)\,dx&=\int_{A_\beta(\lambda)}\int_{\Gamma(x)}\left|f*_1\widetilde{\nabla}_1P_{t_1}*\cdots*_m\widetilde{\nabla}_mP_{t_m}\right|^2(x')\frac{\mathbf{t}d\mathbf{t}dx'}{|R(0,\mathbf{t})|}\,dx\\&\lesssim\int_{W_\beta}\left|f*_1\widetilde{\nabla}_1P_{t_1}*\cdots*_m\widetilde{\nabla}_mP_{t_m}\right|^2(x')\;\mathbf{t}d\mathbf{t}dx'\\&=\int_{W_\beta}|\nabla_1U_1(x,\mathbf{t})|^2\,\mathbf{t}d\mathbf{t}dx\\&\le\int_{\mathbb{R}^n\times(\mathbb{R}_+)^m}|\nabla_1U_1(x,\mathbf{t})|^2\,\phi^2(P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)\,\mathbf{t}d\mathbf{t}dx.\end{aligned}\tag{4.5}$$

4.2. **Estimates of integrals by differential identities.** To prove the Fefferman-Stein type inequality of Theorem 4.1, we need to estimate the integral in the R. H. S. of (4.5) in terms of integrals only involving $U_1$, instead of gradients of $U_1$. This can be done by using differential identities to integrate by parts. We will at first estimate the integral in (4.5) with $\chi_{E_\beta(\lambda)}$ and $f$ replaced by their smoothing $\chi_\varepsilon := P_{\boldsymbol{\varepsilon}}(\chi_{E_\beta(\lambda)})$ and $f_\varepsilon := P_{\boldsymbol{\varepsilon}}(f)$, respectively, and then taking the limit $\varepsilon \to 0$.

**Lemma 4.1.** *Let $\varepsilon > 0$. Suppose that $f_\varepsilon := f * P_{\boldsymbol{\varepsilon}}$ for some real valued function $f \in L^1(\mathbb{R}^n)$, $\chi_\varepsilon := \chi * P_{\boldsymbol{\varepsilon}}$ with $\|\chi\|_{L^\infty} \leq 1$ and $1 - \chi \in L^1(\mathbb{R}^n)$, and*

$$u := f_\varepsilon *_\mu P_{t_\mu}, \qquad v := \chi_\varepsilon *_\mu P_{t_\mu}. \tag{4.6}$$

*Then, we have*

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_\mu u|^2 \phi^2(v) t_\mu dt_\mu dx \lesssim \int_{\mathbb{R}^n} u^2(x,0)\phi^2(v(x,0))dx + \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_1^2(v) |\nabla_\mu v|^2 t_\mu dt_\mu dx, \tag{4.7}$$

*where $\Phi_1$ is a $\mathscr{C}^{2m+2}[0,1]$ function satisfying $\Phi_1(a) = 0$ for $a \in [0, C_1]$ and for $a \in [0,1]$,*

$$\Phi_1(a) \geq |\phi(a)| + |\phi'(a)| + |\phi''(a)|. \tag{4.8}$$

*Proof.* By direct differentiation, we see that

$$\begin{aligned}(X_\mu^{(a)})^2(u^2\phi^2(v)) =& 2u(X_\mu^{(a)})^2 u \cdot \phi^2(v) + 2|X_\mu^{(a)} u|^2 \phi^2(v) + 8u\phi(v)\phi'(v) X_\mu^{(a)} u \cdot X_\mu^{(a)} v \\ &+ 2u^2\phi(v)\phi'(v)(X_\mu^{(a)})^2 v + 2u^2(\phi'(v)^2 + \phi(v)\phi''(v))|X_\mu^{(a)} v|^2,\end{aligned} \tag{4.9}$$

for $a = 1, 2$. Since $u$ and $v$ satisfy partial Laplacian equation

$$\triangle_\mu u = 0, \qquad \triangle_\mu v = 0, \tag{4.10}$$

on $\mathbb{R}^n \times \mathbb{R}_+$, by Proposition 2.1 (2), the summation of (4.9) over $a = 1, 2$ gives us

$$\begin{aligned}|\nabla_\mu u|^2 \phi^2(v) =& \frac{1}{2}\triangle_\mu \left(u^2\phi^2(v)\right) - 4u\phi(v)\phi'(v)\nabla_\mu u \cdot \nabla_\mu v - u^2(\phi'(v)^2 + \phi(v)\phi''(v))|\nabla_\mu v|^2 \\ =:& T_1 + T_2 + T_3.\end{aligned} \tag{4.11}$$

But

$$|T_2| \leq \frac{1}{10} |\nabla_\mu u|^2 \phi^2(v) + 20u^2\phi'(v)^2|\nabla_\mu v|^2 := T_{21} + T_{22}.$$

The integral $\int_{\mathbb{R}^n \times \mathbb{R}_+} T_{21}(x, t_\mu) t_\mu dt_\mu dx$ can be absorbed by the integral of the L.H.S in (4.11), while

$$T_{22} + |T_3| \leq 40u^2\Phi_1^2(v) |\nabla_\mu v|^2 .$$

Then

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_\mu u|^2 \phi^2(v) t_\mu dt_\mu dx \leq \frac{5}{9}\int_{\mathbb{R}^n \times \mathbb{R}_+} \triangle_\mu \left(u^2\phi^2(v)\right) t_\mu dt_\mu dx + \frac{400}{9}\int_{\mathbb{R}^n \times \mathbb{R}_+} u^2\Phi_1^2(v) |\nabla_\mu v|^2 t_\mu dt_\mu dx.$$

Since $U = u^2\phi^2(v)$ satisfies assumptions in Proposition 2.3 by Corollary 3.1, we can apply Proposition 2.3 to $U = u^2\phi^2(v)$ to get

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} \triangle_\mu \left(u^2\phi^2(v)\right) t_\mu dt_\mu dx = \int_{\mathbb{R}^n} u^2(x,0)\phi^2(v(x,0))dx.$$

The estimate follows. $\square$

If we apply Lemma 4.1 to $u = U_1$ in (4.4) with $f$ replaced by $f_\varepsilon$, there appears a term like

$$\int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla_2 U_2|^2 \Phi_1^2(v) |\nabla_1 v|^2 t_\mu dt_\mu dx. \tag{4.12}$$

Here for $k = 1, \ldots, m$, we define functions

$$U_k(x,\mathbf{t}) := f_\varepsilon *_1 P_{t_1} * \cdots *_k P_{t_k} *_{k+1} \widetilde{\nabla}_{k+1} P_{t_{k+1}} * \cdots *_m \widetilde{\nabla}_m P_{t_m}(x), \tag{4.13}$$

on $\mathbb{R}^n \times (\mathbb{R}_+)^m$, which satisfy

$$\nabla_k U_k(x,\mathbf{t}) = U_{k-1}(x,\mathbf{t}), \tag{4.14}$$

by Proposition 2.1 (3). So we need the following estimate further to estimate the term (4.12).

**Lemma 4.2.** *Assume as in Lemma 4.1. Let*

$$u(x,t_\mu,t_\nu) = f_\varepsilon *_\mu P_{t_\mu} *_\nu P_{t_\nu}, \qquad v(x,t_\mu,t_\nu) = \chi_\varepsilon *_\mu P_{t_\mu} *_\nu P_{t_\nu},$$

*for $\mu \neq \nu$. Then for fixed $t_\mu$, we have,*

$$\begin{aligned} \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla_\nu u|^2 \Phi_1^2(v) |\nabla_\mu v|^2 \Big|_{(x,t_\mu,t_\nu)} t_\nu dt_\nu dx \lesssim & \int_{\mathbb{R}^n} u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \Big|_{(x,t_\mu,0)} dx \\ & + \int_{\mathbb{R}^n\times\mathbb{R}_+} u^2 \Phi_2^2(v) \Sigma_{\mu\nu}(v) \Big|_{(x,t_\mu,t_\nu)} t_\nu dt_\nu dx, \end{aligned} \tag{4.15}$$

*where*

$$\Sigma_{\mu\nu}(v) := |\nabla_\mu v|^2 |\nabla_\nu v|^2 + |\nabla_\mu \nabla_\nu v|^2,$$

*and $\Phi_2$ is a $\mathscr{C}^{2m}[0,1]$ function satisfying $\Phi_2(a) = 0$ for $a \in [0, C_1]$, and for $a \in [0,1]$,*

$$\Phi_2(a) \geq |\Phi_1(a)| + |\Phi_1'(a)| + |\Phi_1''(a)|.$$

*Proof.* By direct differentiation similar to (4.11), we have the following identity:

$$\begin{aligned} |\nabla_\nu u|^2 \Phi_1^2(v) |\nabla_\mu v|^2 =& \frac{1}{2} \triangle_\nu \left( u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \right) \\ & - 4u\Phi_1(v)\Phi_1'(v) \nabla_\nu u \cdot \nabla_\nu v |\nabla_\mu v|^2 - 4u\Phi_1^2(v) \nabla_\nu u \cdot (\nabla_\mu v \cdot \nabla_\nu \nabla_\mu v) \\ & - u^2 (\Phi_1'(v)^2 + \Phi_1(v)\Phi_1''(v)) |\nabla_\nu v|^2 |\nabla_\mu v|^2 \\ & - 2u^2 \Phi_1(v) \Phi_1'(v) \nabla_\nu v \cdot (\nabla_\mu v \cdot \nabla_\nu \nabla_\mu v) - u^2 \Phi_1^2(v) |\nabla_\nu \nabla_\mu v|^2 \\ =:& \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3 + \tilde{T}_4 + \tilde{T}_5 + \tilde{T}_6, \end{aligned} \tag{4.16}$$

where $\nabla_\nu u \cdot (\nabla_\mu v \cdot \nabla_\nu \nabla_\mu v) := \sum_{a,b=1,2} X_\nu^{(a)} u \cdot X_\mu^{(b)} v \cdot X_\nu^{(a)} X_\mu^{(b)} v$. Since $U = u^2 \Phi_1^2(v) |\nabla_\mu v|^2$ satisfies assumptions in Proposition 2.3 by the following Lemma 4.4, we can apply Proposition 2.3 to get

$$\int_{\mathbb{R}^n\times\mathbb{R}_+} \triangle_\nu \left( u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \right) t_\nu dt_\nu dx = \int_{\mathbb{R}^n} u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \Big|_{(x,t_\mu,0)} dx.$$

Apply Cauchy-Schwarz inequality to get

$$\left| \int_{\mathbb{R}^n\times\mathbb{R}_+} \tilde{T}_2 t_\nu dt_\nu dx \right| \leq \frac{1}{10} \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla_\nu u|^2 \Phi_1^2(v) |\nabla_\mu v|^2 t_\nu dt_\nu dx + C \int_{\mathbb{R}^n\times\mathbb{R}_+} u^2 \Phi_2^2(v) |\nabla_\nu v|^2 |\nabla_\mu v|^2 t_\nu dt_\nu dx.$$

by definition of $\Phi_2$, and similarly

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}_+}\tilde{T}_3 t_\nu dt_\nu dx\right| \leq \frac{1}{10}\int_{\mathbb{R}^n\times\mathbb{R}_+}|\nabla_\nu u|^2\,\Phi_1^2(v)\,|\nabla_\mu v|^2\,t_\nu dt_\nu dx + C\int_{\mathbb{R}^n\times\mathbb{R}_+}u^2\Phi_1^2(v)|\nabla_\nu\nabla_\mu v|^2 t_\nu dt_\nu dx.$$

The first terms in the R. H. S. above are absorbed by the integral of the L.H.S. of (4.16). The integrals of $\tilde{T}_4, \tilde{T}_5, \tilde{T}_6$ are directly controlled by the second terms in the R. H. S. of (4.15). The Lemma is proved. □

4.3. **Estimates in the general case.** If we apply Lemma 4.1 to the term (4.12), there will appear a term like

$$\int_{\mathbb{R}^n\times\mathbb{R}_+}|\nabla_3 U_3|^2\,\Phi_2^2(v)\Sigma_{12}(v)\,t_3 dt_3 dx. \tag{4.17}$$

Thus we need the following general estimates inductively. For a subset $\mathfrak{j}$ of $\mathfrak{m}$, denote

$$\Sigma_{\mathfrak{j}}(v) := \sum_{\mathfrak{j}_1,\cdots,\mathfrak{j}_l}|\nabla_{\mathfrak{j}_1}v|^2\cdots|\nabla_{\mathfrak{j}_l}v|^2, \tag{4.18}$$

where the summation is taken over all partitions $\mathfrak{j}_1,\cdots,\mathfrak{j}_l$ of $\mathfrak{j}$, i.e. $\mathfrak{j}$ is the disjoint union of $\mathfrak{j}_1,\cdots,\mathfrak{j}_l$.

**Lemma 4.3.** *Assume as in Lemma 4.1. For a subset* $\mathfrak{j}$ *of* $\{1,2,\ldots,k-1\}$ *with* $a=|\mathfrak{j}|$*, let*

$$u(x,\mathbf{t}_{\mathfrak{j}},t_k) = f_\varepsilon *_{\mathfrak{j}} P_{\mathbf{t}_{\mathfrak{j}}} *_k P_{t_k}(x), \qquad v(x,\mathbf{t}_{\mathfrak{j}},t_k) = \chi_\varepsilon *_{\mathfrak{j}} P_{\mathbf{t}_{\mathfrak{j}}} *_k P_{t_k}(x). \tag{4.19}$$

*Then for fixed* $\mathbf{t}_{\mathfrak{j}}\in(\mathbb{R}_+)^a$*, we have*

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}_+}|\nabla_k u|^2\Phi_a^2(v)\Sigma_{\mathfrak{j}}(v)\Big|_{(x,\mathbf{t}_{\mathfrak{j}},t_k)} t_k dt_k dx \lesssim& \int_{\mathbb{R}^n}u^2\Phi_a^2(v)\Sigma_{\mathfrak{j}}(v)\Big|_{(x,\mathbf{t}_{\mathfrak{j}},0)}dx\\ &+\int_{\mathbb{R}^n\times\mathbb{R}_+}u^2\Phi_{a+1}(v)^2\Sigma_{\mathfrak{j}\cup\{k\}}(v)\Big|_{(x,\mathbf{t}_{\mathfrak{j}},t_k)}t_k dt_k dx,\end{aligned} \tag{4.20}$$

*where* $\Phi_{a+1}$ *is a* $\mathscr{C}^{2m+2-2a}[0,1]$ *function satisfying* $\Phi_{a+1}|_{[0,C_1]}=0$ *and* $\Phi_{a+1}\geq|\Phi_a|+|\Phi_a'|+|\Phi_a''|$ *on* $[0,1]$.

*Proof.* Note that $\frac{1}{2}\triangle_k(u^2)=|\nabla_k u|^2$ and

$$\begin{aligned}\frac{1}{2}\triangle_k(\Phi_a^2(v)) &= (\Phi_a(v)\Phi_a''(v)+\Phi_a'(v)^2)\,|\nabla_k v|^2,\\ \frac{1}{2}\triangle_k(|\nabla_{\mathfrak{j}_\alpha}v|^2) &= 2\nabla_{\mathfrak{j}_\alpha}v\cdot\triangle_k\nabla_{\mathfrak{j}_\alpha}v+|\nabla_k\nabla_{\mathfrak{j}_\alpha}v|^2=|\nabla_k\nabla_{\mathfrak{j}_\alpha}v|^2,\end{aligned}$$

by $\triangle_k u=0=\triangle_k v$ and $\triangle_k\nabla_{\mathfrak{j}_\alpha}v=\nabla_{\mathfrak{j}_\alpha}\triangle_k v=0$ for any $\mathfrak{j}_\alpha$ and $k$. It follows from direct differentiation that

$$|\nabla_k u|^2\Phi_a^2(v)\prod_\beta|\nabla_{\mathfrak{j}_\beta}v|^2 = \frac{1}{2}\triangle_k\left(u^2\Phi_a^2(v)\prod_\beta|\nabla_{\mathfrak{j}_\beta}v|^2\right)+\breve{T}_2+\breve{T}_3 =: \breve{T}_1+\breve{T}_2+\breve{T}_3, \tag{4.21}$$

where

$$\begin{aligned}\breve{T}_2 :=& -4u\Phi_a(v)\Phi_a'(v)\nabla_k u\cdot\nabla_k v\prod_\beta|\nabla_{\mathfrak{j}_\beta}v|^2\\ &-4u\Phi_a^2(v)\sum_\alpha\nabla_k u\cdot(\nabla_k\nabla_{\mathfrak{j}_\alpha}v\cdot\nabla_{\mathfrak{j}_\alpha}v)\prod_{\beta\neq\alpha}|\nabla_{\mathfrak{j}_\beta}v|^2 =: \breve{T}_{21}+\breve{T}_{22},\end{aligned} \tag{4.22}$$

are terms involving first-order derivatives of $u$, and

$$\begin{aligned}
\breve{T}_3 :=& - u^2(\Phi_a(v)\Phi_a''(v) + \Phi_a'(v)^2)|\nabla_k v|^2 \prod_\beta \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 \\
& - 4u^2\Phi_a(v)\Phi_a'(v) \sum_\alpha \nabla_k v \cdot (\nabla_k \nabla_{\mathfrak{j}_\alpha} v \cdot \nabla_{\mathfrak{j}_\alpha} v) \prod_{\beta\neq\alpha} \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 \\
& - u^2\Phi_a^2(v) \sum_\alpha |\nabla_k \nabla_{\mathfrak{j}_\alpha} v|^2 \prod_{\beta\neq\alpha} \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 \qquad (4.23)\\
& - 4u^2\Phi_a^2(v) \sum_{\alpha\neq\gamma} (\nabla_k \nabla_{\mathfrak{j}_\alpha} v \cdot \nabla_{\mathfrak{j}_\alpha} v) \cdot (\nabla_k \nabla_{\mathfrak{j}_\gamma} v \cdot \nabla_{\mathfrak{j}_\gamma} v) \prod_{\beta\neq\alpha,\gamma} \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 \\
=:& \breve{T}_{31} + \breve{T}_{32} + \breve{T}_{33} + \breve{T}_{34},
\end{aligned}$$

are terms only involving derivatives of $v$.

Since $U = u^2\Phi_a^2(v)\prod_\beta \left|\nabla_{\mathfrak{j}_\beta} v\right|^2$ satisfies assumptions in Proposition 2.3 by the ensuing Lemma 4.4, we can apply Proposition 2.3 to get the integral of $\breve{T}_1$ is controlled by the first term in R.H.S. of (4.20). Similarly, by applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^n\times\mathbb{R}_+} |\breve{T}_{21}(x, \mathbf{t}_\mathfrak{j}, t_k)| t_k \, dt_k dx \leq& \frac{1}{10} \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla_k u|^2 \Phi_a^2(v) \prod_\beta \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 t_k dt_k dx \\
&+ C \int_{\mathbb{R}^n\times\mathbb{R}_+} u^2 \Phi_{a+1}^2(v) |\nabla_k v|^2 \prod_\beta \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 t_k dt_k dx, \qquad (4.24)\\
\int_{\mathbb{R}^n\times\mathbb{R}_+} |\breve{T}_{22}(x, \mathbf{t}_\mathfrak{j}, t_k)| t_k dt_k dx \leq& \frac{1}{10} \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla_k u|^2 \Phi_a^2(v) \prod_\beta \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 t_k \, dt_k dx \\
&+ C \int_{\mathbb{R}^n\times\mathbb{R}_+} u^2 \Phi_a^2(v) \sum_\alpha |\nabla_k \nabla_{\mathfrak{j}_\alpha} v|^2 \prod_{\beta\neq\alpha} \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 t_k dt_k dx.
\end{aligned}$$

The first terms in the R.H.S. of (4.24) can be absorbed by the integral of the L.H.S. of (4.21). The second terms in the R.H.S. of (4.24) can be controlled by the second terms in the R.H.S. of (4.20), by definition of $\Sigma_{\mathfrak{j}\cup\{k\}}$.

The integrals of $\breve{T}_{3j}$'s are controlled directly by the second term in R.H.S. of (4.20) by applying Cauchy-Schwarz inequality. The estimate (4.20) follows. □

**Lemma 4.4.** *Let $\varepsilon > 0$ and let $u, v$ be given as in Lemma 4.3. Then $U = u^2\Phi_a^2(v)\prod_\beta \left|\nabla_{\mathfrak{j}_\beta} v\right|^2$ satisfies conditions in Proposition 2.3.*

*Proof.* Although there may be $\chi_\varepsilon := \chi * P_{\boldsymbol{\varepsilon}} \notin L^1(\mathbb{R}^n)$, we must have $\nabla_l \chi_\varepsilon = (1-\chi) * \widetilde{\nabla}_l P_{\boldsymbol{\varepsilon}} \in L^1(\mathbb{R}^n)$. But $v \in L^\infty(\mathbb{R}^n)$. Thus by the boundary growth estimate in Proposition 3.4, we see that for any nonempty subset $\mathfrak{l}$ of $\mathfrak{m}$, $|\nabla_\mathfrak{l} v| \in L^1 \cap L^\infty(\mathbb{R}^n)$ with norms only depending on $\varepsilon$. Note that

$$u^2, \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 \text{ satisfy conditions in Proposition 2.3,} \qquad (4.25)$$

except for $\left|\nabla_{\mathfrak{j}_\beta} v\right|^2(\cdot, t_k) \to 0$ as $t_k \to 0$, by Corollary 3.1 (1)-(2) for $\mu = k$, and

$$\nabla_k U = \Big(\nabla_k u^2 \Phi_a^2(v) + 2u^2\Phi_a(v)\Phi_a'(v)\nabla_k v\Big) \prod_\beta \left|\nabla_{\mathfrak{j}_\beta} v\right|^2 + \sum_\alpha u^2\Phi_a^2(v)\nabla_k(|\nabla_{\mathfrak{j}_\alpha} v|^2) \prod_{\beta\neq\alpha} \left|\nabla_{\mathfrak{j}_\beta} v\right|^2.$$

Then $\nabla_k U(\cdot, t_k) \in L^1(\mathbb{R}^n)$ because $\Phi, \Phi'$ are bounded, and all factors in the R.H.S. above is both in $L^1 \cap L^\infty(\mathbb{R}^n)$ by (4.25). Similarly, $\nabla^a U(\cdot, t_k) \in L^1(\mathbb{R}^n)$ for $a = 0, 2$.

Note that $\lim_{t_k\to+\infty}\|t_k^a\partial_{t_k}^a U(\cdot,t_k)\|_{L^1(\mathbb{R}^n)}=0$ for $a=0,1$, because $t_\mu^a\partial_{t_k}^a(u^2)$, $t_k^a\partial_{t_k}^a(|\nabla_{\mathfrak{j}_\alpha}v|^2)\to 0$ in $L^1(\mathbb{R}^n)$ by (4.25), and other factors are all bounded by a constant only depending on $\varepsilon$.

By (4.25) again, as $t_k\to 0$, $u^2(\cdot,t_k)\to u^2(\cdot,0)$, $\left|\nabla_{\mathfrak{j}_\beta}v\right|^2(\cdot,t_k)\to\left|\nabla_{\mathfrak{j}_\beta}v\right|^2(\cdot,0)$ in $L^1(\mathbb{R}^n)$, and $\Phi_a^2(v)(\cdot,t_k)\to\Phi_a^2(v)(\cdot,0)$ a.e. Consequently, $U(\cdot,t_k)\to U(\cdot,0)$. Similarly, $t_k\partial_{t_k}U(\cdot,t_k)\to 0$ in $L^1(\mathbb{R}^n)$ as $t_k\to 0$. □

### 4.4. **Proof of the Fefferman-Stein type good-$\lambda$ inequality.**

**Proposition 4.2.** *Let $\varepsilon>0$. Suppose that $f_\varepsilon:=f*P_{\boldsymbol{\varepsilon}}$ for some $f\in L^1(\mathbb{R}^n)$, $\chi_\varepsilon:=\chi_E*P_{\boldsymbol{\varepsilon}}$ with $E^c$ having bounded measure, and $v(x,\mathbf{t})=\chi_\varepsilon*P_{\mathbf{t}}$. Then, we have the estimate*

$$I=\int_{\mathbb{R}^n\times(\mathbb{R}_+)^m}|\nabla_{\mathfrak{m}}(f_\varepsilon*P_{\mathbf{t}})|^2\phi^2(v)\mathbf{t}d\mathbf{t}dx\lesssim\sum_{\mathfrak{j}}\int_{\mathbb{R}^n\times(\mathbb{R}_+)^{|\mathfrak{j}|}}P_{\mathbf{t}_{\mathfrak{j}}}(f_\varepsilon)^2\Phi_{|\mathfrak{j}|}^2(v)\Sigma_{\mathfrak{j}}(v)\bigg|_{(x,\mathbf{0}_{\mathfrak{j}^c},\mathbf{t}_{\mathfrak{j}})}\mathbf{t}_{\mathfrak{j}}d\mathbf{t}_{\mathfrak{j}}dx,\tag{4.26}$$

*where the summation is taken over subsets $\mathfrak{j}$ of $\mathfrak{m}$, $\Phi_0=\phi$, $\Sigma_{\mathfrak{j}}(v)$ is given by (4.18), and $\mathfrak{j}^c:=\mathfrak{m}\setminus\mathfrak{j}$.*

*Proof.* Applying estimate (4.7) in Lemma 4.1 to real and imaginary parts of $U_1(x,t_1,\mathbf{t}_{\mathfrak{m}_2})$ given by (4.13)-(4.14) for fixed $\mathbf{t}_{\mathfrak{m}_2}$, we get the integral $I$ in (4.26) can be estimated as

$$\begin{aligned}I&=\int_{(\mathbb{R}_+)^{m-1}}\int_{\mathbb{R}^n\times\mathbb{R}_+}|\nabla_1U_1(x,t_1,\mathbf{t}_{\mathfrak{m}_2})|^2\phi^2(v)t_1dt_1dx\,\mathbf{t}_{\mathfrak{m}_2}d\mathbf{t}_{\mathfrak{m}_2}\\&\lesssim\int_{(\mathbb{R}_+)^{m-1}}\left(\int_{\mathbb{R}^n\times\mathbb{R}_+}|U_1|^2\Phi_1^2(v)\left|\nabla_1v\right|^2\bigg|_{(x,\mathbf{t})}t_1dt_1dx+\int_{\mathbb{R}^n}|U_1|^2\phi^2(v)\bigg|_{(x,0,\mathbf{t}_{\mathfrak{m}_2})}dx\right)\mathbf{t}_{\mathfrak{m}_2}d\mathbf{t}_{\mathfrak{m}_2}\\&=\int_{\mathbb{R}^n\times(\mathbb{R}_+)^m}|\,U_1|^2\,\Phi_1^2(v)\left|\nabla_1v\right|^2\bigg|_{(x,\mathbf{t})}\mathbf{t}d\mathbf{t}dx+\int_{\mathbb{R}^n\times(\mathbb{R}_+)^{m-1}}|U_1|^2\,\phi^2(v)\bigg|_{(x,0,\mathbf{t}_{\mathfrak{m}_2})}\mathbf{t}_{\mathfrak{m}_2}d\mathbf{t}_{\mathfrak{m}_2}dx.\end{aligned}\tag{4.27}$$

We claim that

$$I\lesssim\sum_{a=0}^{k-1}\sum_{|\mathfrak{j}|=a}S_{\mathfrak{j}}^{(k)},\tag{4.28}$$

where the summation is taken over all subsets $\mathfrak{j}$ of $\{1,2,\dots,k-1\}$, and

$$S_{\mathfrak{j}}^{(k)}:=\int_{\mathbb{R}^n\times\mathbb{R}_+^{m-k+1+|\mathfrak{j}|}}|U_{k-1}|^2\,\Phi_{|\mathfrak{j}|}^2(v)\Sigma_{\mathfrak{j}}(v)\bigg|_{(x,\mathbf{0}_{\mathfrak{j}_k^c},\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k})}\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k}d\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k}dx,\tag{4.29}$$

$k=2,\dots,m+1$. Here $\Sigma_{\mathfrak{j}}(v)=1$ if $\mathfrak{j}=\emptyset$, $U_k$ is given by (4.13), $U_{k-1}=\nabla_kU_k$, $\mathfrak{m}_k=\{k,\dots,m\}$, and

$$\mathfrak{j}_k^c:=\{1,2,\dots,k-1\}\setminus\mathfrak{j}.$$

Let us prove the claim (4.28) inductively. (4.27) implies that the claim (4.28) already holds for $k=2$. Assume that (4.28) holds for positive integer $k\le m+1$. Let

$$\hat{u}(x,\mathbf{t}_{\mathfrak{j}},t_k)=\hat{f}_\varepsilon*_{\mathfrak{j}}P_{\mathbf{t}_{\mathfrak{j}}}*_kP_{t_k}(x),\qquad\hat{v}(x,\mathbf{t}_{\mathfrak{j}},t_k)=\hat{\chi}_\varepsilon*_{\mathfrak{j}}P_{\mathbf{t}_{\mathfrak{j}}}*_kP_{t_k}(x),$$

for fixed $\mathbf{t}_{\mathfrak{m}_{k+1}}$, where

$$\hat{f}(x)=f*_{k+1}\widetilde{\nabla}_{k+1}P_{t_{k+1}}*\cdots*_m\widetilde{\nabla}_mP_{t_m}(x),\qquad\hat{\chi}(x)=\chi*_{k+1}P_{t_{k+1}}*\cdots*_mP_{t_m}(x).$$

Note that $\hat{f} \in L^1(\mathbb{R}^n)$ by applying (3.12) repeatedly, and similarly, $\|\hat{\chi}\|_{L^\infty} \leq 1$ and $1 - \hat{\chi} \in L^1(\mathbb{R}^n)$. Moreover their $L^1$-norms are independent of $\mathbf{t}$. Then, by definition,

$$U_k|_{(x,\mathbf{0}_{\mathfrak{j}_k^c},\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k})} = \hat{u}(x,\mathbf{t}_{\mathfrak{j}},t_k), \qquad v|_{(x,\mathbf{0}_{\mathfrak{j}_k^c},\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k})} = \hat{v}(x,\mathbf{t}_{\mathfrak{j}},t_k)$$

for fixed $\mathbf{t}_{\mathfrak{m}_{k+1}}$. Applying estimate (4.20) in Lemma 4.3 to $u = \hat{u}$ and $v = \hat{v}$ for fixed $\mathbf{t}_{\mathfrak{j}}$ and $\mathbf{t}_{\mathfrak{m}_{k+1}}$ to get

$$\begin{aligned} S_{\mathfrak{j}}^{(k)} &= \int_{(\mathbb{R}_+)^{m-k+|\mathfrak{j}|}} \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla_k \hat{u}|^2\, \Phi_{|\mathfrak{j}|}^2(\hat{v})\Sigma_{\mathfrak{j}}(\hat{v})\Big|_{(x,\mathbf{t}_{\mathfrak{j}},t_k)} t_k dt_k dx\, \mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_{k+1}} d\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_{k+1}} \\ &\lesssim \int_{(\mathbb{R}_+)^{m-k+|\mathfrak{j}|}} \Bigg( \int_{\mathbb{R}^n} |\hat{u}|^2\, \Phi_{|\mathfrak{j}|}^2(\hat{v})\Sigma_{\mathfrak{j}}(\hat{v})|_{(x,\mathbf{t}_{\mathfrak{j}},0)} dx \\ &\qquad + \int_{\mathbb{R}^n\times\mathbb{R}_+} |\hat{u}|^2\, \Phi_{|\mathfrak{j}|+1}^2(\hat{v})\Sigma_{\mathfrak{j}\cup\{k\}}(\hat{v})\Big|_{(x,\mathbf{t}_{\mathfrak{j}},t_k)} t_k dt_k dx \Bigg) \mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_{k+1}} d\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_{k+1}} \\ &= \int_{(\mathbb{R}_+)^{m-k+|\mathfrak{j}|}} |U_k|^2\, \Phi_{|\mathfrak{j}|}^2(v)\Sigma_{\mathfrak{j}}(v)\Big|_{(x,\mathbf{0}_{\mathfrak{j}_{k+1}^c},\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_{k+1}})} \mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_{k+1}} d\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_{k+1}} dx \\ &\qquad + \int_{\mathbb{R}^n\times(\mathbb{R}_+)^{m-k+|\mathfrak{j}|+1}} |U_k|^2\, \Phi_{|\mathfrak{j}|+1}^2(v)\Sigma_{\mathfrak{j}\cup\{k\}}(v)\Big|_{(x,\mathbf{0}_{\mathfrak{j}_k^c},\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k})} \mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k} d\mathbf{t}_{\mathfrak{j}\cup\mathfrak{m}_k} dx \\ &= S_{\mathfrak{j}}^{(k+1)} + S_{\mathfrak{j}\cup\{k\}}^{(k+1)}. \end{aligned} \tag{4.30}$$

It follows that

$$\sum_{a=0}^{k-1} \sum_{|\mathfrak{j}|=a} S_{\mathfrak{j}}^{(k)} \lesssim \sum_{a=0}^{k} \sum_{|\mathfrak{l}|=a} S_{\mathfrak{l}}^{(k+1)}$$

by definition, where the summation in the right hand side is taken over subsets $\mathfrak{l}$ of $\{1, 2, \ldots, k\}$. The R.H.S. of the estimate (4.26) is exactly (4.28) for $k = m+1$, since $U_m|_{(x,\mathbf{0}_{\mathfrak{j}^c},\mathbf{t}_{\mathfrak{j}})} = P_{\mathbf{t}_{\mathfrak{j}}}(f)$ by definition (4.13). □

Integrals in Proposition 4.2 can be further estimated as follows.

**Lemma 4.5.** *Assume as in Proposition 4.2. Then for a nonempty subset* $\mathfrak{j}$ *of* $\mathfrak{m}$, $E = E_\beta(\lambda)$, *and* $\lambda > 0$, *we have*

$$\int_{\mathbb{R}^n\times(\mathbb{R}_+)^{|\mathfrak{j}|}} P_{\mathbf{t}_{\mathfrak{j}}}(f_\varepsilon)^2 \Phi_{|\mathfrak{j}|}(v)^2 \Sigma_{\mathfrak{j}}(v)\Big|_{(x,\mathbf{0}_{\mathfrak{j}^c},\mathbf{t}_{\mathfrak{j}})} \mathbf{t}_{\mathfrak{j}} d\mathbf{t}_{\mathfrak{j}} dx \lesssim \lambda^2 |E_\beta(\lambda)^c|. \tag{4.31}$$

*Proof.* For $\Phi_a(P_{\mathbf{t}_{\mathfrak{j}}}(\chi_\varepsilon))(x) \neq 0$, we have

$$P_{\mathbf{t}_{\mathfrak{j}}+\boldsymbol{\varepsilon}}(\chi)(x) = P_{\mathbf{t}_{\mathfrak{j}}}(\chi_\varepsilon)(x) > C_1$$

by definition of $\Phi_a$. Here by abusing of notations, we denote $(\mathbf{0}_{\mathfrak{j}^c}, \mathbf{t}_{\mathfrak{j}})$ by $\mathbf{t}_{\mathfrak{j}}$ briefly. Consequently, $(x, \mathbf{t}_{\mathfrak{j}} + \boldsymbol{\varepsilon}) \in \widetilde{W}_\beta$ by Proposition 4.1. Hence, there exists $x' \in E_\beta(\lambda)$ such that $(x, \mathbf{t}_{\mathfrak{j}} + \boldsymbol{\varepsilon}) \in \Gamma_\beta(x')$, and so

$$P_{\mathbf{t}_{\mathfrak{j}}}(f_\varepsilon)(x) = P_{\mathbf{t}_{\mathfrak{j}}+\boldsymbol{\varepsilon}}(f)(x) \leq N^\beta(f)(x') \leq \lambda.$$

Therefore,

$$\int_{\mathbb{R}^n\times(\mathbb{R}_+)^{|\mathfrak{j}|}} P_{\mathbf{t}_{\mathfrak{j}}}(f_\varepsilon)^2 \Phi_{|\mathfrak{j}|}^2(v)\Sigma_{\mathfrak{j}}(v)\Big|_{(x,\mathbf{t}_{\mathfrak{j}})} \mathbf{t}_{\mathfrak{j}} d\mathbf{t}_{\mathfrak{j}} dx \leq C \sum_{\mathfrak{j}_1,\cdots,\mathfrak{j}_l} \lambda^2 \int_{\mathbb{R}^n\times(\mathbb{R}_+)^{|\mathfrak{j}|}} |\nabla_{\mathfrak{j}_1} v|^2 \cdots |\nabla_{\mathfrak{j}_l} v|^2\Big|_{(x,\mathbf{t}_{\mathfrak{j}})} \mathbf{t}_{\mathfrak{j}} d\mathbf{t}_{\mathfrak{j}} dx$$

where $C = \max_a \max_{[0,1]} \Phi_a^2$, the summation is taken over partitions $\mathfrak{j}_1, \cdots, \mathfrak{j}_l$ of $\mathfrak{j}$, and

$$v(x, \mathbf{t}_\mathfrak{j}) = P_{\mathbf{t}_\mathfrak{j}}(\chi_\varepsilon)(x) = P_{\mathbf{t}_\mathfrak{j}+\boldsymbol{\varepsilon}}(\chi)(x).$$

Recall that $|P_{t_\lambda}(h)| \lesssim M_\lambda(h), |t_\lambda \nabla_\lambda P_{t_\lambda}(h)| \lesssim M_\lambda(h)$ for $h \in L^1(\mathbb{R})$. This together with

$$\nabla_{\mathfrak{j}_\alpha} v(x, \mathbf{t}_\mathfrak{j}) = \chi_\varepsilon *_{\mathfrak{j}_1} P_{\mathbf{t}_{\mathfrak{j}_1}} \cdots *_{\mathfrak{j}_\alpha} \widetilde{\nabla}_{\mathfrak{j}_\alpha} P_{\mathbf{t}_{\mathfrak{j}_\alpha}} * \cdots *_{\mathfrak{j}_l} P_{\mathbf{t}_{\mathfrak{j}_l}} = \nabla_{\mathfrak{j}_\alpha}\left(\chi_\varepsilon *_{\mathfrak{j}_\alpha} P_{\mathbf{t}_{\mathfrak{j}_\alpha}}\right) *_{\mathbf{t}_{\mathfrak{j}\setminus\mathfrak{j}_\alpha}} P_{\mathbf{t}_{\mathfrak{j}\setminus\mathfrak{j}_\alpha}},$$

by the commutativity (2.4) of partial convolutions and (3.1), implies

$$|\mathbf{t}_{\mathfrak{j}_\alpha} \nabla_{\mathfrak{j}_\alpha} v(x, \mathbf{t}_\mathfrak{j})|^2 \lesssim \left| M_{\mathfrak{j}\setminus\mathfrak{j}_\alpha}\left(\mathbf{t}_{\mathfrak{j}_\alpha} \nabla_{\mathfrak{j}_\alpha} P_{\mathbf{t}_{\mathfrak{j}_\alpha}}(\chi_\varepsilon)\right)\right|^2,$$

where $\mathfrak{j} \setminus \mathfrak{j}_\alpha$ is the complement of $\mathfrak{j}_\alpha$ in $\mathfrak{j}$, and $M_{\mathfrak{j}\setminus\mathfrak{j}_\alpha} = M_{\lambda_a} \circ \cdots \circ M_{\lambda_1}$ is an iterated maximal function if $\mathfrak{j} \setminus \mathfrak{j}_\alpha = \{\lambda_1, \cdots, \lambda_a\}$. For a subset $\mathfrak{j} = \{j_1, \ldots, j_a\}$ of $\mathfrak{m}$, denote $\frac{d\mathbf{t}_\mathfrak{j}}{\mathbf{t}_\mathfrak{j}} := \frac{dt_{j_1} \cdots dt_{j_a}}{t_{j_1} \cdots t_{j_a}}$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|\mathfrak{j}|}} |\nabla_{\mathfrak{j}_1} v|^2 \cdots |\nabla_{\mathfrak{j}_l} v|^2 \, \mathbf{t}_\mathfrak{j} d\mathbf{t}_\mathfrak{j} dx &\lesssim \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|\mathfrak{j}|}} \prod_{\alpha=1}^{l} \left| M_{\mathfrak{j}\setminus\mathfrak{j}_\alpha}\left(\mathbf{t}_{\mathfrak{j}_\alpha} \nabla_{\mathfrak{j}_\alpha} P_{\mathbf{t}_{\mathfrak{j}_\alpha}}(\chi_\varepsilon)\right)\right|^2 dx \frac{d\mathbf{t}_\mathfrak{j}}{\mathbf{t}_\mathfrak{j}} \\
&= \int_{\mathbb{R}^n} \prod_{\alpha=1}^{l} \int_{(\mathbb{R}_+)^{|\mathfrak{j}_\alpha|}} \left| M_{\mathfrak{j}\setminus\mathfrak{j}_\alpha}\left(\mathbf{t}_{\mathfrak{j}_\alpha} \nabla_{\mathfrak{j}_\alpha} P_{\mathbf{t}_{\mathfrak{j}_\alpha}}(\chi_\varepsilon)\right)\right|^2 \frac{d\mathbf{t}_{\mathfrak{j}_\alpha}}{\mathbf{t}_{\mathfrak{j}_\alpha}} dx \\
&\leq \prod_{\alpha=1}^{l} \left| \int_{\mathbb{R}^n} \left( \int_{(\mathbb{R}_+)^{|\mathfrak{j}_\alpha|}} \left| M_{\mathfrak{j}\setminus\mathfrak{j}_\alpha}\left(\mathbf{t}_{\mathfrak{j}_\alpha} \nabla_{\mathfrak{j}_\alpha} P_{\mathbf{t}_{\mathfrak{j}_\alpha}}(1-\chi_\varepsilon)\right)\right|^2 \frac{d\mathbf{t}_{\mathfrak{j}_\alpha}}{\mathbf{t}_{\mathfrak{j}_\alpha}} \right)^l dx \right|^{\frac{1}{l}} \\
&\lesssim \prod_{\alpha=1}^{l} \left| \int_{\mathbb{R}^n} g_{\mathfrak{j}_\alpha}^{2l}(1-\chi_\varepsilon) dx \right|^{\frac{1}{l}} \\
&\lesssim \|1-\chi_\varepsilon\|_{L^{2l}(\mathbb{R}^n)}^{2l} \to \|1-\chi\|_{L^{2l}(\mathbb{R}^n)}^{2l} = |E_\beta(\lambda)^c|.
\end{aligned}$$

by using Hölder inequality, the vector-valued inequality for the Hardy-Littlewood maximal functions, and the $L^{2l}$ boundedness of the partial Littlewood-Paley $g$-function in Proposition 2.2 . □

*Proof of Theorem 4.1.* Since $f_\varepsilon = f * P_{\boldsymbol{\varepsilon}}$ is smooth on $\overline{T_\Omega}$, we see that $f_\varepsilon(x) \leq N^\beta(f_\varepsilon)(x) \leq N^\beta(f)(x)$ by the definition of nontangential maximal function, and similarly, $v(x, \mathbf{0}) = (\chi_{E_\beta(\lambda)})_\varepsilon$. Hence,

$$\begin{aligned}
I &= \int_{\mathbb{R}^n \times (\mathbb{R}_+)^m} |\nabla_\mathfrak{m}(f_\varepsilon * P_\mathbf{t})|^2 \phi^2(v) \mathbf{t} d\mathbf{t} dx \\
&\lesssim \int_{\mathbb{R}^n} f_\varepsilon^2 \phi^2(v)(x) dx + \sum_{a=1}^{m} \sum_{|\mathfrak{j}|=a} \int_{\mathbb{R}^n \times (\mathbb{R}_+)^a} P_{\mathbf{t}_\mathfrak{j}}(f_\varepsilon)^2 \Phi_a^2(v) \Sigma_\mathfrak{j}(v) \Big|_{(x, \mathbf{0}_{\mathfrak{j}^c}, \mathbf{t}_\mathfrak{j})} \mathbf{t}_\mathfrak{j} d\mathbf{t}_\mathfrak{j} dx \qquad (4.32) \\
&\lesssim \int_{\mathbb{R}^n} N^\beta(f)^2 \phi^2\left((\chi_{E_\beta(\lambda)})_\varepsilon\right)(x) dx + \lambda^2 |E_\beta(\lambda)^c|,
\end{aligned}$$

by Proposition 4.2 and Lemma 4.5. As $\varepsilon \to 0$, $\nabla_\mathfrak{m}(f_\varepsilon * P_\mathbf{t}) \to \nabla_\mathfrak{m}(f * P_\mathbf{t})$ and $v(\cdot, \mathbf{t}) \to \chi * P_\mathbf{t}$ a.e. The resulting inequality (1.11) follows from the inequality (4.5) and the inequality by taking limit $\varepsilon \to 0$ in (4.32) and using Fatou's lemma and Lebesgues' dominated convergence theorem. □

## 5. Littlewood-Paley $g$ function and $S$ functions

5.1. **The Plancherel-Pólya type inequality for $\mathfrak{H}$-valued functions.** Let $\mathfrak{H}$ be a separable Hilbert space and let $\{h_1, h_2, \dots\}$ be an orthonormal basis of $\mathfrak{H}$. A $\mathfrak{H}$-valued function $\mathfrak{f}$ called *measurable* if $\mathfrak{f} = \sum_{j=1}^{\infty} f_j(x)h_j$ converges a.e. with each $f_j$ measurable. A measurable function $\mathfrak{f} \in L^p(\mathbb{R}, \mathfrak{H})$ if $\|\mathfrak{f}\|_{L^p(\mathbb{R},\mathfrak{H})} := (\int_{\mathbb{R}} |\mathfrak{f}(x)|_{\mathfrak{H}}^p dx)^{\frac{1}{p}} < \infty$. In particular, $\int_{\mathbb{R}} |\mathfrak{f}(x)|_{\mathfrak{H}}^2 dx = \sum_{j=1}^{\infty} \int_{\mathbb{R}} |f_j(x)|^2 dx$.

Fix $0 < \beta < 1$ and $\gamma > 0$, we say that $\mathfrak{f}$ defined on $\mathbb{R}^1$ belongs to $\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})$ with $r > 0$ and $x_0 \in \mathbb{R}^1$ if it satisfies $\int_{\mathbb{R}^1} \mathfrak{f}(x)dx = 0$ and

$$
\begin{aligned}
|\mathfrak{f}(x)|_{\mathfrak{H}} &\le C \frac{r^{\gamma}}{(r + |x - x_0|)^{1+\gamma}}, \\
|\mathfrak{f}(x) - \mathfrak{f}(x')|_{\mathfrak{H}} &\le C \left( \frac{|x - x'|}{r + |x - x_0|} \right)^{\beta} \frac{r^{\gamma}}{(r + |x - x_0|)^{1+\gamma}},
\end{aligned} \tag{5.1}
$$

for $|x - x'| \le \frac{r+|x-x_0|}{2}$. Denote $\|\mathfrak{f}\|_{\mathcal{M}(\beta,\gamma,r,x_0;\mathfrak{H})} := \inf\{C; (5.1) \text{ hold}\}$. It is a Banach space as in the scalar case. In particular, $\mathcal{M}(\beta, \gamma, r, x_0) = \mathcal{M}(\beta, \gamma, r, x_0; \mathbb{R})$.

Denote $\rho_t := t\frac{\partial P_t}{\partial t}$. Recall (cf. e.g. [19, Section 2.3]) that there exists function $\varphi \in \mathcal{S}(\mathbb{R})$ such that
(1) supp $\varphi \subseteq [-1, 1]$;
(2) $\int_{\mathbb{R}} s^a \varphi(s)ds = 0$ for $a = 0, 1, 2$;
(3) $\int_0^{+\infty} e^{-\xi} \widehat{\varphi}(\xi)\, d\xi = 1$.

Then we have the Calderón reproducing formula for $\mathfrak{H}$-valued functions on $\mathbb{R}$:

$$
\mathfrak{f}(x) = \lim_{\substack{T\to+\infty \\ \varepsilon\to+0}} \int_{\varepsilon}^{T} \varphi_t * \rho_t * \mathfrak{f}(x) \,\frac{dt}{t} = \int_0^{+\infty} \varphi_t * \rho_t * \mathfrak{f}(x) \,\frac{dt}{t}, \tag{5.2}
$$

for $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$. This is because $\mathfrak{f} = \sum_i f_i h_i$, and each entry $f_i \in L^2(\mathbb{R})$ satisfies the usual Calderón reproducing formula on $\mathbb{R}$. Here $\phi * \mathfrak{f} = \sum_i \phi * f_i h_i$ for $\phi \in L^1(\mathbb{R})$ decaying like the Poisson kernel. $\phi * \mathfrak{f} \in L^p(\mathbb{R}, \mathfrak{H})$ if $\mathfrak{f} \in L^p(\mathbb{R}, \mathfrak{H})$. This is because

$$
\begin{aligned}
\|\phi * \mathfrak{f}\|_{L^p(\mathbb{R},\mathfrak{H})} &= \left( \int_{\mathbb{R}} \left( \sum_i |\phi * f_i|^2(x) \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \left( \int_{\mathbb{R}} \left( \sum_i |M(f_i)|^2(x) \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \\
&\lesssim \left( \int_{\mathbb{R}} \left( \sum_i |f_i|^2(x) \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} = \|\mathfrak{f}\|_{L^p(\mathbb{R},\mathfrak{H})},
\end{aligned}
$$

by the vector-valued inequality for the Hardy-Littlewood maximal functions.

We have the wavelet Calderón reproducing formula for $\mathfrak{H}$-valued functions. See the appendix for its proof. In the sequel, for $j \in \mathbb{Z}$, a *dyadic interval $I$ with side-length $\ell(I) = 2^{-\alpha(j+N)}$* means $I = (0, 2^{-\alpha(j+N)}] + l2^{-\alpha(j+N)}$ for some $l \in \mathbb{Z}$.

**Proposition 5.1.** *There exist a fixed small $\alpha > 0$ and a large integer $N$ such that for each dyadic interval $I$ with side-length $\ell(I) = 2^{-\alpha(j+N)}$, $j \in \mathbb{Z}$, and any fixed point $x_I$ in $I$, there exists $\phi_j(x, x_I) \in \mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})$ satisfying*

$$
\mathfrak{f}(x) = \sum_{j\in\mathbb{Z}} \sum_{I} \mathfrak{f}_{j;I} \phi_j(\cdot, x_I), \tag{5.3}
$$

*where the summation is taken over dyadic intervals with side-length $\ell(I)=2^{-\alpha(j+N)}$, and*

$$\mathfrak{f}_{j;I}=c_\alpha \ell(I)|\widetilde{\rho}_j*\mathfrak{f}(x_I)|_{\mathfrak{H}}, \qquad \widetilde{\rho}_j:=\frac{1}{c_\alpha}\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}\rho_t\frac{dt}{t}, \qquad c_\alpha=\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}\frac{dt}{t}=\alpha\ln 2. \tag{5.4}$$

*The series in* (5.3) *converges in $L^2(\mathbb{R},\mathfrak{H})$ and in the Banach space $M(\beta,\gamma,r,x_0;\mathfrak{H})$.*

We also need the Calderón reproducing formula with compactly supported $\psi$:

$$\mathfrak{f}(x)=\int_0^{+\infty}\psi_t*\psi_t*\mathfrak{f}(x)\frac{dt}{t},$$

for any $\mathfrak{f}\in L^2(\mathbb{R},\mathfrak{H})$. Here we can require $\psi$ to be smooth and compactly supported. The $S$-function associated $\psi$ and $g$-function associated to the Poisson kernel for $\mathfrak{H}$-valued functions on $\mathbb{R}$ are defined as

$$S_\psi(\mathfrak{f})(x):=\left(\int_0^{+\infty}\int_{|x-x'|<t}|\psi_t*\mathfrak{f}(x')|^2_{\mathfrak{H}}\frac{dtdx'}{t}\right)^{\frac12} \quad\text{and}\quad g_P(\mathfrak{f})(x):=\left(\int_0^{+\infty}|\rho_t*\mathfrak{f}|^2_{\mathfrak{H}}(x)\frac{dt}{t}\right)^{\frac12},$$

respectively, where $\psi_t(x)=\frac1t\psi(x/t)$. See e.g. [21] for vector valued Littlewood-Paley $g$ function and $S$ functions. The wavelet Calderón reproducing formula (5.1) yields the following Plancherel-Pólya type inequality (cf. e.g. [19, Theorem 2.16-2.17] for the scalar case).

**Proposition 5.2.** *Let $\alpha>0$ and $N$ as in Proposition 5.1. For a fixed $C_0>0$ and any $\mathfrak{f}\in L^2(\mathbb{R},\mathfrak{H})$,*

$$\left\|\left(\sum_{j\in\mathbb{Z}}\sum_I\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}\sup_{u\in C_0I}|\psi_t*\mathfrak{f}(u)|^2_{\mathfrak{H}}\frac{dt}{t}\chi_I(\cdot)\right)^{\frac12}\right\|_{L^1}\lesssim\left\|\left(\sum_{j\in\mathbb{Z}}\sum_I\inf_{u\in I}\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}|\rho_t*\mathfrak{f}(u)|^2_{\mathfrak{H}}\frac{dt}{t}\chi_I(\cdot)\right)^{\frac12}\right\|_{L^1},$$

*where the summation is taken over all dyadic intervals $I$ with side-lengths $\ell(I)=2^{-\alpha(j+N)}$, and $\chi_I$ is the indicator function of $I$. The implicit constant depends on $\alpha,N$ and $C_0$.*

This inequality actually holds for certain distributions, but here the inequality for $L^2$ functions is sufficient for our purposes. See the appendix for a proof.

5.2. **The estimate $\|f\|_{L^1}\lesssim\|g(f)\|_{L^1}$.** At first, we use the Plancherel-Pólya type inequality to prove the control the $S$-function by $g$-function for $\mathfrak{H}$-valued functions.

**Proposition 5.3.** *For $\mathfrak{f}\in L^2(\mathbb{R},\mathfrak{H})$ with $g_P(\mathfrak{f})\in L^1(\mathbb{R})$, we have $S_\psi(\mathfrak{f})\in L^1(\mathbb{R})$ and $\|S_\psi(\mathfrak{f})\|_{L^1(\mathbb{R})}\lesssim\|g_P(\mathfrak{f})\|_{L^1(\mathbb{R})}$.*

*Proof.* We can write

$$S_\psi(\mathfrak{f})(x)=\left(\sum_{j\in\mathbb{Z}}\sum_I\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}\int_{\mathbb{R}}\chi_t(x-x')|\psi_t*\mathfrak{f}(x')|^2_{\mathfrak{H}}\chi_I(x)\frac{dx'dt}{t}\right)^{\frac12},$$

where the summation is taken over all dyadic intervals $I$ with side-lengths $\ell(I)=2^{-\alpha(j+N)}$. Note that there exists a fixed constant $C_0$ depending only on $\alpha,N$ such that for $2^{-\alpha j}\le t\le 2^{-\alpha(j-1)}$ and $x'\in\mathbb{R}$,

$$\chi_t(x-x')|\psi_t*\mathfrak{f}(x')|_{\mathfrak{H}}\chi_I(x)\le\chi_t(x-x')\sup_{u\in C_0I}|\psi_t*\mathfrak{f}(u)|_{\mathfrak{H}}\chi_I(x).$$

Therefore, we have

$$
\begin{aligned}
\|S_\psi(\mathfrak{f})\|_{L^1(\mathbb{R})} &\le \left\| \left( \sum_{j\in\mathbb{Z}} \sum_{I} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \int_{\mathbb{R}} \chi_t(\cdot - x') \sup_{u\in C_0 I} |\psi_t * \mathfrak{f}(u)|_{\mathfrak{H}}^2 \chi_I(\cdot) \frac{dx'dt}{t} \right)^{\frac12} \right\|_{L^1(\mathbb{R})} \\
&= \sqrt{2} \left\| \left( \sum_{j\in\mathbb{Z}} \sum_{I} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \sup_{u\in C_0 I} |\psi_t * \mathfrak{f}(u)|_{\mathfrak{H}}^2 \chi_I(\cdot) \frac{dt}{t} \right)^{\frac12} \right\|_{L^1(\mathbb{R})} \\
&\lesssim \left\| \left( \sum_{j\in\mathbb{Z}} \sum_{I} \inf_{u\in I} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} |\rho_t * \mathfrak{f}(u)|_{\mathfrak{H}}^2 \frac{dt}{t} \chi_I(\cdot) \right)^{\frac12} \right\|_{L^1(\mathbb{R})} \\
&\le \left\| \left( \sum_{j\in\mathbb{Z}} \sum_{I} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} |\rho_t * \mathfrak{f}(\cdot)|_{\mathfrak{H}}^2 \frac{dt}{t} \chi_I(\cdot) \right)^{\frac12} \right\|_{L^1(\mathbb{R})} \\
&= \left\| \left( \int_0^{+\infty} |\rho_t * \mathfrak{f}(\cdot)|_{\mathfrak{H}}^2 \frac{dt}{t} \right)^{\frac12} \right\|_{L^1(\mathbb{R})} = \|g_P(\mathfrak{f})\|_{L^1(\mathbb{R})},
\end{aligned}
$$

by the Plancherel-Pólya type inequality in Proposition 5.2. □

**Proposition 5.4.** *For any $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$ with $S_\psi(\mathfrak{f}) \in L^1(\mathbb{R})$, we have $\mathfrak{f} \in L^1(\mathbb{R}, \mathfrak{H})$ and $\|\mathfrak{f}\|_{L^1(\mathbb{R},\mathfrak{H})} \lesssim \|S_\psi(\mathfrak{f})\|_{L^1(\mathbb{R})}$.*

*Proof.* It follows from atomic decomposition for $\mathfrak{H}$-valued $L^2$ functions with $L^1$ integrable $S$-functions. This can be established in a standard way exactly as the scalar case (cf. e.g. [8, 42]). We omit details. □

Proposition 5.3 and 5.4 imply the following corollary.

**Corollary 5.1.** *For $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$ with $g_P(\mathfrak{f}) \in L^1(\mathbb{R})$, we have $\mathfrak{f} \in L^1(\mathbb{R}, \mathfrak{H})$ and $\|\mathfrak{f}\|_{L^1(\mathbb{R},\mathfrak{H})} \lesssim \|g_P(\mathfrak{f})\|_{L^1(\mathbb{R})}$.*

**Proposition 5.5.** *For $f \in L^2(\mathbb{R}^n)$ with $g(f) \in L^1(\mathbb{R}^n)$, we have $f \in L^1(\mathbb{R}^n)$ and $\|f\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}$.*

*Proof.* Let $\mathfrak{H} := L^2\big((\mathbb{R}_+)^{m-1}, \frac{d\mathbf{t}_{\mathfrak{m}_2}}{\mathbf{t}_{\mathfrak{m}_2}}\big)$ with the usual $L^2$-norm. It is a separable Hilbert space. Given $f \in L^2(\mathbb{R}^n)$ with $g(f) \in L^1(\mathbb{R}^n)$, define measurable functions

$$\mathfrak{f}_{x^\perp}(s)(t_2, \ldots, t_m) := f *_2 t_2\partial_{t_2} P_{t_2} * \cdots *_m t_m \partial_{t_m} P_{t_m}(x^\perp + s e_1).$$

For almost all $x^\perp \in e_1^\perp$, $f_{x^\perp}$ is $\mathfrak{H}$-valued because

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |\mathfrak{f}_{x^\perp}(s)|_{\mathfrak{H}}^2 ds dx^\perp &= \int_{\mathbb{R}^n} \int_{(\mathbb{R}_+)^{m-1}} \big| f *_2 t_2\partial_{t_2} P_{t_2} * \cdots *_m t_m\partial_{t_m} P_{t_m}(x^\perp + s e_1)\big|^2 \frac{d\mathbf{t}_{\mathfrak{m}_2}}{\mathbf{t}_{\mathfrak{m}_2}} ds dx^\perp \\
&\le \|g_{\mathfrak{m}_2}(f)\|_{L^2}^2 \lesssim \|f\|_{L^2}^2,
\end{aligned}
\tag{5.5}
$$

by Proposition 2.2. Consequently, $\mathfrak{f}_{x^\perp} \in L^2(\mathbb{R}, \mathfrak{H})$ for almost all $x^\perp \in e_1^\perp$. On the other hand,

$$\begin{aligned} g_P(\mathfrak{f}_{x^\perp})(s) &= \left( \int_{\mathbb{R}_+} |\mathfrak{f}_{x^\perp} * t_1 \partial_{t_1} P_{t_1}(s)|^2_{\mathfrak{H}} \frac{dt_1}{t_1} \right)^{\frac{1}{2}} \\ &= \left( \int_{(\mathbb{R}_+)^m} \left| f *_2 t_2 \partial_{t_2} P_{t_2} * \cdots *_m t_m \partial_{t_m} P_{t_m} *_1 t_1 \partial_{t_1} P_{t_1}(x^\perp + s e_1) \right|^2 \frac{d\mathbf{t}_{\mathfrak{m}_2}}{\mathbf{t}_{\mathfrak{m}_2}} \frac{dt_1}{t_1} \right)^{\frac{1}{2}} \\ &= \left( \int_{(\mathbb{R}_+)^m} |f *_1 \partial_{t_1} P_{t_1} *_2 \cdots *_m \partial_{t_m} P_{t_m}|^2 (x^\perp + s e_1) \mathbf{t} d\mathbf{t} \right)^{\frac{1}{2}} \leq g(f)(x^\perp + s e_1), \end{aligned} \tag{5.6}$$

by the commutativity (2.4) of partial convolutions along lines. Consequently, by $g(f) \in L^1(\mathbb{R}^n)$, we see that $g_P(\mathfrak{f}_{x^\perp}) \in L^1(\mathbb{R})$ for almost all $x^\perp \in e_1^\perp$. Then applying Corollary 5.1 , we get $\mathfrak{f}_{x^\perp} \in L^1(\mathbb{R}, \mathfrak{H})$ for almost all $x^\perp \in e_1^\perp$, and

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |\mathfrak{f}_{x^\perp}|_{\mathfrak{H}}(s) \, ds dx^\perp \lesssim \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} g_P(\mathfrak{f}_{x^\perp})(s) \, ds dx^\perp = \int_{\mathbb{R}^n} g(f)(x^\perp + s e_1) \, ds dx^\perp.$$

It can be rewritten as

$$\int_{\mathbb{R}^n} \left( \int_{(\mathbb{R}_+)^{m-1}} |f *_2 t_2 \partial_{t_2} P_{t_2} * \cdots *_m t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{d\mathbf{t}_{\mathfrak{m}_2}}{\mathbf{t}_{\mathfrak{m}_2}} \right)^{\frac{1}{2}} dx \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

If we define $\mathfrak{H}_1 := L^2\big((\mathbb{R}_+)^{m-2}, \frac{d\mathbf{t}_{\mathfrak{m}_3}}{\mathbf{t}_{\mathfrak{m}_3}}\big)$ with the usual norm, and define functions $\mathfrak{f}_{1;x'} : \mathbb{R} \longrightarrow \mathfrak{H}_1$ for almost all $x' \in e_2^\perp$ by $\mathfrak{f}_{1;x'}(s) = f *_3 t_3 \partial_{t_3} P_{t_3} * \cdots *_m t_m \partial_{t_m} P_{t_m}(x' + s e_2)$. Similarly, as in (5.5)-(5.6), we get $\mathfrak{f}_{1;x'} \in L^2(\mathbb{R}, \mathfrak{H}_1)$ and $g_P(\mathfrak{f}_{1;x'}) \in L^1(\mathbb{R})$ for almost all $x' \in e_2^\perp$, and

$$\begin{aligned} &\int_{\mathbb{R}^n} \left( \int_{(\mathbb{R}_+)^{m-2}} |f *_3 t_3 \partial_{t_3} P_{t_3} * \cdots *_m t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{d\mathbf{t}_{\mathfrak{m}_3}}{\mathbf{t}_{\mathfrak{m}_3}} \right)^{\frac{1}{2}} dx \\ \lesssim &\int_{\mathbb{R}^n} \left( \int_{(\mathbb{R}_+)^{m-1}} |f *_2 t_2 \partial_{t_2} P_{t_2} * \cdots *_m t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{d\mathbf{t}_{\mathfrak{m}_2}}{\mathbf{t}_{\mathfrak{m}_2}} \right)^{\frac{1}{2}} dx. \end{aligned}$$

Repeating this procedure, we finally get

$$\|f\|_{L^1(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} |f *_m t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{dt_m}{t_m} \right)^{\frac{1}{2}} dx \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

The proposition is proved. □

**Corollary 5.2.** *For $f \in L^2(\mathbb{R}^n)$ with $g(f) \in L^1(\mathbb{R}^n)$, we have*

$$\sup_{\mathbf{t} \in (\mathbb{R}_+)^m} \|f * P_{\mathbf{t}}\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

*Proof.* For fixed $\mathbf{t} \in (\mathbb{R}_+)^m$, $f * P_{\mathbf{t}} \in L^2(\mathbb{R}^n)$ by the $L^2$-boundedness of the maximal function, and

$$g(f * P_{\mathbf{t}})(x) = \left( \int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}(f * P_{\mathbf{t}} * P_{\mathbf{t}'})(x)|^2 \, \mathbf{t}' d\mathbf{t}' \right)^{\frac{1}{2}} = \left( \int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}(f * P_{\mathbf{t}+\mathbf{t}'})(x)|^2 \, \mathbf{t}' d\mathbf{t}' \right)^{\frac{1}{2}}$$

by $P_t * P_{t'} = P_{t+t'}$. Thus, it equals to

$$\left(\int_{\mathbf{t}+(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}(f * P_{\mathbf{t}'})(x)|^2 (\mathbf{t}' - \mathbf{t}) d\mathbf{t}'\right)^{\frac{1}{2}} \leq \left(\int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}(f * P_{\mathbf{t}'})(x)|^2 \mathbf{t}' d\mathbf{t}'\right)^{\frac{1}{2}} = g(f)(x),$$

i.e. $g(f * P_{\mathbf{t}}) \in L^1(\mathbb{R}^n)$. So we can apply Proposition 5.5 to get $f * P_{\mathbf{t}} \in L^1(\mathbb{R}^n)$ and

$$\|f * P_{\mathbf{t}}\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)},$$

where the implicit constant is independent of $\mathbf{t}$. □

## 6. Proof of the equivalence of various characterizations

In this section, we prove the equivalence of various characterizations in Theorem 1.2.

**(1) implies (2)**. Let $f_\varepsilon := F(\cdot + \mathbf{i}\pi(\boldsymbol{\varepsilon})) \in L^1(\mathbb{R}^n)$. Apply Proposition 3.6 to get

$$|F(x + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\varepsilon}))|^q \leq |f_\varepsilon|^q * P_{\mathbf{t}}(x)$$

for $(x, \boldsymbol{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$, if we choose $0 < q < 1$. Then, $|f_\varepsilon|^q \in L^r(\mathbb{R}^n)$ with $r = \frac{1}{q} > 1$, and

$$\sup_{(x',\mathbf{t})\in\Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\varepsilon}))|^q \leq \sup_{(x',\mathbf{t})\in\Gamma_\beta(x)} |f_\varepsilon|^q * P_{\mathbf{t}}(x') \lesssim M_{it}\left(|f_\varepsilon|^q\right)(x),$$

by using (3.1), where $M_{it}$ is the iterated maximal function on $\mathbb{R}^n$. Therefore,

$$\left\|\sup_{(x',\mathbf{t})\in\Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\varepsilon}))|^q\right\|^r_{L^r(\mathbb{R}^n,dx)} \lesssim \|M_{it}\left(|f_\varepsilon|^q\right)\|^r_{L^r(\mathbb{R}^n)} \lesssim \|\,|f_\varepsilon|^q\,\|^r_{L^r(\mathbb{R}^n)} = \|f_\varepsilon\|_{L^1(\mathbb{R}^n)}, \tag{6.1}$$

where implicit constants are independent of $F$ and $\varepsilon$. Letting $\varepsilon \to 0$ in (6.1), we obtain

$$\begin{aligned}
\left\|\sup_{(x',\mathbf{t})\in\Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t}))|\right\|_{L^1(\mathbb{R}^n,dx)} &= \left\|\lim_{\varepsilon\to 0} \sup_{(x',\mathbf{t})\in(0,\boldsymbol{\varepsilon})+\Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t}))|^q\right\|^r_{L^r(\mathbb{R}^n,dx)} \\
&\leq \liminf_{\varepsilon\to 0} \left\|\sup_{(x',\mathbf{t})\in\Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\varepsilon}))|^q\right\|^r_{L^r(\mathbb{R}^n,dx)} \\
&\lesssim \liminf_{\varepsilon\to 0} \|f_\varepsilon\|_{L^1(\mathbb{R}^n)}
\end{aligned} \tag{6.2}$$

by Fatou's theorem, since domains $(0, \boldsymbol{\varepsilon}) + \Gamma_\beta(x)$ are increasing when $\varepsilon \downarrow 0$ and the limit is the set $\Gamma_\beta(x)$. Therefore, $\|\mathbb{N}^\beta(F)\|_{L^1(\mathbb{R}^n)} \lesssim \|F\|_{H^1(T_\Omega)}$.

**(2) implies (3)**. At first, $F(x + \mathbf{i}y) \to 0$ as $|y| \to +\infty$ in $y_0 + \Omega$ for any fixed $y_0 \in \Omega$, by the boundary growth in Proposition 3.4. Denote $F_\varepsilon(x + \mathbf{i}y) := F(x + \mathbf{i}y + \mathbf{i}\pi(\boldsymbol{\varepsilon}))$. Then $F_\varepsilon$ is smooth on $\overline{T_\Omega}$. By the estimate in Proposition 3.4,

$$|F_\varepsilon(x + \mathbf{i}\pi(\mathbf{t}))| \lesssim \frac{\|F\|_{H^1(T_\Omega)}}{|R(0, \boldsymbol{\varepsilon})|}, \quad \text{for} \quad (x, \boldsymbol{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m,$$

i.e. $F_\varepsilon$ is bounded on $\overline{T_\Omega}$, and so it belongs to $H^p(T_\Omega)$ for any $p > 1$. Then by Theorem 1.1, we get $F_\varepsilon(x + \mathbf{i}\pi(\mathbf{t})) = f_\varepsilon * P_{\mathbf{t}}$ with $f_\varepsilon(\cdot) = F(\cdot + \mathbf{i}\pi(\boldsymbol{\varepsilon})) \in L^p(\mathbb{R}^n)$. Apply Corollary 4.1 to get

$$\|S(f_\varepsilon)\|_{L^1(\mathbb{R}^n)} \lesssim \|N^\beta(f_\varepsilon)\|_{L^1(\mathbb{R}^n)} = \|\mathbb{N}^\beta(F_\varepsilon)\|_{L^1(\mathbb{R}^n)} \leq \|\mathbb{N}^\beta(F)\|_{L^1(\mathbb{R}^n)}. \tag{6.3}$$

On the other hand, $\|S(f_\varepsilon)\|_{L^1(\mathbb{R}^n)} = \|\mathbb{S}(F_\varepsilon)\|_{L^1(\mathbb{R}^n)}$ by definition, and

$$\begin{aligned}\mathbb{S}^2(F_\varepsilon)(x) &= \int_{\Gamma(0)} \left|\nabla_{\mathfrak{m}}(F(x+x'+\mathbf{i}\pi(\mathbf{t})+\mathbf{i}\pi(\boldsymbol{\varepsilon}))\right|^2 \frac{\mathbf{t}d\mathbf{t}dx'}{|R(0,\mathbf{t})|} \\ &= \int_{\mathbb{R}^n\times(\mathbb{R}_+)^m} \left|\nabla_{\mathfrak{m}}(F(x+x'+\mathbf{i}\pi(\mathbf{t}))\right|^2 \chi_{(0,\boldsymbol{\varepsilon})+\Gamma_\beta(x)}(x',\mathbf{t}) \frac{(t_1-\varepsilon)\cdots(t_m-\varepsilon)d\mathbf{t}dx'}{|R(0,\mathbf{t}-\boldsymbol{\varepsilon})|}.\end{aligned} \tag{6.4}$$

Given $\mathbf{t}$, let $\mathfrak{l}$ be $\{l_1,\ldots,l_n\}$ such that $t_{l_1},\ldots,t_{l_n}$ are the largest $n$ numbers among $\{t_1,\ldots,t_m\}$. Then, $(t_{l_1}-\varepsilon,\ldots,t_{l_m}-\varepsilon)$ are also the largest $n$ numbers among $\{t_1-\varepsilon,\ldots,t_n-\varepsilon\}$. Hence,

$$\frac{(t_1-\varepsilon)\cdots(t_m-\varepsilon)}{|R(0,\mathbf{t}-\boldsymbol{\varepsilon})|} \to \frac{t_1\cdots t_m}{|R(0,\mathbf{t})|},$$

as $\varepsilon\to 0$, by Proposition 3.1 and (3.5). Thus,

$$\mathbb{S}^2(F_\varepsilon)(x) \to \int_{\Gamma(0)} \left|\nabla_{\mathfrak{m}}(F(x+x'+\mathbf{i}\pi(\mathbf{t}))\right|^2 \frac{\mathbf{t}d\mathbf{t}dx'}{|R(0,\mathbf{t})|} = \mathbb{S}^2(F)(x),$$

by Lebesgues' dominated convergence theorem. The result follows by taking limit $\varepsilon\to 0$ in (6.3).

**(3) implies (4).**

**Proposition 6.1.** *For $f\in L^1(\mathbb{R}^n)$, we have $g(f)(x)\lesssim S(f)(x)$. In particular, if $F$ is holomorphic on $T_\Omega$, then $\mathbb{G}(F)(x)\lesssim \mathbb{S}(F)(x)$.*

*Proof.* By translations, it is sufficient to prove the inequality for $x=0$. By Proposition 2.1, $\triangle_\mu u(x,\mathbf{t})=0$ for $u(x,\mathbf{t})=f_\varepsilon * P_{\mathbf{t}}(x)$ with $\varepsilon>0$, where $f_\varepsilon = f*P_{\boldsymbol{\varepsilon}}\in L^1(\mathbb{R}^n)$ and is smooth. Thus $\triangle_\mu\nabla_\nu u(x,\mathbf{t})=0$, i.e. $\nabla_\nu u(x)$ is also harmonic in the half plane spanned by $e_\mu$ and $t_\mu$. Then for given $\mathbf{t}\in(\mathbb{R}_+)^m$, we get

$$\nabla_{\mathfrak{m}}u(0,\mathbf{t}) = \frac{2^{2m}}{\pi^m\gamma_0^{2m}t_1^2\cdots t_m^2}\int_{D_1(\gamma_0t_1/2)\times\cdots\times D_m(\gamma_0t_m/2)} \nabla_{\mathfrak{m}}u\left(\sum_{j=1}^m s_j'e_j,\mathbf{t}'\right) d\mathbf{s}'d\mathbf{t}'. \tag{6.5}$$

by repeatedly using the mean value formula for harmonic functions, where disc $D_j(r) := \{(s_j',t_j'); |s_j'|^2 + |t_j'-t_j|^2\le r^2\}$. Note that $t_j'/2\le t_j\le 2t_j'$ when $(s_j',t_j')\in D_j(\gamma_0t_j/2)$. Apply Cauchy-Schwarz inequality to (6.5) to get

$$\begin{aligned} g^2(f_\varepsilon)(0) &= \int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}u(0,\mathbf{t})|^2\mathbf{t}\,d\mathbf{t} \\ &\lesssim \int_{(\mathbb{R}_+)^m}\int_{D_1(\gamma_0t_1/2)\times\cdots\times D_m(\gamma_0t_m/2)} \left|\nabla_{\mathfrak{m}}u\left(\sum_{j=1}^m s_j'e_j,\mathbf{t}'\right)\right|^2 d\mathbf{s}'d\mathbf{t}'\frac{d\mathbf{t}}{t_1\cdots t_m} \\ &\lesssim \int_{(\mathbb{R}_+)^m}\int_{|s_1'|\le\gamma_0t_1'}\cdots\int_{|s_m'|\le\gamma_0t_m'} \left|\nabla_{\mathfrak{m}}u\left(\sum_{j=1}^m s_j'e_j,\mathbf{t}'\right)\right|^2 ds_m'\cdots ds_1'\,d\mathbf{t}' \\ &\lesssim \int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}u(\cdot,\mathbf{t}')|^2 * \chi_{\gamma_0\mathbf{t}'}(0)\,\mathbf{t}'d\mathbf{t}' \\ &\lesssim \int_{(\mathbb{R}_+)^m}\int_{R(0,\mathbf{t}')} |\nabla_{\mathfrak{m}}u(x',\mathbf{t}')|^2dx'\frac{\mathbf{t}'d\mathbf{t}'}{|R(0,\mathbf{t}')|} \\ &= \int_{\Gamma(0)} |\nabla_{\mathfrak{m}}u(x',\mathbf{t}')|^2\frac{\mathbf{t}'d\mathbf{t}'dx'}{|R(0,\mathbf{t}')|} = S^2(f_\varepsilon)(0). \end{aligned}$$

by using Proposition 3.2. The result follows by taking limit $\varepsilon \to 0$. □

**(4) implies (1)**. For $0 < \varepsilon < M$, let

$$\begin{aligned} F_{\varepsilon M}(x + \mathbf{i}\pi(\mathbf{t})) &:= \sum_{\mathfrak{j}} (-1)^{|\mathfrak{j}|} F(x + \mathbf{i}\pi(\mathbf{t} + \mathbf{M}_{\mathfrak{j}})) \\ &= \int_{t_1+\varepsilon}^{t_1+M} \cdots \int_{t_m+\varepsilon}^{t_m+M} \partial_{t_1'} \cdots \partial_{t_m'}(F)(x + \mathbf{i}\pi(\mathbf{t}'))d\mathbf{t}', \end{aligned} \tag{6.6}$$

where the summation is taken over subsets $\mathfrak{j}$ of $\{1, 2, \ldots, m\}$, and

$$\mathbf{M}_{\mathfrak{j}} = (M_1, \ldots, M_m), \qquad M_j = \begin{cases} M, & j \notin \mathfrak{j}, \\ \varepsilon, & j \in \mathfrak{j}. \end{cases}$$

By applying Cauchy-Schwarz inequality to (6.6), we get

$$\begin{aligned} |F_{\varepsilon M}(x + \mathbf{i}\pi(\mathbf{t}))| &\leq \left(\ln \frac{M}{\varepsilon}\right)^{\frac{m}{2}} \left( \int_{t_1+\varepsilon}^{t_1+M} \cdots \int_{t_m+\varepsilon}^{t_m+M} |\partial_{t_1'} \cdots \partial_{t_m'}(F)(x + \mathbf{i}\pi(\mathbf{t}'))|^2 \mathbf{t}' d\mathbf{t}' \right)^{\frac{1}{2}} \\ &\leq \left(\ln \frac{M}{\varepsilon}\right)^{\frac{m}{2}} \mathbb{G}(F)(x) < \infty. \end{aligned}$$

Thus,

$$\sup_{\mathbf{t} \in (\mathbb{R}_+)^m} \|F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\mathbf{t}))\|_{L^1(\mathbb{R}^n)} \leq \left(\ln \frac{M}{\varepsilon}\right)^{\frac{m}{2}} \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)} < +\infty,$$

i.e. $F_{\varepsilon M} \in H^1(T_\Omega)$. Consequently, as in (2) implies (3), $F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\boldsymbol{\varepsilon}))$ is smooth on $\overline{T_\Omega}$ by definition (6.6), and by the estimate in Proposition 3.4, it is bounded on $\overline{T_\Omega}$, and so it belongs to $H^2(T_\Omega)$. Then by Proposition 1.1, we get

$$F_{\varepsilon M}(x + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\varepsilon})) = f_{\varepsilon M} * P_{\mathbf{t}}, \tag{6.7}$$

where $f_{\varepsilon M}(\cdot) = F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\boldsymbol{\varepsilon})) \in L^2(\mathbb{R}^n)$. Moreover, $\|g(f_{\varepsilon M})\|_{L^1(\mathbb{R}^n)} = \|\mathbb{G}(F_{\varepsilon M})\|_{L^1(\mathbb{R}^n)}$ by (6.7), and so

$$\begin{aligned} g(f_{\varepsilon M})(x) &= \mathbb{G}(F_{\varepsilon M}(\cdot))(x + \mathbf{i}\pi(\boldsymbol{\varepsilon})) \\ &= \left( \int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}(F_{\varepsilon M}(x + \mathbf{i}\pi(\mathbf{t}) + \mathbf{i}\pi(\boldsymbol{\varepsilon}))|^2 \, \mathbf{t} d\mathbf{t} \right)^{\frac{1}{2}} \\ &\lesssim \left( \sum_{\mathfrak{j}} \int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}(F(x + \mathbf{i}\pi(\mathbf{t} + \mathbf{M}_{\mathfrak{j}}) + \mathbf{i}\pi(\boldsymbol{\varepsilon}))|^2 \, \mathbf{t} d\mathbf{t} \right)^{\frac{1}{2}} \\ &\lesssim \left( \int_{(\mathbb{R}_+)^m} |\nabla_{\mathfrak{m}}(F(x + \mathbf{i}\pi(\mathbf{t}))|^2 \, \mathbf{t} d\mathbf{t} \right)^{\frac{1}{2}} \approx \mathbb{G}(F)(x), \end{aligned} \tag{6.8}$$

by definition (6.6) of $F_{\varepsilon M}$ and changing coordinates as in (6.4). Thus, by Proposition 5.5, we see that $f_{\varepsilon M} \in L^1(\mathbb{R}^n)$ and $\|f_{\varepsilon M}\|_{L^1(\mathbb{R}^n)} \lesssim \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)}$.

On the other hand, by Corollary 5.2 and (6.7), we have

$$\|F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\varepsilon}))\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f_{\varepsilon M})\|_{L^1(\mathbb{R}^n)} \lesssim \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)},$$

for any $\mathbf{t} \in (\mathbb{R}_+)^m$. Since for $y \in \Omega$, $F(x+\mathbf{i}y+\mathbf{i}y') \to 0$ as $|y'| \to +\infty$ by the assumption, it follows from definition (6.6) that

$$F_{\varepsilon M}(x+\mathbf{i}y) \to (-1)^m F(x+\mathbf{i}y)$$

as $\varepsilon \to 0$ and $M \to +\infty$. So we conclude that

$$\|F(\cdot+\mathbf{i}y)\|_{L^1(\mathbb{R}^n)} \leq \liminf_{\substack{\varepsilon\to 0\\ M\to+\infty}} \|F_{\varepsilon M}(\cdot+\mathbf{i}y+\mathbf{i}\pi(\boldsymbol{\varepsilon}))\|_{L^1(\mathbb{R}^n)} \lesssim \|\mathbb{G}(f)\|_{L^1(\mathbb{R}^n)},$$

by Fatou's theorem, where the implicit constants are independent of $y \in \Omega$. This complete the proof.

## Appendix A. The Plancherel-Pólya type inequality for $\mathfrak{H}$-valued functions

See e.g. [12, 17, 18, 19] for Plancherel-Pólya type inequality. For $\mathfrak{f} = \sum_{j=1}^{\infty} f_j(x)h_j$, by definition, $\mathfrak{f}_k = \sum_{j=1}^{k} f_j(x)h_j$ converges to $\mathfrak{f}$ under the norm $L^p(\mathbb{R},\mathfrak{H})$, i.e. $\int_{\mathbb{R}} |\mathfrak{f}-\mathfrak{f}_k|_{\mathfrak{H}}^p\, dx = \int_{\mathbb{R}} (\sum_{j=k+1}^{\infty} |f_j(x)|^2)^{\frac{p}{2}} dx \to 0$.

By the wavelet Calderón reproducing formula (5.2) on $\mathbb{R}$, we get

$$\begin{aligned}\mathfrak{f}(x) = &\int_{\mathbb{R}_+} \varphi_t * \rho_t * \mathfrak{f}(x)\frac{dt}{t} = \sum_{j\in\mathbb{Z}} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \varphi_t * \rho_t * \mathfrak{f}(x)\frac{dt}{t}\\ =&c_\alpha \sum_{j} \widetilde{\varphi}_j * \widetilde{\rho}_j * \mathfrak{f}(x) + \sum_{j\in\mathbb{Z}} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} (\varphi_t * \rho_t - \widetilde{\varphi}_j * \widetilde{\rho}_j) * \mathfrak{f}(x)\frac{dt}{t} := \mathcal{T}_\alpha\mathfrak{f}(x) + \mathcal{R}_\alpha\mathfrak{f}(x),\end{aligned}$$

for $\mathfrak{f} \in L^2(\mathbb{R},\mathfrak{H})$ where

$$\widetilde{\varphi}_j := \frac{1}{c_\alpha}\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \varphi_t \frac{dt}{t}. \tag{A.1}$$

Now decompose $\mathbb{R}$ into disjoint dyadic intervals $I$ with $\ell(I) = 2^{-\alpha(j+N)}$, where $N$ is a large positive integer fixed later. Then

$$\begin{aligned}T_\alpha\mathfrak{f}(x) := c_\alpha \sum_{j\in\mathbb{Z}} \widetilde{\varphi}_j * \widetilde{\rho}_j * \mathfrak{f}(x) &= c_\alpha \sum_{j\in\mathbb{Z}}\sum_{I} \int_I \widetilde{\varphi}_j(x-x')\widetilde{\rho}_j * \mathfrak{f}(x')dx'\\ &= c_\alpha \sum_{j\in\mathbb{Z}}\sum_{I} \ell(I)\left(\frac{1}{\ell(I)}\int_I \widetilde{\varphi}_j(x-x')dx'\right)\widetilde{\rho}_j * \mathfrak{f}(x_I) + \mathcal{R}'_{\alpha,N}\mathfrak{f}(x)\end{aligned}$$

with $x_I$ any fixed point in $I$, and

$$\mathcal{R}'_{\alpha,N}\mathfrak{f}(x) := c_\alpha \sum_{j\in\mathbb{Z}}\sum_{I} \int_I \widetilde{\varphi}_j(x-x')\Big(\widetilde{\rho}_j * \mathfrak{f}(x') - \widetilde{\rho}_j * \mathfrak{f}(x_I)\Big)dx'.$$

Now we can write

$$\mathfrak{f}(x) = \mathcal{T}\mathfrak{f}(x) + \mathcal{R}_\alpha\mathfrak{f}(x) + R'_{\alpha,N}\mathfrak{f}(x), \qquad \text{where} \qquad \mathcal{T}\mathfrak{f}(x) = \sum_{j}\sum_{I} \mathfrak{f}_{j;I}\widetilde{\phi}_j(\cdot,x_I), \tag{A.2}$$

where $\mathfrak{f}_{j;I}$'s are given by (5.4), and

$$\widetilde{\phi}_j(x,x_I) := \frac{1}{\ell(I)}\int_I \widetilde{\varphi}_j(x-x')dx'\frac{\widetilde{\rho}_j * \mathfrak{f}(x_I)}{|\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}} \in \mathcal{M}(\beta,\gamma,2^{-\alpha j},x_I;\mathfrak{H}). \tag{A.3}$$

Here we take $\widetilde{\phi}_j(\cdot, x_I) \equiv 0$ if $|\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}} = 0$. Since $\varphi$ is compactly supported and smooth, it is easy to see that $\widetilde{\phi}_j(\cdot, x_I)$ have uniformly bounded norm in $\mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})$ by the scalar case. To invert $\mathcal{T}$, we now need to show $\mathcal{R}'_{\alpha,N}$ and $\mathcal{R}_\alpha$ bounded on $M(\beta, \gamma, r, x; \mathfrak{H})$ with small norms. Here $\mathcal{R}'_{\alpha,N}$ has kernel

$$\mathcal{R}'_{\alpha,N}(x,u) = c_\alpha \sum_j \sum_I \int_I \widetilde{\varphi}_j(x - x') \left[\widetilde{\rho}_j(x' - u) - \widetilde{\rho}_j(x_I - u)\right] dx'. \tag{A.4}$$

Let $T$ be a bounded linear operator on $L^2(\mathbb{R})$ associated with a scalar kernel $K(x, y)$ defined on $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y\}$, given initially by $Tf(x) = \int_{\mathbb{R}} K(x, y)f(y)dy$ or $x \in$ supp $f$ for $f \in \mathscr{C}_c^\infty(\mathbb{R})$, where $K(x, y)$ satisfies the following conditions: there exists a constant $C > 0$ such that for all $x \neq y$,
(i) $|K(x, y)| \leq C|x - y|^{-1}$;
(ii) $|K(x, y) - K(x', y)| \leq C|x - x'||x - y|^{-2}$, if $|x - x'| \leq |x - y|/2$;
(iii) $|K(x, y) - K(x, y')| \leq C|y - y'||x - y|^{-2}$, if $|y - y'| \leq |x - y|/2$;
(iv) $|K(x, y) - K(x', y) - K(x, y') + K(x', y')| \leq C|x - x'||y - y'||x - y|^{-3}$, if $|x - x'|$, $|y - y'| \leq |x - y|/2$;
(v) $T(1) = T^*(1) = 0$.

We denote by $\|K\|_{\mathbb{R}}$ the smallest constant $C$ that satisfies (i)-(iv) above. The operator norm of $T$ is defined by $\|T\|_* := \|T\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} + \|K\|_{\mathbb{R}}$. It is known that $\|T\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \to 0$ as $C \to 0$.

**Proposition A.1.** *Suppose that $T$ is an operator as above. Then $T$ is bounded on the test function space $M(\beta, \gamma, r, x_0; \mathfrak{H})$ for $\beta, \gamma \in (0, 1), r > 0$ and $x_0 \in \mathbb{R}$. Moreover, there exists a constant $C$ independent of $\beta, \gamma, r, x_0$ such that*

$$\|T(f)\|_{M(\beta,\gamma,r,x_0;\mathfrak{H})} \leq C\|T\|_*\|f\|_{M(\beta,\gamma,r,x_0;\mathfrak{H})}.$$

*Proof.* For the scalar case, this is [19, Lemma 2.9], i.e. the theorem holds for $M(\beta, \gamma, r, x_0)$. For the $\mathfrak{H}$-valued function $\mathfrak{f} = \sum_{j=1}^\infty f_j(x)h_j \in M(\beta, \gamma, r, x_0; \mathfrak{H})$, we must have $f_i \in M(\beta, \gamma, r, x_0)$ for each $i$, and $\|\mathfrak{f}\|_{M(\beta,\gamma,r,x_0;\mathfrak{H})} = (\sum_i \|f_i\|^2_{M(\beta,\gamma,r,x_0)})^{\frac{1}{2}}$ by definition. Because $T\mathfrak{f} = \sum_{j=1}^\infty Tf_jh_j$, we see that

$$\begin{aligned}
\|T(\mathfrak{f})\|_{M(\beta,\gamma,r,x_0;\mathfrak{H})} &= \left(\sum_i \|T(f_i)\|^2_{M(\beta,\gamma,r,x_0)}\right)^{\frac{1}{2}} \\
&\leq C\|T\|_* \left(\sum_i \|f_i\|^2_{M(\beta,\gamma,r,x_0)}\right)^{\frac{1}{2}} \\
&= C\|T\|_* \|\mathfrak{f}\|_{M(\beta,\gamma,r,x_0;\mathfrak{H})}.
\end{aligned}$$

The result follows. □

*Proof of Proposition 5.1.* Noting that for $x' \in I$, $\widetilde{\rho}_j(x' - u) - \widetilde{\rho}_j(x_I - u) \sim 2^{-\alpha N}\widetilde{\rho}_j(x' - u)$ in terms of the size and smoothness condition, we get

$$R'_{\alpha,N}(x,u) \sim 2^{-\alpha N} \sum_j \widetilde{\varphi}_j * \widetilde{\rho}_j(x - u) \tag{A.5}$$

by definition (A.4), where $\sim$ denotes the equivalence in terms of the size and smoothness conditions of $R'_{\alpha,N}(x, u)$. It is straightforward to verify that the kernel $R'_{\alpha,N}(x, u)$ satisfies conditions (i)-(iv) and the cancellation condition, e.g. if $|s| \approx 2^{-\alpha l}$ for some $l \in \mathbb{Z}$, then

$$\sum_j |\widetilde{\varphi}_j * \widetilde{\rho}_j|(s) \lesssim \sum_j \frac{2^{-\alpha j\gamma}}{(2^{-\alpha j} + |s|)^{1+\gamma}} \leq \sum_{j>l} \frac{2^{-\alpha j\gamma}}{|s|^{1+\gamma}} + \sum_{j \leq l} 2^{\alpha j} \approx \frac{2^{-\alpha l\gamma}}{|s|^{1+\gamma}} + 2^{\alpha l} \approx \frac{1}{|s|}.$$

Similarly, $\mathcal{R}_\alpha$ has the similar expression as (A.5) with $\alpha$ instead of $2^{-\alpha N}$ and satisfies the same conditions.

Hence, applying Proposition A.1, we obtain that

$$\|\mathcal{R}_\alpha\|_{M(\beta,\gamma,r,x_0;\mathfrak{H})} \lesssim \alpha \leq \frac{1}{3}, \tag{A.6}$$

provided we choose $\alpha$ sufficiently small, and then

$$\|R'_{\alpha,N}\|_{M(\beta,\gamma,r,x_0;\mathfrak{H})} \lesssim 2^{-\alpha N} \leq \frac{1}{3}, \tag{A.7}$$

if we choose $N$ sufficiently large. Therefore, $\mathcal{T} = id - R'_{\alpha,N} - \mathcal{R}_\alpha$ is invertible and its inverse $\mathcal{T}^{-1}$ is also bounded on $M(\beta,\gamma,r,x_0;\mathfrak{H})$, with a bound independent of $r, x_0$. Consequently, for $\widetilde{\phi}_j(\cdot,x_I)$ given by (A.3), $\mathcal{T}^{-1}(\widetilde{\phi}_j(\cdot,x_I))$ is also in $M(\beta,\gamma,2^{-\alpha j},x_I;\mathfrak{H})$. Denote it by $\phi_j(x,x_I)$. Also, $\mathcal{T}$ is invertible and its inverse $\mathcal{T}^{-1}$ is also bounded on $L^2(\mathbb{R},\mathfrak{H})$. Then applying $\mathcal{T}^{-1}$ to (A.2), we get

$$\mathfrak{f}(x) = \mathcal{T}^{-1}\mathcal{T}\mathfrak{f} = \sum_j \sum_I c_\alpha |I| |\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}} \widetilde{\phi}_j(x,x_I). \tag{A.8}$$

Here $\|\phi_j(\cdot,x_I)\|_{M(\beta,\gamma,2^{-\alpha j},x_I;\mathfrak{H})}$ are uniformly bounded by Proposition A.1. □

*Proof of Proposition 5.2.* Convolving the wavelet Calderón reproducing formula (5.3) with $\psi_t$ we obtain

$$\psi_t * \mathfrak{f}(x) = \sum_j \sum_I \mathfrak{f}_{j;I} \psi_t * \widetilde{\phi}_j(\cdot,x_I)(x). \tag{A.9}$$

But for $2^{-\alpha k} \leq t < 2^{-\alpha(k-1)}$, we have the following almost orthogonality estimate for $\widetilde{\phi}_j(\cdot,x_I)$ and $\psi_t$ (cf. [21, Lemma 6]),

$$\left|\psi_t * \widetilde{\phi}_j(\cdot,x_I)(x)\right|_{\mathfrak{H}} \leq C 2^{-\alpha|k-j|\beta} \frac{2^{-\alpha(j\wedge k)\gamma}}{(2^{-\alpha(j\wedge k)} + |x-x_I|)^{1+\gamma}}. \tag{A.10}$$

Let $\mathfrak{T}_0 := \left\{I; \ell(I) = 2^{-\alpha(j+N)}, \frac{|x-x_I|}{2^{-\alpha(j\wedge k)}} \leq 1\right\}$, and for $l \in \mathbb{N}$, let

$$\mathfrak{T}_l := \left\{I; \ell(I) = 2^{-\alpha(j+N)}, 2^{\alpha(l-1)} \leq \frac{|x-x_I|}{2^{-\alpha(j\wedge k)}} \leq 2^{\alpha l}\right\}. \tag{A.11}$$

Then $\bigcup_{I\in T_l} I$ is contained in an interval $\widetilde{I}$ centered at $x$ with length $\left|\bigcup_{I\in\mathfrak{T}_l} I\right| \lesssim 2^{-\alpha(j\wedge k)}2^{\alpha l}$. Therefore,

$$\begin{aligned}
&\sum_{\ell(I)=2^{-\alpha(j+N)}} \frac{2^{-\alpha(j\wedge k)\gamma}}{(2^{-\alpha(j\wedge k)} + |x-x_I|)^{1+\gamma}} c_\alpha \ell(I) |\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}} \\
&\lesssim \sum_{l=0}^{\infty} 2^{-\alpha l(1+\gamma)} 2^{\alpha(j\wedge k-j)} \sum_{I\in\mathfrak{T}_l} |\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}} \\
&\lesssim \sum_{l=0}^{\infty} 2^{-\alpha l(1+\gamma)} 2^{\alpha(j\wedge k-j)} \left(\sum_{I\in\mathfrak{T}_l} |\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}^r\right)^{\frac{1}{r}} \\
&= \sum_{l=0}^{\infty} 2^{-\alpha l(1+\gamma)} 2^{\alpha(j\wedge k-j)} \left(\frac{1}{\ell(I)} \int_{\mathbb{R}} \sum_{I\in\mathfrak{T}_l} |\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}^r \chi_I\right\}^{\frac{1}{r}} \\
&\lesssim \sum_{l=0}^{\infty} 2^{-\alpha l(1+\gamma-\frac{1}{r})} 2^{\alpha(j\wedge k-j)\left(1-\frac{1}{r}\right)} \left\{M\left(\sum_{\ell(I)=2^{-\alpha(j+N)}} |\widetilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}^r \chi_I\right)(x^*)\right\}^{\frac{1}{r}}
\end{aligned} \tag{A.12}$$

for any $|x^*-x|\le 2C_0 2^{-\alpha(k+N)}$ and $0<r<1$. In the last inequality, we replace the integral by the average over the interval $C_0\widetilde{I}$ and then by the maximal function. The implicit constants above depend on $\alpha, N, C_0$.

Note that the summation in (A.12) over $l$ converges if we choose $1>r>\frac{1}{1+\gamma}$. If substituting (A.10) and (A.12) into (A.9), we get for given dyadic interval $J$ with $\ell(J)=2^{-\alpha(k+N)}$,

$$\sup_{v\in C_0J}|\psi_t*\mathfrak{f}(v)|_{\mathfrak{H}}\chi_J(u)\lesssim\sum_j 2^{-\alpha|k-j|\beta}2^{\alpha(j\wedge k-j)\left(1-\frac{1}{r}\right)}\left\{M\left(\sum_{\ell(I)=2^{-\alpha(j+N)}}|\widetilde{\rho}_j*\mathfrak{f}(x_I)|^r_{\mathfrak{H}}\chi_I\right)(u)\right\}^{\frac{1}{r}}\chi_J(u)$$

for any $u\in J$. Using Hölder inequality as in the proof of the discrete Young inequality $\|a*b\|_{l^2(\mathbb{Z})}\le\|a\|_{l^1(\mathbb{Z})}\|b\|_{l^2(\mathbb{Z})}$, and the fact that

$$\sum_k 2^{-\alpha|k-j|\beta}2^{\alpha(j\wedge k-j)\left(1-\frac{1}{r}\right)}<C<\infty$$

for some $C>0$ independent of $j$, we obtain

$$\sum_k\sum_J\int_{2^{-\alpha k}}^{2^{-\alpha(k-1)}}\sup_{v\in C_0J}|\psi_t*\mathfrak{f}(v)|^2_{\mathfrak{H}}\chi_J(u)\frac{dt}{t}\lesssim\sum_j\left\{M\left(\sum_I\inf_{x\in I}|\widetilde{\rho}_j*\mathfrak{f}(x)|^r_{\mathfrak{H}}\chi_I\right)(u)\right\}^{\frac{2}{r}},$$

where the summations are taken over dyadic interval $J$ and $I$ with $\ell(J)=2^{-\alpha(k+N)}$ and $\ell(I)=2^{-\alpha(j+N)}$, respectively. Then

$$\begin{aligned}\left\|\left(\sum_k\sum_{j\in\mathbb{Z}}\int_{2^{-\alpha k}}^{2^{-\alpha(k-1)}}\sup_{v\in C_0J}|\psi_t*\mathfrak{f}(v)|^2_{\mathfrak{H}}\chi_J\frac{dt}{t}\right)^{\frac12}\right\|_{L^1}&\lesssim\left\|\left(\sum_j\left\{M\left(\sum_I\inf_{x\in I}|\widetilde{\rho}_j*\mathfrak{f}(x)|^r_{\mathfrak{H}}\chi_I\right)\right\}^{\frac{2}{r}}\right)^{\frac{r}{2}}\right\|^{\frac1r}_{L^{\frac1r}}\\&\lesssim\left\|\left(\sum_j\sum_I\inf_{x\in I}|\widetilde{\rho}_j*\mathfrak{f}(x)|^2_{\mathfrak{H}}\chi_I\right)^{\frac{r}{2}}\right\|^{\frac1r}_{L^{\frac1r}}\\&=\left\|\left(\sum_j\sum_I\inf_{x\in I}|\widetilde{\rho}_j*\mathfrak{f}(x)|^2_{\mathfrak{H}}\chi_I\right)^{\frac12}\right\|_{L^1}\end{aligned}$$

by the Fefferman-Stein vector-valued maximal function inequality [15] for $1/r>1$. We derive the required conclusion by

$$\begin{aligned}\inf_{x\in I}|\widetilde{\rho}_j*\mathfrak{f}(x)|^2_{\mathfrak{H}}&=\frac{1}{c_\alpha^2}\inf_{x\in I}\left|\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}\rho_t*\mathfrak{f}(x)\frac{dt}{t}\right|^2_{\mathfrak{H}}\le\frac{1}{c_\alpha^2}\inf_{x\in I}\left\{\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}|\rho_t*\mathfrak{f}(x)|_{\mathfrak{H}}\frac{dt}{t}\right\}^2\\&\le\frac{1}{c_\alpha}\inf_{x\in I}\int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}}|\rho_t*\mathfrak{f}(x)|^2_{\mathfrak{H}}\frac{dt}{t},\end{aligned}$$

using the Minkowski inequality and the Cauchy-Schwarz inequality. □

## References

[1] Békollé, D. and Sehba, B., Some Carleson measures for the Hilbert-Hardy space of tube domains over symmetric cones, *Eur. J. Math.*. **5** (2019), 585-610.

[2] Calzi, M. and Peloso, M., Holomorphic function spaces on homogeneous Siegel domains, *Diss. Math.* **563** (2021), 1-168.

[3] Chang, D.-C., Han, Y. S. and Wu, X. Hardy spaces associated with two-flag kernels, *Math. Res. Lett.* **32** (2025), 81-134.

[4] Chang, S.-Y. and Fefferman, R., Some recent developments in Fourier analysis and $H^p$-theory on product domains, *Bull. Amer. Math. Soc.* **12** (1985), 1-43.

[5] Chen, P., Cowling, M., Lee, M.-Y., Li, J., and Ottazzi, A., Flag Hardy space theory on Heisenberg groups and applications, arXiv:2102.07371v3.

[6] Chen, P., Duong, X. T., Li, J., Ward, L. and Yan, L., Product Hardy spaces associated to operators with heat kernel bounds on spaces of homogeneous type, *Math. Z.* **282** (2016), 1033-1065.

[7] Cowling, M., Fan, Z., Li, J. and Yan, L., Characterizations of product Hardy spaces on stratified groups by singular integrals and maximal functions, in *The mathematical heritage of Guido Weiss*, edit by Hernández, E. et al., Birkhüser, (2025), 193-227.

[8] Duong, X., Li, J., Wick, B. D. and Yang, Do., Characterizations of product Hardy spaces in the Bessel setting, *J. Fourier Anal. Appl.* **27** (2021), No. 2, Paper No. 24.

[9] Fefferman, R. and Stein, E.M., Singular integrals on product spaces, *Adv. Math.* **45** (1982), 117-143.

[10] Fefferman, C. and Stein, E. M., $H^p$ spaces of several variables, *Acta Math.* **129** (1972), 137-193.

[11] Folland, G. and Stein, E. M., *Hardy spaces on homogeneous groups*, Mathematical Notes **28**, Princeton University Press, Princeton, N. J., 1982.

[12] Frazier, M. and Jawerth, B., A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.* **93** (1990), No. 1, 34-170.

[13] Geller, D., Some results in $H^p$ theory for the Heisenberg group, *Duke Math. J.* **47** (1980), 365-390.

[14] Grafakos, L., Liu L., Yang D., Maximal function characterizations of Hardy spaces on RD-spaces and their applications, *Sci. China Ser. A* **51** (2008), 2253-2284.

[15] Grafakos, L., *Fundamentals of Fourier analysis*, Graduate Texts in Mathematics **302**, Springer-Verlag, Berlin, 2024.

[16] Gundy, R. and Stein, E. M., $H^p$ theory for the poly-disc, *Proc. Natl. Acad. Sci. USA* **76** (1979), 1026-1029.

[17] Han, Y. S., Calderón-type reproducing formula and the $T_b$ theorem, *Rev. Mat. Iberoam.* **10** (1994), 51-91.

[18] Han, Y. S., Plancherel-Pólya type inequality on spaces of homogeneous type and its applications, *Proc. Amer. Math. Soc.* **126** (1998), No. 11, 3315-3327.

[19] Han, Y. S., Lee, M. Y., Li, J. and Wick, B. D., Maximal function, Littlewood-Paley theory, Riesz transform and atomic decomposition in the multi-parameter flag setting, *Mem. Amer. Math. Soc.* **279** (2023), No. 1373, 106 pp.

[20] Han, Y. S., Lee, M. Y., Li, J. and Wick, B. D., Endpoint estimate for Calderó-Zygmund operators in the multiparameter flag setting, *Adv. Math.* **482** (2025), Article ID 110629.

[21] Han, Y. S., Lu, G. and Sawyer, E., Flag Hardy spaces and Marcinkiewicz multipliers on the Heisenberg group, *Analysis PDE* **7** (2014), 1465-1534.

[22] He, S, and Chen, J., Duality of three-parameter Hardy spaces associated with a sum of two flag singular integrals, *Front. Math.* **18** (2023), No. 4, 837-862.

[23] Journé, J.-L., A covering lemma for product spaces, *Proc. Amer. Math. Soc.* **96** (1986), 593-598.

[24] Korányi, A., Boundary behavior of Poisson integrals on symmetric spaces, *Trans. Am. Math. Soc.* **140** (1969), 393-409.

[25] Krantz, S. and Li, S.-Y., Area integral characterizations of functions in Hardy spaces on domains in $\mathbb{C}^n$, *Complex Var. Theory Appl.* **32** (1997), 373-399.

[26] Lacey, M. T., Petermichl, S. Pipher, J. and Wick, B. D., Multi-parameter Div-Curl lemmas, *Bull. Lond. Math. Soc.* **44** (2012), 1123-1131.

[27] Li, J., On Fefferman-Stein type inequality on Shilov boundaries and applications, to appear in *Ann. Sc. Norm. Super. Pisa*.

[28] Malliavin, M.-P. and Malliavin, P., Intégrales de Lusin-Calderón pour les fonctions biharmoniques, *Bull. Sci. Math.* **101** (1977), 357-384.

[29] Merryfield, K., On the area integral, Carleson measures and $H^p$ in the polydisc, *Indiana Univ. Math. J.* **34**, (1985), 663-685.

[30] Müller, D. Ricci, F. and Stein, E. M., Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups I, *Invent. Math.* **119** (1995), 199-233.

[31] Nagel, A., Ricci, F. and Stein E. M., Singular integrals with flag kernels and analysis on quadratic CR manifolds, *J. Funct. Anal.* **181** (2001), 29-118.

[32] Pipher, J., Journé's covering lemma and its extension to higher dimensions, *Duke Math. J.* **53** (1986), 683-690.

[33] Sato, S., Lusin functions and nontangential maximal functions in the $H^p$ theory on the product of upper half-spaces, *Tohoku Math. J.* **37** (1985), 1-13.

[34] Sjögren, P., Admissible convergence of Poisson integrals in symmetric spaces, *Ann. of Math.* **124** (1986), 313-335.

[35] Stout, E., $H^p$-functions on strictly pseudoconvex domains, *Amer. J. Math.* **98** (1976), 821-852.

[36] Stein, E. M., *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series **30**, Princeton University Press, Princeton, N. J., 1970.

[37] Stein, E. M., Boundary behavior of harmonic functions on symmetric spaces: maximal estimates for Poisson integrals, *Invent. Math.* **74** (1983), 63-83.

[38] Stein, E. M., Singular integrals: the roles of Calderón and Zygmunds, *Notices Amer. Math. Soc.* **45** (1998), 1130-1140.

[39] Stein, E. M. and Weiss, G., On the theory of harmonic functions of several variables, *Acta Math.* **103** (1960), 25-62.

[40] Stein, E. M. and Weiss, G., *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series **32**, Princeton University Press, Princeton, N. J., 1971.

[41] Wang, W. and Wu, Q. Y., Atomic decomposition theorem for Hardy spaces on products of Siegel upper half spaces and bi-parameter Hardy spaces, *J. Geom. Anal.* **33** (2023), No. 11, Paper No. 351.

[42] Wang, W. and Wu, Q. Y., Flag-like singular integrals and associated Hardy spaces on a kind of nilpotent Lie groups of step two, *J. Geom. Anal.* **34** (2024), No. 12, Paper No. 373.

[43] Yang, D. and Zhou, Y., Radial maximal function characterizations of Hardy spaces on RD-spaces and their applications, *Math. Ann.* **346** (2010), No. 2, 307-333.