MULTILINEAR CALDERÓN-ZYGMUND THEORY

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Abstract. A systematic treatment of multilinear Calderón-Zygmund operators is presented. The theory developed includes strong type and endpoint weak type estimates, interpolation, the multilinear $T1$ theorem, and a variety of results regarding multilinear multiplier operators.

1. Introduction

The classical Calderón-Zygmund theory and its ramifications have proved to be a powerful tool in many aspects of harmonic analysis and partial differential equations. The main thrust of the theory is provided by the Calderón-Zygmund decomposition, whose impact is deep and far-reaching. This decomposition is a crucial tool in obtaining weak type $(1, 1)$ estimates and consequently L^p bounds for a variety of operators acting on function spaces on $\mathbf{R}^n$ and taking values in some Banach spaces. The Littlewood-Paley theory, with its incisive characterizations of function spaces on $\mathbf{R}^n$, can also be obtained as a consequence of the Calderón-Zygmund theory. The realization of pseudodifferential operators in terms of singular integrals of Calderón-Zygmund type is yet another accomplishment of the theory that brings up intimate connections with operator theory and partial differential equations.

The study of multilinear operators is not motivated by a mere quest to generalize the theory of linear operators but rather by their natural appearance in analysis. Coifman and Meyer were one of the first to adopt a multilinear point of view in their study of certain singular integral operators, such as the Calderón commutators, paraproducts, and pseudodifferential operators. The remarkable proof of the boundedness of the bilinear Hilbert transform by Lacey and Thiele [16], [17] provides, in our view, a further motivation for the systematic development and study of multilinear singular integrals. Within this framework, the bilinear Hilbert transform naturally arises in the context of the bilinear method of rotations, as described at the end of this article.

In this work we prove a variety of theorems regarding what we call multilinear Calderón-Zygmund operators. Their name is justified by the fact that these operators have kernels which satisfy standard estimates and bear boundedness properties analogous to those of the classical linear ones. Particular examples of these operators

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have been previously studied by Coifman and Meyer [6], [7], [8], [9], [18], assuming sufficient smoothness on their symbols and kernels. Our approach provides a systematic treatment of general m-linear Calderón-Zygmund operators on $\mathbf{R}^n$ under minimal smoothness assumptions on their kernels.

Our first result, Theorem 1, is concerned with the natural weak type endpoint estimate of Calderón-Zygmund operators on the m-fold product of L^1 spaces. This theorem improves on results of Coifman and Meyer [6], [7], and Kenig and Stein [15]. A further refinement of Theorem 1, needed for interpolation purposes, is presented in Theorem 2. Next, we show that boundedness of multilinear Calderón-Zygmund operators on one product of L^p spaces implies boundedness on all suitable products of Lebesgue spaces. The precise statement of this result is given in Theorem 3. The situation is then in complete analogy with the linear case, where boundedness on just one L^p space implies boundedness on all L^p spaces for $1 < p < \infty$; the characteristic feature of Calderón-Zygmund operators. Other results we discuss include the multilinear version of the Peetre-Spanne-Stein theorem [19], [20], [21], on the action of singular integrals on L^∞ and the multilinear $T1$ theorem, Theorem 4. This last result provides a powerful characterization of boundedness of multilinear singular integrals on products of L^p spaces in the spirit of the celebrated theorem of David and Journé [10]. Our characterization says that an m-linear Calderón-Zygmund singular integral operator T is bounded on products of Lebesgue spaces if and only if

$$\sup_{\xi_1\in\mathbf{R}^n} \dots \sup_{\xi_m\in\mathbf{R}^n} \|T(e^{2\pi i\xi_1\cdot(\,\cdot\,)}, \dots, e^{2\pi i\xi_m\cdot(\,\cdot\,)})\|_{BMO} < \infty,$$

and similarly for the m transposes of T. A different formulation of a $T1$ theorem for multilinear forms was given before by Christ and Journé [5]. We apply our version of the multilinear $T1$ theorem to obtain some new continuity results for multilinear translation invariant operators and multilinear pseudodifferential operators. We devote the last section of this article to a further analysis of multilinear multipliers.

2. Notation and preliminaries

We will be working on n-dimensional space $\mathbf{R}^n$. We denote by $\mathcal{S}(\mathbf{R}^n)$ the space of all Schwartz functions on $\mathbf{R}^n$ and by $\mathcal{S}'(\mathbf{R}^n)$ its dual space, the set of all tempered distributions on $\mathbf{R}^n$. Similarly we denote by $\mathcal{D}(\mathbf{R}^n)$ the set of all C^∞ functions with compact support on $\mathbf{R}^n$ and by $\mathcal{D}'(\mathbf{R}^n)$ the set of all distributions on $\mathbf{R}^n$. We denote by $L^p = L^p(\mathbf{R}^n)$ the classical Lebesgue spaces of measurable functions whose modulus to the p^{th} power is integrable, with the usual modification when $p = \infty$. We also denote by $L^{p,q} = L^{p,q}(\mathbf{R}^n)$ the Lorentz spaces defined by

$$(1)\qquad \|f\|_{L^{p,q}} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty \text{ and } 0 < p \le \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty \text{ and } 0 < p \le \infty, \end{cases}$$

where f^* is the nonincreasing rearrangement of f on $(0, \infty)$. Clearly $L^{p,p} = L^p$ and $L^{p,\infty} =$ weak L^p. $BMO = BMO(\mathbf{R}^n)$ denotes the usual space of functions with bounded mean oscillation. We use the notation $p' = p/(p-1)$ for $1 < p < \infty$, $1' = \infty$, and $\infty' = 1$.

We will use the following definition for the Fourier transform in n-dimensional euclidean space

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx,$$

while $f^{\vee}(\xi) = \widehat{f}(-\xi)$ will denote the inverse Fourier transform.

The action of a distribution u on a test function f will be denoted by $\langle u\,,\, f\rangle$. Let T be an m-linear operator from $\mathcal{S}(\mathbf{R}^n)\times\cdots\times\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$ which is continuous with respect to the natural topologies of these spaces. A version of the Schwartz kernel theorem (c.f. [13]) gives that any such T has a kernel K, which is a tempered distribution on $(\mathbf{R}^n)^{m+1}$, such that for all $f_1,\dots,f_m,g$ in $\mathcal{S}(\mathbf{R}^n)$

$$\langle T(f_1,\dots,f_m), g\rangle = \langle K\,,\, g\otimes f_1\otimes\cdots\otimes f_m\rangle. \tag{2}$$

Here $g\otimes f_1\otimes\cdots\otimes f_m$ denotes the function

$$(x, y_1,\dots,y_m) \to g(x) f_1(y_1)\dots f_m(y_m).$$

Conversely every tempered distribution K on $(\mathbf{R}^n)^{m+1}$ defines a continuous m-linear map from $\mathcal{S}(\mathbf{R}^n)\times\cdots\times\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$ whose kernel is K. We will occasionally write $K(x, y_1,\dots,y_m)$ for the distribution K to indicate the variables on which it acts.

In this work we study m-linear operators defined on products of test functions and we seek conditions to extend them as bounded operators on certain products of Banach spaces. We will use the notation

$$\|T\|_{X_1\times\cdots\times X_m\to X} = \sup_{\substack{\|f\|_{X_j}=1\\ 1\le j\le m}} \|T(f_1,\dots,f_m)\|_X$$

to denote the *norm* of an m-linear operator T from a product of Banach spaces of functions $X_1\times\cdots\times X_m$ into a quasi-Banach space X. We say that T is *bounded* from $X_1\times\cdots\times X_m$ into X when the norm above is finite.

An m-linear operator $T:\ \mathcal{S}(\mathbf{R}^n)\times\cdots\times\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}'(\mathbf{R}^n)$ is linear in every entry and consequently it has m formal transposes. The j^{th} *transpose* T^{*j} of T is defined via

$$\langle T^{*j}(f_1,\dots,f_m)\,,\, h\rangle = \langle T(f_1,\dots,f_{j-1}, h, f_{j+1},\dots,f_m)\,,\, f_j\rangle,$$

for all $f_1,\dots,f_m,g$ in $\mathcal{S}(\mathbf{R}^n)$.

It is easy to check that the kernel K^{*j} of T^{*j} is related to the kernel K of T via

$$K^{*j}(x, y_1,\dots,y_{j-1}, y_j, y_{j+1},\dots,y_m) = K(y_j, y_1,\dots,y_{j-1}, x, y_{j+1},\dots,y_m). \tag{3}$$

Note that if a multilinear operator T maps a product of Banach spaces $X_1\times\cdots\times X_m$ into another Banach space X, then the transpose T^{*j} maps the product of Banach spaces $X_1\times\dots X_{j-1}\times X^*\times X_{j+1}\times\cdots\times X_m$ into X_j^*. Moreover, the norms of T and T^{*j} are equal.

It is sometimes customary to work with the adjoints of an m-linear operator T whose kernels are the complex conjugates of the kernels K^{*j} defined above. In this paper we choose to work with the transposes, as defined above, to simplify the notation. This choice presents no differences in the study of these operators.

Let $K(x, y_1, \dots, y_m)$ be a locally integrable function defined away from the *diagonal* $x = y_1 = \cdots = y_m$ in $(\mathbf{R}^n)^{m+1}$, which satisfies the *size estimate*

$$|K(x, y_1, \dots, y_m)| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{nm}} \tag{4}$$

for some $A > 0$ and all $(x, y_1, \dots, y_m) \in (\mathbf{R}^n)^{m+1}$ with $x \ne y_j$ for some j. Furthermore, assume that for some $\varepsilon > 0$ we have the *smoothness estimates*

$$\begin{aligned} &|K(x, y_1, \dots, y_j, \dots, y_m) - K(x', y_1, \dots, y_j, \dots, y_m)| \\ &\qquad \le \frac{A|x-x'|^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{nm+\varepsilon}} \end{aligned} \tag{5}$$

whenever $|x - x'| \le \frac{1}{2} \max_{1\le j\le n} |x - y_j|$ and also that for each j,

$$\begin{aligned} &|K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y_j', \dots, y_m)| \\ &\qquad \le \frac{A|y_j-y_j'|^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{nm+\varepsilon}} \end{aligned} \tag{6}$$

whenever $|y_j - y_j'| \le \frac{1}{2} \max_{1\le j\le n} |x - y_j|$. Note that condition (5) is a regularity condition for K^{*j} defined in (3) in terms of K.

For convenience in the notation, we will assume a more symmetric form of the above estimates. It is easy to see that with an appropriate constant $c_{n,m} > 0$, condition (4) can also be written as

$$|K(y_0, y_1, \dots, y_m)| \le \frac{c_{n,m}A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}; \tag{7}$$

while conditions (5) and (6) follow from the more concise estimate

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y_j', \dots, y_m)| \le \frac{c_{n,m}A|y_j - y_j'|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}, \tag{8}$$

whenever $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2} \max_{0\le k\le m} |y_j - y_k|$.

We also note that condition (8) with $\varepsilon = 1$ is a consequence of

$$|\nabla K(y_0, y_1, \dots, y_m)| \le \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{nm+1}},$$

where ∇ denotes the gradient in all possible variables.

We will reserve the letter A for the constant that appears in the size and regularity estimates of K. (If these numbers are different, we will take A to be the largest of all these constants.)

In this article we study m-linear operators $T : \mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ for which there is a function K defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbf{R}^n)^{m+1}$ satisfying (7) and (8) and such that

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, dy_1 \dots dy_m, \tag{9}$$

whenever $f_1, \dots, f_m \in \mathcal{D}(\mathbf{R}^n)$ and $x \notin \cap_{j=1}^m \text{supp } f_j$.

Let $\widetilde{K}$ be the Schwartz kernel of T. Note that if (7), (8), and (9) are satisfied, then for $f_1, \dots, f_m, g$ in $\mathcal{D}(\mathbf{R}^n)$ with $\cap_{j=1}^m \text{supp } f_j \cap \text{supp } g = \emptyset$, we have that

$$\langle \widetilde{K}, g \otimes f_1 \otimes \cdots \otimes f_m \rangle = \int_{\mathbf{R}^n} \int_{(\mathbf{R}^n)^m} K(x, y_1, \dots, y_m) f(y_1) \dots f_m(y_m)\, dy_1 \dots dy_m\, dx,$$

as an absolutely convergent integral. For this reason, we will say that T is an m-linear operator with *Calderón-Zygmund kernel* K. The class of all functions satisfying (7) and (8) with parameters m, A, and ε will be denoted by $m\text{-}CZK(A, \varepsilon)$.

We plan to investigate boundedness properties of operators T with kernels in the class $m\text{-}CZK(A, \varepsilon)$ from a product of L^p spaces into another Lebesgue space. Since kernels satisfying condition (4) include certain distributions which are homogeneous of degree $-mn$, if the corresponding operator maps $L^{p_1} \times \cdots \times L^{p_m} \to L^p$, then the equation

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$$

must hold, as dictated by homogeneity.

For notational convenience, we will occasionally write

$$(y_1, \dots, y_m) = \vec{y},$$
$$K(x, y_1, \dots y_m) = K(x, \vec{y}),$$
$$dy_1 \dots dy_m = d\vec{y}.$$

As in the linear case, the smoothness assumption on the kernel allows us to extend the action of an operator T with kernel in $m\text{-}CZK(A, \varepsilon)$ to functions in $(C^\infty \cap L^\infty)$. To achieve this, let us fix a C^∞ function ψ supported in the ball of radius two in $\mathbf{R}^n$ and satisfying $0 \le \psi(x) \le 1$ and $\psi(x) = 1$ when $0 \le |x| \le 1$. Let $\psi_k(x) = \psi(2^{-k}x)$. We have the following.

Lemma 1. *Every multilinear operator T with kernel K in $m\text{-}CZK(A, \varepsilon)$ can be extended to $(C^\infty \cap L^\infty) \times \cdots \times (C^\infty \cap L^\infty)$ as an element of $\mathcal{D}'(\mathbf{R}^n)$ via*

$$T(f_1, \dots, f_m) = \lim_{k \to \infty} \big(T(\psi_k f_1, \dots, \psi_k f_m) + G(\psi_k f_1, \dots, \psi_k f_m) \big),$$

where

$$G(\psi_k f_1, \dots, \psi_k f_m) = - \int_{\min\limits_{1 \le j \le m} |y_j| > 1} K(0, \vec{y}) (\psi_k f_1)(y_1) \dots (\psi_k f_m)(y_m)\, d\vec{y},$$

and the limit above is taken in the weak-topology of $\mathcal{D}'(\mathbf{R}^n)$.*

Proof. Let $f_j \in L^\infty \cap C^\infty$ for $1 \le j \le m$. Set

$$F_k = T(\psi_k f_1, \dots, \psi_k f_m) + G(\psi_k f_1, \dots, \psi_k f_m).$$

Since $\mathcal{D}'(\mathbf{R}^n)$ is sequentially complete, it is enough to show that for each function $g \in \mathcal{D}(\mathbf{R}^n)$ the $\lim_{k \to \infty} \langle F_k, g \rangle$ exists. Let $B(0, R)$ be the ball centered at zero of radius $R > 0$. Fix $g \in \mathcal{D}(\mathbf{R}^n)$, select a positive integer k_0 so that $\text{supp}\, g \subset B(0, 2^{k_0})$, and write for $k > k_0$

$$\langle F_k, g \rangle = \langle F_{k_0}, g \rangle + \langle F_k - F_{k_0}, g \rangle.$$

Any multilinear operator L satisfies the identity

$$L(f_1,\dots,f_m)-L(h_1,\dots,h_m)=\sum_{j=1}^m L(h_1,\dots,h_{j-1},f_j-h_j,f_{j+1},\dots,f_m), \tag{10}$$

with the obvious interpretations when $j=1$ or $j=m$. Using (10) we can write

$$\begin{aligned}
&\langle F_k-F_{k_0},g\rangle=\\
&\sum_{j=1}^m\langle T(\psi_{k_0}f_1,\dots,\psi_{k_0}f_{j-1},(\psi_k-\psi_{k_0})f_j,\psi_k f_{j+1},\dots,\psi_k f_m),g\rangle+\\
&\sum_{j=1}^m\int_{\mathbf{R}^n}G(\psi_{k_0}f_1,\dots,\psi_{k_0}f_{j-1},(\psi_k-\psi_{k_0})f_j,\psi_k f_{j+1},\dots,\psi_k f_m)g(x)\,dx.
\end{aligned} \tag{11}$$

We need to show that each term in (11) has a limit as $k\to\infty$. Note that for all $k\ge k_0$,

$$\cap_{j=1}^m\mathrm{supp}\,(\psi_k-\psi_{k_0})f_j\cap\mathrm{supp}\,g=\emptyset,$$

so using (9), we control the two terms for $j=1$ in (11) by

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\min\limits_{2\le l\le m}|y_l|>1}\int_{|y_1|>2^{k_0}}|K(x,\vec{y})-K(0,\vec{y})|\\
&\qquad\qquad|(\psi_k(y_1)-\psi_{k_0}(y_1))f_1(y_1)\psi_k(y_2)f_2(y_2)\cdots\psi_k(y_m)f_m(y_m)g(x)|\,d\vec{y}\,dx\\
&+\int_{\mathbf{R}^n}\int_{\min\limits_{2\le l\le m}|y_l|\le 1}\int_{|y_1|>2^{k_0}}|K(x,\vec{y})|\\
&\qquad\qquad|(\psi_k(y_1)-\psi_{k_0}(y_1))f_1(y_1)\psi_k(y_2)f_2(y_2)\cdots\psi_k(y_m)f_m(y_m)g(x)|\,d\vec{y}dx,
\end{aligned}$$

and similarly for $j\ge 2$. Using the smoothness and size conditions on the kernel, it is an easy consequence of the Lebesgue dominated convergence theorem that (11) converges to

$$\begin{aligned}
&\sum_{j=1}^m\int_{\mathbf{R}^n}\int_{\substack{\min\limits_{\substack{1\le l\le m\\ l\ne j}}|y_l|>1}}\int_{|y_j|>2^{k_0}}\big(K(x,\vec{y})-K(0,\vec{y})\big)\\
&\qquad\qquad(1-\psi_{k_0}(y_j))f_1(y_1)f_2(y_2)\cdots f_m(y_m)g(x)\,d\vec{y}\,dx\\
&+\sum_{j=1}^m\int_{\mathbf{R}^n}\int_{\substack{\min\limits_{\substack{1\le l\le m\\ l\ne j}}|y_l|>1}}\int_{|y_j|>2^{k_0}}K(x,\vec{y})\\
&\qquad\qquad(1-\psi_{k_0}(y_j))f_1(y_1)f_2(y_2)\cdots f_m(y_m)g(x)\,d\vec{y}dx,
\end{aligned} \tag{12}$$

as $k\to\infty$. We conclude that $\langle F_k,g\rangle=\langle F_{k_0},g\rangle+\langle F_k-F_{k_0},g\rangle$ has a limit as $k\to\infty$. $\square$

If the functions f_j have compact support, then by choosing k_0 large enough (12) becomes zero. Thus for $f_1, \ldots, f_m$ in $\mathcal{D}(\mathbf{R}^n)$ the actual value of $T(f_1, \ldots, f_m)$ is different from the value given in the above lemma by the constant

$$G(f_1, \ldots, f_m) = -\int_{\min\limits_{1\le j\le m} |y_j|>1} K(0, \vec{y}) f_1(y_1) \ldots f_m(y_m)\, d\vec{y}. \tag{13}$$

This slight discrepancy, however, will cause no ambiguities when $T(f_1, \ldots, f_m)$ is seen as an element of BMO.

We will later need to apply induction on the degree m of multilinearity of an operator T. We will then need to consider $(m-1)$-linear operators obtained by *freezing* one of the functions on which T acts. The following lemma will be useful for this purpose.

Lemma 2. *Let K be in m-$CZK(A, \varepsilon)$, let $f_m \in L^\infty$, and for $(x, y_1, \ldots, y_{m-1})$ not in the diagonal of $(\mathbf{R}^n)^m$ define*

$$K_{f_m}(x, y_1, \ldots, y_{m-1}) = \int_{\mathbf{R}^n} K(x, y_1, \ldots, y_{m-1}, y_m) f_m(y_m)\, dy_m. \tag{14}$$

Then for some constant $c_{n,m} > 0$ we have that K is in $(m-1)$-$CZK(c_{n,m}\|f_m\|_{L^\infty} A, \varepsilon)$.

Proof. Using estimate (4) we obtain

$$\begin{aligned}|K_{f_m}(x, y_1, \ldots, y_{m-1})| &\le \|f_m\|_{L^\infty} A \int_{\mathbf{R}^n} (|x-y_1| + \cdots + |x-y_m|)^{-nm}\, dy_m \\ &\le c_{n,m}\|f_m\|_{L^\infty} A(|x-y_1| + \cdots + |x-y_{m-1}|)^{-n(m-1)},\end{aligned}$$

which gives the size estimate for K_{f_m}.

To verify the smoothness conditions, set $x = y_0$ and assume that for some $0 \le j \le m-1$ we have

$$|y_j - y_j'| \le \frac{1}{2} \max_{0\le k\le m-1} |y_j - y_k| \le \frac{1}{2} \max_{0\le k\le m} |y_j - y_k|.$$

Then, using (8), we obtain

$$\begin{aligned}&|K_{f_m}(y_0, \ldots, y_j, \ldots, y_{m-1}) - K_{f_m}(y_0, \ldots, y_j', \ldots, y_{m-1})| \\ \le& \|f_m\|_{L^\infty} \int_{\mathbf{R}^n} \frac{c_{n,m} A |y_j - y_j'|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}\, dy_m \\ \le& \|f_m\|_{L^\infty} \int_{\mathbf{R}^n} \frac{c_{n,m} A |y_j - y_j'|^\varepsilon}{\big(|y_0 - y_m| + \sum_{k,l=0}^{m-1} |y_k - y_l|\big)^{mn+\varepsilon}}\, dy_m \\ \le& \|f_m\|_{L^\infty} \frac{c_{n,m} A |y_j - y_j'|^\varepsilon}{\big(\sum_{k,l=0}^{m-1} |y_k - y_l|\big)^{(m-1)n+\varepsilon}}.\end{aligned}$$

□

By symmetry, Lemma 2 is also true if we freeze any other variable in K instead of y_m. Moreover, given an m–linear operator T and a fixed function f_j for some $1 \le j \le m$, we can construct the following $(m-1)$–linear operator

$$T_{f_j}(f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_m) = T(f_1, \dots, f_{j-1}, f_j, f_{j+1}, \dots, f_m). \tag{15}$$

It is easy to check that the transposes of the operators defined this way are

$$\begin{aligned} (T_{f_j})^{*k} =& (T^{*k})_{f_j}, \qquad \text{when} \quad k = 1, \dots, j-1, \\ (T_{f_j})^{*k} =& (T^{*(k+1)})_{f_j}, \qquad \text{when} \quad k = j, \dots, m-1. \end{aligned} \tag{16}$$

Lemma 3. *Assume that T is a multilinear operator with kernel K in m-$CZK(A, \varepsilon)$ which extends to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for some indices $1 \le p_1, \dots, p_m, p < \infty$ and $1 \le p_m \le \infty$. Fix a compactly supported bounded function f_m and let T_{f_m} be as in (15). Then T_{f_m} is an $(m-1)$-linear operator with Calderón-Zygmund kernel K_{f_m} given by (14).*

Proof. Let $f_1, \dots, f_{m-1}$ in $\mathcal{D}$ and let f_m be a compactly supported function in L^∞. We need to show that

$$\begin{aligned} &T_{f_m}(f_1, \dots, f_{m-1})(x) \\ &= \int_{(\mathbf{R}^n)^{m-1}} K_{f_m}(x, y_1, \dots, y_{m-1}) f_1(y_1) \dots f_{m-1}(y_{m-1})\, dy_1 \dots dy_{m-1} \end{aligned} \tag{17}$$

for $x \notin \cap_{j=1}^{m-1} \text{supp } f_j$. We will prove (17) by testing against smooth functions h supported in the complement of $\big(\cap_{j=1}^{m-1} \text{supp } f_j \big)$. Duality gives

$$\langle T(f_1, \dots, f_{m-1}, f_m), h \rangle = \langle T^{*m}(f_1, \dots, f_{m-1}, h), f_m \rangle, \tag{18}$$

where $T^{*m}(f_1, \dots, f_{m-1}, h)$ is a well-defined function in $L^{p'_m}$, if $p_m < \infty$, and in L^1, if $p_m = \infty$. Moreover, since $\cap_{j=1}^{m-1} \text{supp } f_j \cap \text{supp } h = \emptyset$, this function is given by the absolutely convergent integral

$$z \to \int_{(\mathbf{R}^n)^m} K(x, y_1, \dots, y_{m-1}, z) f_1(y_1) \dots f_{m-1}(y_{m-1}) h(x)\, dy_1 \dots dy_{m-1} dx$$

for all $z \in \mathbf{R}^n$. It follows that (18) is given by the absolutely convergent integral

$$\int_{\mathbf{R}^n} \int_{(\mathbf{R}^n)^{m-1}} K_{f_m}(x, y_1, \dots, y_{m-1}) f_1(y_1) \dots f_{m-1}(y_{m-1})\, dy_1 \dots dy_{m-1}\, h(x)\, dx$$

which implies (17). □

Remark. If f_m does not have compact support, then we cannot conclude from (9) that T_{f_m} is associated to the Calderón-Zygmund kernel K_{f_m} as defined in (14) (though the function K_{f_m} satisfies the right estimates). We therefore choose to work with L^∞ functions with compact support when we consider the operators T_{f_m}, to ensure that their kernels are indeed given by (14) and are a fortiori in the class $(m-1)$-$CZK(\|f_m\|_{L^\infty} A, \varepsilon)$.

We also observe that if an m-linear operator with kernel in m-$CZK(A, \varepsilon)$ extends to a bounded operator on a product of L^{p_j} spaces with $p_j < \infty$, then the integral representation (9) still holds for compactly supported and bounded functions f_j. This last statement can be easily shown using an elementary limiting argument.

3. An endpoint weak type estimate

The Calderón-Zygmund decomposition is the key tool used in obtaining weak type $(1,1)$ boundedness for classical linear singular integral operators. In this section we use this decomposition to obtain endpoint weak type results for the multilinear operators discussed in Section 1.

Since L^1 is a natural endpoint space for boundedness of singular integrals in the scale of L^p spaces $(1 < p < \infty)$, it is not surprising that the corresponding endpoint result for m-linear operators is attained on the m-fold product $L^1 \times \cdots \times L^1$. By homogeneity this product should be mapped into $L^{1/m,\infty}$. In fact, for operators given by homogeneous kernels, such weak type estimates have been recently obtained by Kenig and Stein [15], building on previous work by Coifman and Meyer [6].

The theorem bellow is sharp in the sense that the space $L^{1/m,\infty}$ cannot be replaced by $L^{1/m}$, as indicated by an example given in Section 6.

Theorem 1. *Let T be a multilinear operator with kernel K in m-$CZK(A,\varepsilon)$. Assume that for some $1 \le q_1, q_2, \dots, q_m \le \infty$ and some $0 < q < \infty$ with*

$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q},$$

T maps $L^{q_1} \times \cdots \times L^{q_m}$ into $L^{q,\infty}$. Then T can be extended to a bounded operator from the m-fold product $L^1 \times \cdots \times L^1$ into $L^{1/m,\infty}$. Moreover, for some constant $C_{n,m}$ (that depends only on the parameters indicated) we have that

$$\|T\|_{L^1\times\cdots\times L^1 \to L^{1/m,\infty}} \le C_{n,m}\big(A + \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}\big). \tag{19}$$

Proof. Set $B = \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}$. Fix an $\alpha > 0$ and consider functions $f_j \in L^1$ for $1 \le j \le m$. Without loss of generality we may assume that $\|f_1\|_{L^1} = \cdots = \|f_m\|_{L^1} = 1$. Setting $E_\alpha = \{x : \ |T(f_1, \dots, f_m)(x)| > \alpha\}$, we need to show that for some constant $C = C_{m,n}$ we have

$$|E_\alpha| \le C(A+B)^{1/m}\alpha^{-1/m}. \tag{20}$$

(Once (20) has been established for f_j's with norm one, the general case follows immediately by scaling.) Let γ be a positive real number to be determined later. Apply the Calderón-Zygmund decomposition to the function f_j at height $(\alpha\gamma)^{1/m}$ to obtain 'good' and 'bad' functions g_j and b_j, and families of cubes $\{Q_{j,k}\}_k$ with disjoint interiors such that

$$f_j = g_j + b_j$$

and

$$b_j = \sum_k b_{j,k}$$

where

$$\begin{gathered}
\operatorname{support}(b_{j,k}) \subset Q_{j,k} \\
\int b_{j,k}(x)dx = 0 \\
\int |b_{j,k}(x)|dx \le C(\alpha\gamma)^{1/m}|Q_{j,k}| \\
|\cup_k Q_{j,k}| \le C(\alpha\gamma)^{-1/m} \\
\|b_j\|_{L^1} \le C \\
\|g_j\|_{L^s} \le C(\alpha\gamma)^{1/ms'}
\end{gathered}$$

for all $j = 1, 2, \dots, m$ and any $1 \le s \le \infty$; (s' is here the dual exponent of s). Now let

$$\begin{aligned}
E_1 =& \{x : \ |T(g_1, g_2, \dots, g_m)(x)| > \alpha/2^m\} \\
E_2 =& \{x : \ |T(b_1, g_2, \dots, g_m)(x)| > \alpha/2^m\} \\
E_3 =& \{x : \ |T(g_1, b_2, \dots, g_m)(x)| > \alpha/2^m\} \\
& \dots \\
E_{2^m} =& \{x : \ |T(b_1, b_2, \dots, b_m)(x)| > \alpha/2^m\},
\end{aligned}$$

where each $E_s = \{x : \ |T(h_1, h_2, \dots, h_m)(x)| > \alpha/2^m\}$ with $h_j \in \{g_j, b_j\}$ and all the sets E_s are distinct. Since $|\{x : \ |T(f_1, \dots, f_m)(x)| > \alpha\}| \le \sum_{s=1}^{2^m} |E_s|$, it will suffice to prove estimate (20) for each of the 2^m sets E_s.

Let us start with set E_1 which is the easiest. Chebychev's inequality and the $L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}$ boundedness give

$$\begin{aligned}
|E_1| \le& \frac{(2^m B)^q}{\alpha^q} \|g_1\|_{L^{q_1}}^q \dots \|g_m\|_{L^{q_m}}^q \le \frac{CB^q}{\alpha^q} \prod_{j=1}^m (\alpha\gamma)^{\frac{q}{mq_j'}} \\
=& \frac{C'B^q}{\alpha^q} (\alpha\gamma)^{(m-\frac{1}{q})\frac{q}{m}} = C'B^q \alpha^{-\frac{1}{m}} \gamma^{q-\frac{1}{m}}.
\end{aligned} \tag{21}$$

Consider a set E_s as above with $2 \le s \le 2^m$. Suppose that for some $1 \le l \le m$ we have l bad functions and $m-l$ good functions appearing in $T(h_1, \dots, h_m)$, where $h_j \in \{g_j, b_j\}$ and assume that the bad functions appear at the entries $j_1, \dots, j_l$. We will show that

$$|E_s| \le C\alpha^{-1/m}\big(\gamma^{-1/m} + \gamma^{-1/m}(A\gamma)^{1/l}\big). \tag{22}$$

Let $l(Q)$ denote the side-length of a cube Q and let Q^* be a certain dimensional dilate of Q with the same center. Fix an $x \notin \cup_{j=1}^m \cup_k (Q_{j,k})^*$. Also fix for the moment the cubes $Q_{j_1,k_1}, \dots, Q_{j_l,k_l}$ and without loss of generality suppose that Q_{j_1,k_1} has the smallest size among them. Let c_{j_1,k_1} be the center of Q_{j_1,k_1}. For fixed $y_{j_2}, \dots, y_{j_l} \in \mathbf{R}^n$, the mean value property of the function b_{j_1,k_1} gives

$$
\begin{aligned}
&\left|\int_{Q_{j_1,k_1}} K(x,y_1,\dots,y_{j_1},\dots,y_m) b_{j_1,k_1}(y_{j_1})\,dy_{j_1}\right| \\
=&\left|\int_{Q_{j_1,k_1}} \big(K(x,y_1,\dots,y_{j_1},\dots,y_m)-K(x,y_1,\dots,c_{j_1,k_1},\dots,y_m)\big) b_{j_1,k_1}(y_{j_1})\,dy_{j_1}\right| \\
\le&\int_{Q_{j_1,k_1}} |b_{j_1,k_1}(y_{j_1})|\,\frac{A\,|y_{j_1}-c_{j_1,k_1}|^{\varepsilon}}{(|x-y_1|+\dots+|x-y_m|)^{mn+\varepsilon}}\,dy_{j_1} \\
\le&\int_{Q_{j_1,k_1}} |b_{j_1,k_1}(y_{j_1})|\,\frac{C\,A\,l(Q_{j_1,k_1})^{\varepsilon}}{(|x-y_1|+\dots+|x-y_m|)^{mn+\varepsilon}}\,dy_{j_1},
\end{aligned}
$$

where the previous to last inequality above is due to the fact that

$$
|y_{j_1}-c_{j_1,k_1}|\le c_n\,l(Q_{j_1,k_1})\le \frac{1}{2}|x-y_{j_1}|\le \frac{1}{2}\max_{1\le j\le m}|x-y_j|.
$$

Multiplying the just derived inequality

$$
\left|\int_{Q_{j_1,k_1}} K(x,\vec{y}) b_{j_1,k_1}(y_{j_1})\,dy_{j_1}\right| \le \int_{Q_{j_1,k_1}} \frac{C\,A\,|b_{j_1,k_1}(y_{j_1})|\,l(Q_{j_1,k_1})^{\varepsilon}}{(|x-y_1|+\dots+|x-y_m|)^{mn+\varepsilon}}\,dy_{j_1}
$$

by $\prod_{i\notin\{j_1,\dots,j_l\}}|g_i(y_i)|$ and integrating over all y_i with $i\notin\{j_1,\dots,j_l\}$, we obtain the estimate

$$
\begin{aligned}
&\int_{(\mathbf{R}^n)^{m-l}} \prod_{i\notin\{j_1,\dots,j_l\}}|g_i(y_i)|\left|\int_{Q_{j_1,k_1}} K(x,\vec{y}) b_{j_1,k_1}(y_{j_1})\,dy_{j_1}\right|\prod_{i\notin\{j_1,\dots,j_l\}}dy_i \\
&\le \prod_{i\notin\{j_1,\dots,j_l\}}\|g_i\|_{L^\infty}\int_{Q_{j_1,k_1}}|b_{j_1,k_1}(y_{j_1})|\,\frac{A\,C\,l(Q_{j_1,k_1})^{\varepsilon}}{(\sum_{j=1}^{l}|x-y_j|)^{mn-(m-l)n+\varepsilon}}\,dy_{j_1} \\
&\le C\,A\prod_{i\notin\{j_1,\dots,j_l\}}\|g_i\|_{L^\infty}\,\|b_{j_1,k_1}\|_{L^1}\,\frac{l(Q_{j_1,k_1})^{\varepsilon}}{\big(\sum_{j=1}^{l}(l(Q_{i,k_i})+|x-c_{i,k_i}|)\big)^{nl+\varepsilon}} \\
&\le C\,A\prod_{i\notin\{j_1,\dots,j_l\}}\|g_i\|_{L^\infty}\,\|b_{j_1,k_1}\|_{L^1}\prod_{i=1}^{l}\frac{l(Q_{j_i,k_i})^{\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}}.
\end{aligned}
\tag{23}
$$

The penultimate inequality above is due to the fact that for $x\notin\cup_{j=1}^{m}\cup_k(Q_{j,k})^*$ and $y_j\in Q_{j,k}$ we have that $|x-y_j|\approx l(Q_{j,k_j})+|x-c_{j,k_j}|$, while the last inequality is due to our assumption that the cube Q_{j_1,k_1} has the smallest side-length. It is now a simple consequence of (23) that for $x \notin \cup_{j=1}^m \cup_k (Q_{j,k})^*$ we have

$$
\begin{aligned}
&|T(h_1,\dots,h_m)(x)| \\
\leq & CA \int_{(\mathbf{R}^n)^{m-1}} \prod_{i\notin\{j_1,\dots,j_l\}} |g_i(y_i)| \prod_{i=2}^{l} \Big(\sum_{k_i} |b_{j_i,k_i}(y_{j_i})|\Big) \left| \int_{Q_{j_1,k_1}} K(x,\vec{y})\, b_{j_1,k_1}(y_{j_1})\, dy_{j_1} \right| \prod_{i\neq j_1} dy_i \\
\leq & CA \prod_{i\notin\{j_1,\dots,j_l\}} \|g_i\|_{L^\infty} \prod_{i=1}^{l} \frac{l(Q_{j_i,k_i})^{\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}} \int_{(\mathbf{R}^n)^{l-1}} \prod_{i=2}^{l} \Big(\sum_{k_i} |b_{j_i,k_i}(y_{j_i})|\Big) dy_{i_2}\dots dy_{i_l} \\
\leq & CA \prod_{i\notin\{j_1,\dots,j_l\}} \|g_i\|_{L^\infty} \prod_{i=2}^{l} \left(\sum_{k_i} \frac{\|b_{j_i,k_i}\|_{L^1}\, l(Q_{j_i,k_i})^{\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}} \right) \\
\leq & C' A (\alpha\gamma)^{\frac{m-l}{m}} \prod_{i=1}^{l} \left(\sum_{k_i} \frac{(\alpha\gamma)^{1/m}\, l(Q_{j_i,k_i})^{n+\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}} \right) = C'' A\, \alpha\, \gamma \prod_{i=1}^{l} M_{i,\varepsilon/l}(x),
\end{aligned}
$$

where

$$M_{i,\varepsilon/l}(x) = \sum_{k_i} \frac{l(Q_{j_i,k_i})^{n+\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}}$$

is the Marcinkiewicz function associated with the union of the cubes $\{Q_{i,k_i}\}_k$. It is a known fact [22] that

$$\int_{\mathbf{R}^n} M_{i,\varepsilon/l}(x)\, dx \leq C|\cup_{k_i} Q_{i,k_i}| \leq C'(\alpha\gamma)^{-1/m}.$$

Now, since

$$|\cup_{j=1}^m \cup_k (Q_{j,k})^*| \leq C(\alpha\gamma)^{-1/m},$$

inequality (22) will be a consequence of the estimate

$$|\{x \notin \cup_{j=1}^m \cup_k (Q_{j,k})^* :\ |T(h_1,\dots,h_m)(x)| > \alpha/2^m\}| \leq C(\alpha\gamma)^{-1/m}(A\gamma)^{1/l}. \tag{24}$$

We prove (24) using an $L^{1/l}$ estimate outside $\cup_{j=1}^m \cup_k (Q_{j,k})^*$; recall here that we are considering the situation where l is not zero. Using the size estimate derived above for $|T(h_1,\dots,h_m)(x)|$ outside the exceptional set, we obtain

$$
\begin{aligned}
&|\{x \notin \cup_{j=1}^m \cup_k (Q_{j,k})^* :\ |T(h_1,\dots,h_m)(x)| > \alpha/2^m\}| \\
\leq & C\alpha^{-1/l} \int_{\mathbf{R}^n - \cup_{j=1}^m \cup_k (Q_{j,k})^*} \big(\alpha\gamma A\, M_{1,\varepsilon/l}(x)\dots M_{l,\varepsilon/l}(x)\big)^{1/l}\, dx \\
\leq & C(\gamma A)^{1/l} \left(\int_{\mathbf{R}^n} M_{1,\varepsilon/l}(x)dx \right)^{1/l} \dots \left(\int_{\mathbf{R}^n} M_{l,\varepsilon/l}(x)dx \right)^{1/l} \\
\leq & C'(\gamma A)^{1/l} \big((\alpha\gamma)^{-1/m} \dots (\alpha\gamma)^{-1/m} \big)^{1/l} = C' \alpha^{-1/m} (A\gamma)^{1/l} \gamma^{-1/m},
\end{aligned}
$$

which proves (24) and thus (22).

We have now proved (22) for any $\gamma > 0$. Selecting $\gamma = (A+B)^{-1}$ in both (21) and (22) we obtain that all the sets E_s satisfy (20). Summing over all $1 \leq s \leq 2^m$ we obtain the conclusion of the theorem. □

For purposes of interpolation that will become apparent in the next section, we will need the following strengthening of Theorem 1.

Let us denote by $L^{p,1}_c$ the space of all compactly supported functions in $L^{p,1}$ for $0 < p < \infty$. Also set $L^{\infty,1}_c = L^\infty_c$, the set of all compactly supported functions in L^∞. (Recall that the definition in (1) gives $L^{\infty,1} = \{0\}$ but, to simplify the statement in the next theorem, we set $L^{\infty,1}_c = L^\infty_c$.)

Theorem 2. *Let T be a multilinear operator with kernel K in m-$CZK(A,\varepsilon)$. Assume that for some $1 \le q_1, q_2, \dots, q_m \le \infty$ and some $0 < q < \infty$ with*

$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q}$$

T maps $L^{q_1,1}_c \times \cdots \times L^{q_m,1}_c$ into $L^{q,\infty}$. Then T is a bounded operator from the m-fold product $L^1 \times \cdots \times L^1$ into $L^{1/m,\infty}$. Moreover, for some constant $C_{n,m}$ (that depends only on the parameters indicated) we have that

$$\|T\|_{L^1\times\cdots\times L^1 \to L^{1/m,\infty}} \le C_{n,m}\big(A + \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}\big). \tag{25}$$

Proof. The proof of this theorem only requires some minor modifications in the proof of the previous result. Let $f_j \in L^1$. Without loss of generality, we may assume that they have compact support and norm one. It follows that the g_j's obtained from the f_j's using the Calderón-Zygmund decomposition, $f_j = g_j + b_j$, must also have compact support. Moreover, it is easy to see that

$$\|g_j\|_{L^{q_j,1}} \le C(\alpha\gamma)^{1/mq_j'}$$

when $q_j < \infty$, while $\|g_j\|_{L^\infty} \le C(\alpha\gamma)^{1/m}$ as before. These estimates are sufficient to deduce (21) while the rest of the arguments remain unchanged. □

4. Multilinear interpolation

In this section we show how to obtain strong type $L^{p_1}\times\cdots\times L^{p_m} \to L^p$ boundedness results for multilinear Calderón-Zygmund operators starting from a single estimate. To avoid unnecessary technical complications (see the remark at the end of Lemma 3) we will be working with L^∞_c instead of L^∞ .

Theorem 3. *Let T be a multilinear operator with kernel K in m-$CZK(A,\varepsilon)$. Let $1 \le q_1, q_2, \dots, q_m, q < \infty$ be given numbers with*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}.$$

Suppose that either (i) or (ii) below hold:
(i) T maps $L^{q_1,1}\times\cdots\times L^{q_m,1}$ into $L^{q,\infty}$ if $q > 1$,
(ii) T maps $L^{q_1,1}\times\cdots\times L^{q_m,1}$ into L^1 if $q = 1$.
Let p, p_j be numbers satisfying $1/m \le p < \infty$, $1 \le p_j \le \infty$, and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}.$$

Then all the statements below are valid:
(iii) when all $p_j > 1$, then T can be extended to a bounded operator from $L^{p_1}\times\cdots\times L^{p_m}$

into L^p, where L^{p_k} should be replaced by L^∞_c if some $p_k = \infty$;
(iv) when some $p_j = 1$, then T can be extended to a bounded map from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^{p,\infty}$, where again L^{p_k} should be replaced by L^∞_c if some $p_k = \infty$.
(v) when all $p_j = \infty$, then T can be extended to a bounded map from the m-fold product $L^\infty_c \times \cdots \times L^\infty_c$ into BMO.

Moreover, there exists a constant C_{n,m,p_j,q_i} such that under either assumption (i) or (ii), we have the estimate

$$\|T\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} \le C_{n,m,p_j,q_i}\big(A + B\big), \tag{26}$$

where $B = \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}$ if $q > 1$, and $B = \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^1}$ if $q = 1$.

Furthermore, conclusions (iii), (iv), and (v) as well estimate (26) are also valid for all the transposes T^{*j}, $1 \le j \le m$.

Remark. Hypothesis (i) is not strong enough to imply (iii) nor (iv) when $q = 1$. The reason is that $L^{1,\infty}$ does not have a predual and its dual is not useful in interpolation.

Before we prove the theorem we set up some notation. We will identify exponents $p_1, \dots, p_m, p$ for which T maps $L^{p_1} \times \cdots \times L^{p_m}$ into L^p with points $(1/p_1, \dots, 1/p_m, 1/p)$ in $\mathbf{R}^{m+1}$. We need to show that T is bounded for $(1/p_1, \dots, 1/p_m, 1/p)$ in the convex hull of the $m+2$ points $E = (1, 1, \dots, 1, m)$, $O = (0, 0, \dots, 0, 0)$, $C_1 = (1, 0, \dots, 0, 1)$, $C_2 = (0, 1, \dots, 0, 1)$, $\dots$, and $C_m = (0, 0, \dots, 1, 1)$ We will denote this set by $EC_1 \dots C_m O$. Observe that the simplex $C_1 C_2 \dots C_m$ is contained in the $(m-1)$-dimensional plane

$$\mathcal{P} = \{(1/p_1, \dots, 1/p_m, 1/p) : \ 1/p_1 + \cdots + 1/p_m = 1/p = 1\}$$

and splits $EOC_1 \dots C_m$ into two simplices $EC_1 \dots C_m$ and $OC_1 \dots C_m$ based on the equilateral polygon $C_1 \dots C_m$ in $\mathcal{P}$. See Figure 1. Let q_j, q be as in the statement of the theorem and let $Q = (1/q_1, \dots, 1/q_m, 1/q)$. In a geometric language, assumption (i) is saying that Q lies in the interior of $OC_1 \dots C_m$ while assumption (ii) is saying that Q lies in the interior of $C_1 \dots C_m$. Geometrically speaking, conclusion (iii) is saying that T satisfies a strong type bound in the closure of the simplex $OC_1C_2 \dots C_m$ minus its vertices union the interior of the simplex $EC_1C_2 \dots C_m$. Conclusion (iv) is saying that T satisfies a weak type bound on the vertices $C_1, \dots, C_m$ and on the exterior faces of the simplex $EC_1 \dots C_m$.

Let M_j be the midpoints of the line segments OC_j. Then the $(m-1)$-dimensional simplices $\mathcal{P}_j = C_1 \dots C_{j-1} M_j C_{j+1} \dots C_m$ determine the planes of symmetry with respect to the transposes T^{*j}. See Figure 1. This means that the reflection of the point

$$P = (1/p_1, \dots, 1/p_m, 1/p)$$

in $OC_1 \dots C_m$ with the respect to $\mathcal{P}_j$ is the point

$$(1/p_1, \dots, 1/p_{j-1}, 1/p', 1/p_{j+1}, \dots, 1/p_m, 1/p_j').$$

Observe that the boundedness of the transpose T^{*j} at the latter point is equivalent to the boundedness of T at P.

The main idea of the proof of the theorem is to obtain appropriate bounds in each of the faces of the polyhedron $EC_1 \dots C_m O$ by reducing matters to $(m-1)$-linear operators. Induction on m will then be used to obtain the required bounds on the faces of this polyhedron which will imply strong type bounds on its interior, by interpolation. In the case $m = 1$ (required by the induction) all the statements of the theorem are known classical results about linear Calderón-Zygmund operators which we therefore omit.

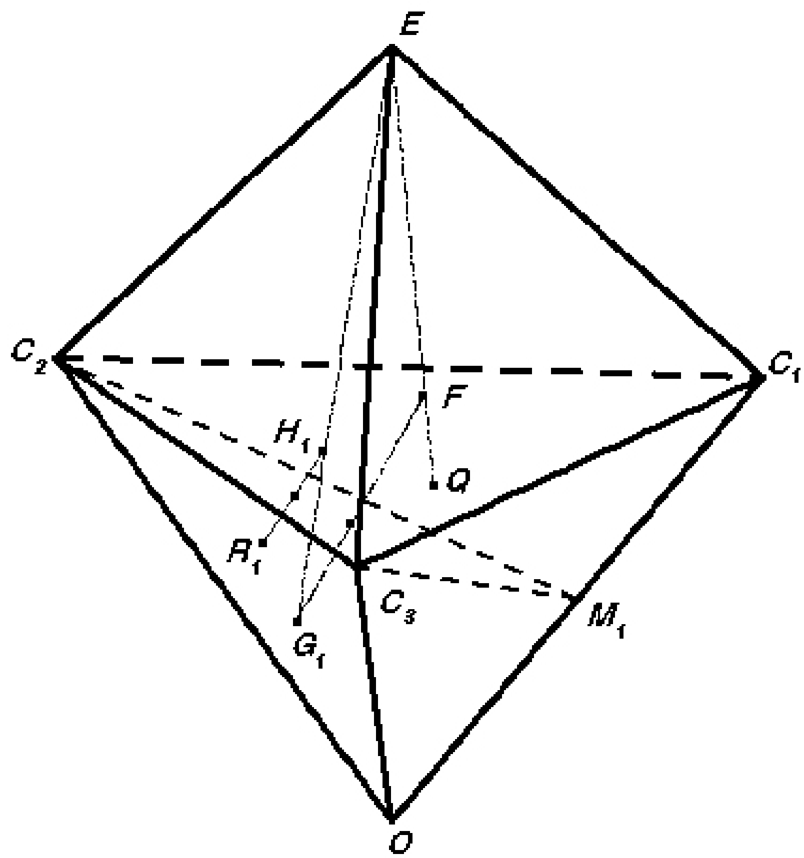

FIGURE 1. A geometric description of the proof of Lemma 4 for trilinear operators.

We will need to use the following version of the multilinear Marcinkiewicz interpolation theorem, Theorem A below, obtained by Grafakos and Kalton [11]. Other versions of the multilinear Marcinkiewicz interpolation theorem can be found in Janson [14] and Strichartz [24].

We say that a finite subset Θ of $[0,\infty)^m$ is *affinely independent* if the conditions

$$\sum_{\theta\in\Theta}\lambda_\theta\theta=(0,\dots,0)\qquad\text{and}\qquad\sum_{\theta\in\Theta}\lambda_\theta=0$$

imply $\lambda_\theta=0$, for all $\theta\in\Theta$.

Theorem A. ([11] Theorem 4.6) *Let $0<p_{j,k},p_j\le\infty$ for $1\le j\le m+1$ and $1\le k\le m$, suppose that for $j=1,\dots,m+1$ we have*

$$\sum_{k=1}^m\frac{1}{p_{j,k}}=\frac{1}{p_j},$$

and that the set $\Theta=\{(1/p_{j,1},\dots,1/p_{j,m}):\ j=1,\dots,m+1\}$ is affinely independent, that is

$$\det\begin{pmatrix}1/p_{1,1} & 1/p_{1,2} & \dots & 1/p_{1,m} & 1\\ 1/p_{2,1} & 1/p_{2,2} & \dots & 1/p_{2,m} & 1\\ \dots & \dots & \dots & \dots & \dots\\ 1/p_{m+1,1} & 1/p_{m+1,2} & \dots & 1/p_{m+1,m} & 1\end{pmatrix}\neq 0.$$

Assume that an m-linear map T satisfies

$$\|T(\chi_{E_1},\dots,\chi_{E_m})\|_{L^{p_j,\infty}}\le M|E_1|^{1/p_{j,1}}\dots|E_m|^{1/p_{j,m}}\tag{27}$$

for all sets E_j of finite measure and all $1\le j\le m+1$. Suppose that the point $(1/q_1,\dots,1/q_m,1/q)$ lies in the open convex hull of the points $(1/p_{j,1},\dots,1/p_{j,m},1/p_j)$ in $\mathbf{R}^{m+1}$. Then T extends to a bounded m-linear map from $L^{q_1}\times\cdots\times L^{q_k}$ into L^q with constant a multiple of M.

Remark. If (27) is assumed to hold on characteristic functions of compact sets, then it follows that T maps $L^{q_1}_c\times\cdots\times L^{q_m}_c$ into L^q and a simple density argument gives the conclusion of Theorem A. Naturally, the same conclusion holds if (27) is valid for general compactly supported functions f_j (instead of χ_{E_j}).

The proof of Theorem 3 will be a consequence of Theorem A and of the following lemma.

Lemma 4. *Under either hypothesis (i) or (ii) there exists a point V in the interior of the $(m-1)$-dimensional simplex $C_1C_2\dots C_m$ at which T satisfies a strong type bound with constant a multiple of $(A+B)$. Similarly for every $1\le j\le m$ there exists a point V^{*j} in the interior of $C_1C_2\dots C_m$ at which T^{*j} satisfies a strong type bound with constant also a multiple of $(A+B)$.*

Let us now prove Theorem 3 assuming Lemma 4.

Proof. Using Theorem 2, we obtain a weak type estimate for T at the point E. Duality and Lemma 4 imply that T satisfies a strong type bound at certain points V_j which lie in the interior of each of the m faces

$$\mathcal{S}_j=OC_1\dots C_{j-1}C_{j+1}\dots C_m$$

of the simplex $OC_1\dots C_m$. (V_j is the reflection of V^{*j} with respect to $\mathcal{P}_j$.) We will use this information and induction on m to obtain strong type bounds for T in the closure of each face $\mathcal{S}_j$ minus its vertices. At the vertices $C_1, \dots, C_{j-1}, \dots, C_{j+1}, \dots, C_m$ of $\mathcal{S}_j$ we will prove weak type bounds. We achieve this by using the inductive hypothesis that the theorem is true for $(m-1)$-linear operators.

For simplicity let us only work with the face $\mathcal{S}_j$ for $j = m$. Fix a function f_m in $L^\infty_c(\mathbf{R}^n)$ and define the $(m-1)$-linear operator

$$T_{f_m}(f_1, \dots, f_{m-1}) = T(f_1, f_2, \dots, f_m).$$

By Lemma 3, it follows that T_{f_m} has a kernel in $(m-1)$-$CZK(c_{n,m}\|f_m\|_{L^\infty}A, \varepsilon)$. Moreover, T_{f_m} satisfies a strong type estimate at a point V_m in the interior of the face $\mathcal{S}_m$ with bound a constant multiple of $(A+B)\|f_m\|_{L^\infty}$ since T satisfies a strong type estimate at the point V_m with bound a constant multiple $(A+B)$. The induction hypothesis now gives that T_{f_m} is bounded on the closure of the $(m-1)$-dimensional simplex $OC_1 \dots C_{m-1}$ minus its vertices. It also gives that T_{f_m} satisfies weak type estimates at the vertices $C_1, \dots, C_{m-1}$. Moreover, all the bounds in the estimates are constant multiples of $A\|f_m\|_{L^\infty} + (A+B)\|f_m\|_{L^\infty}$. It follows that T satisfies a strong type estimate on the interior of the face $\mathcal{S}_m$ with bound a multiple of $(A+B)$ with the restriction that its last argument lies in L^∞_c. Similarly T satisfies a weak type estimate at the vertices $C_1, \dots, C_{m-1}$ with bound a multiple of $(A+B)$ with the same restriction on its last argument.

Next we observe that if D_j are points in the interior of the faces $\mathcal{S}_j$ then the points $D_1, \dots, D_m$ and E are affinely independent in the sense of Theorem A.

Once strong type estimates have been obtained on the faces $\mathcal{S}_j$, Theorem A implies strong type estimates in the closure of the simplex $OC_1 \dots C_m$ minus its $m+1$ vertices, union the interior of the simplex $EC_1 \dots C_m$. Note that we are using here the remark after Theorem A since in one of the arguments of T only compactly supported functions appear. The weak type bounds on the sides of the simplex $EC_1 \dots C_m$ follow by interpolation between E and the points C_j at which we already know that a weak type estimate holds. The weak type estimates on each of the edges EC_j are obtained by complex interpolation. This concludes the proof of (iii) and (iv) for T.

To obtain conclusion (v) observe that the induction hypothesis gives that T_{f_m} maps the $(m-1)$-fold product $L^\infty_c \times \cdots \times L^\infty_c$ into BMO with bound a multiple of $(A+B)\|f_m\|_{L^\infty}$. Since f_m is an arbitrary element of L^∞_c, assertion (v) follows for T.

Since Lemma 4 gives the same conclusion for all the transposes T^{*j} of T, it follows that the same result is also valid for all the of T^{*j}'s as claimed in the statement of the Theorem 3. □

We now prove Lemma 4.

Proof. Let F be the point of intersection of the line segment QE with the simplex $C_1C_2 \dots C_m$. Under hypothesis (i) pick $\widetilde{F}$ on the line QE, close to F and in the interior of the simplex $OC_1 \dots C_m$. Under assumption (ii) just let $\widetilde{F} = F = Q$. Multilinear complex interpolation between the points E and Q implies that T satisfies a weak type estimate at the point $\widetilde{F}$ with constant bounded by a multiple of $(A+B)$. The estimate at the point $\widetilde{F}$ implies in particular that

$$\|T(f_1,\dots,f_m)\|_{L^{\alpha,\infty}} \le C(A+B)\prod_{k=1}^{m}\|f_k\|_{L^{\alpha_k,1}} \tag{28}$$

for some $\alpha_1,\dots\alpha_m,\alpha>1$. Under assumption (ii) this step is vacuous since estimate (28) already holds. Now let G_j and $\widetilde{G}_j$ be the reflections of F and $\widetilde{F}$ about $\mathcal{P}_j$. Duality gives the following estimates at the points $\widetilde{G}_j$

$$\|T^{*j}(f_1,\dots,f_m)\|_{L^{\alpha',1}} \le C(A+B)\prod_{k\neq j}\|f_k\|_{L^{\alpha_k,1}}\|f_j\|_{L^{\alpha_j',1}}. \tag{29}$$

This is because for $s>1$, $L^{s,\infty}$ is the dual space of $L^{s',1}$ and these two spaces are 'norming duals' of each other (i.e. each of the two norms can be realized as a supremum of integrals against functions in the unit ball of the other space). Now let H_j be the intersection of the line $\widetilde{G}_jE$ with the $(m-1)$-dimensional simplex $C_1C_2\dots C_m$. Pick a point $\widetilde{H}_j$ on the line $\widetilde{G}_jE$ but inside $OC_1\dots C_m$ and near H_j. See Figure 1. Theorem 2 implies that T^{*j} satisfies a weak type estimate at the point E. Multilinear complex interpolation between the points E and $\widetilde{G}$ gives the following Lorentz space estimate at the point $\widetilde{H}_j$

$$\|T^{*j}(f_1,\dots,f_m)\|_{L^{\gamma_j,\infty}} \le C(A+B)\|f_j\|_{L^{\beta_{jj},1}}\prod_{k\neq j}\|f_k\|_{L^{\beta_{kj},1}},$$

for some $1<\gamma_j,\beta_{kj}<\infty$. Now reflect the points H_j and $\widetilde{H}_j$ about $\mathcal{P}_j$ to obtain points R_j and $\widetilde{R}_j$ at which T satisfies the Lorentz space estimates

$$\|T(f_1,\dots,f_m)\|_{L^{\beta_{jj}',\infty}} \le C(A+B)\|f_j\|_{L^{\gamma_j',1}}\prod_{k\neq j}\|f_k\|_{L^{\beta_{kj},1}}.$$

We now have $m+1$ points $\widetilde{R}_1,\dots,\widetilde{R}_m$, and E at which T satisfies restricted weak type estimates of the form $L^{s_1,1}\times\cdots\times L^{s_m,1}\to L^{s,\infty}$ with constant bounded by a multiple of $(A+B)$. We observed earlier that if D_j are points in the interior of the faces $\mathcal{S}_j$, then the points $D_1,\dots,D_m,E$ are affinely independent in the sense of Theorem A. Since the notion of affine independence is stable under small perturbations, we conclude that the $m+1$ points $\widetilde{R}_1,\dots,\widetilde{R}_m$, and E are affinely independent when the $\widetilde{R}_j$'s are very close to the R_j's. Theorem A implies that T satisfies a strong type estimate at every point $P=(1/p_1,\dots,1/p_m,1/p)$ in the interior of the polyhedron $\widetilde{R}_1\dots\widetilde{R}_mE$ with bound a multiple of $(A+B)$. In this way we obtain a point V in the interior of the $(m-1)$-dimensional simplex $C_1C_2\dots C_m$ at which T satisfies a strong type bound with constant a multiple of $(A+B)$.

To obtain the points V^{*j} as in the statement of the lemma we argue as follows. Let W_j be the reflections of Q with respect to $\mathcal{P}_j$. Then T^{*j} satisfies a strong type estimate at the point W_j. Repeat the argument above with T^{*j} playing the role of T and W_j playing the role of the starting point Q. We find points V^{*j} in the interior of the $(m-1)$-dimensional simplex $C_1C_2\dots C_m$ at which T^{*j} satisfies a strong type bound with constant a multiple of $(A+B)$. This concludes the proof of the Lemma. $\square$

Remark. As mentioned earlier, to avoid technical complications we have obtained estimates only on L^∞_c. In many instances, such estimates and certain ad hoc procedures allow extensions of linear or multilinear operators to all of L^∞. One way to achieve this is described in the book of Meyer and Coifman [18], Section 13.3, in their treatment of multilinear multiplier operators.

Remark. Extensions to L^∞ can also be obtained in certain cases by duality. Notice that Theorem 3 gives that T^{*j} maps the m-fold product $L^m \times \cdots \times L^m$ into L^1. Using duality, we can then extend T as a bounded operator

$$L^m \times \cdots \times L^m \times L^\infty \times L^m \times \cdots \times L^m \to L^{m'}.$$

We obtain similar extensions for all points in the boundary of $OC_1 \dots C_m$, except at the vertices $O, C_1, \dots, C_m$.

We now discuss how to achieve this extension at the vertex O. This will allow us to obtain a multilinear version of the theorem of Peetre, Spanne, and Stein on the boundedness of a linear Calderón-Zygmund operators from L^∞ to BMO.

Proposition 1. *Under either hypothesis (i) or (ii) of Theorem 3, T has an extension that maps*

$$L^\infty \times \cdots \times L^\infty \to BMO$$

with bound a constant multiple of $(A + B)$. By duality, T also maps

$$L^\infty \times \cdots \times H^1 \times \cdots \times L^\infty \to L^1.$$

(where H^1 is the Hardy space predual of BMO).

Proof. Fix a C^∞ function ψ supported in the ball of radius two in $\mathbf{R}^n$ and satisfying $0 \le \psi(x) \le 1$ and $\psi(x) = 1$ when $0 \le |x| \le 1$ and let $\psi_k(x) = \psi(2^{-k}x)$ as in Lemma 1. Theorem 3 gives that T maps

$$L^\infty_c \times \cdots \times L^\infty_c \times L^2 \to L^2.$$

Since T is well defined on this product of spaces the expression $T(\psi_k f_1, \dots \psi_k f_m)$ is a well defined L^2 function whenever $f_j \in L^\infty$. For $f_1, \dots, f_m \in L^\infty$ let

$$G(\psi_k f_1, \dots, \psi_k f_m) = -\int_{\min\limits_{1\le j\le m} |y_j| > 1} K(0, \vec{y})(\psi_k f_1)(y_1) \dots (\psi_k f_m)(y_m)\, d\vec{y}.$$

Lemma 1 implicitly contains a proof that the limit

$$\lim_{k\to\infty} \big(T(\psi_k f_1, \dots, \psi_k f_m) + G(\psi_k f_1, \dots, \psi_k f_m)\big) = T(f_1, \dots, f_m),$$

exists pointwise almost everywhere and defines a locally integrable function. (The smoothness of the functions f_j in the proof this lemma was needed to make sense of the expression $T(\psi_k f_1, \dots, \psi_k f_m)$ when T was only defined on $\mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n)$.)

To check that this extension of T maps $L^\infty \times \cdots \times L^\infty$ into BMO, let us observe that

$$\begin{aligned}\|T(f_1,\dots,f_m)\|_{BMO} \leq& \limsup_{k\to\infty} \big\|T(\psi_k f_1,\dots,\psi_k f_m)+G(\psi_k f_1,\dots,\psi_k f_m)\big\|_{BMO}\\ =&\limsup_{k\to\infty} \|T(\psi_k f_1,\dots,\psi_k f_m)\|_{BMO}\\ \leq& c_{n,m}(A+B)\limsup_{k\to\infty}\|\psi_k f_1\|_{L^\infty}\dots\|\psi_k f_m\|_{L^\infty}\\ =&c_{n,m}(A+B)\|f_1\|_{L^\infty}\dots\|f_m\|_{L^\infty},\end{aligned}$$

where the last inequality follows by assertion (v) of Theorem 3. □

5. The multilinear $T1$ theorem

As we saw in Section 4, if a multilinear operator with kernel in m-$CZK(A,\varepsilon)$ maps $L^{q_1}\times\cdots\times L^{q_m}$ into L^q for a single point $(1/q_1,\dots,1/q_m,1/q)$ with $q>1$, then T maps $L^{p_1}\times\cdots\times L^{p_m}$ into L^p in the full range of possible exponents. It is therefore natural to ask under what conditions T maps $L^{q_1}\times\cdots\times L^{q_m}$ into L^q for one $(m+1)$-tuple $(1/q_1,\dots,1/q_m,1/q)$. A necessary and sufficient condition for this to happen is given by the multilinear $T1$ theorem discussed in this section.

The linear $T1$ theorem was obtained by David and Journé [10]. Its original formulation involves three conditions equivalent to L^2 boundedness. These conditions are that $T1\in BMO$, $T^*1\in BMO$, and that a certain weak boundedness property, which we do not need to state here, holds. An equivalent formulation of the $T1$ theorem, also found in [10] and better suited for our purposes, is the following: A linear operator T with kernel in 1-$CZK(A,\varepsilon)$ maps $L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$ if and only if

$$\sup_{\xi\in\mathbf{R}^n}\big(\|T(e^{2\pi i\xi\cdot(\,\cdot\,)})\|_{BMO}+\|T^*(e^{2\pi i\xi\cdot(\,\cdot\,)})\|_{BMO}\big)<\infty.$$

In this section we will state and prove a multilinear version of the $T1$ theorem using the characterization stated above. We will base some of our arguments on yet another formulation of the $T1$ theorem given by Stein [23]. Let us consider the set of all C^∞ functions supported in the unit ball of $\mathbf{R}^n$ satisfying

$$\|\partial^\alpha\phi\|_{L^\infty}\leq 1,$$

for all multiindices $|\alpha|\leq[n/2]+1$. Such functions are called *normalized bumps.* For a normalized bump ϕ, $x_0\in\mathbf{R}^n$, and $R>0$, define the function

$$\phi^{R,x_0}(x)=\phi\left(\tfrac{x-x_0}{R}\right).$$

The formulation in [23], Theorem 3, page 294, says that a necessary and sufficient condition for an operator T with kernel in 1-$CZK(A,\varepsilon)$ to be L^2-bounded is that for some constant $B>0$ we have

$$\|T(\phi^{R,x_0})\|_{L^2}+\|T^*(\phi^{R,x_0})\|_{L^2}\leq B\,R^{n/2}$$

for all normalized bumps ϕ, all $R>0$ and all $x_0\in\mathbf{R}^n$. Moreover, the norm of the operator T on L^2 (and therefore on L^p) is bounded by a constant multiple of $(A+B)$.

We are now in a position to state the multilinear $T1$ theorem. Recall that in view of Lemma 1, $T(e^{2\pi i\xi_1\cdot(\,\cdot\,)},\dots e^{2\pi i\xi_m\cdot(\,\cdot\,)})$ is a well defined element of $\mathcal{D}'(\mathbf{R}^n)$.

Theorem 4. *Fix* $1 < q_1, \dots, q_m, q < \infty$ *with*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}. \tag{30}$$

Let T *be a continuous multilinear operator from* $\mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ *with kernel* K *in* m*-*$CZK(A, \varepsilon)$*. Then* T *has a bounded extension from* $L^{q_1} \times \cdots \times L^{q_m}$ *into* L^q *if and only if*

$$\sup_{\xi_1 \in \mathbf{R}^n} \dots \sup_{\xi_m \in \mathbf{R}^n} \|T(e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, e^{2\pi i \xi_m \cdot (\,\cdot\,)})\|_{BMO} \le B \tag{31}$$

and also

$$\sup_{\xi_1 \in \mathbf{R}^n} \dots \sup_{\xi_m \in \mathbf{R}^n} \|T^{*j}(e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, e^{2\pi i \xi_m \cdot (\,\cdot\,)})\|_{BMO} \le B \tag{32}$$

for all $j = 1, \dots, m$*. Moreover, if (31) and (32) hold then we have that*

$$\|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^q} \le c_{n,m,q_j}(A + B),$$

for some constant c_{n,m,q_j} *depending only on the parameters indicated.*

Proof. We begin the proof by observing that the necessity of conditions (31) and (32) follows from Proposition 1. The thrust of this theorem is provided by their sufficiency, i.e. the fact that if (31) and (32) hold, then T is extends to a bounded operator from $L^{q_1} \times \cdots \times L^{q_m}$ into L^q.

Let us say that T is *BMO-restrictedly bounded with bound* C if

$$\|T(\phi_1^{R_1,x_1}, \dots, \phi_m^{R_m,x_m})\|_{BMO} \le C < \infty,$$

and

$$\|T^{*j}(\phi_1^{R_1,x_1}, \dots, \phi_m^{R_m,x_m})\|_{BMO} \le C < \infty$$

for all $1 \le j \le m$, all ϕ_j normalized bumps, all $R_j > 0$, and all $x_j \in \mathbf{R}^n$.

We will need the following lemma whose proof we postpone until the end of this section.

Lemma 5. *If (31) and (32) are satisfied, then* T *is* BMO*-restrictedly bounded with bound a multiple of* $B > 0$*.*

We will now show by induction on m that if T is BMO-restrictedly bounded with bound $B > 0$, then it must map $L^{q_1} \times \cdots \times L^{q_m} \to L^q$ for some $1 < q, q_j < \infty$ satisfying

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q},$$

with norm controlled by a multiple of $(A + B)$.

To start the induction, we explain why this fact is true when $m = 1$. Note that for a point y outside the ball $B(x, 2R)$, the size estimate on the kernel of T gives

$$|T(\phi^{R,x})(y)| \le CAR^n |x - y|^{-n}. \tag{33}$$

By Lemma 5, the BMO norm of the function $T(\phi^{R,x})$ is bounded by a multiple of B. Pick z at distance $5R$ from x. As usual, let g_E denote the average of the function g on a ball E. Then,

$$\begin{aligned}\|T(\phi^{R,x})\|_{L^2(B(x,2R))} &\le \|T(\phi^{R,x}) - T(\phi^{R,x})_{B(x,2R)}\|_{L^2(B(x,2R))}\\ &\quad + \|T(\phi^{R,x})_{B(x,2R)} - T(\phi^{R,x})_{B(z,R)}\|_{L^2(B(x,2R))}\\ &\quad + \|T(\phi^{R,x})_{B(z,R)}\|_{L^2(B(x,2R))}\\ &\le cBR^{n/2} + cBR^{n/2} + cAR^{n/2},\end{aligned}$$

where we have used (33) and basic properties of BMO functions. The same computations apply to T^*. It follows that

$$\|T(\phi^{R,x})\|_{L^2} + \|T^*(\phi^{R,x})\|_{L^2} \le c(A+B)R^{n/2}. \tag{34}$$

As mentioned before, see Stein [23], this last condition implies that T maps L^2 into L^2 with bound a multiple of $(A+B)$. This completes the case $m=1$ of the induction.

Suppose now that the required conclusion of the BMO-restrictedly boundedness condition is valid for $(m-1)$-linear operators. Let T be an m-linear operator which is BMO-restrictedly bounded with bound $B>0$. Consider the $(m-1)$-linear operator

$$T_{\phi_m^{R_m,x_m}}(f_1,\dots,f_{m-1}) = T(f_1,\dots,f_{m-1},\phi_m^{R_m,x_m})$$

obtained from T by freezing an arbitrary normalized bump in the last entry. It is easy to see that $T_{\phi_m^{R_m,x_m}}$ satisfies the $(m-1)$-linear BMO-restrictedly boundedness condition with bound B, because of identities (16). The induction hypothesis implies that $T_{\phi_m^{R_m,x_m}}$ is bounded from $L^{q_1}\times\cdots\times L^{q_{m-1}}$ into L^q for some $1<q_j,q<\infty$ satisfying $1/q_1+\cdots+1/q_{m-1}=1/q$. Since ϕ is compactly supported, Lemma 3 gives that $T_{\phi_m^{R_m,x_m}}$ has a kernel in $(m-1)$-$CZK(A,\varepsilon)$. Theorem 3 *(v)* now gives that $T_{\phi_m^{R_m,x_m}}$ maps the $(m-1)$-fold product $L^\infty_c\times\cdots\times L^\infty_c$ into BMO with norm at most a multiple of $(A+B)$. Thus the estimate

$$\|T(g,\phi_2^{R_2,x_2},\dots,\phi_m^{R_m,x_m})\|_{BMO} \le c(A+B)\|g\|_{L^\infty} \tag{35}$$

holds for all $g\in L^\infty_c$. Similar estimates hold when the function g above appears in any other entry $2\le j\le m$.

Now for $1\le j\le m$ consider the operators T_{g_j} defined by

$$T_{g_j}(f_1,\dots,f_{m-1}) = T(f_1,\dots,f_{j-1},g_j,f_{j+1},\dots,f_{m-1}),$$

for functions $g_j\in L^\infty_c$. Inequality (35) is saying that T_{g_1} satisfies the $(m-1)$-linear BMO-restrictedly boundedness condition with constant a multiple of $(A+B)\|g_1\|_{L^\infty}$. Similar conclusions are valid for T_{g_j}. The inductive hypothesis implies that T_{g_j} maps

$$L^{q_1}\times\cdots\times L^{q_{j-1}}\times L^{q_{j+1}}\times\cdots\times L^{q_m}\to L^q$$

for some $1<q_k=q_k(j), q=q(j)<\infty$ satisfying

$$\sum_{\substack{1\le k\le m\\ k\ne j}} \frac{1}{q_k} = \frac{1}{q},$$

with bound a multiple of $A\|g_j\|_{L^\infty} + (A+B)\|g_j\|_{L^\infty}$. It follows that

$$T: \ L^{q_1} \times \cdots \times L^{q_{j-1}} \times L_c^\infty \times L^{q_{j+1}} \cdots \times L^{q_m} \to L^q,$$

with norm controlled by a multiple of $(A+B)$. Therefore, for $1 \le j \le m$ there exist points Q_j in the interior of the faces $\mathcal{S}_j$, as defined in Section 4, at which T satisfies strong type estimates with bound a multiple of $(A+B)$. Furthermore, Theorem 2 gives that T maps $L^1 \times \cdots \times L^1$ into $L^{1/m,\infty}$. We have observed that the $m+1$ points $Q_1, \dots, Q_m$, and E are affinely independent. We can then use Theorem A to interpolate between these points and obtain a point Q in the interior of $OC_1C_2\dots C_m$ at which T satisfies a strong type bound with the required constant. This concludes the proof of the theorem modulo the proof of Lemma 5. □

We now prove Lemma 5.

Proof. At a formal level the proof of this lemma is clear since we can write each bump as the inverse Fourier transform of its Fourier transform and interchange the integrations with the action of T to obtain

$$\begin{aligned} &T(\phi_1^{R_1,x_1}, \dots, \phi_m^{R_m,x_m}) \\ &= \int_{\mathbf{R}^n} \widehat{\phi_1^{R_1,x_1}}(\xi_1) \dots \int_{\mathbf{R}^n} \widehat{\phi_m^{R_1,x_m}}(\xi_m) T(e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, e^{2\pi i \xi_m \cdot (\,\cdot\,)})\, d\vec{\xi}. \end{aligned} \tag{36}$$

To justify this identity we provide the following argument.

Let us set $\phi_j^{R_j,x_j} = \phi_j$. Pick a smooth and compactly supported function g with mean value zero and let ψ_k be as in Lemma 1. Observe that $(\psi_k\phi_1, \dots, \psi_k\phi_m)$ converges to $(\phi_1, \dots, \phi_m)$ in $\mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n)$ and therefore

$$\lim_{k\to\infty} \langle T(\psi_k\phi_1, \dots, \psi_k\phi_m), g\rangle = \langle T(\phi_1, \dots, \phi_m), g\rangle.$$

The continuity and multilinearity of T also allow us to write

$$\begin{aligned} &\langle T(\phi_1, \dots, \phi_m), g\rangle \\ &= \lim_{k\to\infty} \int_{\mathbf{R}^n} \cdots \int_{\mathbf{R}^n} \widehat{\phi_1}(\xi_1) \dots \widehat{\phi_m}(\xi_m) \left\langle T\big(\psi_k e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, \psi_k e^{2\pi i \xi_m \cdot (\,\cdot\,)}\big), g\right\rangle d\vec{\xi}. \end{aligned} \tag{37}$$

Pick k_0 so that the support of g is contained in ball of radius 2^{k_0} centered at the origin. Set

$$F_k = T(\psi_k e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, \psi_k e^{2\pi i \xi_m \cdot (\,\cdot\,)}) + G(\psi_k e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, \psi_k e^{2\pi i \xi_m \cdot (\,\cdot\,)})$$

where G is defined in Lemma 1. Using that g has mean-value zero we obtain

$$\langle T(\psi_k e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, \psi_k e^{2\pi i \xi_m \cdot (\,\cdot\,)}), g\rangle = \langle F_k, g\rangle = \langle F_{k_0}, g\rangle + \langle F_k - F_{k_0}, g\rangle. \tag{38}$$

The proof of Lemma 1 gives that $|\langle F_k - F_{k_0}, g\rangle|$ is bounded uniformly on $\xi_1, \dots, \xi_m$ by a constant that depends on g. On the other hand

$$\begin{aligned} &\langle T(\psi_{k_0} e^{2\pi i \xi_1 \cdot (\,\cdot\,)}, \dots, \psi_{k_0} e^{2\pi i \xi_m \cdot (\,\cdot\,)}), g\rangle \\ =&\langle K, g \otimes \psi_{k_0} e^{2\pi i \xi_1 \cdot (\,\cdot\,)} \otimes \cdots \otimes \psi_{k_0} e^{2\pi i \xi_m \cdot (\,\cdot\,)}\rangle, \end{aligned} \tag{39}$$

where K is the Schwartz kernel of T. It follows that the expression in (39) is controlled by a finite sum of the L^∞ norms of derivatives of $g\,\psi_{k_0}e^{2\pi i\xi_1\cdot(\,\cdot\,)}\dots\psi_{k_0}e^{2\pi i\xi_m\cdot(\,\cdot\,)}$ on a compact set (that depends on g). This is in turn bounded by

$$C_g(1+|\xi_1|)^N\dots(1+|\xi_m|)^N$$

for some $N>0$ and some constant C_g depending on g. The Lebesgue dominated convergence theorem allows us to pass the limit inside the integrals in (37) to obtain

$$\langle T(\phi_1,\dots,\phi_m),g\rangle=\int_{\mathbf{R}^n}\dots\int_{\mathbf{R}^n}\widehat{\phi_1}(\xi_1)\dots\widehat{\phi_m}(\xi_m)\,\langle T\big(e^{2\pi i\xi_1\cdot(\,\cdot\,)},\dots,e^{2\pi i\xi_m\cdot(\,\cdot\,)}\big),g\rangle d\vec{\xi}.$$

Using the H^1-BMO duality we obtain that the distribution $T(\phi_1,\dots,\phi_m)$ can be identified with a BMO function satisfying

$$\|T(\phi_1,\dots,\phi_m)\|_{BMO}\le B\|\widehat{\phi_1}\|_{L^1}\dots\|\widehat{\phi_m}\|_{L^1}\le cB.$$

In the last inequality we used the fact that all the derivatives of the normalized bumps up to order $[n/2]+1$ are bounded and also the fact that $\|\widehat{\phi_j}\|_{L^1}=\|\widehat{\phi_j^{R_j,x_j}}\|_{L^1}$ is independent of $R_j>0$ and of $x_j\in\mathbf{R}^n$. □

We now mention another characterization of boundedness of multilinear operators with Calderón-Zygmund kernels. This formulation can be used in specific applications to justify formal computations involving the action of an operator on an m-tuple of characters.

Proposition 2. *Let ψ_k be as in Lemma 1. Fix $1<q_1,\dots,q_m,q<\infty$ with*

$$\frac{1}{q_1}+\dots+\frac{1}{q_m}=\frac{1}{q}. \tag{40}$$

Let T be a continuous multilinear operator from $\mathcal{S}(\mathbf{R}^n)\times\dots\times\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}'(\mathbf{R}^n)$ with kernel K in m-$CZK(A,\varepsilon)$. Then T has a bounded extension from $L^{q_1}\times\dots\times L^{q_m}$ into L^q if and only if

$$\sup_{k>0}\sup_{\xi_1\in\mathbf{R}^n}\dots\sup_{\xi_m\in\mathbf{R}^n}\|T(\psi_k e^{2\pi i\xi_1\cdot(\,\cdot\,)},\dots,\psi_k e^{2\pi i\xi_m\cdot(\,\cdot\,)})\|_{BMO}\le B \tag{41}$$

and also

$$\sup_{k>0}\sup_{\xi_1\in\mathbf{R}^n}\dots\sup_{\xi_m\in\mathbf{R}^n}\|T^{*j}(\psi_k e^{2\pi i\xi_1\cdot(\,\cdot\,)},\dots,\psi_k e^{2\pi i\xi_m\cdot(\,\cdot\,)})\|_{BMO}\le B \tag{42}$$

for all $j=1,\dots,m$. Moreover, if (41) and (42) hold then we have that

$$\|T\|_{L^{q_1}\times\dots\times L^{q_m}\to L^q}\le c_{n,m,q_j}(A+B),$$

for some constant c_{n,m,q_j} depending only on the parameters indicated.

Proof. Using our definition for $T(e^{2\pi i\xi_1\cdot(\,\cdot\,)},\dots,e^{2\pi i\xi_m\cdot(\,\cdot\,)})$, it follows that (41) and (42) imply (31) and (32) respectively. □

It is possible to obtain a version of Theorem 4 involving a certain multilinear weak boundedness property and the action of T and its transposes on the m-tuple $(1,\dots,1)$. In fact, using other methods, Christ and Journé established in [5] a multilinear $T1$

theorem for forms. Consider the $(m+1)$-linear form defined on functions in $\mathcal{D}(\mathbf{R}^n)$ via

$$U(f_1,\dots,f_m,f_{m+1}) = \langle T(f_1,\dots,f_m), f_{m+1}\rangle.$$

It is proved in [5] that the estimates

$$|U(f_1,\dots,f_{m+1})| \le C\big(\prod_{j\neq k,l}\|f_j\|_{L^\infty}\big)\|f_k\|_{L^2}\|f_l\|_{L^2}$$

are equivalent to imposing an appropriate multilinear weak boundedness condition on U, together with the hypotheses $U_j(1)\in BMO$. The distributions $U_j(1)$ are defined by $\langle U_j(1), g\rangle = U(1,\dots,1,g,1,\dots,1)$, with g, a test function with mean zero, in the j-position.

The version of the the $T1$ theorem we gave is more suitable for some applications. We end this section with an example. Other applications of our multilinear $T1$ theorem are given in the next section.

Example. Consider the class of multilinear pseudodifferential operators

$$T(f_1,\dots,f_m)(x) = \int_{\mathbf{R}^n}\cdots\int_{\mathbf{R}^n}\sigma(x,\vec{\xi})\widehat{f_1}(\xi_1)\dots\widehat{f_m}(\xi_m)\,e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}d\xi_1\dots d\xi_m$$

with symbols σ satisfying

$$|\partial_x^\alpha\partial_{\xi_1}^{\beta_1}\dots\partial_{\xi_m}^{\beta_m}\sigma(x,\xi_1,\dots,\xi_m)| \le C_{\alpha,\beta}(1+|\xi_1|+\cdots+|\xi_m|)^{|\alpha|-(|\beta_1|+\cdots+|\beta_m|)},$$

for all $\alpha,\beta_1,\dots,\beta_m$ n-tuples of nonnegative integers. We will denote the class of all such symbols by m-$S^0_{1,1}$. It is easy to see that such operators have kernels in m-CZK. For these operators we have that

$$T(e^{2\pi i\eta_1\cdot(\,\cdot\,)},\dots,e^{2\pi i\eta_m\cdot(\,\cdot\,)}) = \sigma(x,\eta_1,\dots,\eta_m)e^{2\pi i x\cdot(\eta_1+\cdots+\eta_m)},$$

which is uniformly bounded in $\eta_j\in\mathbf{R}^n$. It follows from Theorem 4 that a necessary and sufficient condition for T to map a product of L^p spaces into another Lebesgue space with the usual relation on the indices, is that $T^{*j}(e^{2\pi i\eta_1\cdot(\,\cdot\,)},\dots,e^{2\pi i\eta_m\cdot(\,\cdot\,)})$ are in BMO uniformly in $\eta_k\in\mathbf{R}^n$. In particular this is the case if all the transposes of T have symbols in m-$S^0_{1,1}$. Therefore we have obtained the following multilinear extension of a result of Bourdaud [2].

Corollary 1. *Let T be a multilinear pseudodifferential operator with symbol in the class m-$S^0_{1,1}$. Suppose that all of the transposes T^{*j} also have symbols in m-$S^0_{1,1}$. Then T extends as bounded operator from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p, when $1<p_j<\infty$ and*

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}. \tag{43}$$

Moreover, if one $p_j=1$, then T maps $L^{p_1}\times\cdots\times L^{p_m}$ into $L^{p,\infty}$ and in particular it maps $L^1\times\cdots\times L^1\to L^{1/m,\infty}$.

In general the symbols of the transposes of an operator with symbol in m-$S^0_{1,1}$ are hard to compute. Nevertheless, this can be explicitly achieved for the class of operators studied in the next section.

6. Translation invariant multilinear operators

Let $\tau_h(f)(x) = f(x-h)$ be the translation of a function f on $\mathbf{R}^n$ by $h \in \mathbf{R}^n$. We say that a multilinear operator T from $\mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ *commutes with translations*, or that it is *translation invariant*, if for all $f_1, \dots, f_m \in \mathcal{S}(\mathbf{R}^n)$ and all $h \in \mathbf{R}^n$ we have

$$\tau_h(T(f_1, \dots, f_m)) = T(\tau_h f_1, \dots, \tau_h f_m). \tag{44}$$

When $m = 1$, an operator that satisfies (44) and maps $L^p \to L^q$ for some $1 \leq p, q \leq \infty$ must be given by convolution with a tempered distribution K_0 on $\mathbf{R}^n$, i.e. it has the form

$$Tf(x) = (K_0 * f)(x).$$

An analogous result is true for multilinear operators.

Proposition 3. *Let T be a continuous multilinear operator originally defined from $\mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$. Assume that T commutes with translations and that it extends to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for some indices $1 \leq p_1, \dots, p_m, p \leq \infty$. Then there exists a tempered distribution K_0 on $(\mathbf{R}^n)^m$ such that for all $f_1, \dots, f_m$ in $\mathcal{S}(\mathbf{R}^n)$ we have*

$$T(f_1, \dots, f_m)(x) = \big(K_0 * (f_1 \otimes \cdots \otimes f_m)\big)(x, \dots, x), \tag{45}$$

where $*$ denotes convolution on $(\mathbf{R}^n)^m$, and

$$(f_1 \otimes \cdots \otimes f_m)(y_1, \dots, y_m) = f_1(y_1) \dots f_m(y_m).$$

Formally speaking, this proposition is saying that the Schwartz kernel K of T has the special form $K(x, y_1, \dots, y_m) = K_0(x - y_1, \dots, x - y_m)$.

Proof. We indicate the main ideas. Fix $f_1, \dots, f_m \in \mathcal{S}(\mathbf{R}^n)$. Using identity (10) and the property that T commutes with translations we obtain that

$$\frac{\partial}{\partial x_k} T(f_1, \dots, f_m) = \sum_{j=1}^{m} T(f_1, \dots, f_{j-1}, \frac{\partial}{\partial x_k} f_j, f_{j+1}, \dots, f_m),$$

and hence any distributional partial derivative of $T(f_1, \dots, f_m)$ is an L^p function. Then $T(f_1, \dots, f_m)$ agrees almost everywhere with a continuous function whose value at zero is controlled by a finite sum of L^p norms of derivatives of $T(f_1, \dots, f_m)$. Define a continuous multilinear functional L on $\mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n)$ by setting

$$L(f_1, \dots, f_m) = T(f_1, \dots, f_m)(0).$$

Since $(\mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n))'$ can be identified with $\mathcal{S}'((\mathbf{R}^n)^m)$, there exists a distribution $u \in \mathcal{S}'((\mathbf{R}^n)^m)$ such that

$$L(f_1, \dots, f_m) = \langle u, f_1 \otimes \cdots \otimes f_m \rangle.$$

Let $\widetilde{u}$ be the reflection of u, i.e. $\widetilde{u}(F) = u(\widetilde{F})$, where $\widetilde{F}(z) = F(-z)$. Then $K_0 = \widetilde{u}$ is the required distribution. $\square$

Using the Fourier transform we can write (at least in the distributional sense) the multilinear operator

$$T(f_1,\dots,f_m)(x) = \int_{(\mathbf{R}^n)^m} K_0(x-y_1,\dots,x-y_m) f_1(y_1)\dots f_m(y_m)\,d\vec{y},$$

as

$$T(f_1,\dots,f_m)(x) = \int_{(\mathbf{R}^n)^m} \sigma(\xi_1,\dots,\xi_m)\widehat{f_1}(\xi_1)\dots\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}, \tag{46}$$

where σ is the Fourier transform of K_0 in $(\mathbf{R}^n)^m$. Under this general setting, σ may be a distribution but we are only interested here in the case where σ is a function. We want to consider translation invariant operators which are given by (46), with σ a function, and which extend to bounded operators from some product of L^p spaces into another Lebesgue space. Observe that (46) is a priori well defined for $f_1,\dots,f_m$ in $\mathcal{S}(\mathbf{R}^n)$ when the function σ is locally integrable and has some tempered growth at infinity, i.e. it satisfies an estimate of the form

$$|\sigma(\xi_1,\dots,\xi_m)| \le C(|\xi_1|+\cdots+|\xi_m|)^N \tag{47}$$

when $|\xi_1|+\cdots+|\xi_m| > R$ for some $C,N,R>0$. In the sequel, whenever we write (46), we will assume that σ is locally integrable and satisfies (47).

Definition. A locally integrable function σ defined on $(\mathbf{R}^n)^m$ and satisfying (47) is called a $(p_1,\dots,p_m,p)$ *multilinear multiplier* if the corresponding operator T given by (46) extends to a bounded operator from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$. We denote by $\mathcal{M}_{p_1,\dots,p_m,p}(\mathbf{R}^n)$ the space of all $(p_1,\dots,p_m,p)$ multilinear multipliers on $\mathbf{R}^n$. We define the norm of σ in $\mathcal{M}_{p_1,\dots,p_m,p}(\mathbf{R}^n)$ to be the norm of the corresponding operator T from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p, i.e.

$$\|m\|_{\mathcal{M}_{p_1,\dots,p_m,p}} = \|T_m\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p}.$$

In view of the correspondence between kernels K_0 and multipliers σ, multilinear operators which commute with translations will also be called multilinear multiplier operators. It is natural to ask whether the symbols of multilinear multiplier operators which are bounded from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p, where the indices satisfy

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}, \tag{48}$$

are themselves bounded functions. This is of course the case when $m=1$, since such operators are always L^2 bounded. The following theorem gives some basic properties of multilinear multipliers and in particular answers this question.

Proposition 4. *The following are true:*

(i) If $\lambda\in\mathbf{C}$, σ,σ_1 and σ_2 are in $\mathcal{M}_{p_1,\dots,p_m,p}$, then so are $\lambda\sigma$ and $\sigma_1+\sigma_2$, and

$$\|\lambda\sigma\|_{\mathcal{M}_{p_1,\dots,p_m,p}} = |\lambda|\|\sigma\|_{\mathcal{M}_{p_1,\dots,p_m,p}},$$

$$\|\sigma_1+\sigma_2\|_{\mathcal{M}_{p_1,\dots,p_m,p}} \le C_p\big(\|\sigma_1\|_{\mathcal{M}_{p_1,\dots,p_m,p}}+\|\sigma_2\|_{\mathcal{M}_{p_1,\dots,p_m,p}}\big).$$

(ii) If $\sigma(\xi_1,\dots,\xi_m)\in\mathcal{M}_{p_1,\dots,p_m,p}$ and $\tau_1,\dots,\tau_m\in\mathbf{R}^n$, then $\sigma(\xi_1+\tau_1,\dots,\xi_m+\tau_m)$ is in $\mathcal{M}_{p_1,\dots,p_m,p}$ with the same norm.

(iii) *If* $\sigma(\xi_1,\dots,\xi_m) \in \mathcal{M}_{p_1,\dots,p_m,p}$ *and* $\delta > 0$, *then* $\delta^{n(\frac{1}{p_1}+\dots+\frac{1}{p_m}-\frac{1}{p})}\sigma(\delta\xi_1,\dots,\delta\xi_m)$ *is in* $\mathcal{M}_{p_1,\dots,p_m,p}$ *with the same norm.*

(iv) *If* $\sigma(\xi_1,\dots,\xi_m) \in \mathcal{M}_{p_1,\dots,p_m,p}$ *and* A *is an orthogonal matrix in* $\mathbf{R}^n$, *then* $\sigma(A\xi_1,\dots,A\xi_m)$ *is in* $\mathcal{M}_{p_1,\dots,p_m,p}$ *with the same norm.*

(v) *Let* σ_j *be a sequence of functions in* $\mathcal{M}_{p_1,\dots,p_m,p}$ *such that* $\|\sigma_j\|_{\mathcal{M}_{p_1,\dots,p_m,p}} \le C$ *for all* $j = 1, 2, \dots$. *If* σ_j *are uniformly bounded by a locally integrable function on* $\mathbf{R}^n$, *they satisfy (47) uniformly in* j, *and they converge pointwise to* σ *a.e. as* $j \to \infty$, *then* σ *is in* $\mathcal{M}_{p_1,\dots,p_m,p}$ *with norm bounded by* C.

(vi) *Assume that for all* ξ_j *and some* $N > 0$ *we have*

$$|\sigma(\xi_1,\dots,\xi_m)| \le C(1 + |\xi_1| + \dots + |\xi_m|)^N$$

and that σ *is in* $\mathcal{M}_{p_1,\dots,p_m,p}$, *for some* $1 \le p_1,\dots,p_m \le \infty$ *and* $0 < p < \infty$ *satisfying (48). Then* σ *is a bounded function with* L^∞ *norm less than its* $\mathcal{M}_{p_1,\dots,p_m,p}$ *norm and thus* $\mathcal{M}_{p_1,\dots,p_m,p}$ *can be naturally embedded in* L^∞.

(vii) *Let* $1 \le p_1,\dots,p_m \le \infty$ *and* $0 < p < \infty$ *satisfying (48). Then the spaces* $\mathcal{M}_{p_1,\dots,p_m,p}(\mathbf{R}^n)$ *are complete, and thus they are Banach spaces when* $p \ge 1$ *and quasi-Banach spaces when* $p < 1$.

Proof. *(i)-(iv)* are straightforward. *(v)* easily follows from the Lebesgue dominated convergence theorem and Fatou's lemma, while *(vii)* is a consequence of *(v)*. We prove *(vi)*. Let B be the norm of $T : L^{p_1} \times \dots \times L^{p_m} \to L^p$. Let us first assume that σ is a C^∞ function. This assumption can be disposed using suitable regularization. For fixed $\vec{a} = (a_1,\dots,a_m) \in (\mathbf{R}^n)^m$ and $f_1,\dots,f_m \in \mathcal{S}(\mathbf{R}^n)$ we have that

$$\int_{(\mathbf{R}^n)^m} \widehat{f_1}(\xi_1)\dots\widehat{f_m}(\xi_m)\sigma(\vec{a}+\varepsilon\vec{\xi})e^{2\pi i x\cdot(\xi_1+\dots+\xi_m)}d\vec{\xi}$$

converges to $\sigma(\vec{a})f_1(x)\dots f_m(x)$ as $\varepsilon \to 0$. Moreover the functions $\sigma(\vec{a}+\varepsilon\vec{\xi})$ are in $\mathcal{M}_{p_1,\dots,p_m,p}$ uniformly in $\vec{a}$ and $\varepsilon > 0$. Fatou's lemma (recall $p < \infty$) and the fact that σ is in $\mathcal{M}_{p_1,\dots,p_m,p}$ give that

$$|\sigma(\vec{a})|\,\|f_1\dots f_m\|_{L^p} \le B\|f_1\|_{L^{p_1}}\dots\|f_m\|_{L^{p_m}}.$$

Picking $f_1 = \dots = f_m$ we obtain the required conclusion. □

We also have the following result, whose linear version was obtained by Hörmander [12].

Proposition 5. *Suppose that a multilinear multiplier operator* T *has a compactly supported kernel and maps the* m*-fold product* $L^{p_1} \times \dots \times L^{p_m}$ *into* L^p, *where* $1 < p_j < \infty$ *and* $0 < p < \infty$. *Then*

$$p \ge \left(\tfrac{1}{p_1} + \dots + \tfrac{1}{p_m}\right)^{-1}. \tag{49}$$

Proof. Fix $f_1,\dots,f_m \in \mathcal{D}(\mathbf{R}^n)$. Then,

$$\begin{aligned} T(f_1+\tau_h f_1,\dots,f_1+\tau_h f_m) &= T(f_1,\dots,f_m) + T(\tau_h f_1,\dots,\tau_h f_m) \\ &= T(f_1,\dots,f_m) + \tau_h(T(f_1,\dots,f_m)) \end{aligned}$$

for h sufficiently large. Taking L^p norms and letting h goes to infinity we obtain

$$2^{\frac{1}{p}}\|T(f_1,\dots,f_m)\|_{L^p} \le 2^{\frac{1}{p_1}+\dots+\frac{1}{p_m}}\|T\|\,\|f_1\|_{L^{p_1}}\dots\|f_m\|_{L^{p_m}},$$

which implies (49). □

As examples of operators that map $L^{p_1}\times L^{p_2}\to L^p$ when $p>(\frac{1}{p_1}+\frac{1}{p_2})^{-1}$, we mention the bilinear fractional integrals

$$I_\alpha(f_1,f_2)(x) = \int_{|t|\le 1} f(x+t)g(x-t)\,|t|^{\alpha-n}\,dt.$$

These operators map $L^{p_1}(\mathbf{R}^n)\times L^{p_2}(\mathbf{R}^n)\to L^p(\mathbf{R}^n)$ when $0<\alpha<n$, $1<p_1,p_2<\infty$, and

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{\alpha}{n}+\frac{1}{p}.$$

See the articles by Grafakos and Kalton [11] and also by Kenig and Stein [15] for details.

It is very natural to ask for sufficient conditions on bounded functions σ on $(\mathbf{R}^n)^m$ so that the corresponding operators are continuous from $L^{p_1}\times\dots\times L^{p_m}$ into L^p, when the indices satisfy (48). When $m=1$, the classical Hörmander-Mihlin multiplier theorem says that if a function σ on $\mathbf{R}^n$ satisfies

$$|\partial^\alpha\sigma(\xi)|\le C_\alpha|\xi|^{-|\alpha|}$$

for $|\alpha|\le[n/2]+1$, then σ is an L^p multiplier for $1<p<\infty$. The multilinear analogue of the Hörmander-Mihlin multiplier theorem was obtained by Coifman and Meyer when $p>1$. The point of the next proposition is the extension of this result to the range $p>1/m$.

Proposition 6. *Suppose that $a(\xi_1,\dots,\xi_m)$ is a C^∞ function on $(\mathbf{R}^n)^m-\{0\}$ which satisfies*

$$|\partial_{\xi_1}^{\beta_1}\dots\partial_{\xi_m}^{\beta_m}a(\xi_1,\dots,\xi_m)|\le C_{\beta_1,\dots,\beta_m}(|\xi_1|+\dots+|\xi_m|)^{-(|\beta_1|+\dots+|\beta_m|)} \tag{50}$$

for all multiindices $\beta_1,\dots,\beta_m$. Let T be as in (46). Then T is a bounded operator from $L^{p_1}\times\dots\times L^{p_m}$ into L^p, when $1<p_j<\infty$ and

$$\frac{1}{p_1}+\dots+\frac{1}{p_m}=\frac{1}{p}. \tag{51}$$

Moreover, if one $p_j=1$, then T maps $L^{p_1}\times\dots\times L^{p_m}$ into $L^{p,\infty}$ and in particular it maps $L^1\times\dots\times L^1\to L^{1/m,\infty}$.

Proof. First we observe that conditions (50) easily imply that the inverse Fourier transform of a, satisfies

$$|\partial_{\xi_1}^{\beta_1}\dots\partial_{\xi_m}^{\beta_m}a^\vee(x_1,\dots,x_m)|\le C_{\beta_1,\dots,\beta_m}(|x_1|+\dots+|x_m|)^{-(mn+|\beta_1|+\dots+|\beta_m|)} \tag{52}$$

for all multiindices $\beta_1,\dots,\beta_m$. It follows that the kernel

$$K(x,y_1,\dots,y_m)=a^\vee(x-y_1,\dots,x-y_m)$$

of the operator T satisfies the required size and smoothness conditions (4), (5), and (6). The $L^{p_1}\times\dots\times L^{p_m}\to L^p$ boundedness of T for a fixed point $(1/p_1,\dots,1/p_m,1/p)$

satisfying (51) will follow from the multilinear $T1$ theorem (Theorem 4) once we have verified the required BMO conditions. As in the example in the previous section, we have that

$$T(e^{2\pi i\eta_1\cdot(\,\cdot\,)},\dots,e^{2\pi i\eta_m\cdot(\,\cdot\,)})(x)=a(\eta_1,\dots,\eta_m)e^{2\pi i x\cdot(\eta_1+\cdots+\eta_m)}$$

which is in L^∞ and thus in BMO uniformly in $\eta_1,\dots\eta_m$. The same calculation is valid for the m transposes of T since their corresponding multipliers also satisfy (50). The weak type results follow from Theorem 1. □

We now turn our attention to sufficient conditions on a singular kernel K_0 so that the corresponding translation invariant operator

$$T(f_1,\dots,f_m)(x)=\int_{\mathbf{R}^n}\dots\int_{\mathbf{R}^n}K_0(x-y_1,\dots,x-y_m)f_1(y_1)\dots f_m(y_m)d\vec{y} \tag{53}$$

maps $L^{p_1}\times\cdots\times L^{p_m}$ into L^p when the indices satisfy (51). The next theorem gives a satisfactory sufficient condition. In what follows $|(u_1,\dots,u_m)|$ will denote the euclidean norm of $\vec{u}=(u_1,\dots,u_m)$ thought as an element in $\mathbf{R}^{nm}$.

Theorem 5. *Let $K_0(u_1,\dots,u_m)$ be a locally integrable function on $(\mathbf{R}^n)^m-\{0\}$ which satisfies the size estimate*

$$|K_0(u_1,\dots,u_m)|\le A|(u_1,\dots,u_m)|^{-nm}, \tag{54}$$

the cancellation condition

$$\left|\int_{R_1<|(u_1,\dots,u_m)|<R_2}K_0(u_1,\dots,u_m)\,d\vec{u}\right|\le A<\infty, \tag{55}$$

for all $0<R_1<R_2<\infty$, and the smoothness condition

$$|K_0(u_1,\dots,u_j,\dots,u_m)-K_0(u_1,\dots,u_j',\dots,u_m)|\le A\frac{|u_j-u_j'|^\varepsilon}{|(u_1,\dots,u_m)|^{nm+\varepsilon}}, \tag{56}$$

whenever $|u_j-u_j'|<\frac{1}{2}|u_j|$. Suppose that for some sequence $\varepsilon_j\downarrow 0$ the limit

$$\lim_{j\to\infty}\int_{\varepsilon_j<|\vec{u}|\le 1}K_0(u_1,\dots,u_m)\,d\vec{u}$$

exists, and therefore K_0 extends to a tempered distribution on $(\mathbf{R}^n)^m$. Then the multilinear operator T given by (53) maps $L^{p_1}\times\cdots\times L^{p_m}$ into L^p when $1<p_j<\infty$ and (51) is satisfied. Moreover, if one $p_j=1$, then it maps $L^{p_1}\times\cdots\times L^{p_m}$ into $L^{p,\infty}$.

Proof. We will use the following well-known result (see for instance the article of Benedek, Calderón, and Panzone [1]).

Let L be a locally integrable function on $\mathbf{R}^N-\{0\}$ with the following properties:

(i) $|L(z)|\le A|z|^{-N}$,
(ii) $\left|\int_{R_1\le|z|\le R_2}L(z)\,dz\right|\le A$ uniformly in $0<R_1<R_2<\infty$,
(iii) $\int_{|z|\ge 2|w|}|L(z-w)-L(z)|\,dz\le A$
(iv) $\lim\limits_{j\to\infty}\int_{\varepsilon_j<|z|\le 1}L(z)\,dz$ exists.

Then L extends to a distribution on $\mathbf{R}^N$ whose Fourier transform is a bounded function (with L^∞ norm controlled by a multiple of A).

We will prove this theorem using Theorem 4. As in the previous application of this theorem, we have (with some formal computations that are easily justified using Proposition 2)

$$T(e^{2\pi i\eta_1\cdot(\,\cdot\,)},\dots,e^{2\pi i\eta_m\cdot(\,\cdot\,)})(x)=e^{2\pi ix\cdot(\eta_1+\cdots+\eta_m)}\widehat{K_0}(\eta_1,\dots,\eta_m),$$

which is a bounded function, hence in BMO. The calculations with the transposes are similar; for example

$$T^{*1}(e^{2\pi i\eta_1\cdot(\,\cdot\,)},\dots,e^{2\pi i\eta_m\cdot(\,\cdot\,)})(x)=e^{2\pi ix\cdot(\eta_1+\cdots+\eta_m)}\widehat{K_0}(-\eta_1-\cdots-\eta_m,\eta_2,\dots,\eta_m)$$

which is in BMO. □

The case $n=1$ and $m=2$ in the corollary below was studied by Coifman and Meyer [6].

Corollary 2. *The result of Theorem 5 holds if K_0 has the form*

$$K_0(u_1,\dots,u_m)=\frac{\Omega\left(\dfrac{(u_1,\dots,u_m)}{|(u_1,\dots,u_m)|}\right)}{|(u_1,\dots,u_m)|^{mn}},$$

where Ω is an integrable function with mean value zero on the sphere $\mathbf{S}^{nm-1}$ which is Lipschitz of order $\varepsilon>0$.

Example. Let R_1 be *the bilinear Riesz transform in the first variable*

$$R_1(f_1,f_2)(x)=\text{p.v.}\int_{\mathbf{R}}\int_{\mathbf{R}}\frac{x-y_1}{|(x-y_1,x-y_2)|^3}f_1(y_1)f_2(y_2)\,dy_1dy_2.$$

By Corollary 2, this operator maps $L^{p_1}(\mathbf{R})\times L^{p_2}(\mathbf{R})$ into $L^p(\mathbf{R})$ for $1/p_1+1/p_2=1/p$, $1<p_1,p_1<\infty$, $1/2<p<\infty$. It also maps $L^1\times L^1$ into $L^{1/2,\infty}$. However, it does not map $L^1\times L^1$ into $L^{1/2}$. In fact, letting $f_1=f_2=\chi_{[0,1]}$, an easy computation shows that $R_1(f_1,f_2)(x)$ behaves at infinity like $|x|^{-2}$.

It is also natural to ask whether the corollary above is true under less stringent conditions on the function Ω. For instance, is the conclusion of Corollary 2 true when Ω is an odd function in $L^1(\mathbf{S}^{nm-1})$? It is a classical result obtained by Calderón and Zygmund [4] using the method of rotations, that homogeneous linear singular integrals with odd kernels are always L^p bounded for $1<p<\infty$.

We now indicate what happens if the method of rotation is used in the multilinear setting. Let Ω be an odd integrable function on $\mathbf{S}^{nm-1}$. Using polar coordinates in $\mathbf{R}^{nm}$ we can write

$$T(f_1,\dots,f_m)(x)=\int_{\mathbf{S}^{mn-1}}\Omega(\theta_1,\dots,\theta_m)\left\{\int_0^{+\infty}f_1(x-t\theta_1)\dots f_m(x-t\theta_m)\frac{dt}{t}\right\}d\vec{\theta}.$$

Replacing θ by $-\theta$, changing variables, and using that Ω is odd we obtain

$$T(f_1,\dots,f_m)(x)=\int_{\mathbf{S}^{mn-1}}\Omega(\theta_1,\dots,\theta_m)\left\{\int_0^{+\infty}f_1(x+t\theta_1)\dots f_m(x+t\theta_m)\frac{dt}{t}\right\}d\vec{\theta}.$$

Averaging these two identities we conclude that

$$T(f_1,\dots,f_m)(x) = \frac{1}{2}\int_{\mathbf{S}^{mn-1}} \Omega(\theta_1,\dots,\theta_m)\left\{\int_{-\infty}^{+\infty} f_1(x-t\theta_1)\dots f_m(x-t\theta_m)\frac{dt}{t}\right\} d\vec{\theta}.$$

To be able to complete the method of rotations we need to know whether the operator inside the curly brackets above is uniformly bounded in $\vec{\theta}\in\mathbf{S}^{mn-1}$. We call the operator

$$\mathcal{H}_{\vec{\theta}}(f_1,\dots,f_m)(x) = \int_{-\infty}^{+\infty} f_1(x-t\theta_1)\dots f_m(x-t\theta_m)\frac{dt}{t}$$

the *directional m-linear Hilbert transform* (in the direction $\vec{\theta}$).

The observations above involving the method of rotations motivate the following

Question. Is the operator $\mathcal{H}_{\vec{\theta}}$ bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ uniformly in $\vec{\theta}$ when $1<p_1,\dots,p_m,p<\infty$ satisfy (51)?

Some progress has been recently achieved on this question by Thiele and independently by Grafakos and Li, for $m=2$ and $n=1$.

The boundedness of the directional bilinear Hilbert transforms in dimension one was recently obtained by Lacey and Thiele [16], [17] with constants depending on the direction.

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