

A SHARP ESTIMATE FOR THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

LOUKAS GRAFAKOS, STEPHEN MONTGOMERY-SMITH, AND OLEXEI MOTRUNICH

University of Missouri, Columbia and Princeton University

ABSTRACT. The best constant in the usual L^p norm inequality for the centered Hardy-Littlewood maximal function on \mathbb{R}^1 is obtained for the class of all “peak-shaped” functions. A function on the line is called “peak-shaped” if it is positive and convex except at one point. The techniques we use include variational methods.

0. Introduction.

Let

$$(0.1) \quad (Mf)(x) = \sup_{\delta > 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |f(t)| dt$$

be the centered Hardy-Littlewood maximal operator on the line. This paper grew out from our attempt to find the operator norm of M on $L^p(\mathbb{R}^1)$.

Since M is a positive operator, we may restrict our attention to positive functions. Let \mathcal{P} be the set of all positive functions f on \mathbb{R}^1 , which are convex except at one point (where we also allow f to be discontinuous). We call such functions “peak-shaped.”

We were able to find the best constant in the inequality

$$(0.2) \quad \|Mf\|_{L^p} \leq C(p) \|f\|_{L^p} \quad \text{for } f \text{ in } \mathcal{P} \cap L^p.$$

for $1 < p < \infty$. It turns out that the best such $C(p)$ is the unique number c_p which satisfies the equality

$$(0.3) \quad M(|x|^{-1/p}) = c_p |x|^{-1/p}.$$

Note that the function $|x|^{-1/p}$ is locally integrable, so the left-hand side of (0.3) is well-defined. Strictly speaking, the function $|x|^{-1/p}$ doesn't belong to the space $\mathcal{P} \cap L^p$. It is, however, the pointwise limit of a sequence of functions in $\mathcal{P} \cap L^p$ and the norm ratio of this sequence converges to c_p .

Here is our main result.

Research of L. Grafakos was partially supported by the University of Missouri Research Board.

Research of S. Montgomery-Smith was partially supported by the National Science Foundation.

Research of O. Motrunich was partially supported by the University of Missouri-Columbia Research Council.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

Theorem. *The smallest possible constant $C(p)$ in the inequality*

$$\|Mf\|_{L^p} \leq C(p)\|f\|_{L^p} \quad \text{for } f \text{ in } \mathcal{P} \cap L^p,$$

is

$$(0.4) \quad c_p = \sup_{\tau > 1} \frac{(\tau + 1)^{\frac{p-1}{p}} + (\tau - 1)^{\frac{p-1}{p}}}{2\tau^{\frac{p-1}{p}}},$$

that is, the constant in (0.3).

One may ask the corresponding question when $p = 1$. It was communicated to us by José Barrionuevo [Ba] that the best constant C_1 in the weak type inequality

$$|\{x : (Mf)(x) > \lambda\}| \leq C_1 \frac{\|f\|_{L^1}}{\lambda}$$

for f in $\mathcal{P} \cap L^1$ is $C_1 = 1$. This result is sharp and is analogous to our result when $p = 1$. (In fact, this result is valid for the wider class of positive functions that are increasing on $(-\infty, c)$, and decreasing on (c, ∞) for some number c .)

It is still a mystery what happens for general functions f . It is conjectured in [DGS] that c_p is the operator norm of the Hardy-Littlewood maximal function on $L^p(\mathbb{R}^1)$. Our methods will not work for arbitrary functions and we will point out during the proof where they break down. For general functions $f \in L^1$, the conjecture used to be that $C_1 = 3/2$. However, it has recently been shown by Aldaz [Al] that C_1 lies between $3/2$ and 2 . This result tends to suggest that the value c_p given by our Theorem is not the best constant for general $f \in L^p$.

The authors would like to thank the anonymous referee for many valuable comments, and for pointing out errors in the original version of the manuscript.

1. Some preliminary Lemmas.

Throughout this paper we fix a p with $1 < p < \infty$, and a positive function f in $\mathcal{P} \cap L^p$. Since M commutes with translations, we may assume that f is convex except at 0. By a density argument, we may also make the following assumptions:

- (1) f is smooth everywhere except at 0;
- (2) f has compact support;
- (3) $f(x) = |x|^{-1/2p}$ for $|x|$ sufficiently small.

We remark that the third condition may seem unnatural, but it simplifies some of the technicalities of the proof. Equivalently, this condition may be replaced by a general assumption that f is “spiky” enough near the origin but $f(0) < \infty$.

For every real $x \neq 0$, define the function $\xi_x(t) = \frac{1}{2t} \int_{x-t}^{x+t} f(u) du$ if $t > 0$, and $\xi_x(0) = f(x)$. It can be seen that $\xi_x(t)$ is a C^∞ function of $t > 0$ (except at $t = |x|$, where it is merely continuous) and that it tends to zero as $t \rightarrow \infty$.

Furthermore, we see that

$$\xi'_x(t) = \frac{\frac{1}{2}(f(x+t) + f(x-t)) - \xi_x(t)}{t}.$$

Convexity shows us that $\xi'_x(t) \geq 0$ for $t \in (0, |x|)$, and the third condition on f shows us that $\xi'_x(t) > 0$ for t close to $|x|$. Thus we see that $\xi_x(t)$ is non-decreasing for t in some open neighborhood of $(0, |x|]$.

The global maximum of ξ_x over $[0, \infty)$ is equal to $(Mf)(x)$. This maximum is attained on some set of real numbers $B_x = \{t : \xi_x(t) = \sup_{u \geq 0} \xi_x(u)\}$. Set $\delta(x) = \max B_x$. Since B_x is a closed set, it contains $\delta(x)$. Note that $\delta(x) > |x|$ for $x \neq 0$. Thus

$$(1.1) \quad (Mf)(x) = \frac{1}{2\delta(x)} \int_{x-\delta(x)}^{x+\delta(x)} f(t) dt.$$

Since $\delta(x)$ is a critical point of ξ_x , it follows that $\xi'_x(\delta(x)) = 0$. A simple calculation and (1.1) give formula (1.2) below.

Now fix $x_0 \neq 0$. By the Implicit Function Theorem, the equation $\xi'_x(\delta) = 0$ can be solved for δ as a smooth function of x in the vicinity of any point $(x_0, \delta(x_0))$, as long as $\frac{\partial \xi'_x(\delta)}{\partial \delta} \neq 0$ at $(x_0, \delta(x_0))$. This condition is equivalent to $f'(x_0 + \delta(x_0)) \neq f'(x_0 - \delta(x_0))$, which follows from the fact that $x_0 + \delta(x_0)$ and $x_0 - \delta(x_0)$ lie on opposite sides of the origin and that f has different kind of monotonicity on each side. Therefore δ coincides with a smooth function in the neighborhood of every point $x_0 \neq 0$, which implies that $\delta(x)$ is a smooth function of $x \neq 0$. As a consequence $(Mf)(x)$ is also smooth for $x \neq 0$.

We notice that for sufficiently small $|x|$ that $\delta(x) = (1 + \tau_{2p})|x|$ for a fixed value τ_{2p} , and that $(Mf)(x) = c_{2p}|x|^{-1/2p}$. Thus $\delta(x)$ is a continuous function of x .

Lemma 1. *For $x \neq 0$, we have*

$$(1.2) \quad (Mf)(x) = \frac{f(x + \delta(x)) + f(x - \delta(x))}{2},$$

and

$$(1.3) \quad (Mf)'(x) = \frac{f(x + \delta(x)) - f(x - \delta(x))}{2\delta(x)}.$$

PROOF. (1.2) is proved as indicated above. To prove (1.3), differentiate the identity (1.1) and use (1.2). This completes the proof of Lemma 1. *QED.*

Formula (1.3) indicates that the points $x + \delta(x)$ and $x - \delta(x)$ are the x -coordinates of some two points of intersection of the graph of f with the tangent line to the graph of Mf at $(x, f(x))$.

Lemma 2. For $x \neq 0$ we have that $|\delta'(x)| > 1$. Moreover Mf is in \mathcal{P} with its maximum at 0.

PROOF. We begin by showing that Mf has no inflection points away from 0. Differentiating (1.2) and (1.3) we obtain that for $x \neq 0$, we have

$$(1.4) \quad (Mf)'(x) = f'(x + \delta(x)) \frac{(1 + \delta'(x))}{2} + f'(x - \delta(x)) \frac{(1 - \delta'(x))}{2}$$

(1.5)

$$(Mf)'(x)\delta'(x) + \delta(x)(Mf)''(x) = f'(x + \delta(x)) \frac{(1 + \delta'(x))}{2} - f'(x - \delta(x)) \frac{(1 - \delta'(x))}{2}.$$

If $q \neq 0$ were an inflection point, then $(Mf)''(q) = 0$, and by (1.4) and (1.5) it would follow that

$$\begin{aligned} f'(q + \delta(q))(1 + \delta'(q)) &= (1 + \delta'(q))(Mf)'(q) \\ f'(q - \delta(q))(1 - \delta'(q)) &= (1 - \delta'(q))(Mf)'(q). \end{aligned}$$

Then $(Mf)'(q)$ would be equal to either $f'(q + \delta(q))$ or $f'(q - \delta(q))$. By Lemma 1, $(Mf)'(q)$ is the slope of the line segment that joins $(q - \delta(q), f(q - \delta(q)))$ to $(q + \delta(q), f(q + \delta(q)))$. By the convexity conditions on f , this line would then necessarily lie on or below the graph of f . By (1.2), this would imply that $(Mf)(q) \leq f(q)$, a contradiction. Therefore Mf has no inflection points away from 0, hence it is either concave or convex there. Since $(Mf)(x)$ looks like $\frac{1}{x}$ near $\pm\infty$, it follows that Mf is convex on $(-\infty, 0)$ and on $(0, +\infty)$.

We now show that if $x \neq 0$, then $|\delta'(x)| > 1$. Let $x_1 < x_2 < 0$ and let L_i be the tangent line to the graph of Mf at x_i , $i = 1, 2$. L_i passes through the point $(x_i + \delta(x_i), f(x_i + \delta(x_i)))$. Since Mf is convex on $(-\infty, 0)$, the line L_1 lies lower than the line L_2 to the right of x_2 . Since f is decreasing on $(0, \infty)$, it follows that L_2 intersects the graph of f on $(0, \infty)$ at a point with x -coordinate bigger than the x -coordinate of the intersection of L_1 with the graph of f . This implies that $x_1 + \delta(x_1) > x_2 + \delta(x_2)$ which proves that $x + \delta(x)$ is decreasing on $(-\infty, 0)$. Therefore $\delta'(x) < -1$ on $(-\infty, 0)$. Likewise one can show that $\delta'(x) > 1$ on $(0, +\infty)$. This completes the proof of Lemma 2. *QED.*

2. The variational functional.

It will be convenient to have $\delta(x) \leq 0$ for $x < 0$. To achieve this we define $s(x)$ to be equal to $\delta(x)$ for $x > 0$ and $s(x)$ to be $-\delta(x)$ for $x < 0$. We observe that (1.1), (1.2), and (1.3) remain valid for $s(x)$. We also observe that $s'(x) > 1$ for $x \neq 0$ and that

- (1) $\lim_{x \rightarrow +\infty} x + s(x) = +\infty$,
- (2) $\lim_{x \rightarrow -\infty} x + s(x) = -\infty$,
- (3) $\lim_{x \rightarrow +\infty} x - s(x) = \inf(\text{support}(f))$,
- (4) $\lim_{x \rightarrow -\infty} x - s(x) = \sup(\text{support}(f))$.

For simplicity, we denote by $g = Mf$ the maximal function of f . It turns out that a suitable convex combination of the integrals of the functions $(g(x) - g'(x)s(x))^p(s'(x) - 1)$ and $(g(x) + g'(x)s(x))^p(s'(x) + 1)$ will give rise to a functional related to $\|f\|_{L^p}^p$. Our goal will be to minimize this functional by selecting a suitable $s(x)$. To find such a minimizer, we solve the corresponding Euler-Lagrange equations.

By Lemma 1 we have that

$$(2.1) \quad f(x + s(x)) = g(x) + g'(x)s(x),$$

and

$$(2.2) \quad f(x - s(x)) = g(x) - g'(x)s(x).$$

Raise both sides of (2.1) to the power p , multiply them by $1 + s'(x)$, and integrate from $-\infty$ to ∞ to obtain

$$(2.3) \quad \begin{aligned} & \int_{-\infty}^{+\infty} (g(x) + g'(x)s(x))^p(s'(x) + 1) dx \\ &= \int_{-\infty}^{+\infty} f(x + s(x))^p(s'(x) + 1) dx \\ &= \int_{-\infty}^{+\infty} f(x)^p dx = \|f\|_{L^p}^p \end{aligned}$$

Similarly, raise both sides of (2.2) to the power p , multiply them by $s'(x) - 1$, and integrate from $-\infty$ to ∞ . We obtain

$$(2.4) \quad \begin{aligned} & \int_{-\infty}^{+\infty} (g(x) - g'(x)s(x))^p(s'(x) - 1) dx \\ &= \int_{-\infty}^{+\infty} f(x - s(x))^p(s'(x) - 1) dx \\ &= \int_{-\infty}^{+\infty} f(-x)^p dx = \|f\|_{L^p}^p \end{aligned}$$

At this point, we remark that the calculation above will not work for general functions f , because in that case the function $s(x)$ will have many discontinuities, and above formulae will have to include terms needed to account for these discontinuities. These discontinuities are generally rather unpredictable, and we have not been able to find a way to deal with this problem.

Let $\frac{1}{2} < \alpha < 1$ be a real number to be selected later to depend on p only. Let F be the following function of three variables:

$$(2.5) \quad F(x, y, z) = \alpha(g(x) + g'(x)y)^p(z + 1) + (1 - \alpha)(g(x) - g'(x)y)^p(z - 1).$$

The domain of F is the set of all (x, y, z) which satisfy

- (1) $-\infty < x < \infty$
- (2) $-\frac{g(x)}{|g'(x)|} < y < \frac{g(x)}{|g'(x)|}$
- (3) $-\infty < z < \infty$

Clearly $(x, s(x), s'(x))$ lies in the domain of definition of F , and combining (2.3) and (2.4) we obtain

$$(2.6) \quad \|f\|_{L^p}^p = \int_{-\infty}^{+\infty} F(x, s(x), s'(x)) dx.$$

Now we define a functional $I(\phi)$ that we would like to minimize. The domain of I will be those functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ that are smooth except possibly at 0, such that $F(x, \phi(x), \phi'(x))$ is integrable on compact subsets of $\mathbb{R} \setminus \{0\}$, and such that the following improper integral converges:

$$\begin{aligned} I(\phi) &= \int_{-\infty}^{+\infty} F(x, \phi(x), \phi'(x)) dx \\ &= \lim_{\substack{a \rightarrow -\infty \\ b \rightarrow 0^- \\ c \rightarrow 0^+ \\ d \rightarrow +\infty}} \left(\int_a^b + \int_c^d \right) F(x, \phi(x), \phi'(x)) dx. \end{aligned}$$

Denote by $\partial_1 F$, $\partial_2 F$, and $\partial_3 F$ the partial derivatives of $F(x, y, z)$ with respect to x , y , and z respectively. To minimize I , we consider the associated Euler-Langrange equations:

$$(2.7) \quad \frac{d}{dx} [(\partial_3 F)(x, \phi(x), \phi'(x))] = (\partial_2 F)(x, \phi(x), \phi'(x)).$$

We now have the following result.

Lemma 3. *The function*

$$(2.8) \quad s_0(x) = -\beta \frac{g(x)}{g'(x)}, \quad \text{where} \quad \beta = \beta(\alpha) = \frac{\alpha^{\frac{1}{p-1}} - (1-\alpha)^{\frac{1}{p-1}}}{\alpha^{\frac{1}{p-1}} + (1-\alpha)^{\frac{1}{p-1}}},$$

is an exact solution of equation (2.7) on $\mathbb{R}^1 \setminus \{0\}$.

REMARK: Note that $0 < \beta(\alpha) < 1$, since $\alpha > \frac{1}{2}$.

PROOF. To prove Lemma 3, rewrite (2.7) as

$$[\alpha(g(x) + g'(x)\phi(x))^{p-1} - (1 - \alpha)p(g(x) - g'(x)\phi(x))^{p-1}]g''(x)\phi(x) = 0.$$

Then substituting for $\phi = s_0$, we obtain the result. *QED*.

We would like to be able to directly deduce that $I(s) \geq I(s_0)$. Unfortunately, general theorems from calculus of variations (see for example [Br]) are not directly applicable here since $F(x, y, z)$ does not satisfy the usual convexity conditions needed. As it turns out, the desired inequality $I(s) \geq I(s_0)$ will be a consequence of the key inequality below which is true because of the very specific structure of the function $F(x, y, z)$.

Lemma 4. *For all $x \neq 0$ we have*

$$(2.9) \quad F(x, s(x), s'(x)) - F(x, s_0(x), s'_0(x)) \geq (\partial_2 F)(x, s_0(x), s'_0(x))(s(x) - s_0(x)) + (\partial_3 F)(x, s_0(x), s'_0(x))(s'(x) - s'_0(x))$$

PROOF. We observe that if h is a convex function on an interval J , then for all x, y in J we have

$$(2.10) \quad h(x) - h(y) \geq h'(y)(x - y)$$

irrespectively of the order of x and y . Next observe that for all x and all $z > 1$ the function $F(x, y, z)$ is convex in y . This is because when $z > 1$, $(\partial_2^2 F)(x, y, z) > 0$ for all x, y .

Let $x \neq 0$. Then by Lemma 2, $s'(x) = |\delta'(x)| > 1$ and

$$(2.11) \quad F(x, s(x), s'(x)) - F(x, s_0(x), s'_0(x)) = [F(x, s(x), s'(x)) - F(x, s_0(x), s'(x))] + [F(x, s_0(x), s'(x)) - F(x, s_0(x), s'_0(x))] \geq (\partial_2 F)(x, s_0(x), s'(x))(s(x) - s_0(x)) + (\partial_3 F)(x, s_0(x), s'_0(x))(s'(x) - s'_0(x)),$$

by convexity of F in the second variable, (2.10), and linearity of F in the third variable.

Calculation gives

$$(2.12) \quad (\partial_3 \partial_2 F)(x, y, z) = pg'(x)[\alpha(g(x) + g'(x)y)^{p-1} - (1 - \alpha)(g(x) - g'(x)y)^{p-1}].$$

Setting $y = s_0(x)$ in (2.12) we obtain that

$$(\partial_3 \partial_2 F)(x, s_0(x), z) = pg'(x)[\alpha(1 - \beta)^{p-1} - (1 - \alpha)(1 + \beta)^{p-1}] = 0,$$

since by the definition of β , it follows that

$$\alpha = \frac{(1 + \beta)^{p-1}}{(1 + \beta)^{p-1} + (1 - \beta)^{p-1}}.$$

We have now proved that the function $(\partial_2 F)(x, s_0(x), z)$ is constant in z . The proof of (2.9) is now complete if we replace $s'(x)$ by $s'_0(x)$ in the first summand of (2.11). *QED*.

3. The core of the proof.

Next we have the following.

Lemma 5. *Both s and s_0 lie in the domain of the functional I . We have the equality*

$$(3.1) \quad I(s_0) = r(\alpha) \|g\|_{L^p}^p,$$

where $r(\alpha) = \gamma_1(\alpha) + p\beta(\alpha)\gamma_2(\alpha)$ with $\gamma_1(\alpha) = \alpha(1 - \beta)^p - (1 - \alpha)(1 + \beta)^p$ and $\gamma_2(\alpha) = \alpha(1 - \beta(\alpha))^p + (1 - \alpha)(1 + \beta(\alpha))^p$. We also have the inequality

$$(3.2) \quad I(s) = \|f\|_{L^p}^p \geq I(s_0).$$

PROOF. First, it is clear that s is in the domain of I , by the calculations at the beginning of the previous section. Let us work with s_0 . We see that if $0 < a < b < \infty$, then integrating by parts we get

$$\begin{aligned} \int_a^b F(x, s_0(x), s'_0(x)) dx &= \int_a^b \gamma_1 g(x)^p - \gamma_2 g(x)^p \frac{d}{dx} \left(\beta \frac{g(x)}{g'(x)} \right) dx \\ &= r(\alpha) \int_a^b g(x)^p dx - \frac{g(b)^{p+1}}{g'(b)} + \frac{g(a)^{p+1}}{g'(a)}. \end{aligned}$$

As $a \rightarrow 0$, we have the explicit formula for $g(a) = c_{2p}|a|^{-1/2p}$ which tells us that $\frac{g(a)^{p+1}}{g'(a)} \rightarrow 0$. When b is very large, we note that $f(b + \delta(b)) = 0$ (because f has compact support), and hence by (1.2) and (1.3) we have that $\left| \frac{g(b)}{g'(b)} \right| = \delta(b)$. Furthermore, $|b + \inf(\text{support}(f))| \leq \delta(b) \leq |b + \sup(\text{support}(f))|$, and $g(b) = O\left(\frac{1}{b}\right)$. Thus $\frac{g(b)^{p+1}}{g'(b)} \rightarrow 0$ as $b \rightarrow +\infty$. We obtain a similar result if $-\infty < a < b < 0$, and thus we obtain that s_0 is in the domain of I , and that $I(s_0) = r(\alpha) \|g\|_{L^p}^p$.

Now let us consider $I(s)$, which we already know is equal to $\|f\|_{L^p}^p$. It is here that estimate (2.9) plays its crucial role. If $0 < a < b < \infty$, then

$$\begin{aligned} &\int_a^b F(x, s(x), s'(x)) dx \\ &= \int_a^b [F(x, s(x), s'(x)) - F(x, s_0(x), s'_0(x))] dx + \int_a^b F(x, s_0(x), s'_0(x)) dx \\ &\geq \int_a^b [(\partial_2 F)(x, s_0(x), s'_0(x))(s(x) - s_0(x)) + (\partial_3 F)(x, s_0(x), s'_0(x))(s'(x) - s'_0(x))] dx \\ (3.3) \quad &+ \int_a^b F(x, s_0(x), s'_0(x)) dx, \end{aligned}$$

where we used Lemma 4 in the inequality above. Next, we integrate by parts, and (3.3) is now equal to

$$\begin{aligned} & \int_a^b \left[(\partial_2 F)(x, s_0(x), s'_0(x)) - \frac{d}{dx} ((\partial_3 F)(x, s_0(x), s'_0(x))) \right] (s(x) - s_0(x)) dx \\ & + (\partial_3 F)(b, s_0(b), s'_0(b))(s(b) - s_0(b)) - (\partial_3 F)(a, s_0(a), s'_0(a))(s(a) - s_0(a)) \\ & + \int_a^b F(x, s_0(x), s'_0(x)) dx. \end{aligned}$$

First note that the first integral evaluates to 0 by Lemma 3. Now, as $a \rightarrow 0$, we may explicitly calculate to see that $(\partial_3 F)(a, s_0(a), s'_0(a))(s(a) - s_0(a)) \rightarrow 0$. Also, if b is very large, using the fact that $f(b + \delta(b)) = 0$, (1.2), (1.3), and (2.8), we see that $s_0(b) = \beta s(b)$, and hence $(\partial_3 F)(b, s_0(b), s'_0(b))(s(b) - s_0(b))$ is a constant multiple of $g(b)^p s(b)$. Arguing as above, we see that $(\partial_3 F)(b, s_0(b), s'_0(b))(s(b) - s_0(b))$ tends to 0 as $b \rightarrow \infty$. We obtain a similar result when $-\infty < a < b < 0$, hence we conclude that $I(s) \geq I(s_0)$. *QED.*

To finish the proof, we only need the following result.

Lemma 6. *There exists $\frac{1}{2} < \alpha < 1$ such that*

$$r(\alpha) = c_p^{-p},$$

where c_p is the constant in (0.4).

REMARK: In fact it is true that c_p^{-p} is the absolute minimum of $r(\alpha)$ for $\frac{1}{2} < \alpha < 1$.

PROOF. We have that

$$r(\alpha) = \alpha(1 - \beta)^p - (1 - \alpha)(1 + \beta)^p + p\beta\gamma = \frac{2^p(p-1)\alpha(1-\alpha)(\alpha^{\frac{1}{p-1}} - (1-\alpha)^{\frac{1}{p-1}})}{(\alpha^{\frac{1}{p-1}} + (1-\alpha)^{\frac{1}{p-1}})^p}.$$

For $t \in (1, +\infty)$, define

$$h(t) = \frac{(t+1)^{\frac{p-1}{p}} + (t-1)^{\frac{p-1}{p}}}{2^{\frac{p-1}{p}} t}.$$

We see that $h'(t)$ is monotonically decreasing on $(1, +\infty)$ and attains its only zero at the unique τ satisfying

$$(3.4) \quad \left(\frac{p+\tau}{p-\tau} \right)^p = \frac{\tau+1}{\tau-1}.$$

Let

$$(3.5) \quad \alpha_0 = \frac{(p + \tau)^{p-1}}{(p + \tau)^{p-1} + (p - \tau)^{p-1}}$$

where τ satisfies (3.4). It is clear that $\frac{1}{2} < \alpha_0 < 1$. We have

$$(3.6) \quad \begin{aligned} r(\alpha_0) &= \frac{2(p-1)\tau(p+\tau)^{p-1}(p-\tau)^{p-1}}{p^p[(p+\tau)^{p-1} + (p-\tau)^{p-1}]} \\ &= \frac{2^p(p-1)\tau}{p\left(\frac{p-\tau}{p+\tau} + 1\right)^{p-1}\left(\left(\frac{p+\tau}{p-\tau}\right)^{p-1} + 1\right)} \\ &= \frac{2^p(p-1)\tau}{p\left[(\tau-1)^{-\frac{1}{p}} + (\tau+1)^{-\frac{1}{p}}\right]^{p-1}\left[(\tau+1)^{\frac{p-1}{p}} + (\tau-1)^{\frac{p-1}{p}}\right]} \\ &= \left(\frac{2\tau\frac{p-1}{p}}{(\tau-1)^{\frac{p-1}{p}} + (\tau+1)^{\frac{p-1}{p}}}\right)^p = h(\tau)^{-p} = c_p^{-p}, \end{aligned}$$

where c_p is the constant in (0.4). *QED*.

REFERENCES

- [Al] J.M. Aldaz, *Remarks on the Hardy-Littlewood maximal function*, Proc. Roy. Soc. Endib. **128** (1998), 1–9.
- [Ba] José Barrionuevo, *personal communication*.
- [Br] U. Brechtken-Manderscheid, *Introduction to the Calculus of Variations*, Chapman & Hall, 1991, London, UK.
- [CG] M. Christ and L. Grafakos, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123** (1995), 1687–1693.
- [DGS] R. Dror, S. Ganguli, and R. Strichartz, *A search for best constants in the Hardy-Littlewood maximal theorem*, J. Fourier Anal. Appl. **2** (1996), 473–486.
- [GM] L. Grafakos and S. Montgomery-Smith, *Best constants for uncentered maximal functions*, Bull. London Math. Soc. **29** (1997), 60–64.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211,
 DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211,
 DEPARTMENT OF PHYSICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544

E-mail address:

loukas@math.missouri.edu, stephen@math.missouri.edu, motrunich@princeton.edu