A MULTILINEAR SCHUR TEST AND MULTIPLIER OPERATORS

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Abstract. A multilinear version of Schur's test is obtained for products of L^p spaces and is used to derive boundedness for multilinear multiplier operators acting on Sobolev and Besov spaces.

1. Introduction

The classical Schur test provides a criterion for boundedness of positive operators. We extend this result to the multilinear setting. As an application we prove boundedness for certain multilinear multiplier operators acting on products of Besov spaces. These operators are not positive, but appropriate discretization techniques reduce their study to positive tensors acting on spaces of sequences. In that setting Schur's test can be applied. This application extends a result of Coifman and Meyer for multilinear multipliers [3], [15], to diagonal Besov spaces (and in particular Sobolev spaces). Related results have been recently obtained in [2], [20], and [9] using different techniques.

The arguments related to the multilinear Schur test are elementary, yet powerful, since they provide nontrivial necessary and sufficient conditions for positive multilinear operators to be bounded on products of L^p spaces. These results are discussed in Section 2. In Section 3 we set up the background for the aforementioned application. The details of the proof are given in Section 4.

The authors started this collaboration while they were at the Mathematical Sciences Research Institute, Berkeley, during the program in Harmonic Analysis, Fall 1997. They want to thank the MSRI and the organizers of the program for providing a very inspiring research atmosphere.

2. A multilinear Schur test

Let us recall a known version of Schur's test. Let X and Y be measure spaces equipped with nonnegative, σ-finite measures and let T be a linear operator taking measurable functions on Y to measurable functions on X. We assume that T is an

Date: February 9, 2000.

1991 *Mathematics Subject Classification.* Primary 47G99, 42B99. Secondary 42B25, 47B65, 42B15, 42B20.

Key words and phrases. Schur's test, multilinear operators, pseudodifferential operators, singular integrals, Littlewood-Paley theory, wavelets, Sobolev space, Besov spaces.

Grafakos' research partially supported by the NSF under grant DMS 9623120.

Torres' research partially supported by the NSF under grant DMS 9696267.

integral operator which can be written in the form

$$Tf(x) = \int K(x,y)f(y)\,dy,$$

for some nonnegative kernel $K(x,y) \geq 0$. We denote by T^* be the formal transpose operator

$$T^*f(y) = \int K(x,y)f(x)\,dx.$$

Also, for $1 < p < \infty$, we denote by $p' = p/(p-1)$ the dual exponent.

Theorem. (Schur's test). *Let $A > 0$. The following are equivalent.*
(a) T maps $L^p(Y)$ to $L^p(X)$ with norm less than or equal to A.
(b) For all $B > A$ there exists a measurable function h on Y, $0 < h < \infty$ a.e., such that

$$T^*((Th)^{p-1}) \leq B^p\, h^{p-1} \qquad a.e.$$

(c) For all $B > A$ there exist measurable functions u on Y and w on X, $0 < u,\ w < \infty$ a.e., such that

$$T(u^{p'}) \leq B\, w^{p'} \qquad a.e.$$
$$T^*(w^p) \leq B\, u^p \qquad a.e.$$

Before we discuss the multilinear case we mention some related history. The test is named after I. Schur who gave a sufficient condition for a square matrix to map $l^2(\mathbf{Z})$ to $l^2(\mathbf{Z})$, see [17]. This result was extended by Hardy, Littlewood, and Polya [10] on l^p, for $1 < p < \infty$; see also [11]. In 1959 Karlin [13] proved that (a) implies (c) above when $p = 2$. In 1963 Aronszajn, Mulla, and Szeptycki [1] proved that (c) implies (a) for all $1 < p < \infty$, and Gagliardo [8] established the equivalence between (a) and (c) for all $1 < p < \infty$ (his paper was published two years later). In 1990 Howard and Schep [12] introduced the equivalent condition (b) involving only one function.

In the multilinear setting, a version of Schur's test for weighted L^p spaces was proved by Cwikel and Kerman [4]. They showed that a positive n-linear operator maps a product of weighted L^p spaces into a weighted L^r space if and only if a set of $3n + 5$ conditions involving $(n + 1)(n + 2)$ functions hold. In this article we give a new set of $n + 1$ conditions involving only $n + 1$ functions to characterize boundedness of such operators. We work with unweighted Lebesgue spaces since we can always incorporate the weights with the measures, when the resulting measures are also σ-finite. We also obtain a version of Schur's test in the off-diagonal case $1/r > \sum_{j=1}^{n} 1/p_j$. Here we need a set of $n+2$ conditions involving $n+1$ functions to characterize boundedness. These versions of Schur's test are better suited for certain applications as indicated in Example 1 and Theorem 3.

We now set up the background for the multilinear version of Schur's test. Let $X_1, \ldots, X_n$ be measure spaces equipped with nonnegative, σ-finite measures μ_j, $j =$

$1, \dots, n$. Also let X be another measure space with nonnegative measure μ. Let

$$K(x, x_1, \dots, x_n) \geq 0 \tag{1}$$

be a nonnegative measurable function on the product space $X \times X_1 \times \cdots \times X_n$. Consider the n-linear operator T with kernel K, that is

$$T(f_1, \dots, f_n)(x) = \int_{X_1} \cdots \int_{X_n} K(x, x_1, \dots, x_n) f_1(x_1) \dots f_n(x_n)\, d\mu_1(x_1) \dots d\mu_n(x_n),$$

defined for suitable measurable functions f_j on X_j. $T(f_1, \dots, f_n)$ is then a measurable function on X. Since T is n-linear it has n transposes. The j^{th} transpose T^{*j} of T is the transpose of the linear operator

$$g \to T(f_1, \dots, f_{j-1}, g, f_{j+1}, \dots, f_n)$$

with the functions f_k fixed for $k \neq j$. It is easy to check that the kernel K_j of the operator T^{*j} is

$$K_j(x, x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n) = K(x_j, x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n).$$

Fix indices $1 < p_1, \dots, p_n, r < \infty$ satisfying

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}. \tag{2}$$

We are interested in finding a necessary and sufficient condition for T to map the product of Lebesgue spaces $L^{p_1}(X_1) \times \cdots \times L^{p_n}(X_n)$ into $L^r(X)$. We have the following.

Theorem 1. *Let $A > 0$. The following are equivalent.*
(a) T maps $L^{p_1}(X_1) \times \cdots \times L^{p_n}(X_n)$ to $L^r(X)$ with norm less than or equal to A.
(b) For all $B > A$ there exist measurable functions h_j on X_j with $0 < h_1, \dots, h_n < \infty$ a.e., such that

$$T^{*j}(h_1, \dots, h_{j-1}, T(h_1, \dots, h_n)^{r-1}, h_{j+1}, \dots, h_n) \leq B^r h_j^{p_j - 1} \qquad \text{a.e.} \tag{3}$$

for all $1 \leq j \leq n$.
(c) For all $B > A$ there exist measurable functions u_j on X_j and w on X with $0 < u_1, \dots, u_N, w < \infty$ a.e., such that

$$\begin{aligned} T(u_1^{p_1'}, u_2^{p_2'}, \dots, u_n^{p_n'}) &\leq B\, w^{r'} && \text{a.e.} \\ T^{*1}(w^r, u_2^{p_2'}, \dots, u_n^{p_n'}) &\leq B\, u_1^{p_1} && \text{a.e.} \\ &\cdots \\ T^{*n}(u_1^{p_1'}, u_2^{p_2'}, \dots, w^r) &\leq B\, u_n^{p_n} && \text{a.e.} \end{aligned} \tag{4}$$

We now prove this result. For notational simplicity we only give the proof in the case $n = 2$. The general case presents no differences, only notational inconveniences. Set $p_1 = p$, $p_2 = q$, $f_1 = f$, $f_2 = g$, $u_1 = u$, and $u_2 = v$ below.

Proof. Let us start by proving that (b) implies (c). We are given h_1 and h_2 satisfying (3). Define u, v, and w via $u^{p'} = h_1$, $v^{q'} = h_2$ and $Bw^{r'} = T(h_1, h_2)$. Then (4) is clearly satisfied for this choice of u, v, and w.

We now prove that (c) implies (a). We will estimate the L^r norm of $T(f, g)$ using duality. Let $f \in L^p(X_1)$, $g \in L^q(X_2)$, and $h \in L^{r'}(X)$ which we can assume to be nonnegative. Then,

$$\int_X T(f,g)h\,d\mu = \int_{X_1}\int_{X_2}\int_X K(x,x_1,x_2)h(x)f(x_1)g(x_2)\,d\mu(x)d\mu_2(x_2)d\mu_1(x_1). \tag{5}$$

Write the integrand above as $L(x, x_1, x_2)M(x, x_1, x_2)N(x, x_1, x_2)$ where

$$\begin{aligned} L(x,x_1,x_2) &= h(x)\frac{u(x_1)^{p'/r'}v(x_2)^{q'/r'}}{w(x)}K(x,x_1,x_2)^{1/r'},\\ M(x,x_1,x_2) &= f(x_1)\frac{w(x)^{r/p}v(x_2)^{q'/p}}{u(x_1)}K(x,x_1,x_2)^{1/p}, \quad \text{and}\\ N(x,x_1,x_2) &= g(x_2)\frac{u(x_1)^{p'/q}w(x)^{r/q}}{v(x_2)}K(x,x_1,x_2)^{1/q}. \end{aligned}$$

Here we used the facts that $1/p+1/q = 1/r$, $1/r'+1/q = 1/p'$ and $1/p+1/r' = 1/q'$. We now apply Hölder's inequality with exponents r', p, q to the functions L, M, N with respect to the measure $d\mu_1(x_1)d\mu_2(x_2)d\mu(x)$ in $X_1 \times X_2 \times X$ to control (5) by the product

$$\left(\int_X \frac{h^{r'}}{w^{r'}}T(u^{p'},v^{q'})d\mu\right)^{1/r'}\left(\int_{X_1}\frac{f^p}{u^p}T^{*1}(w^r,v^{q'})d\mu_1\right)^{1/p}\left(\int_{X_2}\frac{g^q}{v^q}T^{*2}(u^{p'},w^r)d\mu_2\right)^{1/q}.$$

Fubini's theorem above is justified by the σ-finiteness of the spaces. Now using (4) we conclude that the above (and hence (5)) is bounded by

$$B^{\frac{1}{r'}+\frac{1}{p}+\frac{1}{q}}\|h\|_{L^{r'}}\|f\|_{L^p}\|g\|_{L^q},$$

and invoking duality this implies (a).

We now concentrate on the third part of the equivalence, the fact that (a) implies (b). Without loss of generality we assume that $A = \|T\| = \|T^{*1}\| = \|T^{*2}\| = 1$, where the norms are taken on the correct spaces. We therefore take $B > 1$ in the argument below. We introduce operators $R(f, g)$ and $S(f, g)$ acting on functions f on X_1 and g on X_2 as follows

$$R(f,g) = T^{*1}(T(f,g)^{r/r'},g)^{p'/p}, \qquad S(f,g) = T^{*2}(f,T(f,g)^{r/r'})^{q'/q}.$$

Observe that for $f \in L^p(X_1)$ and $g \in L^q(X_2)$ the following estimates are valid.

$$\|R(f,g)\|_{L^p(X_1)} \le \|f\|_{L^p(X_1)}^{rp'/r'p}\|g\|_{L^q(X_2)}^{rp'/p}, \tag{6}$$

$$\|S(f,g)\|_{L^q(X_2)} \le \|f\|_{L^p(X_1)}^{rq'/q}\|g\|_{L^q(X_2)}^{rq'/r'q}. \tag{7}$$

In fact, to verify (6) we use that T maps $L^p \times L^q \to L^r$ and that T^{*1} maps $L^{r'} \times L^q \to L^{p'}$ in the sequence of inequalities below.

$$\begin{aligned}\|R(f,g)\|_{L^p} &= \|T^{*1}(T(f,g)^{r/r'},g)\|_{L^{p'}}^{p'/p} \\ &\le \|T\|^{p'/p}\|T(f,g)^{r/r'}\|_{L^{r'}}^{p'/p}\|g\|_{L^q}^{p'/p} \\ &= \|T\|^{p'/p}\|T(f,g)\|_{L^r}^{rp'/r'p}\|g\|_{L^q}^{p'/p} \\ &\le \|T\|^{rp'/p}\|f\|_{L^p}^{rp'/r'p}\|g\|_{L^q}^{rp'/p} = \|f\|_{L^p}^{rp'/r'p}\|g\|_{L^q}^{rp'/p}.\end{aligned}$$

Likewise for $S(f,g)$. Now set $B_1 = B^{rp'/p} > 1$ and $B_2 = B^{rq'/q} > 1$. Use the fact that X_1 and X_2 are σ-finite to select functions $f_1 > 0$ a.e. on X_1 and $g_1 > 0$ a.e. on X_2 such that $\|f_1\|_{L^p} \le (B_1-1)/B_1$ and $\|g_1\|_{L^q} \le (B_2-1)/B_2$. Define sequences f_n on X_1 and g_n on X_2 inductively by setting

$$f_{n+1} = f_1 + \frac{1}{B_1}R(f_n,g_n), \qquad g_{n+1} = g_1 + \frac{1}{B_2}S(f_n,g_n). \tag{8}$$

We claim that $\|f_n\|_{L^p(X_1)} \le 1$ and similarly $\|g_n\|_{L^q(X_2)} \le 1$ for all n. This is best seen by induction. Clearly $\|f_1\|_{L^p} \le 1$ and $\|g_1\|_{L^q} \le 1$. If we have $\|f_n\|_{L^p} \le 1$ and $\|g_n\|_{L^q} \le 1$ for some integer n, then

$$\|f_{n+1}\|_{L^p} \le \|f_1\|_{L^p} + \frac{1}{B_1}\|R(f_n,g_n)\|_{L^p} \le \frac{B_1-1}{B_1} + \frac{1}{B_1}\|f_n\|_{L^p}^{rp'/r'p}\|g_n\|_{L^q}^{p'/p} \le 1$$

and similarly for g_{n+1}.

Since the kernel $K \ge 0$ we have that T, T^{*1}, and T^{*2} are increasing functionals in every argument and thus so are R and S. This implies that the sequences f_n and g_n are increasing. Let h_1 be the pointwise limit of f_n as $n \to \infty$ and h_2 be the pointwise limit of g_n as $n \to \infty$. Fatou's Lemma implies that $\|h_1\|_{L^p} \le 1$ and $\|h_2\|_{L^q} \le 1$ which tell us that h_1 and h_2 are finite a.e. Clearly $h_1 \ge f_1 > 0$ a.e and $h_2 \ge g_1 > 0$ a.e.

Next we will show that $R(f_n,g_n)$ and $S(f_n,g_n)$ converge to $R(h_1,h_2)$ and $S(h_1,h_2)$ pointwise. Observe that the Lebesgue dominated convergence theorem implies that f_n converges to h_1 in $L^p(X_1)$ and g_n converges to h_2 in $L^q(X_2)$. Then $T(f_n,g_n)$ converges to $T(h_1,h_2)$ in $L^r(X)$ and thus $T(f_n,g_n)^{r/r'}$ converges to $T(h_1,h_2)^{r/r'}$ in $L^{r'}(X)$. The continuity of T^{*1} implies that $T^{*1}(T(f_n,g_n)^{r/r'},g_n)$ converges to $T^{*1}(T(h_1,h_2)^{r/r'},h_2)$ in $L^{p'}(X_1)$ and hence $R(f_n,g_n)$ converges to $R(h_1,h_2)$ in $L^p(X_1)$. Hence some subsequence of $R(f_n,g_n)$ converges to $R(h_1,h_2)$ a.e. Here we are using again the fact that the underlying spaces are σ-finite. However, since $R(f_n,g_n)$ is increasing it follows that the whole sequence converges to $R(h_1,h_2)$ a.e. Similarly we prove that $S(f_n,g_n)$ converges to $S(h_1,h_2)$ a.e.

Now letting $n \to \infty$ in (8) we obtain that

$$\begin{aligned} h_1 &= f_1 + \frac{1}{B_1}R(h_1,h_2), \qquad \text{a.e} \\ h_2 &= g_1 + \frac{1}{B_2}S(h_1,h_2), \qquad \text{a.e} \end{aligned}$$

These two equations imply that

$$\begin{aligned} T^{*1}(T(h_1,h_2)^{r/r'},h_2) \leq & B^r h_1^{p/p'}, \qquad \text{a.e} \\ T^{*2}(h_1,T(h_1,h_2)^{r/r'}) \leq & B^r h_2^{q/q'}, \qquad \text{a.e} \end{aligned}$$

which is the required conclusion since we showed that $0 < h_1, h_2 < \infty$ a.e. □

We now give a concrete application of Theorem 1.

Example 1. Let $X_1 = X_2 = \cdots = X_n = (0, \infty)$ with the usual Lebesgue measure and let T be the n-linear Hilbert operator

$$T(f_1, \dots, f_n)(x) = \int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1) \dots f_n(x_n)}{(x + x_1 + \cdots + x_n)^n} \, dx_1 \dots dx_n.$$

Observe that T coincides with all of its transposes. Let $1 < p_1, \dots, p_n, r < \infty$ satisfy (2) as before. To show that T maps $L^{p_1} \times \cdots \times L^{p_n}$ into L^r it suffices to find $u_1, \dots, u_n, w$ satisfying condition (4). For $1 \leq j \leq n$ set

$$u_j(x_j) = x_j^{-1/p_j p_j'}, \qquad \text{and} \qquad w(x) = x^{-1/rr'}.$$

Using induction, it is not hard to see that all the conditions in (4) are satisfied with equality and appropriate constants for this choice of u_j and w. This implies that T maps $L^{p_1}(0,\infty) \times \cdots \times L^{p_n}(0,\infty)$ into $L^r(0,\infty)$.

Next we discuss how to modify conditions (b) and (c) in Theorem 1 to characterize boundedness of positive multilinear operators in the off-diagonal case

$$\frac{1}{r} > \sum_{j=1}^n \frac{1}{p_j}. \tag{9}$$

For the corresponding result in the linear case ($n = 1$ below) see [8] and [18]. We have the following.

Theorem 2. *Let $1 < p_1, p_2, ..., p_n, r < \infty$ satisfy (9) and let $A > 0$. The following are equivalent.*
(a) T maps $L^{p_1}(X_1) \times \cdots \times L^{p_n}(X_n)$ to $L^r(X)$ with norm less than or equal to A.
(b) For all $B > A$ there exist measurable functions h_j on X_j which satisfy $0 < h_1, \dots, h_n < \infty$ a.e. such that

$$T^{*j}(h_1, \dots, h_{j-1}, T(h_1, \dots, h_n)^{r-1}, h_{j+1}, \dots, h_n) \leq B^r h_j^{p_j - 1} \qquad \text{a.e.}$$

for all $1 \leq j \leq n$, and

$$\int_X \big(T(h_1, ..., h_n)(x)\big)^r d\mu(x) \leq B^r. \tag{10}$$

(c) For all $B > A$ there exist measurable functions u_j on X_j and w on X with $0 < u_1, \dots, u_N, w < \infty$ a.e., such that

$$
\begin{aligned}
T(u_1^{p_1'}, u_2^{p_2'}, \dots, u_n^{p_n'}) &\le B\, w^{r'} \qquad && a.e. \\
T^{*1}(w^r, u_2^{p_2'}, \dots, u_n^{p_n'}) &\le B\, u_1^{p_1} \qquad && a.e. \\
&\dots \\
T^{*n}(u_1^{p_1'}, u_2^{p_2'}, \dots, w^r) &\le B\, u_n^{p_n} \qquad && a.e.
\end{aligned}
$$

and

$$\int_X T(v_1^{p_1'}, \dots, v_n^{p_n'})(x) w^r(x)\, d\mu(x) \le B. \tag{11}$$

Proof. The proof follows from a minor modification of the proof of Theorem 1. As before we take $n = 2$ for simplicity.

We show that (a) implies (b) by exactly repeating the corresponding argument in the proof of Theorem 1. It is noteworthy to observe that nowhere in that argument we used that $1/r = 1/p + 1/q$. (In particular, this part of the proof holds for any exponents $1 < p, q, r < \infty$.) The new condition (10) also follows because

$$
\begin{aligned}
\int_X T(h_1, h_2)^r\, d\mu &= \int_X T(h_1, h_2) T(h_1, h_2)^{r/r'}\, d\mu = \int_{X_1} T^{*1}(T(h_1, h_2)^{r/r'}, h_2) h_1\, d\mu_1 \\
&\le \int_{X_1} B^r h_1^{p/p'} h_1\, d\mu_1 = B^r \int_{X_1} h_1^p\, d\mu_1 \le B^r,
\end{aligned}
$$

since we proved in the previous theorem that $\|h_1\|_{L^p} \le 1$.

We show that (b) implies (c) by defining u, v, and w as before.

To see that (c) implies (a), modify the argument in the proof of Theorem 1 as follows. In (5) write the integrand as

$$L(x, x_1, x_2) M(x, x_1, x_2) N(x, x_1, x_2) O(x, x_1, x_2),$$

where the new factor is

$$O(x, x_1, x_2) = u(x_1)^{1-p'/r'-p'/q} v(x_2)^{1-q'/r'-q'/p} w^{1-r/p-r/q}(x) K(x_1, x_2, x_3)^{1/r-1/p-1/q}.$$

Now apply Hölder's inequality with exponents r', p, q, and $(1/r - 1/p - 1/q)^{-1}$. The new factor is controlled using condition (11). Here we use the assumption that $1/r > 1/p + 1/q$. □

3. Wavelet discretization of bilinear operators and Besov spaces

We shall use the discrete Littlewood-Paley definition of Besov spaces (see [16] and [19] for details). We fix a function ϕ in the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ whose Fourier transform satisfies $|\widehat{\phi}(\xi)| > 0$ in the annulus $\pi/4 < |\xi| < \pi$ and is zero everywhere else. Set $\phi_\nu(x) = 2^{\nu n}\phi(2^\nu x)$. For $0 < p, s < \infty$ and any real α, the homogeneous Besov space $\dot{B}_p^{\alpha,s}(\mathbf{R}^n)$ can be defined to be the collection of all tempered distributions modulo polynomials ($\mathcal{S}'/\mathcal{P}$) such that

$$\|f\|_{\dot B_p^{\alpha,s}} = \Big(\sum_{\nu} \big(2^{\nu\alpha}\|f*\phi_\nu\|_{L^p}\big)^s \Big)^{1/s} < \infty. \tag{12}$$

We will consider the "diagonal" case $s = p$ which we will simply denote by $\dot B_p^\alpha$. These spaces measure oscillatory properties of functions both at large and small scales. In particular for $p = 2$, these spaces coincide with the (homogeneous) Sobolev spaces. It is true that $(\dot B_p^\alpha)^* = \dot B_{p'}^{-\alpha}$, for $1 \le p < \infty$.

Based on the work of Frazier and Jawerth [5] the function ϕ in (12) can be chosen to generate an almost orthogonal wavelet (ϕ–transform) decomposition of the Besov spaces. That is, every $f \in \dot B_p^\alpha$ can be written in the form

$$f = \sum_{\nu,k} \langle f, \phi_{\nu k}\rangle \phi_{\nu k}, \tag{13}$$

and

$$\|f\|_{\dot B_p^\alpha} \approx \Big(\sum_{\nu,k} \big(|\langle f,\phi_{\nu k}\rangle| 2^{\nu(\alpha+n/2-n/p)}\big)^p \Big)^{1/p}; \tag{14}$$

where ν ranges over $\mathbf{Z}$, k over $\mathbf{Z}^n$,

$$\phi_{\nu k}(x) = 2^{\nu n/2}\phi(2^\nu x - k),$$

and $\langle\cdot,\cdot\rangle$ stands for the pairing of distributions and test functions. See also [14] and [7].

Let $\dot b_p^\alpha$ be the space of all sequences $s = \{s_{\nu k}\}$ for which

$$\|s\|_{\dot b_p^\alpha} = \Big(\sum_{\nu,k} \big(|s_{\nu k}| 2^{\nu(\alpha+n/2-n/p)}\big)^p \Big)^{1/p} < \infty. \tag{15}$$

Using (13) we can associate to every bilinear operator a discrete tensor

$$A = \{a(\lambda m, \nu k, \mu l)\} = \{\langle T(\phi_{\nu k}, \phi_{\mu l}), \phi_{\lambda m}\rangle\};$$

that is,

$$T(f,g) = \sum_{\lambda,m}\sum_{\mu,l}\sum_{\nu,k} a(\lambda m, \nu k, \mu l)\langle f,\phi_{\nu k}\rangle\langle g,\phi_{\mu l}\rangle \phi_{\lambda m}. \tag{16}$$

We consider also the associated trilinear form

$$\Lambda(f,g,h) = \sum_{\lambda,m}\sum_{\mu,l}\sum_{\nu,k} a(\lambda m, \nu k, \mu l)\langle f,\phi_{\nu k}\rangle\langle g,\phi_{\mu l}\rangle\langle h,\phi_{\lambda m}\rangle. \tag{17}$$

Because of (14), T maps $\dot B_p^{\alpha_1} \times \dot B_q^{\alpha_2}$ into $\dot B_r^{\alpha_3}$ if and only if the tensor A maps $\dot b_p^{\alpha_1} \times \dot b_q^{\alpha_2}$ into $\dot b_r^{\alpha_3}$. Note that the right hand side of (15) is the L^p norm on $\mathbf{Z}\times\mathbf{Z}^n$ with respect to the measure

$$d^{\alpha,p} = \{d_{\nu k}^{\alpha,p}\} = \{2^{\nu(\alpha+n/2-n/p)p}\}.$$

It follows that we can realize the discrete trilinear form in (17) on L^p spaces. More precisely, for three sequences $s = \{s_{\nu k}\}$, $t = \{t_{\mu l}\}$, and $u = \{u_{\lambda m}\}$, we will consider

$$\Lambda(s,t,u) = \sum_{\lambda,m}\sum_{\mu,l}\sum_{\nu,k} a(\lambda m, \nu k, \mu l) s_{\nu k} t_{\mu l} u_{\lambda m}$$
$$= \sum_{\lambda,m}\sum_{\mu,l}\sum_{\nu,k} K(\lambda m, \nu k, \mu l) s_{\nu k} t_{\mu l} \widetilde{u}_{\lambda m} d_{\nu k}^{\alpha_1,p} d_{\mu l}^{\alpha_2,q} d_{\lambda m}^{\alpha_3,r},$$

where

$$\widetilde{u}_{\lambda m} = u_{\lambda m} 2^{-\lambda(\alpha_3+n/2-n/r)r/r'} 2^{\lambda(-\alpha_3+n/2-n/r')} = u_{\lambda m} 2^{-\lambda(\alpha_3 r+(r/r'-1)n/2)}$$

and

$$K(\lambda m, \nu k, \mu l) = a(\lambda m, \nu k, \mu l) 2^{-\nu(\alpha_1+n/2-n/p)p} 2^{-\mu(\alpha_2+n/2-n/q)q}. \tag{18}$$

We will apply then Theorem 1 to the (discrete) bilinear integral operator with kernel $|K(\lambda m, \nu k, \mu l)|$ as an operator from $L^p(\mathbf{Z}\times\mathbf{Z}^n, d^{\alpha_1,p}) \times L^q(\mathbf{Z}\times\mathbf{Z}^n, d^{\alpha_2,q})$ into $L^r(\mathbf{Z}\times\mathbf{Z}^n, d^{\alpha_3,r})$. A similar approach was used by Frazier and Jawerth [6] in the linear case. Note that

$$\|\widetilde{u}\|_{L^{r'}(\mathbf{Z}\times\mathbf{Z}^n, d^{\alpha_3,r})} = \|u\|_{L^{r'}(\mathbf{Z}\times\mathbf{Z}^n, d^{-\alpha_3,r'})} = \|u\|_{\dot{b}_{r'}^{-\alpha_3}}.$$

Therefore, from the wavelet decomposition and the estimate

$$|\Lambda(s,t,u)| \le C\|s\|_{L^p(\mathbf{Z}\times\mathbf{Z}^n, d^{\alpha_1,p})} \|t\|_{L^q(\mathbf{Z}\times\mathbf{Z}^n, d^{\alpha_2,q})} \|\widetilde{u}\|_{L^{r'}(\mathbf{Z}\times\mathbf{Z}^n, d^{\alpha_3,r})},$$

with $s = \{< f, \phi_{\nu k} >\}$, $t = \{< g, \phi_{\mu l} >\}$, and $u = \{< h, \phi_{\lambda m} >\}$, it will follow that

$$| < T(f,g), h > | \le C\|f\|_{\dot{B}_p^{\alpha_1}} \|g\|_{\dot{B}_q^{\alpha_2}} \|h\|_{\dot{B}_{r'}^{-\alpha_3}},$$

and by duality T will map $\dot{B}_p^{\alpha_1} \times \dot{B}_q^{\alpha_2}$ into $\dot{B}_r^{\alpha_3}$.

4. Estimates on the tensors of bilinear multipliers

We consider bilinear multipliers

$$T(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta) e^{ix\cdot(\xi+\eta)} d\xi\, d\eta,$$

with symbols satisfying

$$|\partial_\xi^\beta \partial_\eta^\delta \sigma(\xi,\eta)| \le C_{\beta,\delta}(|\xi|+|\eta|)^{-(|\beta|+|\delta|)} \tag{19}$$

for all $(\xi,\eta) \ne (0,0)$ and all multi-indices β and δ. Such operators are a priori defined for functions in the space $\mathcal{S}_0$ given by

$$\mathcal{S}_0 = \{f \in \mathcal{S} : \partial^\gamma \widehat{f}(0) = 0 \text{ for all } \gamma\}.$$

It is easy to verify that the class of bilinear multipliers with symbols satisfying (19) is closed by taking either transpose. The symbols of the formal transposes are given by $\sigma^{*1}(\xi,\eta) = \sigma(-(\xi+\eta),\eta)$ and $\sigma^{*2}(\xi,\eta) = \sigma(\xi, -(\xi+\eta))$.

In the following lemmata we recall some basic estimates. The first one follows from some usual integration by parts arguments, while the others follow from standard computations using the cancellations involved to subtract appropriate Taylor polynomials. For brevity in the presentation we will not repeat the computations here, but full details are given in [9].

Lemma 1. *Let T be a bilinear operator with symbol $\sigma(\xi,\eta)$ satisfying (19). Then, for any family of almost orthogonal wavelets $\{\phi_{\nu k}\}$ as in Section 3,*

$$|\partial^\gamma T(\phi_{\nu k},\phi_{\mu l})(x)| \le C_{N,\gamma}\frac{2^{\nu n/2}2^{\mu n/2}\max(2^\nu,2^\mu)^{|\gamma|}}{(1+2^\nu|x-2^{-\nu}k|)^N(1+2^\mu|x-2^{-\mu}l|)^N}$$

for all γ and all $N>n$.

Lemma 2. *Let ψ_λ be a function in $\mathbf{R}^n$ that satisfies*

$$|\psi_\lambda(x)| \le C_N\frac{2^{\lambda n/2}}{(1+2^\lambda|x-x_\lambda|)^N} \tag{20}$$

and

$$\int_{\mathbf{R}^n}\psi_\lambda(x)x^\gamma\,dx = 0 \qquad \textit{for all } |\gamma|\le L-1. \tag{21}$$

Let $\psi_{\nu,\mu}$ be another function satisfying

$$|\partial^\gamma\psi_{\nu,\mu}(x)| \le C_N\frac{2^{\nu n/2}2^{\mu n/2}\max(2^\nu,2^\mu)^{|\gamma|}}{(1+2^\nu|x-x_\nu|)^N(1+2^\mu|x-x_\mu|)^N} \qquad \textit{for all } |\gamma|\le L \tag{22}$$

for some x_ν, x_μ in $\mathbf{R}^n$, and all $N>n$. Suppose that $\lambda\ge\max(\nu,\mu)$. Then for all $N>0$ we have

$$\left|\int_{\mathbf{R}^n}\psi_\lambda(x)\psi_{\nu,\mu}(x)\,dx\right| \le$$
$$\frac{C_{n,N,L}\,2^{-(\lambda-\max(\nu,\mu))L}\,2^{-\lambda n/2}\,2^{\nu n/2}\,2^{\mu n/2}}{((1+2^{\min(\lambda,\nu)}|x_\lambda-x_\nu|)(1+2^{\min(\nu,\mu)}|x_\nu-x_\mu|)(1+2^{\min(\mu,\lambda)}|x_\mu-x_\lambda|))^N}.$$

Lemma 3. *Suppose that ψ_ν satisfies*

$$|\partial^\gamma\psi_\nu(x)| \le C_N\frac{2^{\nu n/2}2^{\nu|\gamma|}}{(1+2^\mu|x-x_\nu|)^N} \qquad \textit{for all } |\gamma|\le L \tag{23}$$

for some x_ν in $\mathbf{R}^n$ and all $N>n$. Suppose also that $\psi_{\mu,\lambda}$ is another function satisfying (22) for $\gamma=0$ and also

$$\int_{\mathbf{R}^n}\psi_{\mu,\lambda}(x)x^\gamma\,dx = 0 \qquad \textit{for all } |\gamma|\le L-1. \tag{24}$$

Assume that $\max(\mu,\lambda)\ge\nu$. Then for all $N>0$ we have

$$\left|\int_{\mathbf{R}^n}\psi_\nu(x)\psi_{\mu,\lambda}(x)\,dx\right| \le$$
$$\frac{C_{n,N}\,2^{-(\max(\mu,\lambda)-\nu)L}\,2^{-\max(\mu,\lambda)n/2}\,2^{\min(\mu,\lambda)n/2}\,2^{\nu n/2}}{((1+2^{\min(\nu,\mu)}|x_\nu-x_\mu|)(1+2^{\min(\mu,\lambda)}|x_\mu-x_\lambda|)(1+2^{\min(\lambda,\nu)}|x_\lambda-x_\nu|))^N}.$$

When no cancellation is assumed, we have the following.

Lemma 4. *Suppose that ψ_ν, ψ_μ, ψ_λ are functions defined on $\mathbf{R}^n$ satisfying the following estimates for all $x \in \mathbf{R}^n$*

$$|\psi_\nu(x)| \le C_N \frac{2^{\nu n/2}}{(1+2^\nu|x-x_\nu|)^N} \tag{25}$$

$$|\psi_\mu(x)| \le C_N \frac{2^{\mu n/2}}{(1+2^\mu|x-x_\mu|)^N}, \tag{26}$$

$$|\psi_\lambda(x)| \le C_N \frac{2^{\lambda n/2}}{(1+2^\lambda|x-x_\lambda|)^N}, \tag{27}$$

for some x_ν, x_μ, x_λ in $\mathbf{R}^n$ and all $N > n$. Then the following estimate is valid

$$\int_{\mathbf{R}^n} |\psi_\nu(x)|\,|\psi_\mu(x)|\,|\psi_\lambda(x)|\,dx \le \frac{C_{n,N}\,2^{-\max(\nu,\mu,\lambda)n/2}\,2^{\mathrm{med}(\nu,\mu,\lambda)n/2}\,2^{\min(\nu,\mu,\lambda)n/2}}{((1+2^{\min(\nu,\mu)}|x_\nu-x_\mu|)(1+2^{\min(\mu,\lambda)}|x_\mu-x_\lambda|)(1+2^{\min(\lambda,\nu)}|x_\lambda-x_\nu|))^N},$$

where $\mathrm{med}(\nu,\mu,\lambda)$ *is one of the parameters* (ν,μ,λ) *chosen so that* $\min(\nu,\mu,\lambda) \le \mathrm{med}(\nu,\mu,\lambda) \le \max(\nu,\mu,\lambda)$.

We can now use the above to estimate the entries of the discrete tensor associated to the bilinear multiplier operators under consideration. To simplify the notation we define

$$B(\nu k,\mu l) = (1+2^{\min(\nu,\mu)}|2^{-\nu}k - 2^{-\mu}l|)$$

and similarly $B(\mu l,\lambda m)$ and $B(\lambda m,\nu k)$.

Lemma 5. *Let T be a bilinear operator with symbol $\sigma(\xi,\eta)$ satisfying (19) and let $\{\phi_{\nu k}\}$ be a family of almost orthogonal wavelets. Then the bilinear tensor associated with T, $\{a(\lambda m,\nu k,\mu l)\} = \{\langle T(\phi_{\nu k},\phi_{\mu l}),\phi_{\lambda m}\rangle\}$, satisfies the following estimates with $N > n$:*

$$|a(\lambda m,\nu k,\mu l)| \le \frac{C_{n,N,L}\,2^{-(\lambda-\max(\nu,\mu))L}\,2^{-\lambda n/2}\,2^{\nu n/2}\,2^{\mu n/2}}{(B(\nu k,\mu l)\,B(\mu l,\lambda m)\,B(\lambda m,\nu k))^N}, \tag{28}$$

for all $L \ge 0$ and $\lambda \ge \max(\nu,\mu)$.

$$|a(\lambda m,\nu k,\mu l)| \le \frac{C_{n,N,L}\,2^{-(\nu-\max(\lambda,\mu))L}\,2^{-\nu n/2}\,2^{\lambda n/2}\,2^{\mu n/2}}{(B(\nu k,\mu l)\,B(\mu l,\lambda m)\,B(\lambda m,\nu k))^N}, \tag{29}$$

for all $L \ge 0$ and $\nu \ge \max(\mu,\lambda)$.

$$|a(\lambda m,\nu k,\mu l)| \le \frac{C_{n,N,L}\,2^{-(\mu-\max(\lambda,\nu))L}\,2^{-\mu n/2}\,2^{\lambda n/2}\,2^{\nu n/2}}{(B(\nu k,\mu l)\,B(\mu l,\lambda m)\,B(\lambda m,\nu k))^N}, \tag{30}$$

for all $L \ge 0$ and $\mu \ge \max(\nu,\lambda)$.

$$|a(\lambda m,\nu k,\mu l)| \le \frac{C_{n,N}\,2^{-\max(\nu,\mu,\lambda)n/2}\,2^{\mathrm{med}(\nu,\mu,\lambda)n/2}\,2^{\min(\nu,\mu,\lambda)n/2}}{(B(\nu k,\mu l)\,B(\mu l,\lambda m)\,B(\lambda m,\nu k))^N}, \tag{31}$$

for all ν, μ, λ.

In addition under the indicated cancellation conditions the following estimates hold: if for all $|\gamma| \le L-1$

$$\int T(\phi_{\nu k}, \phi_{\mu l})(x) x^\gamma dx = 0, \tag{32}$$

then

$$|a(\lambda m, \nu k, \mu l)| \le \frac{C_{n,N,L}\, 2^{-(\max(\nu,\mu)-\lambda)L}\, 2^{-\max(\nu,\mu)n/2}\, 2^{\min(\nu,\mu)n/2}\, 2^{\lambda n/2}}{(B(\nu k, \mu l)\, B(\mu l, \lambda m)\, B(\lambda m, \nu k))^N}, \tag{33}$$

when $\lambda \le \max(\nu, \mu)$*;*
if for all $|\gamma| \le L-1$

$$\int T^{*1}(\phi_{\lambda m}, \phi_{\mu l})(x) x^\gamma dx = 0, \tag{34}$$

then

$$|a(\lambda m, \nu k, \mu l)| \le \frac{C_{n,N,L}\, 2^{-(\max(\lambda,\mu)-\nu)L}\, 2^{-\max(\lambda,\mu)n/2}\, 2^{\min(\lambda,\mu)n/2}\, 2^{\nu n/2}}{(B(\nu k, \mu l)\, B(\mu l, \lambda m)\, B(\lambda m, \nu k))^N}, \tag{35}$$

when $\nu \le \max(\lambda, \mu)$*;*
and if for all $|\gamma| \le L-1$

$$\int T^{*2}(\phi_{\nu k}, \phi_{\lambda m})(x) x^\gamma dx = 0, \tag{36}$$

then

$$|a(\lambda m, \nu k, \mu l)| \le \frac{C_{n,N,L}\, 2^{-(\max(\lambda,\nu)-\mu)L}\, 2^{-\max(\lambda,\nu)n/2}\, 2^{\min(\lambda,\nu)n/2}\, 2^{\mu n/2}}{(B(\nu k, \mu l)\, B(\mu l, \lambda m)\, B(\lambda m, \nu k))^N}, \tag{37}$$

when $\mu \le \max(\lambda, \nu)$.

Proof. Using Lemma 1 the estimate (28) follows from Lemma 2 with $\psi_\lambda = \phi_{\lambda m}$ and $\psi_{\nu,\mu} = T(\phi_{\nu k}, \phi_{\mu l})$. Similarly (31) follows from Lemma 4. If we assume the cancellation in the operator stated in (32), then (33) follows from Lemma 3. The other estimates can be obtained in similar fashion reversing the roles of the parameters ν, μ, and λ since the transposes T^{*1} and T^{*2} are in the same class as T.

□

Remark 1.

The cancellation conditions in Lemma 5 are always satisfied when the parameters ν, μ, and λ are far apart. In fact, it is easy to see that the conditions in (32) are equivalent to

$$\int \widehat{\phi}_{\nu k}(\xi) \partial_\eta^\gamma (\sigma(\xi, \eta - \xi) \widehat{\phi}_{\mu l}(\eta - \xi))|_{\eta=0}\, d\xi = 0. \tag{38}$$

Because on the condition on the support of the generating function $\widehat{\phi}$, the above are always satisfied if $|\nu - \mu| > 10$. Similarly with the cancellation conditions involving the transposes of T.

5. Bilinear multipliers on Besov spaces

We will use Schur's test to obtain boundedness results for bilinear multipliers.

Theorem 3. *Let $\alpha_1, \alpha_2 > 0$, $1 < p, q, r < \infty$, $1/p + 1/q = 1/r$. Let T be a bilinear multiplier operator whose symbol satisfies (19). Assume also that T^{*1} and T^{*2} satisfies the cancellation conditions (34) and (36) with $L = L_1 \geq \alpha_1 \frac{r'}{q'} + \alpha_2 \frac{r'}{p}$ and $L = L_2 \geq \alpha_1 \frac{r'}{q} + \alpha_2 \frac{r'}{p'}$. Then T can be extended to be a bounded operator from $\dot{B}_p^{\alpha_1} \times \dot{B}_q^{\alpha_2}$ into $\dot{B}_r^{\alpha_1+\alpha_2}$.*

Proof. We want to apply Theorem 1 to the discrete bilinear integral operator with kernel $|K(\lambda m, \nu k, \mu l)|$ defined in (18) and as explained in Section 3. Thus, with the same notation therein, we need to find three sequences $u = \{u_{\nu k}\}$, $v = \{v_{\mu l}\}$, and $w = \{w_{\lambda m}\}$, such that

$$S^{\lambda m} = \sum_{\mu l} \sum_{\nu k} |K(\lambda m, \nu k, \mu l)| u_{\nu k}^{p'} v_{\mu l}^{q'} d_{\nu k}^{\alpha_1, p} d_{\mu l}^{\alpha_2, q} \leq C w_{\lambda m}^{r'}. \tag{39}$$

$$S^{\nu k} = \sum_{\lambda m} \sum_{\mu l} |K(\lambda m, \nu k, \mu l)| v_{\mu l}^{q'} w_{\lambda m}^{r} d_{\lambda m}^{\alpha_1+\alpha_2, r} d_{\mu l}^{\alpha_1, q} \leq C u_{\nu k}^{p}. \tag{40}$$

$$S^{\mu l} = \sum_{\nu k} \sum_{\lambda m} |K(\lambda m, \nu k, \mu l)| u_{\nu k}^{p'} w_{\lambda m}^{r} d_{\nu k}^{\alpha_1, p} d_{\lambda m}^{\alpha_1+\alpha_2, r} \leq C v_{\mu l}^{q}, \tag{41}$$

Denoting by S the bilinear operator with kernel $|K|$, the above are exactly the conditions

$$S(u^{p'}, v^{q'}) \leq C w^{r'},$$
$$S^{*1}(w^r, v^{q'}) \leq C u^p,$$
$$S^{*2}(u^{p'}, w^r) \leq C v^q,$$

required by Schur's test. We will estimate the left hand sides of (39)–(41) by splitting each of them into six different sums. Each of these sums will be denoted by symbols of the form $S_{\nu k, \mu l, \lambda m}^{\lambda m}$, where the superscripts indicate the parameters that are kept fixed and the subscripts are set so that ν, μ, and λ are in nonincreasing order from left to right. Thus, for example,

$$S_{\nu k, \mu l, \lambda m}^{\lambda m} = \sum_{\nu \geq \mu} \sum_{\mu \geq \lambda} \sum_{k} \sum_{l} |K(\lambda m, \nu k, \mu l)| u_{\nu k}^{p'} v_{\mu l}^{q'} d_{\nu k}^{\alpha_1, p} d_{\mu l}^{\alpha_2, q},$$

where the summations indices are ν, μ, k, and l. We clearly have

$$S^{\lambda m} \leq S_{\nu k, \mu l, \lambda m}^{\lambda m} + S_{\nu k, \lambda m, \mu l}^{\lambda m} + S_{\mu l, \nu k, \lambda m}^{\lambda m} + S_{\lambda m, \nu k, \mu l}^{\lambda m} + S_{\mu l, \lambda m, \nu k}^{\lambda m} + S_{\lambda m, \mu l, \nu k}^{\lambda m}.$$

The roles of the variables ν and μ are similar. Thus, to estimate $S^{\lambda m}$ we only need to discuss the three sums with, say, $\nu \geq \mu$. By reversing the roles of ν and μ, the estimates for $S^{\nu k}$ and $S^{\mu l}$ are also seen to be analogous and therefore we will only treat the latter.

We start with the simpler case $p = q = r' = 3$, $\alpha_1 = \alpha_2 = \alpha$. We choose $0 < \epsilon < 3\alpha/2$ and we claim that

$$u_{\nu k} = 2^{-\nu(3\alpha+n/2-\epsilon)2/3},$$

$$v_{\mu l} = 2^{-\mu(3\alpha+n/2-\epsilon)2/3},$$

$$w_{\lambda m} = 2^{-\lambda(3\alpha+n/4-\epsilon)2/3},$$

do the job. As explained before, it suffices to consider the following nine sums.

5.1. **Estimate for $S^{\mu l}_{\mu l,\nu k,\lambda m}$.** *(No cancellation is needed.)*

We use (29) and bound $B(\mu l, \lambda m)^{-1}$ by 1. Summing in m and then in k produces a constant factor. Hence,

$$S^{\mu l}_{\mu l,\nu k,\lambda m} \leq$$

$$C\sum_{\mu\geq\nu}\sum_{\nu\geq\lambda} 2^{-(\mu-\nu)L}2^{-\mu n/2}2^{\nu n/2}2^{\lambda n/2}2^{\lambda(3\alpha-n/4)}2^{-\mu(3\alpha+n/2)}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\lambda(3\alpha+n/4-\epsilon)}$$

$$\leq C\sum_{\mu\geq\nu} 2^{\nu(L+n/2-3\alpha-n/2+\epsilon)}2^{\mu(-L-n/2-3\alpha-n/2)}\sum_{\nu\geq\lambda}2^{\lambda(n/2+3\alpha-n/4-3\alpha-n/4+\epsilon)}.$$

Since $\epsilon > 0$, we obtain

$$S^{\mu l}_{\mu l,\nu k,\lambda m} \leq C2^{\mu(-L-n-3\alpha)}\sum_{\mu\geq\nu}2^{\nu(L-3\alpha+2\epsilon)}.$$

If we choose $L > 3\alpha - 2\epsilon$ we obtain

$$S^{\mu l}_{\mu l,\nu k,\lambda m} \leq C2^{-\mu(6\alpha+n-2\epsilon)}$$

as desired.

5.2. **Estimate for $S^{\mu l}_{\mu l,\lambda m,\nu k}$.** *(We use the cancellation in T^{*1}.)*

We use (35) with $L > 3\alpha - \epsilon$ and bound $B(\lambda m, \nu k)^{-1}$ by 1. Summing in m and k produces a constant. Hence,

$$S^{\mu l}_{\mu l,\lambda m,\nu k} \leq$$

$$C\sum_{\mu\geq\lambda}\sum_{\lambda\geq\nu} 2^{-(\mu-\nu)L}2^{-\mu n/2}2^{\nu kn/2}2^{\lambda n/2}2^{\lambda(3\alpha-n/4)}2^{-\mu(3\alpha+n/2)}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\lambda(3\alpha+n/4-\epsilon)}$$

$$\leq C\sum_{\mu\geq\lambda} 2^{\mu(-L-n/2-3\alpha-n/2)}2^{\lambda(n/2+3\alpha-n/4-3\alpha-n/4+\epsilon)}\sum_{\lambda\geq\nu}2^{\nu(L+n/2-3\alpha-n/2+\epsilon)}.$$

Since $L - 3\alpha + \epsilon > 0$, we obtain

$$S^{\mu l}_{\mu l,\lambda m,\nu k} \leq C2^{\mu(-L-n-3\alpha)}\sum_{\mu\geq\lambda}2^{\lambda(L-3\alpha+2\epsilon)}.$$

Again using $L > 3\alpha - 2\epsilon$ we obtain the right estimate.

5.3. **Estimate for** $S^{\mu l}_{\nu k,\mu l,\lambda m}$**.** *(No cancellation is needed.)*

Using (31), bounding $B(\lambda m,\nu k)^{-1}$ by 1, and summing in m and k produces a factor of $C2^{(\nu-\mu)n}$. We get,

$$S^{\mu l}_{\nu k,\mu l,\lambda m} \leq$$

$$C\sum_{\nu\geq\mu}\sum_{\mu\geq\lambda} 2^{(\nu-\mu)n}2^{-\nu n/2}2^{\mu n/2}2^{\lambda n/2}2^{\lambda(3\alpha-n/4)}2^{-\mu(3\alpha+n/2)}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\lambda(3\alpha+n/4-\epsilon)}$$

$$\leq C\sum_{\nu\geq\mu} 2^{\mu(-n+n/2-3\alpha-n/2)}2^{\nu(n-n/2-3\alpha-n/2+\epsilon)}\sum_{\mu\geq\lambda}2^{\lambda(n/2+3\alpha-n/4-3\alpha-n/4+\epsilon)}.$$

Since $\epsilon>0$, we obtain

$$S^{\mu l}_{\nu k,\mu l,\lambda m} \leq C2^{\mu(-n-3\alpha)}\sum_{\nu\geq\mu}2^{-\nu(3\alpha-2\epsilon)},$$

which gives the desired estimate because $3\alpha-\epsilon>0$.

5.4. **Estimate for** $S^{\mu l}_{\lambda m,\mu l,\nu k}$**.** *(We use the cancellation in* T^{*1}*.)*

We use the estimate (35) with $L>\max(3\alpha-\epsilon,\epsilon)$ and bound the same factor in the denominator as in the previous case. Summing in m and k produces now a factor of $C2^{(\lambda-\mu)n}$, yielding

$$S^{\mu l}_{\lambda m,\mu l,\nu k} \leq$$

$$C\sum_{\lambda\geq\mu}\sum_{\mu\geq\nu} 2^{(\lambda-\mu)n}2^{-(\lambda-\nu)L}2^{n/2(-\lambda+\mu+\nu)}2^{\lambda(3\alpha-n/4)}2^{-\mu(3\alpha+n/2)}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\lambda(3\alpha+n/4-\epsilon)}$$

$$\leq C\sum_{\lambda\geq\mu} 2^{\mu(-n+n/2-3\alpha-n/2)}2^{\lambda(n-L-n/2+3\alpha-n/4-3\alpha-n/4+\epsilon)}\sum_{\mu\geq\nu}2^{\nu(L+n/2-3\alpha-n/2+\epsilon)}$$

$$\leq C2^{\mu(-n-6\alpha+L+\epsilon)}\sum_{\lambda\geq\mu}2^{-\lambda(L-\epsilon)},$$

and the right estimate follows.

5.5. **Estimate for** $S^{\mu l}_{\nu k,\lambda m,\mu l}$**.** *(No cancellation is needed.)*

We use again (31) as in the estimate for $S^{\mu l}_{\nu k,\mu l,\lambda m}$ and the fact that $\epsilon>0$ to get,

$$S^{\mu l}_{\nu k,\lambda m,\mu l} \leq$$

$$C\sum_{\nu\geq\mu}\sum_{\nu\geq\lambda\geq\mu} 2^{(\nu-\mu)n}2^{-\nu n/2}2^{\mu n/2}2^{\lambda n/2}2^{\lambda(3\alpha-n/4)}2^{-\mu(3\alpha+n/2)}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\lambda(3\alpha+n/4-\epsilon)}$$

$$\leq C\sum_{\nu\geq\mu} 2^{\mu(-n+n/2-3\alpha-n/2)}2^{\nu(n-n/2-3\alpha-n/2+\epsilon)}\sum_{\nu\geq\lambda\geq\mu}2^{\lambda(n/2+3\alpha-n/4-3\alpha-n/4+\epsilon)}.$$

$$\leq C2^{\mu(-n-3\alpha)}\sum_{\nu\geq\mu}2^{-\nu(3\alpha-2\epsilon)},$$

which sums to $C2^{\mu(-n-3\alpha+2\epsilon)}$ because $3\alpha-2\epsilon>0$.

5.6. **Estimate for** $S^{\mu l}_{\lambda m,\nu k,\mu l}$. *(No cancellation is needed.)*

This time we use (28) with $L > 3\alpha - \epsilon$ and bound $B(\nu k, \mu l)^{-1}$ by 1. Summing in k and then in m gives rise to a factor of $C2^{(\lambda-\mu)n}$ and hence

$$S^{\mu l}_{\lambda m,\nu k,\mu l} \leq$$

$$C\sum_{\lambda\geq\mu}\sum_{\lambda\geq\nu\geq\mu} 2^{(\lambda-\mu)n}2^{-(\lambda-\nu)L}2^{n/2(-\lambda+\mu+\nu)}2^{\lambda(3\alpha-n/4)}2^{-\mu(3\alpha+n/2)}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\lambda(3\alpha+n/4-\epsilon)}$$

$$\leq C\sum_{\lambda\geq\mu}2^{\mu(-n+n/2-3\alpha-n/2)}2^{\lambda(-L+n-n/2+3\alpha-n/4-3\alpha-n/4+\epsilon)}\sum_{\lambda\geq\nu\geq\mu}2^{\nu(L+n/2-3\alpha-n/2+\epsilon)}.$$

$$\leq C2^{\mu(-n-3\alpha)}\sum_{\lambda\geq\mu}2^{\lambda(-3\alpha+2\epsilon)} \leq C2^{-\mu(n+6\alpha-2\epsilon)}.$$

5.7. **Estimate for** $S^{\lambda m}_{\nu k,\mu l,\lambda m}$. *(No cancellation is needed.)*

We use (31) and bound $B(\mu l, \lambda m)^{-1}$ by 1. Summing in l and k produces a factor of $C2^{(\nu-\lambda)n}$. We then estimate

$$S^{\lambda m}_{\nu k,\mu l,\lambda m} \leq$$

$$C\sum_{\nu\geq\lambda}\sum_{\nu\geq\mu\geq\lambda}2^{(\nu-\lambda)n}2^{-\nu n/2}2^{\mu n/2}2^{\lambda n/2}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\mu(3\alpha+n/2-\epsilon)}$$

$$\leq C\sum_{\nu\geq\lambda}2^{\lambda(-n+n/2)}2^{\nu(n-n/2-3\alpha-n/2+\epsilon)}\sum_{\nu\geq\mu\geq\lambda}2^{\mu(n/2-3\alpha-n/2+\epsilon)}.$$

$$\leq C2^{\lambda(-n/2-3\alpha+\epsilon)}\sum_{\nu\geq\lambda}2^{\nu(-3\alpha+\epsilon)} \leq C2^{-\lambda(n/2+6\alpha-2\epsilon)}.$$

5.8. **Estimate for** $S^{\lambda m}_{\lambda m,\nu k,\mu l}$. *(We use the cancellation in T^{*2} and Remark 1.)*

By Remark 1, for $\nu << \lambda$ we have as much cancellation as we want in T^{*2}. We use (37) and bound $B(\nu k, \mu l)^{-1}$ by 1. This time summing in k and l produces a factor of $C2^{(\lambda-\nu)n}$. We proceed with

$$S^{\lambda m}_{\lambda m,\nu k,\mu l} \leq$$

$$C\sum_{\lambda\geq\nu}\sum_{\nu\geq\mu}2^{(\lambda-\nu)n}2^{-(\lambda-\mu)L}2^{-\lambda n/2}2^{\mu n/2}2^{\nu n/2}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\mu(3\alpha+n/2-\epsilon)}$$

$$\leq C\sum_{\lambda\geq\nu}2^{\lambda(n-L-n/2)}2^{\nu(-n+L+n/2-3\alpha-n/2+\epsilon)}\sum_{\nu\geq\mu}2^{\mu(L+n/2-3\alpha-n/2+\epsilon)}.$$

If we choose L large enough,

$$S^{\lambda}_{\lambda m,\nu k,\mu l} \leq C2^{\lambda(n/2-L)}\sum_{\lambda\geq\nu}2^{\nu(L-n-6\alpha+2\epsilon)} \leq C2^{-\lambda(n/2+6\alpha-2\epsilon)}.$$

On the other hand if $\lambda \sim \nu$, we use the cancellation in T^{*2}, (37), with $L > 3\alpha - \epsilon$ and we replace ν by λ in the above computations to obtain

$$S^{\lambda m}_{\lambda m,\nu k,\mu l} \leq$$

$$C\sum_{\lambda\geq\mu}2^{-(\lambda-\mu)L}2^{-\lambda n/2}2^{\mu n/2}2^{\lambda n/2}2^{-\lambda(3\alpha+n/2-\epsilon)}2^{-\mu(3\alpha+n/2-\epsilon)}$$

$$\leq C2^{\lambda(-L-3\alpha-n/2+\epsilon)}\sum_{\lambda\geq\mu}2^{\mu(L+n/2-3\alpha-n/2+\epsilon)},$$

and the right estimate follows.

5.9. **Estimate for $S^{\lambda m}_{\nu k,\lambda m,\mu l}$.** *(We use the cancellation in T^{*2}.)*

We use the cancellation in T^{*2} and (37) with $L > 3\alpha - \epsilon$ and $B(\nu k, \mu l)^{-1}$ bounded by 1. Summing in k and l produces

$$S^{\lambda m}_{\nu k,\lambda m,\mu l} \leq$$

$$C\sum_{\nu\geq\lambda}\sum_{\lambda\geq\mu}2^{(\nu-\lambda)n}2^{-(\nu-\mu)L}2^{-\nu n/2}2^{\mu n/2}2^{\lambda n/2}2^{-\nu(3\alpha+n/2-\epsilon)}2^{-\mu(3\alpha+n/2-\epsilon)}$$

$$\leq C\sum_{\nu\geq\lambda}2^{\lambda(-n+n/2)}2^{\nu(-L+n-n/2-3\alpha-n/2+\epsilon)}\sum_{\lambda\geq\mu}2^{\mu(L+n/2-3\alpha-n/2+\epsilon)}.$$

$$\leq C2^{\lambda(L-n/2-3\alpha+\epsilon)}\sum_{\nu\geq\lambda}2^{\nu(-L-3\alpha+\epsilon)}\leq C2^{-\lambda(n/2+6\alpha-2\epsilon)}.$$

This concludes the proof in the case $p = q = r' = 3$ and $\alpha_1 = \alpha_2 = \alpha$.

5.10. **The general case.** ($1/p + 1/q + 1/r' = 1$, $\alpha_1, \alpha_2 > 0$.)

The general case is only notationally more complicated. We want to find three sequences of the form

$$u_{\nu k} = 2^{-\nu x_\nu},$$

$$v_{\mu l} = 2^{-\mu x_\mu},$$

$$w_{\lambda m} = 2^{-\lambda x_\lambda},$$

for some x_ν, x_μ, and x_λ real which satisfy condition (c) of Theorem 1. Homogeneity considerations (counting the powers of 2 in the previous calculations) show that x_ν, x_μ, x_λ must be solutions of the system of linear equations,

$$\begin{aligned} -p'x_\nu + qx_\mu - rx_\lambda &= F(\alpha_2, q) \\ px_\nu - q'x_\mu - rx_\lambda &= F(\alpha_1, p) \\ -p'x_\nu - q'x_\mu - r'x_\lambda &= -n/2 \end{aligned} \tag{42}$$

where

$$F(y,z) = (y + n/2 - n/z)z - (\alpha_1 + \alpha_2 + n/2 - n/r)r.$$

The system (42) has infinitely many solutions which can be written in the form,

$$\begin{aligned} x_\lambda &= \alpha_1 + \alpha_2 + n/2r' - \varepsilon/r \\ x_\nu &= \alpha_1 r'/p'q' + \alpha_2 r'/pp' + n/2p' - \varepsilon r'/pp' \\ x_\mu &= \alpha_1 r'/qq' + \alpha_2 r'/p'q' + n/2q' - \varepsilon r'/qq' \end{aligned} \tag{43}$$

with ε arbitrary. It is rather tedious but completely elementary to check that all the computations carried out in the case $p = q = r' = 3$ can be repeated in the general case if $\varepsilon > 0$ is chosen so that

$$\begin{aligned} L_1 >& \alpha_1 \frac{r'}{q'} + \alpha_2 \frac{r'}{p} - \varepsilon \frac{r'}{p} > \varepsilon > 0 \\ L_2 >& \alpha_1 \frac{r'}{q} + \alpha_2 \frac{r'}{p'} - \varepsilon \frac{r'}{q} > \varepsilon > 0. \end{aligned}$$

Because of the hypotheses of the theorem, these conditions can always be achieved if we choose ε small enough. We spare the reader from these routine computations, but we work out a particular case, say the term $S^{\mu l}_{\lambda m,\mu l,\nu k}$, to illustrate what is needed. Proceeding as in 5.4 we use (34) with

$$L > \max(\varepsilon, \alpha_1 \frac{r'}{q'} + \alpha_2 \frac{r'}{p} - \varepsilon \frac{r'}{p}) = \alpha_1 \frac{r'}{q'} + \alpha_2 \frac{r'}{p} - \varepsilon \frac{r'}{p}$$

and obtain

$$S^{\mu l}_{\lambda m,\mu l,\nu k} \leq$$

$$C \sum_{\lambda \geq \mu} \sum_{\mu \geq \nu} 2^{(\lambda-\mu)n} 2^{-(\lambda-\nu)L} 2^{(-\lambda+\mu+\nu)n/2} 2^{\lambda(\alpha_1+\alpha_2+n/2-n/r)r} 2^{-\mu(\alpha_2+n/2-n/q)q} 2^{-\nu x_\nu p'} 2^{-\lambda x_\lambda r}$$

$$\leq C \sum_{\lambda \geq \mu} 2^{\mu(n/2-nq/2-\alpha_2 q)} 2^{\lambda(-L-n/2+nr/2-nr/2r'+\varepsilon)} \sum_{\mu \geq \nu} 2^{\nu(L-\alpha_1 r'/q'-\alpha_2 r'/p+\varepsilon r'/p)}$$

$$\leq C 2^{\mu(n/2-nq/2-\alpha_2 q+L-\alpha_1 r'/q'-\alpha_2 r'/p+\varepsilon r'/p)} \sum_{\lambda \geq \mu} 2^{-\lambda(L-\epsilon)}$$

$$\leq C 2^{\mu(n/2-nq/2-\alpha_2 q+L-\alpha_1 r'/q'-\alpha_2 r'/p+\varepsilon r'/p-L+\varepsilon)}$$

$$= C 2^{-\mu(\alpha_1 r'/q'+\alpha_2 r'(1/p+q/r')+nq/2-n/2-\varepsilon(r'/p+1)}$$

$$= C 2^{-\mu x_\mu q},$$

where in the last equality we have used the facts that

$$\frac{1}{pq} + \frac{1}{r'} = \frac{1}{p'q'},$$

$$\frac{1}{2} - \frac{1}{2q} = \frac{1}{2q'},$$

and

$$\frac{r'}{pq} + \frac{1}{q} = \frac{r'}{qq'}.$$

□

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