L^p BOUNDS FOR SINGULAR INTEGRALS AND MAXIMAL SINGULAR INTEGRALS WITH ROUGH KERNELS

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Abstract. Convolution type Calderón-Zygmund singular integral operators with rough kernels p.v. $\Omega(x)/|x|^n$ are studied. A condition on Ω implying that the corresponding singular integrals and maximal singular integrals map $L^p \to L^p$ for $1 < p < \infty$ is obtained. This condition is shown to be different from the condition $\Omega \in H^1(\mathbf{S}^{n-1})$.

1. Introduction and statements of results

In this paper, Ω will be a complex-valued integrable function defined on the sphere $\mathbf{S}^{n-1}$ with mean value zero with respect to surface measure. Denote by T_Ω the Calderón-Zygmund singular integral operator defined as follows:

$$(1) \qquad (T_\Omega f)(x) = \lim_{\varepsilon \to 0} \int\limits_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy,$$

for f in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$. The limit in (1) is easily shown to exist for any f continuously differentiable function on $\mathbf{R}^n$ with some decay at infinity.

For $\varepsilon > 0$, denote by

$$(T_\Omega^\varepsilon f)(x) = \int_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy$$

the truncated singular integral associated with T_Ω and by

$$(T_\Omega^* f)(x) = \sup_{\varepsilon>0} |T_\Omega^\varepsilon f(x)|$$

the maximal singular integral operator corresponding to this Ω.

Establishing the a priori bound $\|T_\Omega^\varepsilon f\|_{L^p} \le C\|f\|_{L^p}$ independently of $f \in \mathcal{S}(\mathbf{R}^n)$ and of $\varepsilon > 0$, leads to a (unique) extension of T_Ω^ε on $L^p(\mathbf{R}^n)$. Now, for $f \in L^p(\mathbf{R}^n)$, $T_\Omega^\varepsilon f$ converges in L^p as $\varepsilon \to 0$ to some $T_\Omega f$ (which extends $T_\Omega f$ defined in (1) for $f \in \mathcal{S}(\mathbf{R}^n)$), and by Fatou's lemma, T_Ω is a bounded operator on L^p.

A similar a priori bound for T_Ω^* implies that for $f \in L^p(\mathbf{R}^n)$, $T_\varepsilon f$ converges (to $T_\Omega f$) almost everywhere as $\varepsilon \to 0$.

We now discuss L^p boundedness properties of these operators. It is well known that if Ω has some smoothness, then both T_Ω and T_Ω^* extend to bounded operators on

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$L^p(\mathbf{R}^n)$ for all $1 < p < \infty$. See [11] for details. In this paper we shall be concerned with Ω rough. The method of rotations introduced by Calderón and Zygmund [2] implies that T_Ω and T_Ω^* map $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ for any Ω odd in $L^1(\mathbf{S}^{n-1})$. The situation for general Ω's is significantly more involved. Calderón and Zygmund [2] proved that if

$$\int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \ln(2 + |\Omega(\theta)|)\, d\theta < \infty, \tag{2}$$

then T_Ω and T_Ω^* are bounded operators on L^p for $1 < p < \infty$.

Some years later, condition (2) above was independently improved by Connett [4] and Ricci and Weiss [9] who showed that if

$$\Omega \in H^1(\mathbf{S}^{n-1}), \tag{3}$$

then T_Ω maps $L^p(\mathbf{R}^n)$ into itself for $1 < p < \infty$. $H^1(\mathbf{S}^{n-1})$ here denotes the 1-Hardy space on the unit sphere in the sense of Coifman and Weiss [3]; (this paper contains a proof of this result only in dimension $n = 2$). See also [8] for a simple proof of this result on $\mathbf{R}^n$.

The H^1 condition (3) is also sufficient to imply that T_Ω^* is bounded on L^p for $1 < p < \infty$. For a proof of this fact we refer the reader to [8] and also to Fan and Pan [7] who recently obtained this result independently for a more general class of operators.

The main purpose of this paper is to present alternative conditions that imply L^p boundedness for T_Ω and T_Ω^*. If we examine the proof giving the formula of the Fourier transform of $\text{p.v.}\,\Omega(x)/|x|^n$, we observe that the mild assumption

$$\sup_{\xi \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \ln \frac{1}{|\theta \cdot \xi|}\, d\theta < +\infty, \tag{4}$$

suffices to imply that $(\text{p.v.}\,\Omega(x)/|x|^n)\widehat{\ }$ is a bounded function, which is equivalent to saying that T_Ω maps $L^2(\mathbf{R}^n)$ into itself. It is unknown to us whether condition (4) implies L^p boundedness for some $p \neq 2$.

Motivated by (4) we consider the family of conditions

$$\sup_{\xi \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \left(\ln \frac{1}{|\theta \cdot \xi|}\right)^{1+\alpha} d\theta < +\infty \tag{5}$$

for $\alpha > 0$. We can show that if Ω satisfies condition (5) for some $\alpha > 0$, then T_Ω maps $L^p(\mathbf{R}^n)$ into itself for some $p \neq 2$. More precisely, we have the following theorem:

Theorem 1. *Let Ω be a function in $L^1(\mathbf{S}^{n-1})$ with mean value zero which satisfies condition (5) for some $\alpha > 1$. Then T_Ω extends to a bounded operator from $L^p(\mathbf{R}^n)$ into itself for $(2+\alpha)/(1+\alpha) < p < 2 + \alpha$.*

As a corollary we obtain that if Ω satisfies condition (5) for all $\alpha > 0$, then it maps $L^p(\mathbf{R}^n)$ into itself for all $1 < p < \infty$. Regarding T_Ω^* we can prove the following:

Theorem 2. *Let Ω be a function in $L^1(\mathbf{S}^{n-1})$ with mean value zero which satisfies condition (5) for some $\alpha > 1$. Then T_Ω^* extends to a bounded operator from $L^p(\mathbf{R}^n)$ into itself for $1 + 3/(1+2\alpha) < p < 2(2+\alpha)/3$.*

We conclude that if Ω satisfies condition (5) for all $\alpha > 0$, then T_Ω^* maps L^p to L^p for all $1 < p < \infty$. We don't know whether the ranges of indices in Theorems 1 and 2 are sharp. More fundamentally, we do not know an example of an $\Omega \in L^1(\mathbf{S}^{n-1})$ such that T_Ω maps $L^p \to L^p$ for some given $p = p_0 \geq 2$ but not for some other $p_1 > p_0$.

In section 5, we show that condition (5) for all $\alpha > 0$ is indeed disjoint from the H^1 condition (3).

2. Boundedness of singular integrals

The theme of the proof of Theorem 1 is based on ideas developed by J. Duoandikoetxea and J.-L. Rubio de Francia [6] to treat several other operators of this sort. Define

$$\sigma_k(x) = \frac{\Omega(x)}{|x|^n}\chi_{2^k \leq |x| \leq 2^{k+1}}, \qquad k \in \mathbf{Z}.$$

Observe that $\widehat{\sigma_k}(\xi) = \widehat{\sigma_0}(2^k\xi)$. We calculate $\widehat{\sigma_0}(\xi)$. Set $\xi' = \xi/|\xi|$. Expressing $\widehat{\sigma_0}$ in polar coordinates, we obtain

$$\widehat{\sigma_0}(\xi) = \int_{\mathbf{S}^{n-1}} \Omega(\theta) \left[\int_1^2 e^{2\pi i r|\xi|(\xi'\cdot\theta)}\frac{dr}{r}\right] d\theta. \tag{6}$$

Using that Ω has mean value zero, we deduce that

$$|\widehat{\sigma_0}(\xi)| \leq 2\pi(\ln 2)\|\Omega\|_{L^1}|\xi| = C|\xi|, \tag{7}$$

which is a good estimate for $|\xi| \leq 2$. For $|\xi| \geq 2$ observe the following: The integral inside brackets in (6) is bounded by $\min\left(2, 3|\xi'\cdot\theta|^{-1}|\xi|^{-1}\right)$. (Pick a θ so that $\xi'\cdot\theta \neq 0$.) Therefore it must satisfy the estimate

$$\left|\int_1^2 e^{2\pi i r|\xi|(\xi'\cdot\theta)}\frac{dr}{r}\right| \leq \frac{2\left(\ln(\frac{3}{2}|\xi'\cdot\theta|^{-1})\right)^{1+\alpha}}{(\ln|\xi|)^{1+\alpha}}. \tag{8}$$

It follows from (8) and (5) that

$$|\widehat{\sigma_0}(\xi)| \leq C(\ln|\xi|)^{-1-\alpha} \qquad \text{for } |\xi| \geq 2. \tag{9}$$

Since σ_k is obtained from σ_0 by a suitable dilation, it follows that there exists a constant $C > 0$, such that for all $k \in \mathbf{Z}$ the estimates below are valid:

$$\begin{aligned} |\widehat{\sigma_k}(\xi)| &\leq C(\ln|2^k\xi|)^{-1-\alpha}, \quad && \text{for } 2^k|\xi| \geq 2,\\ |\widehat{\sigma_k}(\xi)| &\leq C2^k|\xi|, && \text{for } 2^k|\xi| \leq 2. \end{aligned} \tag{10}$$

Now let ψ be a C^∞ function supported in $\{x \in \mathbf{R}^n : 3/4 \leq |x| \leq 9/4\}$, such that $\sum_{j\in\mathbf{Z}}(\psi(2^j\xi))^2 = 1$. Let S_j be the operator given on the Fourier transform by multiplication by $\psi_j(\xi) = \psi(2^j\xi)$. Define

$$T_j f = \sum_{k\in\mathbf{Z}} S_{j+k}(\sigma_k * S_{j+k}f).$$

It is easy to see that the identity

$$T_\Omega f = \sum_{j\in\mathbf{Z}} T_j f$$

is valid at least for f in the Schwartz class. Using a Fourier transform calculation, (10), and the fact that ψ_{j+k} is supported near the annulus $|\xi| \sim 2^{-j-k}$, we obtain that T_j are bounded on $L^2(\mathbf{R}^n)$ with bound $C2^{-j}$ for $j \geq 0$ and $C(|j|)^{-1-\alpha}$ for $j \leq -1$. In short

$$\|T_j f\|_{L^2} \leq C(1+|j|)^{-1-\alpha}\|f\|_{L^2} \qquad \text{for all } j \in \mathbf{Z}. \tag{11}$$

We will also need estimates for the following maximal operator

$$f \to \sigma^*(f) = \sup_{k\in\mathbf{Z}}(|\sigma_k| * |f|).$$

Without loss of generality we can assume that $\|\Omega\|_{L^1(\mathbf{S}^{n-1})} = 1$. It follows that $\widehat{|\sigma_0|}(0) = 1$. Introduce a radial function in the Schwartz class Φ, such that $\widehat{\Phi}(\xi) = 1$ for $|\xi| \leq 2$ and $\widehat{\Phi}(\xi) = 0$ for $|\xi| > 3$. Let us also introduce Φ_k defined by $\widehat{\Phi_k}(\xi) = \widehat{\Phi}(2^k\xi)$. Clearly we have

$$\sigma^*(f) \leq \sup_{k\in\mathbf{Z}} |(|\sigma_k| - \Phi_k) * |f|| + \sup_{k\in\mathbf{Z}} |\Phi_k * |f||. \tag{12}$$

Denote $\mu_k = |\sigma_k| - \Phi_k$. Since $\widehat{\mu_k}(0) = 0$, the same proof giving (10) implies that

$$\begin{aligned} |\widehat{\mu_k}(\xi)| &\leq C2^k|\xi|, && \text{for } 2^k|\xi| \leq 2,\\ |\widehat{\mu_k}(\xi)| &\leq C(\log|2^k\xi|)^{-1-\alpha}, && \text{for } 2^k|\xi| \geq 2. \end{aligned} \tag{13}$$

Therefore we obtain from (12) that

$$\sigma^*(f) \leq \sup_{k\in\mathbf{Z}}(\mu_k * |f|) + \mathcal{M}f \leq \left(\sum_k |\mu_k * |f||^2\right)^{1/2} + \mathcal{M}f, \tag{14}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. Since for all $1 < r < \infty$,

$$\Big\|\big(\sum_k (\mu_k * f)^2\big)^{1/2}\Big\|_{L^r}^r = \text{Average} \quad \|\sum_k \varepsilon_k(\mu_k * f)\|_{L^r}^r, \tag{15}$$

over all choices of signs $\varepsilon_k = \pm 1$, estimates for the square function on the right hand side of (14) can be obtained from estimates on integral operators of the form $g \to \sum_k \varepsilon_k(\mu_k * g)$. Now using (13) and (14) we conclude that σ^* maps $L^2 \to L^2$, whenever $\alpha > 0$. At this point we recall the following lemma:

Lemma 1. *(See [6] p. 544) If $\|\sigma^*(f)\|_{L^s} \leq C\|f\|_{L^s}$ and $\dfrac{1}{2s} = \left|\dfrac{1}{2} - \dfrac{1}{q}\right|$, then for arbitrary functions g_k we have*

$$\big\|(\sum_{k\in\mathbf{Z}} |\sigma_k * g_k|^2)^{1/2}\big\|_{L^q} \leq C\big\|(\sum_{k\in\mathbf{Z}} |g_k|^2)^{1/2}\big\|_{L^q}.$$

Applying Lemma 1 with $s = 2$ and $q = q_0 = 4$, we obtain that

$$\|T_j f\|_{L^{q_0}} \leq C\big\|(\sum_{k\in\mathbf{Z}} |\sigma_k * S_{j+k}f|^2)^{1/2}\big\|_{L^{q_0}} \leq C\big\|(\sum_{k\in\mathbf{Z}} |S_{j+k}f|^2)^{1/2}\big\|_{L^{q_0}} \leq C\|f\|_{L^{q_0}}, \tag{16}$$

where the middle inequality is a consequence of Lemma 1 and the first and last inequalities follow from the Littlewood-Paley theorem.

Interpolating between estimates (11) and (16) we obtain that

$$\|T_j f\|_{L^p} \le C(1+|j|)^{-(1+\alpha)\theta_p}\|f\|_{L^p},$$

where $1/p = \theta_p/2 + (1-\theta_p)/q_0$. Now observe that $T_\Omega = \sum_{j\in\mathbf{Z}} T_j$ maps $L^p \to L^p$ for all p's for which $p_1' < p < p_1$, where $p_1 = (4+4\alpha)/(2+\alpha)$ is the unique solution of the equation $(1+\alpha)\theta_p = 1$. The same argument also gives that $T_\varepsilon f = \sum_k \varepsilon_k(\mu_k * f)$ maps $L^p \to L^p$ for $p_1' < p < p_1$ uniformly on the choice of the signs (ε_j), $\varepsilon_j = \pm 1$. It follows that the square function in (15) is also bounded on L^p for this range of p's and hence so is $\sigma^*(f)$ by the estimate in (14). Thus we are in a position to apply Lemma 1 again with s in the interval (p_1', p_1).

Now continue this way. Fix $s_1 \in (2, p_1)$ and let q_1 be the unique number bigger than $q_0 = 4$ which satisfies the equation $1/2s_1' = |1/2 - 1/q_1|$. Apply Lemma 1 with $s = s_1'$ and $q = q_1$. As before we obtain that T_Ω maps $L^p \to L^p$ for $p_2' < p < p_2$, where p_2 is the unique solution of the equation $(1+\alpha)\theta_p = 1$, where θ_p is given by $1/p = \theta_p/2 + (1-\theta_p)/q_1$ now. This bootstrapping argument leads to an inductive definition of three sequences $2 = p_0 < p_1 < \dots$, $2 < s_1 < s_2 < \dots$, and $4 = q_0 < q_1 < \dots$ such that for $k = 1, 2, \dots$

$$p_{k-1} < s_k < p_k, \qquad \frac{1}{p_k} - \frac{1}{q_{k-1}} = \frac{1}{1+\alpha}\left(\frac{1}{2} - \frac{1}{q_{k-1}}\right), \qquad \frac{1}{2s_k'} = \frac{1}{2} - \frac{1}{q_k}.$$

Let $b = \sup_k p_k$. The above equations easily imply that $b = 2+\alpha$. Therefore T_Ω maps L^p to L^p for $2 \le p < 2+\alpha$. The remaining range of p's follows by duality.

3. Boundedness of maximal singular integrals

We now prove Theorem 2. We use below the same notation as in the previous section. Let

$$(T_k f)(x) = \int_{|y|>2^k} \frac{\Omega(y)}{|y|^n} f(x-y)\,dy = \sum_{j=k}^{\infty} (\sigma_j * f)(x),$$
$$(T^* f)(x) = \sup_k |(T_k f)(x)|.$$

If $2^{k-1} \le \varepsilon < 2^k$, then

$$|(T_\Omega^\varepsilon f)(x)| \le |(T_k f)(x)| + \left|\int_{\varepsilon<|y|<2^k} \frac{\Omega(y)}{|y|^n} f(x-y)\,dy\right| \le |(T_k f)(x)| + (|\sigma_k| * |f|)(x).$$

From the proof of Theorem 1 we know that σ^* maps $L^p \to L^p$ for $(2+\alpha)/(1+\alpha) < p < 2+\alpha$. Since

$$|(T_\Omega^* f)(x)| \le |(T^* f)(x)| + \sigma^*(|f|)(x),$$

it suffices to show that $T^*: L^p \to L^p$ for the claimed range of p's, which is contained in the interval $((2+\alpha)/(1+\alpha), 2+\alpha)$.

With Φ as in the previous section, estimate

$$\sup_{k\in\mathbf{Z}} |(T_k f)(x)| \le \sup_{k\in\mathbf{Z}} \left|\Phi_k * \sum_{j=k}^{\infty} \sigma_j * f\right| + \sup_{k\in\mathbf{Z}} \left|(\delta - \Phi_k) * \sum_{j=k}^{\infty} \sigma_j * f\right|, \tag{17}$$

where δ is Dirac mass at the origin. It is easy to see that

$$\sup_{k\in\mathbf{Z}}\left|\Phi_k * \sum_{j=k}^{\infty}\sigma_j * f\right| \le C\left(\mathcal{M}(Tf)+\mathcal{M}(f)\right), \qquad \text{(see [6], p.548)}$$

which implies L^p bounds for the first term on the right hand side of (17) for $(2+\alpha)/(1+\alpha) < p < 2+\alpha$. Control the second term on the right hand side of (17) by

$$\sup_{k\in\mathbf{Z}}\left|(\delta-\Phi_k) * \sum_{j=0}^{\infty}\sigma_{j+k} * f\right| \le \sum_{j=0}^{\infty} Q_j(f),$$

where

$$(Q_j f)(x) = \sup_{k\in\mathbf{Z}}\left|(\delta-\Phi_k) * \sigma_{j+k} * f\right|.$$

To conclude the proof of Theorem 2, it suffices to show that for $j \ge 0$ we have

$$\|Q_j f\|_{L^p} \le C\|f\|_{L^p}, \qquad \text{for } 2 \le p < 2+\alpha, \text{ and} \tag{18}$$

$$\|Q_j f\|_{L^2} \le C(1+j)^{-\alpha}\|f\|_{L^2}. \tag{19}$$

Then, a simple interpolation between (18) and (19) gives that Q_j maps $L^p \to L^p$ with bound $C_\delta(1+j)^{2\alpha(2+\alpha-\delta-p)/p(\alpha-\delta)}$, for any $\delta > 0$ small, and the conclusion of Theorem 2 follows by summing on j.

Now observe that

$$|Q_j f| \le \sup_k |\sigma_{j+k} * f| + \sup_k |\Phi_k * \sigma_{j+k} * f| \le C(\sigma^*(f) + \mathcal{M}(\sigma^*(f))).$$

Therefore Q_j is bounded on L^p whenever σ^* is, that is $\|Q_j f\|_{L^p} \le C\|f\|_{L^p}$ when $2 \le p < 2+\alpha$ and (18) is proved. To prove (19) we need to exploit some orthogonality. We have

$$\|Q_j f\|_{L^2}^2 \le \sum_k \|(\delta-\Phi_k) * \sigma_{j+k} * f\|_{L^2}^2 = \text{Average}\|\sum_k \varepsilon_k\big((\delta-\Phi_k) * \sigma_{j+k} * f\big)\|_{L^2}^2,$$

where $\varepsilon = (\varepsilon_k)_k$ is a sequence of ±1's. For a fixed sequence $\varepsilon_k = \pm1$, let us denote by

$$M_{j,k} f = \varepsilon_k(\delta-\Phi_k) * \sigma_{j+k} * f.$$

We will need the following

Lemma 2. *Let $m \ge 1$, $j \ge 0$, and $k_1 \le \ldots \le k_{2m}$ be integers. Then*

$$\|M_{j,k_1}\ldots M_{j,k_{2m}}\|_{2\to2} \le C^{2m}\prod_{i=1}^{2m}\left(\frac{1}{1+j+k_i-k_1}\right)^{1+\alpha}.$$

Proof. Since $\widehat{\Phi_{k_1}}(\xi)$ vanishes for $2^{k_1}|\xi| \le 2$ we have,

$$\begin{aligned}
\|M_{j,k_1}\dots M_{j,k_{2m}}f\|_{L^2}^2 &= \int_{\mathbf{R}^n}\prod_{i=1}^{2m}\left|1-\widehat{\Phi_{k_i}}(\xi)\right|^2|\widehat{\sigma_{j+k_i}}(\xi)|^2|\widehat{f}(\xi)|^2d\xi \\
&\le (C)^{2m}\int_{|\xi|\ge 2^{1-k_1}}\prod_{i=1}^{2m}\left[\frac{1}{\log\left(2^{j+k_i}|\xi|\right)}\right]^{2+2\alpha}|\widehat{f}(\xi)|^2d\xi \\
&\le C^{2m}\prod_{i=1}^{2m}\left[\frac{1}{1+j+k_i-k_1}\right]^{2+2\alpha}\|f\|_{L^2}^2,
\end{aligned}$$

where we used the first estimate in (10) in the last inequality above. □

Now we return to the proof of Theorem 2. We must show that $\left\|M_j^{\varepsilon,N}\right\|_{2\to 2} \le C(1+j)^{-\alpha}$ uniformly on N and $\varepsilon = (\varepsilon_k)$, where

$$M_j^{\varepsilon,N} = \sum_{k=-N}^{N}\varepsilon_k M_{j,k}.$$

Since $M_j^{\varepsilon,N}$ are self adjoint operators, we have

$$\left\|M_j^{\varepsilon,N}\right\|_{2\to 2}^{2m} = \left\|(M_j^{\varepsilon,N})^{2m}\right\|_{2\to 2} \le \sum_{-N\le k_1\le\dots\le k_{2m}\le N}\|M_{j,k_1}\dots M_{j,k_{2m}}\|_{2\to 2}$$

$$\le \sum_{-N\le k_1\le\dots\le k_{2m}\le N} C^{2m}\prod_{i=1}^{2m}\left(\frac{1}{1+j+k_i-k_1}\right)^{1+\alpha} \le \frac{NC^{2m}}{(1+j)^{1+\alpha}}\left(\frac{1}{(1+j)^{\alpha}}\right)^{2m-1}$$

$$\le N\frac{C^{2m}}{1+j}(1+j)^{-2m\alpha}.$$

Taking $(2m)^{\text{th}}$ roots and letting $m\to\infty$ we obtain

$$\left\|M_j^{\varepsilon,N}\right\|_{2\to 2} \le C(1+j)^{-\alpha}.$$

This concludes the proof of (19) and hence of Theorem 2.

4. Examples

It is easy to see that condition (5) for all $\alpha > 0$ contains the case $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$, considered by several authors, including [6]. However, it does not include the condition $\Omega \in L\log L(\mathbf{S}^{n-1})$ of Calderón and Zygmund. It is therefore natural to ask whether there exist examples of $\Omega \notin L\log L(\mathbf{S}^{n-1})$ which satisfy (5) for all $\alpha > 0$. In this section we prove something more.

We construct an example to show that there exist integrable functions on $\mathbf{S}^{n-1}$ with mean value zero which are not in $H^1(\mathbf{S}^{n-1})$ but which satisfy (5) for all $\alpha > 0$. We also show that there exist functions in $L^1(\mathbf{S}^{n-1})$ which satisfy the converse. The examples are given only when $n = 2$ but they can be easily lifted to higher dimensions.

We begin with the converse which is easier. The function

$$\Omega(\theta) = \sum_{k=2}^{\infty} \frac{e^{ik\theta}}{(\log k)^2}$$

belongs to $H^1(\mathbf{S}^1)$ but it fails to satisfy condition (5) for any $\alpha > 0$. Both assertions follow from the fact that $\Omega(\theta)$ behaves like $\theta^{-1}\log^{-2}(\theta^{-1})$ as $\theta \to 0+$ (See [13] p. 189).

We now construct an $\Omega \in L^1(\mathbf{S}^1) \setminus H^1(\mathbf{S}^1)$ with mean value zero which satisfies condition (5) for all $\alpha > 0$. The example presented below is unavoidably complicated. The problem is that such a function must have an infinite number of spikes which are sufficiently far away from each other and which are (barely) integrable and have mean value zero.

At this point we think of $\mathbf{S}^1$ as the interval $[0,1]$ via the identification

$$\widetilde{\Omega}(x) = \Omega(\cos(2\pi x), \sin(2\pi x)), \tag{20}$$

where Ω is defined on $\mathbf{S}^1$ and $\widetilde{\Omega}$ on $[0,1]$. It is not hard to see that under the identification given in (20), the condition $\Omega \notin H^1(\mathbf{S}^1)$ is equivalent to the fact that the Hilbert transform of $\widetilde{\Omega}\chi_{[0,1]}$ is not in $L^1(\mathbf{R}^1)$, and condition (5) is equivalent to

$$\sup_{0 \le z \le 1} \int_0^1 |\widetilde{\Omega}(x)| \ln^{1+\alpha} \frac{1}{|x-z|} dx \le C_\alpha < \infty. \tag{21}$$

For a detailed justification of these facts see [10]. Now let

$$\begin{aligned} a_n &= (\ln n)^{-1}, & b_n &= e^{-\gamma_n}, \\ \gamma_n &= e^{(\ln n)^{1/2}}, & \delta_n &= e^{-\gamma_n^{1/4}}, \\ d_n &= a_n + \delta_n, & c_n &= a_n - \delta_n. \\ \beta_n &= 1 - (\ln n + \tfrac{3}{2}\ln \gamma_n)\gamma_n^{-1}, \end{aligned}$$

Heuristically speaking, a_n is a sequence that decays slowly to zero, c_n and d_n are symmetric points about a_n at distance δ_n, $(c_n - b_n, c_n)$ and $(d_n - b_n, d_n)$ are small intervals near c_n and d_n with length $b_n = e^{-\gamma_n}$, where $(\ln n)^\varepsilon << \gamma_n << n^\varepsilon$ for all $\varepsilon > 0$, and the β_n's are powers that converge to 1 at a rate $\sim \gamma_n^{-1}$. It is easy to see that

$$\frac{b_n^{1-\beta_n}}{1-\beta_n} = \frac{1}{n\gamma_n^{1/2}(\ln n + \frac{3}{2}(\ln n)^{1/2})} \sim \frac{1}{n\gamma_n^{1/2}\ln n}, \tag{22}$$

for n large. Now let

$$\widetilde{\Omega}(x) = \sum_{n=10^9}^{\infty} \left(\frac{1}{|x-c_n|^{\beta_n}} \chi_{(c_n-b_n,c_n)}(x) - \frac{1}{|x-d_n|^{\beta_n}} \chi_{(d_n-b_n,d_n)}(x) \right).$$

We first verify that condition (21) holds for all $\alpha > 0$. The worst possible z's in (21) are the singularities of $\widetilde{\Omega}$, i.e. the points $z = c_n$, d_n, and $z = 0$. By symmetry we consider only $z = c_n$ and $z = 0$. Fix $N \geq 10^9$ and take $z = c_N$. We have

$$\int_0^1 |\widetilde{\Omega}(x)| \ln^{1+\alpha} \frac{1}{|x - c_N|} dx \leq I_1(N) + I_2(N) + I_3(N) + I_4(N),$$

where

$$\begin{aligned}
I_1(N) &= \sum_{n \neq N} \int\limits_{c_n - b_n}^{c_n} \frac{1}{|x - c_n|^{\beta_n}} \ln^{1+\alpha} \frac{1}{|x - c_N|} dx, \\
I_2(N) &= \sum_{n \neq N} \int\limits_{d_n - b_n}^{d_n} \frac{1}{|x - d_n|^{\beta_n}} \ln^{1+\alpha} \frac{1}{|x - c_N|} dx, \\
I_3(N) &= \int\limits_{c_N - b_N}^{c_N} \frac{1}{|x - c_N|^{\beta_N}} \ln^{1+\alpha} \frac{1}{|x - c_N|} dx, \\
I_4(N) &= \int\limits_{d_N - b_N}^{d_N} \frac{1}{|x - d_N|^{\beta_N}} \ln^{1+\alpha} \frac{1}{|x - c_N|} dx.
\end{aligned}$$

Observe that $I_2(N) \leq I_1(N)$ and that $I_4(N) \leq I_3(N)$. Also, it is easy to see that

$$\sup_{N \geq 10^9} I_3(N) \leq C \sup_{N \geq 10^9} \frac{b_N^{1-\beta_N}}{1 - \beta_N} \ln^{1+\alpha} \frac{1}{b_N} \leq C \sup_{N \geq 10^9} \frac{\gamma_N^{1+\alpha}}{N \gamma_N^{1/2} \ln N} \leq C_\alpha.$$

To control $\sup\limits_{N \geq 10^9} I_1(N)$ we need to show that

$$\sup_{N \geq 10^9} \left[\sum_{n \neq N} \int\limits_{c_n - b_n}^{c_n} \frac{1}{|x - c_n|^{\beta_n}} \ln^{1+\alpha} \frac{1}{|x - c_N|} dx \right] \leq C_\alpha. \tag{23}$$

Using that $|x - c_N| \sim |c_n - c_N| \sim |a_n - a_N|$ in the integrand above and (22), we conclude that (23) will be a consequence of

$$\sup_{N \geq 10^9} \left[\sum_{n \neq N} \frac{b_n^{1-\beta_n}}{1 - \beta_n} \ln^{1+\alpha} \frac{1}{|a_n - a_N|} \right] \leq C_\alpha. \tag{24}$$

We have two cases. For $n > N$, $\quad |a_n - a_N| \geq |a_{N+1} - a_N| \geq (N \ln^2 N)^{-1}$ and therefore

$$\sup_{N \geq 10^9} \left[\sum_{n > N} \frac{b_n^{1-\beta_n}}{1 - \beta_n} \ln^{1+\alpha} \frac{1}{|a_n - a_N|} \right] \leq C \sup_{N \geq 10^9} \sum_{n > N} \frac{\ln^{1+\alpha}(N \ln^2 N)}{n \gamma_n^{1/2} \ln n} \leq C_\alpha,$$

the latter being an easy consequence of the integral test. For $10^9 \le n \le N-1$ we have

$$\begin{aligned} &\sum_{n=10^9}^{N-1} \frac{b_n^{1-\beta_n}}{1-\beta_n} \ln^{1+\alpha} \frac{1}{|a_n - a_N|} \\ \le C &\sum_{n=10^9}^{N-1} \frac{1}{n\gamma_n^{1/2} \ln n} \ln^{1+\alpha} \frac{1}{|(\ln n)^{-1} - (\ln N)^{-1}|} = A(N) + B(N), \end{aligned}$$

where $A(N)$ is the sum above of over the indices $10^9 \le n < \gamma_N$ and $B(N)$ is the sum over the the indices $\gamma_N \le n \le N-1$. On $A(N)$ we have $|(\ln n)^{-1} - (\ln N)^{-1}|^{-1} \le C \ln n$, and thus $A(N)$ is clearly bounded independently of N. On $B(N)$ we have $|(\ln n)^{-1} - (\ln N)^{-1}|^{-1} \le CN(\ln N)^2$. Now estimate $\sup\limits_{N \ge 10^9} B(N)$ by

$$C \sup_{N \ge 10^9} \ln^{1+\alpha}(N^2) \sum_{n \ge \gamma_N} \frac{1}{n\gamma_n^{1/2} \ln n} \le C \sup_{N \ge 10^9} \frac{\ln^{1+\alpha}(N^2)}{\gamma_{\gamma_N}^{1/3}} \le C,$$

where we used the integral test to deduce the first inequality above. This concludes the proof of (21) when $z = c_N$. Condition (21) for $z = 0$ is equivalent to the following inequality

$$\sum_{n=10^9}^{\infty} \frac{\ln^{1+\alpha}(\ln n)}{n\gamma_n^{1/2} \ln n} \le C_\alpha,$$

which is certainly correct by the choice of our parameters. This proves that $\widetilde{\Omega}$ satisfies condition (21) for all $\alpha > 0$.

We now prove that $\widetilde{\Omega}$ is not in the Hardy space H^1. Extend $\widetilde{\Omega}$ to be equal to zero outside the interval $[0, 1]$. Let H be the usual Hilbert transform. Fix $N \ge 10^9$ and $y \in [d_N, d_N + b_N]$. Obviously

$$\pi |(H\widetilde{\Omega})(y)| \ge K_N(y) - L_N(y), \tag{25}$$

where

$$\begin{aligned} K_N(y) &= \left| \int_{c_N - b_N}^{c_N} \frac{1}{|x - c_N|^{\beta_N}} \frac{1}{x-y} dx - \int_{d_N - b_N}^{d_N} \frac{1}{|x - d_N|^{\beta_N}} \frac{1}{x-y} dx \right|, \\ L_N(y) &= \sum_{n \ne N} \left| \int_{c_n - b_n}^{c_n} \frac{1}{|x - c_n|^{\beta_n}} \frac{1}{x-y} dx - \int_{d_n - b_n}^{d_n} \frac{1}{|x - d_n|^{\beta_n}} \frac{1}{x-y} dx \right|. \end{aligned}$$

We first prove that

$$\sup_{N \ge 10^9} \sup_{y \in [d_N, d_N + b_N]} L_N(y) \le C. \tag{26}$$

Indeed,

$$\int_{c_n-b_n}^{c_n} \frac{1}{|x-c_n|^{\beta_n}} \frac{1}{x-y} dx = \frac{b_n^{1-\beta_n}}{1-\beta_n} \frac{1}{(-b_n+c_n-y)} + \text{smaller term},$$
$$\int_{d_n-b_n}^{d_n} \frac{1}{|x-d_n|^{\beta_n}} \frac{1}{x-y} dx = \frac{b_n^{1-\beta_n}}{1-\beta_n} \frac{1}{(-b_n+d_n-y)} + \text{smaller term},$$

where the smaller terms are bounded by $Cb_n^{1-\beta_n}/(1-\beta_n)$ and $\sum_{n\geq 10^9} b_n^{1-\beta_n}/(1-\beta_n) \leq C$. Therefore

$$L_N(y) \leq C \sum_{n\neq N} \frac{b_n^{1-\beta_n}}{1-\beta_n} \frac{|d_n-c_n|}{|a_n-a_N|^2} \leq C \sum_{n\neq N} \frac{b_n^{1-\beta_n}}{1-\beta_n} \frac{\delta_n}{|a_n-a_N|^2},$$

and thus it remains to prove that

$$\sup_{N\geq 10^9} \sum_{n\neq N} \frac{\delta_n}{n\gamma_n^{1/2} \ln n} \frac{1}{((\ln n)^{-1}-(\ln N)^{-1})^2} \leq C. \tag{27}$$

The sum in (27) for $n > N$ is bounded by

$$\sum_{n>N} \frac{\delta_n}{n\gamma_n^{1/2} \ln n} \frac{1}{((\ln n)^{-1}-(\ln N)^{-1})^2} \leq N^2 \ln^4 N \sum_{n>N} \frac{\delta_n}{n\gamma_n^{1/2} \ln n} \leq C,$$

uniformly in $N \geq 10^9$. Split the sum in (27) for $n < N$ into the sum $A'(N)$ over the indices $10^9 \leq n < \gamma_N$ and the sum $B'(N)$ over the indices $\gamma_N \leq n \leq N-1$. Using that when $10^9 \leq n < \gamma_N$ we have $|(\ln n)^{-1}-(\ln N)^{-1}|^{-1} \leq C \ln n$ we conclude that $A'(N)$ is bounded independently of N. When $\gamma_N \leq n \leq N-1$ we have $|(\ln n)^{-1}-(\ln N)^{-1}|^{-1} \leq CN(\ln N)^2$ and hence

$$\sup_{N\geq 10^9} B'(N) \leq C \sup_{N\geq 10^9} N^5 \sum_{n\geq \ln N} \frac{1}{n\gamma_n^{1/2}(\ln n)e^{\gamma_n^{1/4}}} \leq C,$$

which follows from the integral test. This proves (27) and hence $L_N(y)$ is bounded uniformly in N.

Now we turn our attention to $K_N(y)$. Observe that the following inequality holds

$$\int_{d_N-b_N}^{d_N} \frac{1}{|x-d_N|^{\beta_N}} \frac{1}{y-x} dx \geq \frac{3}{2} \int_{c_N-b_N}^{c_N} \frac{1}{|x-c_N|^{\beta_N}} \frac{1}{y-x} dx,$$

because of the proximity of y to the support of the first integral. Therefore

$$|K_N(y)| \geq c \int_{d_N-b_N}^{d_N} \frac{1}{|x-d_N|^{\beta_N}} \frac{1}{y-x} dx - C$$

when $y \in [d_N, d_N + \delta_N]$. Integrate over this set to obtain

(28)
$$\int_{d_N}^{d_N+\delta_N} |K_N(y)|dy \geq$$
$$c\left|\int_{d_N-b_N}^{d_N} \frac{1}{|x-d_N|^{\beta_N}} \ln(d_N+\delta_N-x)dx - \int_{d_N-b_N}^{d_N} \frac{1}{|x-d_N|^{\beta_N}} \ln(d_N-x)dx\right| - C\delta_N.$$

We clearly have that

$$\left|\int_{d_N-b_N}^{d_N} \frac{1}{|x-d_N|^{\beta_N}} \ln(d_N+\delta_N-x)dx\right| \leq C|\ln \delta_N|\frac{b_N^{1-\beta_N}}{1-\beta_N} \leq \frac{\gamma_N^{1/4}}{N\gamma_N^{1/2}\ln N}, \tag{29}$$

while the the crucial fact is that

$$\left|\int_{d_N-b_N}^{d_N} \frac{1}{|x-d_N|^{\beta_N}} \ln(d_N-x)dx\right| \geq C|\ln b_N|\frac{b_N^{1-\beta_N}}{1-\beta_N} \geq \frac{\gamma_N^{1/2}}{N\ln N}. \tag{30}$$

Combining (25), (26), (28), (29), and (30) we obtain

$$\begin{aligned}\left\|H\widetilde{\Omega}\right\|_{L^1} &\geq \sum_{N\geq 10^9} \int_{d_N}^{d_N+\delta_N} |(H\widetilde{\Omega})(y)|dy \\ &\geq c\sum_{N\geq 10^9} \frac{\gamma_N^{1/2}}{N\ln N} - C\sum_{N\geq 10^9} \frac{1}{N\gamma_N^{1/4}\ln N} - C\sum_{N\geq 10^9} \delta_N = \infty.\end{aligned}$$

This proves that $\widetilde{\Omega} \notin H^1([0,1])$.

References

[1] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, *Acta Math.* **88** (1952), 85–139.

[2] A. P. Calderón and A. Zygmund, *On singular integrals*, *Amer. J. Math.* **78** (1956), 289–309.

[3] R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, *Bull. Amer. Math. Soc.* **83** (1977), 569-645.

[4] W. C. Connett, *Singular integrals near L^1*, *Proc. Sympos. Pure Math. of the Amer. Math. Soc.* (S. Wainger and G. Weiss, eds) Vol **35** I (1979), 163–165.

[5] J. Daly, K. Phillips, *On the classification of homogeneous multipliers bounded on $\mathbf{H}^1(\mathbf{R}^2)$*, *Proc. Amer. Math. Soc.* **106** (1989), 685–696.

[6] J. Duoandikoetxea and J.-L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, *Inv. Math.* **84** (1986), 541–561.

[7] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by subvarieties*, *Amer. J. Math.* **119** (1997), 799–839.

[8] L. Grafakos and A. Stefanov, *Convolution Calderón-Zygmund singular integral operators with rough kernels*, to appear, Proc. 7th IWAA, Orono, Maine 1997.

[9] F. Ricci and G. Weiss, *A characterization of* $\mathbf{H}^1(\mathbf{\Sigma}_{n-1})$, *Proc. Sympos. Pure Math. of Amer. Math. Soc.* (S. Wainger and G. Weiss, eds) Vol **35** I (1979), 289–294.

[10] A. Stefanov, *Characterizations of* H^1 *and applications to singular integrals*, (1998), submitted.

[11] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton NJ, 1970.

[12] E. M. Stein, *Harmonic analysis: Real variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton NJ, 1993.

[13] A. Zygmund, *Trigonometric series, Vol I*, Cambridge University Press, Cambridge UK, 1959.

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