COMPENSATED COMPACTNESS AND THE HEISENBERG GROUP

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Abstract. Jacobians of maps on the Heisenberg group are shown to map suitable group Sobolev spaces into the group Hardy space H^1. From this result and a weak* convergence theorem for the Hardy space H^1 of the Heisenberg group, a compensated compactness property for these Jacobians is obtained.

0. Introduction

We investigate compensated compactness properties of Jacobians of maps on the Heisenberg group and we prove results analogous to those for the Jacobians of maps on $\mathbb{R}^n$.

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Lie group with multiplicative structure $(z,t)\cdot(z',t') = (z+z', t+t'+2\,\mathrm{Im}\, z\cdot\bar{z'})$ where $z=(z_1,\dots,z_n)$, $z'=(z'_1,\dots,z'_n)$ and $z\cdot\bar{z'} = \sum_{j=1}^n z_j\bar{z'_j}$. This is the n^{th} order Heisenberg group. It can be shown that the group operation is C^∞ on the manifold $\mathbb{C}^n\times\mathbb{R}$ and hence $\mathbb{H}^n$ is a locally compact Lie group. Let $(z_1,\dots,z_n,t)$ be coordinates on $\mathbb{H}^n$. Write $z_j = x_j + iy_j$ and define the vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t} \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t} \qquad T = \frac{\partial}{\partial t}.$$

It turns out that the X_j, Y_j and T form a basis for the left invariant vector fields on $\mathbb{H}^n$. They satisfy the commutation relations $[X_j, Y_j] = -4T$, $j=1,\dots,n$ and all other

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commutators vanish. The group $\mathbb{H}^n$ is equipped with a natural dilation structure $r(z,t) = (rz, r^2t)$, $r > 0$, which is consistent with the group multiplication. The associated norm $|(z,t)| = (|z|^4 + t^2)^{1/4}$ is homogeneous of degree 1 with respect to this group of dilations. We denote by $B_r(u_0)$ the Heisenberg group ball $\{u \in \mathbb{H}^n \ : \ |u^{-1}u_0| < r\}$.

Haar measure on $\mathbb{H}^n$ is the usual Lebesgue measure on $\mathbb{R}^{2n+1}$. Convolution on the Heisenberg group $\mathbb{H}^n$ is defined by

$$(f * g)(u) = \int f(v)g(v^{-1}u)\ dv = \int f(uv^{-1})g(v)\ dv.$$

Fix φ be a smooth bump on $\mathbb{H}^n$ with $\int \varphi\ du \neq 0$. For a distribution f on $\mathbb{H}^n$ define

$$f^+(u) = \sup_{\delta>0} |(f * \varphi_\delta)(u)| \qquad \text{where} \qquad \varphi_\delta(u) = \delta^{-2n-2}\varphi(\delta^{-1}u).$$

If $\gamma > \frac{2n+2}{2n+3}$, the Hardy space $H^\gamma(\mathbb{H}^n) = H^\gamma$ is the set of all f such that $f^+ \in L^\gamma(\mathbb{H}^n)$. An alternative definition of $H^\gamma(\mathbb{H}^n)$ can be given via the atomic decomposition. See [FOS] or [CW] for details.

Given a C^1 map $F = (f_1, \ldots, f_n)$ from $\mathbb{H}^n$ into $\mathbb{R}^n$, define

$$\text{(0.3)} \qquad \operatorname{Jac}(F) = \det \begin{pmatrix} L_1 f_1 & L_1 f_2 & \ldots & L_1 f_n \\ L_2 f_1 & L_2 f_2 & \ldots & L_2 f_n \\ \vdots & \vdots & & \vdots \\ L_n f_1 & L_n f_2 & \ldots & L_n f_n \end{pmatrix},$$

where L_j is either X_j or Y_j. We would like to show that $\operatorname{Jac}(F)$ maps a product of suitable group Sobolev spaces into the spaces $H^\gamma(\mathbb{H}^n)$ for $1 \geq \gamma > \frac{2n+2}{2n+3}$. The analogous result for the vector fields $\frac{\partial}{\partial x_j}$, $1 \leq j \leq n$ on $\mathbb{R}^n$ has been proved by P. L. Lions and Y. Meyer when $\gamma = 1$ and extended by [CLMS] for $1 \geq \gamma > \frac{n}{n+1}$. Note that the lower bound for γ in both cases is $\frac{d}{d+1}$, where d is the homogeneous dimension of the group. Our first result is

Theorem 1. *Let $n \geq 2$ and for $1 \leq j \leq n$, let $1 < p_j < 2n+2$. Let $\gamma = \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)^{-1}$ be the harmonic mean of the p_j's. We suppose that $\frac{2n+2}{2n+3} < \gamma \leq 1$. Then, there exists* *a constant $C > 0$ that depends only on n and on the p_j's such that for every map $F = (f_1, \dots f_n)$ on $\mathbb{H}^n$ we have:*

$$\| \operatorname{Jac}(F)\|_{H^\gamma} \le C \prod_{j=1}^{n} \Big[\sum_{k=1}^{n} (\|X_k f_j\|_{L^{p_j}} + \|Y_k f_j\|_{L^{p_j}}) \Big]. \tag{0.4}$$

1. Proof of Theorem 1

Note that since γ is the harmonic mean of the p_j's, Hölder's inequality implies that $|\operatorname{Jac}(F)|^\gamma$ is integrable. The novelty provided by our Theorem is that any smooth maximal function of $\operatorname{Jac}(F)$ raised to the power γ is also integrable.

Two basic ingredients are needed for the proof. The first is that whenever $F = (f_1, \dots, f_n)$ is a compactly supported C^1 map on the Heisenberg group, $\operatorname{Jac}(F)$ has integral zero. This is explained in the next section. The second ingredient is the Poincaré (local Sobolev) inequality (1.0) stated below.

Let B be a Heisenberg group ball. We denote by $2B$ its double and by $|B|$ its the Lebesgue measure.

Theorem. *Let q and r be given such that $\frac{1}{2n+2} < \frac{1}{r} < 1$ and $\frac{1}{r} - \frac{1}{2n+2} \le \frac{1}{q} \le 1$. Then there exists a constant $C > 0$ that depends only on n, q and r such that for all Heisenberg group balls B and for all f with mean value zero over B, the following inequality is valid*

$$\left(\int_B |f(u)|^q du \right)^{\frac{1}{q}} \le C|B|^m \sum_{j=1}^{2n} \left[\left(\int_{2B} |(X_j f)(u)|^r du \right)^{\frac{1}{r}} + \left(\int_{2B} |(Y_j f)(u)|^r du \right)^{\frac{1}{r}} \right], \tag{1.0}$$

where we set $m = \frac{1}{2n+2} + \frac{1}{q} - \frac{1}{r} \ge 0$.

The theorem above is true even when the ball $2B$ on the right hand side of the inequality is replaced by the ball B. This more subtle result has been proved by Jerison [JE] when $1 \le q = r < \infty$ and recently by Lu [L1], [L2] in the remaining cases, including the endpoint case $m = 0$. The global form of (1.0) can be found in [FOS] and [VSC]. The local case stated above can be obtained in different ways. We refer the reader to [L2] for a proof.

To prove Theorem 1 we only need the Poincaré inequality above for $m > 0$. We will need the case $m = 0$ to prove the sharper endpoint result that the Jacobian (0.3) maps into the space weak H^γ for $\gamma = \frac{2n+2}{2n+3}$. (See next section).

Let F be a compactly supported C^1 map from the Heisenberg group into $\mathbb{R}^n$. Our estimates will be independent of the function F and a density argument will give the required inequality for general functions F for which the right hand side of (0.4) is finite. Fix ψ a smooth bump with support inside the unit ball $|u| < 1$ and also fix $u_0 = (z_0, t_0) \in \mathbb{H}^n$. We need to show that $\sup_{\delta>0} |\operatorname{Jac}(F) * \psi_\delta| \in L^\gamma$. We have

$$(\operatorname{Jac}(F) * \psi_\delta)(u_0) = \iint_{\mathbb{C}^n \times \mathbb{R}} \operatorname{Jac}(F)(z,t)\ \varphi_\delta(z,t)\ dz d\bar{z} dt,$$

where $\varphi_\delta(z,t) = \varphi_\delta^{u_0}(z,t) = \delta^{-2n-2}\psi\big(\frac{z_0 - z}{\delta}\,\frac{t_0 - t - 2\ Im\ z\cdot\bar{z}_0}{\delta^2}\big)$. Let $F = (f_1, f_2, \dots, f_n)$. Start with the identity:

$$\operatorname{Jac}(F)\,\varphi_\delta = -\ \det\begin{pmatrix} f_1(L_1\varphi_\delta) & L_1 f_2 & \dots & L_1 f_n \\ f_1(L_2\varphi_\delta) & L_2 f_2 & \dots & L_2 f_n \\ \vdots & \vdots & & \vdots \\ f_1(L_n\varphi_\delta) & L_n f_2 & \dots & L_n f_n \end{pmatrix}$$

$$+\ \det\begin{pmatrix} L_1(f_1\varphi_\delta) & L_1 f_2 & \dots & L_1 f_n \\ L_2(f_1\varphi_\delta) & L_2 f_2 & \dots & L_2 f_n \\ \vdots & \vdots & & \vdots \\ L_n(f_n\varphi_\delta) & L_n f_2 & \dots & L_n f_n \end{pmatrix}, \tag{1.1}$$

which follows from the multilinearity of the Jacobian. Let $c_1 = \int_{B_\delta} f_1\ du$ where $B_\delta = B_\delta(u_0) = \{u \in \mathbb{H}^n : |u^{-1}u_0| < \delta\}$. We replace f_1 by $f_1 - c_1$ in (1.1) and we note that $\operatorname{Jac}(F)$ remains unchanged. Next we integrate over the Heisenberg group. Note that the second determinant in (1.1) is the Jacobian of the map $(f_1\varphi_\delta, f_2, \dots, f_n)$. A crucial fact, explained in the next section, is that Jacobians of compactly supported maps have integral zero. Using this fact we conclude that

$$\int_{\mathbb{H}^n} \operatorname{Jac}(F)\varphi_\delta\ du = -\int_{\mathbb{H}^n} \det\begin{pmatrix} (f_1 - c_1)(L_1\varphi_\delta) & L_1 f_2 & \dots & L_1 f_n \\ (f_1 - c_1)(L_2\varphi_\delta) & L_2 f_2 & \dots & L_2 f_n \\ \vdots & \vdots & & \vdots \\ (f_1 - c_1)(L_n\varphi_\delta) & L_n f_2 & \dots & L_n f_n \end{pmatrix} du \tag{1.2}$$

Expand the determinant in (1.2) along its first column. We obtain

$$\left|\int_{\mathbb{H}^n} \mathrm{Jac}(F)\varphi_\delta \; du\right| = \left|\int_{\mathbb{H}^n} \sum_{j=1}^{n} (-1)^{j+1} \, (f_1 - c_1)(L_j\varphi_\delta)M_j(f_2,\ldots,f_n) \; du\right|, \tag{1.3}$$

where M_j is a minor. We treat only one term of the sum in (1.3), say the first one, since the remaining terms are similar. The minor M_1 is a sum of terms of the form $\pm\prod_{j=2}^{n} L_{m_j}f_j$ where $\{m_2,\ldots,m_n\} = \{2,\ldots,n\}$. We need to estimate the L^γ (quasi)norm of the supremum over all $\delta > 0$ of a typical term of the form

$$\int_{\mathbb{H}^n} |f_1 - c_1| \; |L_1\varphi_\delta| \; \prod_{j=2}^{n} |L_{m_j}f_j| \; du. \tag{1.4}$$

We observe that $|(L_1\varphi_\delta)(u)| \le C\delta^{-2n-3}$. To see this, recall that $u_0 = (z_0, t_0)$ and let $z_0 = (x_j^0 + iy_j^0)$ and $z = (x_j + iy_j)$. Then $(X_1\varphi_\delta)(z,t) = \frac{1}{\delta^{2n+2}}\{-\frac{1}{\delta}\frac{\partial\psi}{\partial x_1} + \frac{2y_1^0 - 2y_1}{\delta^2}\frac{\partial\psi}{\partial t}\}$, where the function inside the curly brackets is evaluated at $\left(\frac{z_0 - z}{\delta}, \frac{t_0 - t - 2\ Im\ z\cdot\bar{z}_0}{\delta^2}\right)$. Since ψ is supported in the unit ball we deduce that $|y_1^0 - y_1| \le \delta$ and therefore the expression inside the curly brackets above is bounded by $C\delta^{-1}$. We obtain the estimate $|(X_1\varphi_\delta)(u)| \le C\delta^{-2n-3}$ and similarly $|(Y_1\varphi_\delta)(u)| \le C\delta^{-2n-3}$ for all $u \in \mathbb{H}^n$. Let's now estimate (1.4) by

$$C\delta^{-2n-3}\int_{B_\delta} |f_1 - c_1| \; \prod_{j=2}^{n} |L_{m_j}f_j| \; du. \tag{1.5}$$

For any $1 \le j \le n$ select $1 < s_j < p_j$ and let $q = (1 - \sum_{j=2}^{n}\frac{1}{s_j})^{-1}$. Because of our assumption on γ and on the p_j's , one can check that $0 < \frac{1}{s_1} - \frac{1}{2n+2} < \frac{1}{q} < 1$. We now apply Hölder's inequality to (1.5) with exponents

$$\frac{1}{q} + \frac{1}{s_2} + \ldots + \frac{1}{s_n} = 1.$$

We estimate (1.5) by

$$C\delta^{-2n-3} \; \|f_1 - c_1\|_{L^q(B_\delta)} \prod_{j=2}^{n} \; \|L_{m_j}f_j\|_{L^{s_j}(B_\delta)}. \tag{1.6}$$

We are now ready to use the Poincaré inequality (1.0). Since we have $\frac{1}{2n+2} < \frac{1}{s_1} < 1$ and $\frac{1}{s_1} - \frac{1}{2n+2} < \frac{1}{q}$, the hypotheses are satisfied. We obtain that (1.6) is bounded by

$$C\delta^{-2n-3}(\delta^{2n+2})^{(\frac{1}{2n+2}+\frac{1}{q}-\frac{1}{s_1})}\left[\sum_{k=1}^{n}\left[\|X_k f_1\|_{L^{s_1}(2B_\delta)} + \|Y_k f_1\|_{L^{s_1}(2B_\delta)}\right]\right]\prod_{j=2}^{n}\|L_{m_j} f_j\|_{L^{s_j}(B_\delta)}$$

which equals

$$C\delta^{-(2n+2)\sum_{j=1}^{n}\frac{1}{s_j}}\left[\sum_{k=1}^{n}\left[\|X_k f_1\|_{L^{s_1}(2B_\delta)} + \|Y_k f_1\|_{L^{s_1}(2B_\delta)}\right]\right]\prod_{j=2}^{n}\|L_{m_j} f_j\|_{L^{s_j}(B_\delta)} \tag{1.7}$$

$$\leq C'\prod_{j=1}^{n}\left[\sum_{k=1}^{n}\left[\left((|X_k f_j|^{s_j})^*\right)^{\frac{1}{s_j}}(u_0) + \left((|Y_k f_j|^{s_j})^*\right)^{\frac{1}{s_j}}(u_0)\right]\right], \tag{1.8}$$

where by g^* we denote the Hardy-Littlewood maximal function of g on the Heisenberg group defined as follows:

$$g^*(u_0) = \sup_{\epsilon>0}\frac{1}{|B_\epsilon(u_0)|}\int_{B_\epsilon(u_0)}|g(u)|\,du.$$

(1.8) now controls the supremum over all $\delta > 0$ of (1.4). Summing (1.8) over all possible permutations $\{m_2, \ldots, m_n\} = \{2, \ldots, n\}$ we get the estimate below for the first term of the sum in (1.3):

$$C'(n-1)!\prod_{j=1}^{n}\left[\sum_{k=1}^{n}\left[\left((|X_k f_j|^{s_j})^*\right)^{\frac{1}{s_j}}(u_0) + \left((|Y_k f_j|^{s_j})^*\right)^{\frac{1}{s_j}}(u_0)\right]\right]. \tag{1.9}$$

Similar estimates hold for the other terms of the sum in (1.3). In fact, the proof is exactly the same, except that the index 1 is replaced by some $2 \leq l \leq n$. Now (1.9) majorizes the supremum over all $\delta > 0$ of (1.3). Since $s_j < p_j$, Hölder's inequality with exponents $\frac{1}{p_1} + \ldots + \frac{1}{p_n} = \frac{1}{\gamma}$ and the boundedness of the Hardy-Littlewood maximal function on L^t for $t > 1$ will give that the L^γ (quasi)norm of (1.9) is bounded above by

$$C_{n,p_j}\prod_{j=1}^{n}\left[\sum_{k=1}^{n}\left[\|X_k f_j\|_{L^{p_j}} + \|Y_k f_j\|_{L^{p_j}}\right]\right]. \tag{1.10}$$

Therefore $\| \sup_{\delta>0} | \int \mathrm{Jac}(F)\varphi_\delta \, du| \|_{L^\gamma}$ is dominated by (1.10) and the proof of (0.4) is now complete.

2. Remarks on Theorem 1

We begin this section by indicating why $\mathrm{Jac}(F)$ has integral zero whenever F is a compactly supported C^1 function from the Heisenberg group into $\mathbb{R}^n$.

We expand $\mathrm{Jac}(F)$ along a column, say the first one. We obtain

$$\mathrm{Jac}(F) = \sum_{j=1}^{n} (-1)^{j+1} (L_j f_1) M_j(f_2, \ldots, f_n). \tag{2.1}$$

Next, we integrate over the Heisenberg group. Using that F is compactly supported and that the L_j's are skew-adjoint, ($L_j^* = -L_j$), an integration by parts gives:

$$\int_{\mathbb{H}^n} \mathrm{Jac}(F)\, du = -\int_{\mathbb{H}^n} f_1 \sum_{j=1}^{n} (-1)^{j+1} L_j [M_j(f_2, \ldots, f_n)]\, du. \tag{2.2}$$

Each $L_j[M_j(f_2, \ldots, f_n)]$ is a sum of $(n-1)!$ terms so the sum in (2.2) consists of a total of $n!$ signed terms. Note that since $L_j \in \{X_j, Y_j\}$, all the commutators $[L_j, L_k] = 0$. This means that the vector fields L_j, $1 \le j \le n$ commute with each other and the order of the indices $i_1, \ldots, i_n$ in the expression $L_{i_1} L_{i_2} \ldots L_{i_n}$ is irrelevant. An easy induction argument, shows that the sum in (2.2) is identically equal to zero! We omit the details. Let us state this observation as a proposition.

Proposition. *Let $k \le N$ and suppose that $Z_1, \ldots, Z_k$ are smooth skew adjoint vector fields in $\mathbb{R}^N$ which commute with each other. Then for all $f_1, \ldots, f_k$ compactly supported C^1 functions on $\mathbb{R}^N$, the Jacobian* $\det(Z_j f_l)$ *has integral zero.*

Let us point out that there is a more esoteric reason for the apperently artificial cancellation discussed above. As the reader may have guessed this reason must involve some general way of integrating by parts, i.e. some form of Stokes' Theorem. For instance, in $\mathbb{R}^N$ let $Z_j = \frac{\partial}{\partial x_j}$, for $1 \le j \le N$. Pick a ball B properly containing the support of $(f_1, \ldots, f_N)$. We have

$$\int_{\mathbb{R}^N} \det(Z_j f_l)\ dx = \int_B \det(Z_j f_l)\ dx_1 \wedge \ldots \wedge dx_N = \int_B df_1 \wedge \ldots \wedge df_N =$$

$$\int_B d(f_1 df_2 \wedge \ldots \wedge df_N) = \int_{\partial B} f_1 df_2 \wedge \ldots \wedge df_N = 0,$$

since f_1 vanishes on ∂B. We used Stokes' Theorem in the penultimate equality above.

Finally we note that the proof of Theorem 1 can be modified to give the following endpoint result. We denote by $\|g\|_{H^{\gamma,\infty}}$ the weak $L^\gamma(\mathbb{H}^n)$ (quasi)norm of g^+.

Theorem 1a. *In the endpoint case where the harmonic mean of $p_1, \ldots, p_n$ is $\gamma = \frac{2n+2}{2n+3}$, there exits a constant $C > 0$ that depends only on n and on the p_j's, such that for all $F = (f_1, \ldots, f_n)$ as before, the following estimate holds:*

$$\|\operatorname{Jac}(F)\|_{H^{\gamma,\infty}} \le C \prod_{j=1}^n \big[\sum_{k=1}^n (\|X_k f_j\|_{L^{p_j}} + \|Y_k f_j\|_{L^{p_j}})\big]. \tag{2.3}$$

Sketch of proof: We set $q = (1 - \sum_{j=2}^n \frac{1}{p_j})^{-1}$. Since $p_1 < 2n+2$, we conclude that $1 < q < \infty$. Up to and including estimate (1.5) the proof is similar to the previous section. Then we obtain (1.6) by applying Hölder's inequality to (1.5) with exponents q, $s_2 = p_2, \ldots, s_n = p_n$. The Poincaré inequality (1.0) with $m = 0$ will give (1.8) where the s_j's are replaced by the p_j's. Therefore, the pointwise estimate

$$C_{n,p_j} \prod_{j=1}^n \Big[\sum_{k=1}^n \big[\big((|X_k f_j|^{p_j})^*\big)^{\frac{1}{p_j}}(u_0) + \big((|Y_k f_j|^{p_j})^*\big)^{\frac{1}{p_j}}(u_0)\big]\Big]. \tag{2.4}$$

holds for the supremum over all $\delta > 0$ of (1.3). We obtain the required weak type inequality by applying to (2.4) an argument similar to that in [G], page 77.

We end this section by noting that for some typle $F = (f_1, \ldots, f_n)$ of compactly supported smooth functions, $\operatorname{Jac}(F)$ doesn't have vanishing first order moments and hence it can't lie in H^γ for $\gamma \leq \frac{2n+2}{2n+3}$. Therefore, Theorem 1a is sharp since the weak space $H^{\gamma,\infty}$ cannot be replaced by the (strong) space H^γ.

3. Weak convergence in $H^1(\mathbb{H}^n)$

In this section we wish to extend the theorem of Jones and Journé, [JJ], on weak* convergence in $H^1(\mathbb{R}^n)$. The proof we are going to give follows the ideas of the original proof by [JJ] and our only contribution is its extension to general *normal* spaces of homogeneous type. Following R. Macías and C. Segovia, [MS], we say that a space of homogeneous type (X, d, μ) is *normal* if there exist two positive constants c_1, c_2 satisfying

$$c_1 r \leq \mu(B(x,r)) \leq c_2 r \tag{3.1}$$

for every ball $B(x,r)$ with radius r and $\mu(\{x\}) < r < \mu(X)$. Since $\mathbb{R}^n$, or the Heisenberg group, with the usual structures are not normal spaces in the sense of Macías and Segovia, we need to modify their definition by introducing a power of β in (3.1). We call a space of homogeneous type (X, d, μ) β-normal if for some constants c_1, c_2 and β the following inequality holds

$$c_1 r^\beta \leq \mu(B(x,r)) \leq c_2 r^\beta$$

for every r with $\mu(\{x\}) < r < \mu(X)$. Fortunately, a result of [MS] asserts that a given quasimetric d may always be replaced by another quasimetric δ, which induces the same topology, such that the space (X, δ, μ) is normal. $\delta(x,y)$ is the "measure" distance which is defined as $\delta(x,y) = \inf\{\mu(B) : B \text{ is a d-ball containing } x \text{ and } y\}$ and the topologies induced by d and δ coincide. Then by raising the quasimetric δ to the power $1/\beta$, we obtain a third quasimetric $\delta_\beta = \delta^{1/\beta}$ such that (X, δ_β, μ) is β-normal and the topologies induced by δ_β and δ coincide. Thus for all $\beta > 0$, every space of homogeneous type (X, d, μ) has a topologically equivalent quasimetric δ_β under which it is β-normal. Furthermore by another result of [MS] every d-Lipschitz function of order α is equal *a.e.* to a δ_β-Lipschitz function of order $\beta\alpha$ ([MS] prove this for $\beta = 1$ but raising the metric to the power $1/\beta$ explains the index $\beta\alpha$).

We assume that (X, d, μ) is a β-normal space of homogeneous type. The spaces $H^1(X)$, $VMO(X)$ and $BMO(X)$ are defined in [CW].

Under the assumption of β-normality for X, the theorem of [JJ] on weak* convergence in $H^1(\mathbb{R}^n)$ carries over to $H^1(X)$.

Theorem 2. *Suppose $\{f_n\}$ is a sequence in $H^1(X)$ such that $\|f_n\|_{H^1(X)} \le 1$ for all n and such that $f_n \to f$ a.e. Then f is in $H^1(X)$ and for all $\varphi \in VMO(X)$*

$$\int_X f_n \varphi \, d\mu \to \int_X f \varphi \, d\mu.$$

Proof: We may suppose that $\|\varphi\|_{L^1}, \|\varphi\|_{L^\infty} \le 1$, support$(\varphi)$ is compact and that $\|\varphi\|_{\mathrm{Lip}\,1} \le 1$. VMO is defined as the closure in BMO of the continuous functions with compact support. To see that it is sufficient to suppose $\varphi \in \mathrm{Lip}\,1$, consider a compact set $K \subseteq X$ containing the support of φ and note that by the Stone Weierstrass Theorem $\mathrm{Lip}\,1 \cap C(K)$ is uniformly dense in $C(K)$.

Fix $\varepsilon > 0$. We need to find an n_0 such that for all $n \ge n_0$, $|\int (f_n - f)\varphi \, d\mu| \le \varepsilon$. By Fatou's lemma $\|f\|_{L^1} = \|\underline{\lim}_n |f_n|\|_{L^1} \le \underline{\lim}_n \|f_n\|_{L^1} \le \underline{\lim}_n \|f_n\|_{H^1} \le 1$. Thus $f \in L^1$ and we can therefore find a $\delta > 0$ such that $\int_A |f| \, d\mu \le \varepsilon$ whenever $\mu(A) \le C\delta e^{\varepsilon^{-1}}$. We select δ smaller than $\varepsilon^{\beta+1} e^{-\varepsilon^{-1}}$.

By Egorov's Theorem, there exists a set E such that $f_n \to f$ uniformly on $X - E$ and $\mu(E) < \delta$. Let $\tau = \max(0, 1 + \varepsilon \log(\chi_E)^*)$, where $(\chi_E)^*$ denotes the Hardy-Littlewood maximal function of the characteristic function of the set E. Note that $0 \le \tau \le 1$ and $\tau \equiv 1$ a.e. on E. We will show that $\varphi\tau \in BMO(X)$ and

$$\|\varphi\tau\|_{BMO} \le C\varepsilon. \tag{3.3}$$

Assuming (3.3) we complete the proof of Theorem 2. Clearly E is contained in the support of τ. However, the support of τ is not very much larger than E; in fact, the weak type (1,1) estimate for the maximal function gives that $\mu\,(\text{support}(\tau)) \le Ce^{\varepsilon^{-1}}\mu\,(E) \le Ce^{\varepsilon^{-1}}\delta$. Hence $\int_{\text{support}(\tau)} |f|\,d\mu \le \varepsilon$ by our choice of δ. We now select n_0 such that for all $n \ge n_0$ the uniform norm of $f_n - f$ on $X - E$ is smaller than ε. Then for $n \ge n_0$ we have

$$\begin{aligned}
|\int_X (f_n - f)\varphi\,d\mu| &\le |\int_X (f_n - f)\varphi(1-\tau)\,d\mu| + |\int_X f\varphi\tau\,d\mu| + |\int_X f_n\varphi\tau\,d\mu| \\
&\le \|f_n - f\|_{L^\infty(X-E)}\|\varphi\|_{L^1} + \int_{\text{support}\,(\tau)} |f|\,d\mu + \|f_n\|_{H^1}\|\varphi\tau\|_{BMO} \\
&\le \varepsilon + \varepsilon + C\varepsilon = (C+2)\varepsilon.
\end{aligned}$$

This proves that f_n converges weakly* to f. Let us now show (3.3). The estimate

$$\frac{1}{\mu(B)}\int_B |\varphi\tau|\;d\mu \le \frac{1}{\mu(B)}\|\varphi\|_{L^\infty}\mu\big(\text{support}\,(\tau)\big) \le \frac{1}{\mu(B)}Ce^{\varepsilon^{-1}}\delta \le \frac{C\varepsilon^{\beta+1}}{\mu(B)} \le C\varepsilon$$

holds when B is a ball with $\mu(B) \ge \varepsilon^\beta$. Assume therefore that $\mu(B) \le \varepsilon^\beta$. If φ_B is the average of φ over B, for $x \in B$ we have

$$\begin{aligned}
&|\varphi(x) - \varphi_B| \le \frac{1}{\mu(B)}\int_B |\varphi(x) - \varphi(y)|\,d\mu(y) \le \\
&\frac{C}{\mu(B)}\int_B d(x,y)\,d\mu(y) \le C\,\text{radius}\,(B) \le C\mu(B)^{\frac{1}{\beta}} \le C\varepsilon.
\end{aligned}$$

We used that φ is Lipschitz and that X is β-normal in the inequalities above. Now assume for a moment that

$$\|\tau\|_{BMO} \le C\varepsilon. \tag{3.4}$$

Then (3.3) follows easily, since for all balls B we have:

$$\frac{1}{\mu(B)}\int_B |\varphi\tau - \varphi_B\tau_B|\,d\mu \le \frac{|\tau_B|}{\mu(B)}\int_B |\varphi - \varphi_B|\,d\mu + \frac{|\varphi_B|}{\mu(B)}\int_B |\tau - \tau_B|\,d\mu \le C\varepsilon.$$

It suffices therefore to prove (3.4). Since $\max(0,g) = \frac{g+|g|}{2}$ and $\|\,|g|\,\|_{BMO} \le 2\|g\|_{BMO}$, (3.4) reduces to the estimate

$$\|\log(\chi_E)^*\|_{BMO} \le C. \tag{3.5}$$

For $0 < \delta < 1$, the function $\left((\chi_E)^*\right)^\delta$ is an A_1-weight with a bound B_δ that depends only on δ and not on E. For a proof of this see [S2], Ch. V par. 5.2. The proof presented there for Euclidean spaces can be easily adapted to spaces of homogeneous type. Therefore $\left((\chi_E)^*\right)^\delta$ is an A_2-weight with bound B_δ^2. By the comment in [S2] Ch. V par. 1.8, the logarithm of an A_2-weight is in BMO with norm bounded by the logarithm of the A_2 bound of the weight. The same argument applies in the setting of a space of homogeneous type. We obtain that $\log(\chi_E)^* = \frac{1}{\delta}\log\left((\chi_E)^*\right)^\delta$ is in BMO with norm $\leq \frac{2}{\delta}\log B_\delta$. This concludes the proof of (3.5) and hence of Theorem 2. A different proof of (3.5) can be found in [CR] and [JO].

4. Compensated Compactness properties of Jacobians

In this section, we give a corollary of Theorems 1 and 2. We say that a sequence of functions converges weakly* in H^1 if it converges in the weak* topology of $H^1 = (VMO)^*$. For a sequence of functions $F_k : \mathbb{H}^n \to \mathbb{R}^n$, we use the notation $F_k = (f_1^k, \dots, f_n^k)$. Let L_j be X_j or Y_j as before. We have the following

Corollary. *(Compensated compactness property of Jacobians on the Heisenberg group.) Let $B > 0$ be a constant. Suppose that for some p_j, $1 < p_j < \infty$ with harmonic mean 1, and for some sequence of functions $F_k : \mathbb{H}^n \to \mathbb{R}^n$ the bound below holds:*

$$(4.1) \qquad \sum_{j,l=1}^{n} \|L_j f_l^k\|_{L^{p_l}} \leq B \qquad \textit{for all} \qquad k = 1, 2, 3, \dots .$$

Then some subsequence $\mathrm{Jac}(F_{k'})$ *of* $\mathrm{Jac}(F_k)$ *converges weakly* in $H^1(\mathbb{H}^n)$. Furthermore, if for all $j, l \in \{1, \dots, n\}$ the sequences $L_j f_l^k$ converge to g_{jl} a.e. as $k \to \infty$, then* $\det(g_{jl})$ *is in $H^1(\mathbb{H}^n)$ and* $\mathrm{Jac}(F_{k'})$ *converges weakly* to* $\det(g_{jl})$ *as $k' \to \infty$.*

Proof: By (4.1) and Theorem 1, the sequence $\mathrm{Jac}(F_k)$ satisfies $\|\mathrm{Jac}(F_k)\|_{H^1} \leq CB$ for all k. $H^1(\mathbb{H}^n)$ is the dual of $VMO(\mathbb{H}^n)$. The Banach-Alaoglu Theorem guarantees the existence of a subsequence $\mathrm{Jac}(F_{k'})$ which converges weakly* in H^1. If $L_j f_l^k \to g_{jl}$ a.e.

as $k \to \infty$, then $\mathrm{Jac}(F_k) \to \det(g_{jl})$ a.e. and hence $\mathrm{Jac}(F_{k'}) \to \det(g_{jl})$ a.e. as $k' \to \infty$. By Theorem 2 we conclude that $\det(g_{jl})$ is in H^1 and that the weak* limit of $\mathrm{Jac}(F_{k'})$ is $\det(g_{jl})$. The proof of the Corollary is now complete.

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