Analytic families of multilinear operators

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Abstract

We prove complex interpolation theorems for analytic families of multilinear operators defined on quasi-Banach spaces, with explicit constants on the intermediate spaces. We obtain analogous results for analytic families of operators defined on spaces generated by the Calderón method applied to couples of quasi-Banach lattices with nontrivial lattice convexity. As an application we derive a multilinear version of Stein's classical interpolation theorem for analytic families of operators taking values in Lebesgue, Lorentz, and Hardy spaces. We use this theorem to prove that the biliner Bochner-Riesz operator is bounded from $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ into $L^{p/2}(\mathbb{R}^n)$ for $1 < p < 2$.

1 Introduction

Stein's interpolation theorem [19] for analytic families of operators between L^p spaces ($p \geq 1$) has found several significant applications in harmonic analysis. This theorem provides a generalization of the classical single-operator Riesz-Thorin interpolation theorem to a family $\{T_z\}$ of operators that depend analytically on a complex variable z. This theorem has been extended to analytic families defined on quasi-Banach spaces and taking values in Lebesgue spaces (or more general quasi-Banach function lattices) by Cwikel and Sagher [6].

The aim of this paper is to prove a version of Stein's interpolation theorem for analytic families of multilinear operators defined on products of quasi-Banach spaces and taking values in quasi-Banach function lattices. In the framework of Banach spaces, interpolation for analytic families of multilinear operators can be obtained via duality in a way similar to that used in the linear case [19]. For instance, one may adapt the proofs in Zygmund [22, Chapter XII, (3.3)] and Berg and Löfstrom [3, Theorem 4.4.2] for a single multilinear operator to a family of multilinear operators. However, this duality-based approach is not applicable to quasi-Banach spaces since their topological dual spaces may be trivial. Motivated by important applications of multilinear interpolation in the context of certain

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quasi-Banach spaces, we introduce an appropriate notion of analytic families of multilinear operators taking values in quasi-Banach function lattices on a measure space, which avoids the duality problem. The influence of the work of Calderón [4] and Sagher [17, 18] on this paper is considerable.

We introduce notation from interpolation theory relevant for this work. Throughout this paper, the open strip $\{z;\, 0 < \operatorname{Re} z < 1\}$ in the complex plane is denoted by S, its closure by $\overline{S}$ and its boundary by ∂S. Let $A(S)$ be the space of scalar-valued functions, analytic in S and continuous and bounded in $\overline{S}$. For a given couple (A_0, A_1) of quasi-Banach spaces and A another quasi-Banach space satisfying $A \subset A_0 \cap A_1$, we denote by $\mathcal{F}(A)$ the space of all functions $f\colon S \to A$ that can be written as finite sums of the form

$$f(z) = \sum_{k=1}^{N} \varphi_k(z) a_k, \qquad z \in \overline{S},$$

where $a_k \in A$ and $\varphi_k \in A(S)$. For every $f \in \mathcal{F}(A)$ we set

$$\|f\|_{\mathcal{F}(A)} = \max\big\{\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0},\, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1}\big\}.$$

For every $\theta \in (0,1)$ we define on $A_0 \cap A_1$ the following quasi-seminorm: for $a \in A_0 \cap A_1$ set

$$\|a\|_\theta = \inf\big\{\|f\|_{\mathcal{F}(A_0\cap A_1)};\, f \in \mathcal{F}(A_0 \cap A_1),\, f(\theta) = a\big\}.$$

Clearly we have that $\|a\|_\theta \le \|a\|_{A_0\cap A_1}$ for every $a \in A_0 \cap A_1$, and notice that $\|\cdot\|_\theta$ could be identically zero (see [20, §3]).

A quasi-Banach couple is said to be *admissible* whenever for all $\theta \in (0,1)$, $\|\cdot\|_\theta$ is a quasi-norm on $A_0 \cap A_1$, and in this case, the quasi-normed space $(A_0 \cap A_1, \|\cdot\|_\theta)$ is denoted by $(A_0, A_1)_\theta$.

We will make use the of following result (see [18, Theorem 1]) which states that if A is dense in $A_0 \cap A_1$, then for $a \in A$ we have

$$\|a\|_\theta = \inf\{\|f\|_{\mathcal{F}(A)};\, f \in \mathcal{F}(A),\, f(\theta) = a\}.$$

If in addition there is a completion of $(A_0, A_1)_\theta$ which is set-theoretically contained in $A_0 + A_1$, then it is denoted by $[A_0, A_1]_\theta$. We refer to [7, 10, 17, 18], where complex interpolation of certain quasi-Banach spaces is studied.

Notice that if $f \in \mathcal{F}(A_0 \cap A_1)$, and $0 < \theta < 1$ then the following important estimate is well known in the case where A_0, A_1 are Banach spaces (see [4]),

$$\log \|f(\theta)\|_\theta \le \int_{-\infty}^{\infty} \log\, \|f(it)\|_{A_0} P_0(\theta, t)\, dt + \int_{-\infty}^{\infty} \log\, \|f(1+it)\|_{A_1} P_1(\theta, t)\, dt, \tag{1}$$

where $P_0(\theta, t)$ and $P_1(\theta, t)$ are the values of the Poisson kernels of the strip, on $\operatorname{Re} z = 0$ and $\operatorname{Re} z = 1$, respectively. The same estimate holds in the case of quasi-Banach spaces; the proof is similar to the Banach space case (see [4] or [3, Lemma 4.3.2]).

We recall that the Poisson kernels P_j ($j = 0, 1$) for the strip are obtained from the Poisson kernel for the unit disc by means of a conformal mapping and are given by

$$P_j(x + iy, t) = \frac{e^{-\pi(t-y)} \sin \pi x}{\sin^2 \pi x + (\cos \pi x - (-1)^j e^{-\pi(t-y)})^2}, \quad x + iy \in \overline{S}.$$

Using the fact that the Poisson kernels satisfy

$$\int_{\mathbb{R}} P_0(\theta, t)\, dt = 1 - \theta, \qquad \int_{\mathbb{R}} P_1(\theta, t)\, dt = \theta$$

together with (1), Jensen's inequality, and the concavity of the logarithmic function, we obtain the following result:

Lemma 1.1. *Let (A_0, A_1) be a couple of complex quasi-Banach spaces. For every f in $\mathcal{F}(A_0 \cap A_1)$, $0 < p_0, p_1 < \infty$, and $0 < \theta < 1$ we have*

$$\|f(\theta)\|_\theta \le \Big(\frac{1}{1-\theta} \int_{-\infty}^{\infty} \|f(it)\|_{A_0}^{p_0} P_0(\theta, t)\, dt\Big)^{\frac{1-\theta}{p_0}} \Big(\frac{1}{\theta} \int_{-\infty}^{\infty} \|f(1+it)\|_{A_1}^{p_1} P_1(\theta, t)\, dt\Big)^{\frac{\theta}{p_1}}.$$

Before we begin our discussion of analytic families of multilinear operators we recall (see [9, 19]) that a continuous function $F\colon \overline{S} \to \mathbb{C}$ which is analytic in S is said to be of *admissible growth* if there is $0 \le \alpha < \pi$ such that

$$\sup_{z \in \overline{S}} \frac{\log |F(z)|}{e^{\alpha |\operatorname{Im} z|}} < \infty.$$

The following lemma due to Hirschman [9] (see also [8]) will play a key role hereby.

Lemma 1.2. *If a function $F\colon \overline{S} \to \mathbb{C}$ is analytic in S, continuous on $\overline{S}$, and is of admissible growth, then for all $\theta \in (0, 1)$ we have*

$$\log |F(\theta)| \le \int_{-\infty}^{\infty} \log |F(it)|\, P_0(\theta, t)\, dt + \int_{-\infty}^{\infty} \log |F(1 + it)|\, P_1(\theta, t)\, dt.$$

All measure spaces considered throughout this paper will be complete and σ-finite. Let (Ω, Σ, μ) be a complete σ-finite measure space and let $L^0(\mu)$ (resp., $\widetilde{L}^0(\mu)$) denote the space of all equivalence classes of real-valued (resp., complex-valued) measurable functions on Ω with the topology of convergence in measure on μ-finite sets. A quasi-Banach (function) lattice X on (Ω, Σ, μ) is a subspace of $L^0(\mu)$, which is complete with respect to a quasi-norm $\|\cdot\|$ and it is a solid subspace in $L^0(\mu)$, i.e., it has the property: $f \in L^0(\mu)$, $g \in X$ $|f| \le |g|$

a.e. implies $f \in X$ and $\|f\|_X \leq \|g\|_X$; moreover we will assume that there exists $u \in X$ with $u > 0$ a.e..

If X is a quasi-Banach lattice on (Ω, Σ, μ) and $w \in L^0(\mu)$ is strictly positive a.e., then we define $X(w)$ to be the quasi-Banach lattice of all $f \in L^0(\mu)$ such that $fw \in X$, equipped with the quasi-norm $\|f\|_{X(w)} = \|fw\|_X$.

If X is a quasi-Banach lattice, we will use the same letter X to denote its *complexification*, namely, the space of all $f \in \widetilde{L}^0(\mu)$ such that $|f| \in X$, with the quasi-norm $\|f\|_X = \big\||f|\big\|_X$. A quasi-Banach lattice X is said to be maximal (or X has the Fatou property) whenever $0 \leq f_n \uparrow f$ a.e., $f_n \in X$, and $\sup_{n\geq 1} \|f_n\|_X < \infty$ implies that $f \in X$ and $\|f_n\|_X \to \|f\|_X$.

The Köthe dual space X' of a quasi-Banach lattice X on (Ω, Σ, μ) is defined as the space of all $f \in L^0(\mu)$ such that $\int_\Omega |fg|\, d\mu < \infty$ for every $g \in X$. It is a Banach lattice on (Ω, Σ, μ) when equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \leq 1} \int_\Omega |fg|\, d\mu\,.$$

In certain cases X' could be trivial, for instance, if $X = L^p$ on a nonatomic measure space with $0 < p < 1$, then $(L^p)'$ is trivial. Notice that X is a maximal Banach lattice if and only if $X = X'' := (X')'$ with equality of norms (see, e.g., [14, p. 30]).

If X_0 and X_1 are quasi-Banach lattices on a given measure space (Ω, Σ, μ) and $0 < \theta < 1$, we define the quasi-Banach lattice $X_\theta = X_0^{1-\theta} X_1^\theta$ to be the space of all $f \in L^0(\mu)$ such that $|f| \leq |f_0|^{1-\theta}|f_1|^\theta$ μ-a.e. for some $f_i \in X_i$ $(i = 0, 1)$ and equipped with the quasi-norm

$$\|f\|_{X_\theta} = \inf\big\{\|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^\theta;\ |f| \leq |f_0|^{1-\theta}|f_1|^\theta\ \mu\text{-a.e.}\big\}.$$

A quasi-Banach lattice X is said to be *p-convex* $(0 < p < \infty)$ if there exists a constant $C > 0$ such that for any $f_1,\ldots,f_n \in X$ we have

$$\Big\|\Big(\sum_{k=1}^n |f_k|^p\Big)^{1/p}\Big\|_X \leq C\Big(\sum_{k=1}^n \|f_k\|_X^p\Big)^{1/p}.$$

The optimal constant C in this inequality is called the p-convexity constant of X, and is denoted, by $M^{(p)}(X)$. A quasi-Banach lattice is said to have nontrivial convexity whenever it is p-convex for some $0 < p < \infty$.

We now introduce the concept of analytic families of multilinear operators with respect to a measure. Let (Ω, Σ, μ) be a measure space and let $\mathcal{X}_1,\ldots,\mathcal{X}_m$ be linear spaces. We assume that for every $z \in \overline{S}$ there is a multilinear operator $T_z\colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to \widetilde{L}^0(\mu)$. The family $\{T_z\}_{z\in\overline{S}}$ is said to be *analytic* if for any $(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ and for almost every $\omega \in \Omega$ the function

$$z \mapsto T_z(x_1, \ldots, x_m)(\omega), \quad z \in \overline{S} \tag{2}$$

is analytic in S and continuous on $\overline{S}$. Additionally, if for $j = 0$ and $j = 1$ the function

$$(t, \omega) \mapsto T_{j+it}(x_1, ..., x_n)(\omega), \quad (t, \omega) \in \mathbb{R} \times \Omega$$

is $(\mathcal{L} \times \Sigma)$-measurable for every $(x_1, ..., x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, and for almost every $\omega \in \Omega$ the function (2) is of admissible growth, then the family $\{T_z\}$ is said to be an *admissible analytic family.* Here $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$.

We now state the main result of this article.

Theorem 1.1. *For each $1 \le i \le m$, let $\overline{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j-convex for $j = 0, 1$. Assume that $\mathcal{X}_i$ is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \le i \le m$, and that $\{T_z\}_{z \in \overline{S}}$ is an admissible analytic family of multilinear operators $T_z \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to Y_0 \cap Y_1$. Suppose that for every $(x_1, ..., x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$,*

$$\|T_{j+it}(x_1, ..., x_m)\|_{Y_j} \le K_j(t) \|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot)\, dt)$ for all $\theta \in (0, 1)$.

Then for all $(x_1, ..., x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, all $s \in \mathbb{R}$, and all $0 < \theta < 1$ we have

$$\|T_{\theta+is}(x_1, ..., x_m)\|_{Y_0^{1-\theta} Y_1^{\theta}} \le \big(M^{(p_0)}(Y_0)\big)^{1-\theta} \big(M^{(p_1)}(Y_1)\big)^{\theta} K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta}, \qquad (3)$$

where

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s)\, dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s)\, dt. \qquad (4)$$

2 Quasi-Banach lattices and the proof of the main theorem

Before proving our main theorem, we make some remarks concerning quasi-Banach lattices.

Suppose we are given a quasi-Banach lattice X on (Ω, Σ, μ). For any $0 < s < \infty$ we define X^s to be a quasi-Banach lattice of all $x \in L^0(\mu)$ such that $|x|^s \in X$ equipped with the quasi-norm $\|x\|_{X^s} = \big\||x|^s\big\|_X^{1/s}$. Note that if X is p-convex, then for $f \in X^{1/p}$ and f_1,...,$f_n \in X^{1/p}$ with $|f| \le \sum_{k=1}^n |f_k|$, we have

$$\begin{aligned}\|f\|_{X^{1/p}} &\le \Big\| \sum_{k=1}^{n} (|f_k|^{1/p})^p \Big\|_{X^{1/p}} = \Big\| \Big(\sum_{k=1}^{n} (|f_k|^{1/p})^p \Big)^{1/p} \Big\|_X^p \\ &\le (M^{(p)}(X))^p \sum_{k=1}^{n} \big\||f_k|^{1/p}\big\|_X^p = (M^{(p)}(X))^p \sum_{k=1}^{n} \|f_k\|_{X^{1/p}}.\end{aligned}$$

This shows that the lattice norm $\|\cdot\|^*$ defined for $f \in X^{1/p}$ by

$$\|f\|^* = \inf\Big\{\sum_{k=1}^n \|f_k\|_{X^{1/p}};\ |f| \le \sum_{k=1}^n |f_k|,\ n \in \mathbb{N}\Big\},$$

satisfies

$$(M^{(p)}(X))^{-p}\,\|\cdot\|_{X^{1/p}} \le \|\cdot\|^* \le \|\cdot\|_{X^{1/p}}.$$

In particular this implies the well known fact that if a quasi-Banach lattice X on (Ω, Σ, μ) is p-convex $(0 < p < \infty)$, then there exists a Banach function space Y on (Ω, Σ, μ) such that $f \in X$ if and only if $|f|^p \in Y$ and

$$(M^{(p)}(X))^{-1}\,\|f\|_X \le \big\||f|^p\big\|_Y^{1/p} \le \|f\|_X.$$

We state below a lemma which is well know fact but we do not have references. We only mention that it easily follows by the above estimates in combination with the description of a second Köthe dual space E'' of any Banach lattice E (see [21, pp. 451, 471]), which states that $f \in E''$ if and only if there exists a sequence $(f_n)_{n=1}^\infty$ of elements in E, such that $0 \le f_n \uparrow |f|$ a.e. and $\sup_n \|f_n\|_E < \infty$. Moreover for any $f \in E''$ we have

$$\|f\|_{E''} = \inf\big\{\lim_{n\to\infty} \|f_n\|_E;\ 0 \le f_n \uparrow |f| \text{ a.e.}\big\}.$$

Lemma 2.1. *Let X be a quasi-Banach lattice on (Ω, Σ, μ) which is p-convex for some $0 < p < \infty$. Then there exists a Banach lattice Y on (Ω, Σ, μ) such that $Y^p = X$ and*

$$(M^{(p)}(X))^{-1}\|f\|_X \le \|f\|_{Y^p} \le \|f\|_X, \quad f \in X.$$

If in addition X is maximal, then Y can be chosen also maximal.

We recall a quite general version of Minkowski's inequality, whose proof can be found for instance in [15, pp. 45–46]. Let $(\Omega_1, \Sigma_1, \mu)$, $(\Omega_2, \Sigma_2, \nu)$ be measure spaces and $F\colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ measurable with respect to the σ-algebra of $\nu \times \mu$-measurable sets, and let E be a Banach lattice on $(\Omega_1, \Sigma_1, \mu)$ and assume that $F(\cdot, t) \in E$ for $t \in \Omega_2$ and $t \mapsto \|F(\cdot, t)\|_E$ belongs to $L_1(\nu)$. Then

$$\Big\|\int_{\Omega_2} F(\cdot, t)\, d\nu(t)\Big\|_{E''} \le \int_{\Omega_2} \|F(\cdot, t)\|_E\, d\nu(t).$$

We point out the following fact (see [12]) that will be used in the sequel: if $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 0, 1)$ are measure spaces and X is a Banach lattice on $(\Omega_2, \Sigma_2, \mu_2)$, then for any $\Sigma_1 \times \Sigma_2$-measurable function f defined on $\Omega_1 \times \Omega_2$, the function v_f (given by $v_f(s) := \|f(s, \cdot)\|_X$ and $v_f(s) = \infty$ if $f(s, \cdot) \notin X$ for $s \in \Omega_1$) is Σ_1-measurable if μ_1 is discrete or if μ_1 is arbitrary but X has a norm which satisfies the so called (C)-condition (i.e., $0 \le x_n \uparrow x \in X$

implies $\|x_n\|_X \to \|x\|_X$). Notice that Luxemburg [11] gives a simple example, using the Bohr compactification of the integers, showing that the result need not hold if X does not satisfy the (C)-condition.

Next, we have the following result:

Lemma 2.2. *Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space*[*] *(Ω, Σ, μ) such that X_0 is p_0-convex and X_1 is p_1-convex. Then for every $0 < \theta < 1$ we have*

$$\|x\|_{X_0^{1-\theta}X_1^{\theta}} \leq (M^{(p_0)}(X_0))^{1-\theta}(M^{(p_1)}(X_1))^{\theta}\, \|x\|_{(X_0,X_1)_\theta}, \quad x \in X_0 \cap X_1.$$

In particular (X_0, X_1) is an admissible quasi-Banach couple.

Proof. Fix $\varepsilon > 0$. For a given $x \in X_0 \cap X_1$ there exists a function $f \in \mathcal{F}(X_0 \cap X_1)$ such that the function

$$F(z,\omega) := f(z)(\omega) = \sum_{k=1}^{N} \varphi_k(z)x_k(\omega), \quad z \in \overline{S},\ \omega \in \Omega,$$

$1 \leq k \leq N$, $x_k \in X_0 \cap X_1$, $\varphi_k \in A(S)$, satisfies $F(\theta,\omega) = x(\omega)$ and

$$\|F\|_{\mathcal{F}(X_0 \cap X_1)} \leq (1+\varepsilon)\|x\|_{(X_0,X_1)_\theta}.$$

Applying Lemma 1.1 to the scalar case (i.e., to $A_0 = A_1 = \mathbb{C}$) yields

$$\begin{aligned}|F(\theta,\omega)| &\leq \Big(\frac{1}{1-\theta}\int_{-\infty}^{\infty}|F(it,\omega)|^{p_0}P_0(\theta,t)\,dt\Big)^{\frac{1-\theta}{p_0}}\Big(\frac{1}{\theta}\int_{-\infty}^{\infty}|F(1+it,\omega)|^{p_1}P_1(\theta,t)\,dt\Big)^{\frac{\theta}{p_1}}\\ &= y_0(\omega)^{1-\theta}y_1(\omega)^{\theta},\end{aligned}$$

where for every $\omega \in \Omega$

$$y_0(\omega) = \Big(\frac{1}{1-\theta}\int_{-\infty}^{\infty}|F(it,\omega)|^{p_0}P_0(\theta,t)\,dt\Big)^{\frac{1}{p_0}},$$

$$y_1(\omega) = \Big(\frac{1}{\theta}\int_{-\infty}^{\infty}|F(1+it,\omega)|^{p_1}P_1(\theta,t)\,dt\Big)^{\frac{1}{p_1}}.$$

By Lemma 2.1 there exists a Banach lattice E_j on (Ω, Σ, μ) such that for $j = 0, 1$

$$(M^{(p_j)}(X_j))^{-1}\,\|f\|_{X_j} \leq \|f\|_{E_j^{p_j}} \leq \|f\|_{X_j}, \quad f \in X_j.$$

Since the above identities y_0 and y_1 are given by Riemann integrals of functions with values in Banach lattices E_0 and E_1, respectively, we obtain

$$\|y_0\|_{E_0^{p_0}}^{p_0} = \||y_0|^{p_0}\|_{E_0} \leq \frac{1}{1-\theta}\int_{-\infty}^{\infty}\big\||F(it,\omega)|^{p_0}\big\|_{E_0}P_0(\theta,t)\,dt,$$

[*]Recall all measure spaces in this article are assumed to be complete and σ-finite.

$$\|y_1\|_{E_1^{p_1}}^{p_1} = \||y_1|^{p_1}\|_{E_1} \le \frac{1}{\theta} \int_{-\infty}^{\infty} \big\||F(1+it,\omega)|^{p_1}\big\|_{E_1} P_1(\theta,t)\, dt.$$

Consequently,

$$\begin{aligned}
\|y_0\|_{X_0} &\le M^{(p_0)}(X_0)\|y_0\|_{E_0^{p_0}} \\
&\le M^{(p_0)}(X_0)\Big(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \big\||F(it,\cdot)|^{p_0}\big\|_{E_0} P_0(\theta,t)\, dt\Big)^{1/p_0} \\
&= M^{(p_0)}(X_0)\Big(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \|F(it,\cdot)\|_{E_0^{p_0}}^{p_0} P_0(\theta,t)\, dt\Big)^{1/p_0} \\
&\le M^{(p_0)}(X_0)\Big(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \|F(it,\cdot)\|_{X_0}^{p_0} P_0(\theta,t)\, dt\Big)^{1/p_0} \\
&\le (1+\varepsilon)M^{(p_0)}(X_0)\, \|x\|_{(X_0,X_1)_\theta}.
\end{aligned}$$

Similarly we obtain

$$\|y_1\|_{X_1} \le M^{(p_1)}(X_1)(1+\varepsilon)\|x\|_{(X_0,X_1)_\theta}.$$

Combining $|x| \le |y_0|^{1-\theta}|y_1|^\theta$ with the preceding estimates for the norms of y_0 and y_1 in X_0 and X_1, respectively, yields

$$\|x\|_{X_0^{1-\theta}X_1^\theta} \le (1+\varepsilon)(M^{(p_0)}(X_0))^{1-\theta}(M^{(p_1)}(X_1))^\theta\, \|x\|_{(X_0,X_1)_\theta}.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

We now prove Theorem 1.1.

Proof. Without loss of generality, we may assume that $s = 0$. If this case is proved, then applying the result for $s = 0$ to the admissible analytic family $z \mapsto \widetilde{T}_z = T_{z+is}$ with $\widetilde{K}_j(t) = K_j(t+s)$, $j = 1, 2$, for a fixed real s, we obtain the required estimate (4).

Fix $0 < \theta < 1$ and $(x_1, ..., x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$. Without loss of the generality we may assume that $\|x_i\|_{(X_{0i},X_{1i})_\theta} < 1$ for each $1 \le i \le m$. Once this case is completed, replacing x_i by $(1-\delta)x_i/\|x_i\|_{(X_{0i},X_{1i})_\theta}$, we obtain (3) for general x_i when we let $\delta \to 0$.

There exist finite sequences $\{x_{i\,k}\}_{k=1}^{n_i}$ in $\mathcal{X}_i$ and $\{\varphi_{i\,k}\}_{k=1}^{n_i}$ in $A(S)$ such that for each $1 \le i \le m$,

$$x_i = \sum_{k=1}^{n_i} \varphi_{i\,k}(\theta)\, x_{i\,k}$$

and

$$\Big\|\sum_{k=1}^{n_i} \varphi_{i\,k}\, x_{i\,k}\Big\|_{\mathcal{F}(\mathcal{X}_i)} < 1.$$

For every $z \in \overline{S}$ define $F_z \colon \Omega \to \mathbb{C}$ by

$$F_z(\omega) = T_z\Big(\sum_{k=1}^{n_1} \varphi_{1\,k}(z)\, x_{1\,k}, ..., \sum_{k=1}^{n_m} \varphi_{m\,k}(z)\, x_{m\,k}\Big)(\omega), \quad \omega \in \Omega. \tag{5}$$

Since all functions $\varphi_{i\,k}$ are bounded in $\overline{S}$ and

$$F_z(\omega) = \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} \varphi_{1\,k_1}(z) \cdots \varphi_{m\,k_m}(z)\, T_z(x_{1\,k_1}, ..., x_{m,k_m})(\omega),$$

our hypothesis on the family $\{T_z\}$ yields that for almost every $\omega \in \Omega$, the function $z \mapsto F_z(\omega)$ is analytic in S, continuous in $\overline{S}$, and of admissible growth. Thus applying Lemma 1.2 we obtain that for almost all $\omega \in \Omega$

$$\log |F_\theta(\omega)| \le \int_{-\infty}^{\infty} \log |F_{it}(\omega)|\, P_0(\theta,t)\, dt + \int_{-\infty}^{\infty} \log |F_{1+it}(\omega)|\, P_1(\theta,t)\, dt. \tag{6}$$

Since the Poisson kernels satisfy

$$\int_{\mathbb{R}} P_0(\theta,t)\, dt = 1-\theta \quad \text{ and } \quad \int_{\mathbb{R}} P_1(\theta,t)\, dt = \theta,$$

ν_0 and ν_1 defined by $d\nu_0(t) = (1-\theta)^{-1} P_0(\theta,t)\, dt$ and $d\nu_1(t) = \theta^{-1} P_0(\theta,t)\, dt$ are probability measures on $\mathbb{R}$.

Taking exponentials of both sides of the above inequality we obtain

$$|F_\theta(\omega)| \le \Big(\exp \int_{\mathbb{R}} \log |F_{it}(\omega)|^{p_0}\, d\nu_0(t)\Big)^{(1-\theta)/p_0} \Big(\exp \int_{\mathbb{R}} \log |F_{1+it}(\omega)|^{p_1}\, d\nu_1(t)\Big)^{\theta/p_1}.$$

This implies

$$|F_\theta(\omega)| \le \Big(\int_{\mathbb{R}} |F_{it}(\omega)|^{p_0}\, d\nu_0(t)\Big)^{(1-\theta)/p_0} \Big(\int_{\mathbb{R}} |F_{1+it}(\omega)|^{p_1}\, d\nu_1(t)\Big)^{\theta/p_1} = g_0(\omega)^{1-\theta} g_1(\omega)^{\theta},$$

where for $j = 0, 1$,

$$g_j(\omega) := \Big(\int_{\mathbb{R}} |F_{j+it}(\omega)|^{p_j}\, d\nu_j(t)\Big)^{1/p_j}, \quad \omega \in \Omega.$$

Notice that g_0 and g_1 are measurable functions by Tonelli's theorem. We claim that $g_j \in Y_j$ for $j = 0, 1$. By Lemma 2.1 there exists a maximal Banach lattice E_j on (Ω, Σ, μ) such that for $j = 0, 1$

$$\big(M^{(p_j)}(Y_j)\big)^{-1} \|f\|_{Y_j} \le \|f\|_{E_j^{p_j}} \le \|f\|_{Y_j}, \quad f \in Y_j.$$

Put $C_j = M^{(p_j)}(Y_j)$ for $j = 0, 1$. By Minkowski's inequality we have

$$\begin{aligned}
\|g_j\|_{Y_j} &\le M^{(p_j)}(Y_j) \Big\| \int_{\mathbb{R}} |F_{j+it}(\cdot)|^{p_j}\, d\nu_j(t) \Big\|_{E_j}^{1/p_j} \le C_j \Big(\int_{\mathbb{R}} \big\| |F_{j+it}(\cdot)|^{p_j} \big\|_{E_j}\, d\nu_j(t)\Big)^{1/p_j} \\
&\le C_j \Big(\int_{\mathbb{R}} \big\| F_{j+it}(\cdot) \big\|_{E^{p_j}}^{p_j}\, d\nu_j(t)\Big)^{1/p_j} \le C_j \Big(\int_{\mathbb{R}} \big\| F_{j+it}(\cdot) \big\|_{Y_j}^{p_j}\, d\nu_j(t)\Big)^{1/p_j} \\
&= C_j \Big(\int_{\mathbb{R}} \Big\| T_{j+it}\Big(\sum_{k=1}^{n_1} \varphi_{1\,k}(j+it)\, x_{1\,k}, \ldots, \sum_{k=1}^{n_m} \varphi_{m\,k}(j+it)\, x_{m\,k}\Big) \Big\|_{Y_j}^{p_j} d\nu_j(t)\Big)^{1/p_j} \\
&\le C_j \Big(\int_{\mathbb{R}} K_j(t)^{p_j} \Big(\prod_{\ell=1}^{m} \Big\| \sum_{k=1}^{n_\ell} \varphi_{\ell k}(j+it) x_{\ell k} \Big\|_{X_{j\ell}} \Big)^{p_j} d\nu_j(t)\Big)^{1/p_j} \\
&\le C_j \Big(\int_{\mathbb{R}} K_j(t)^{p_j}\, d\nu_j(t)\Big)^{1/p_j} < \infty.
\end{aligned}$$

This gives that $F_\theta = T_\theta(x_1, \dots, x_m) \in Y_\theta$ with

$$\|T_\theta(x_1, \dots, x_m)\|_{Y_\theta} \le \|g_0\|_{Y_0}^{1-\theta} \, \|g_1\|_{Y_1}^{\theta} \le C_0^{1-\theta} C_1^{\theta} \, K(\theta, p_0, p_1),$$

where we set

$$K(\theta, p_0, p_1) = \Big(\frac{1}{1-\theta} \int_{\mathbb{R}} K_0(t)^{p_0} P_0(\theta, t)\, dt \Big)^{\frac{1-\theta}{p_0}} \, \Big(\frac{1}{\theta} \int_{\mathbb{R}} K_1(t)^{p_1} P_1(\theta, t)\, dt \Big)^{\frac{\theta}{p_1}}.$$

To finish the proof, we use the well known fact that if a quasi-Banach lattice E is r-convex, then it is p-convex for every $0 < p < r$, and $M^{(p)}(E) \le M^{(r)}(E)$ (see, e.g., [5]). Repeating the preceding discussion with p in place of both p_0 and p_1 we obtain for every $0 < p < \min\{p_0, p_1\}$,

$$\|T_\theta(x_1, \dots, x_m)\|_{Y_\theta} \le \|g_0\|_{Y_0}^{1-\theta} \, \|g_1\|_{Y_1}^{\theta} \le C_0^{1-\theta} C_1^{\theta} \, K(\theta, p, p).$$

To conclude the proof, use that ν_0 and ν_1 are probability measures to deduce that

$$\begin{aligned}
\lim_{p\to 0+} K(\theta, p, p) &= \lim_{p\to 0+} \Big[\Big(\frac{1}{1-\theta} \int_{\mathbb{R}} K_0(t)^{p} P_0(\theta, t)\, dt \Big)^{1/p}\Big]^{1-\theta} \, \Big[\Big(\frac{1}{\theta} \int_{\mathbb{R}} K_1(t)^{p} P_1(\theta, t)\, dt \Big)^{1/p}\Big]^{\theta} \\
&= \exp\Big(\int_{\mathbb{R}} P_0(\theta, t) \log K_0(t)\, dt \Big) \, \exp\Big(\int_{\mathbb{R}} P_1(\theta, t) \log K_1(t)\, dt \Big) \\
&= K_\theta(0),
\end{aligned}$$

and this yields (4) when $s = 0$. The case of a general s follows by a translation as discussed at the beginning of the proof. □

3 General applications

We now obtain certain easy consequences of the above theorem. We will need a variant of Calderón's lemma for quasi-Banach lattices. Since this requires a different proof than in the Banach case, we include a proof for the sake of completeness.

Lemma 3.1. *Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space (Ω, Σ, μ). If $x_j \in X_j$ $(j = 0, 1)$ are such that $|x_0|$ and $|x_1|$ are bounded above and their non-zero values have positive lower bounds, then $|x_0|^{1-\theta} |x_1|^{\theta} \in (X_0, X_1)_\theta$ and*

$$\big\| |x_0|^{1-\theta} |x_1|^{\theta} \big\|_{(X_0, X_1)_\theta} \le \|x_0\|_{X_0}^{1-\theta} \|x_1\|_{X_1}^{\theta}.$$

Proof. Fix $\varepsilon > 1$. Without loss of generality we can assume that $\|x_0\|_{X_0} = \|x_1\|_{X_1} = 1$. Let $x = |x_0|^{1-\theta} |x_1|^{\theta}$ and let $E = \operatorname{supp} x$. Our hypotheses imply that there exist integers $0 \le N \le M$ such that

$$\varepsilon^{-N} |x_0(\omega)| \le |x_1(\omega)| < \varepsilon^{M} |x_0(\omega)|, \quad \omega \in E.$$

For each $-N \le k \le M-1$ let $x_k = x\,\chi_{E_k}$, where

$$E_k := \{\omega \in E;\ \varepsilon^k |x_0(\omega)| < |x_1(\omega)| \le \varepsilon^{k+1}|x_0(\omega)|\}.$$

Then we have

$$x_k = |x_0 \chi_{E_k}|^{1-\theta} |x_1 \chi_{E_k}|^{\theta} \le \varepsilon\, \varepsilon^{k\theta} |x_0|\, \chi_{E_k}, \qquad x_k \le \varepsilon^{-k(1-\theta)} |x_1|\, \chi_{E_k},$$

and so for any finite sequence $\{\lambda_k\}$ of complex numbers with $|\lambda_k| \le 1$ we have

$$\Big| \sum_{k=-N}^{M} \lambda_k \varepsilon^{-k\theta} x_k \Big| \le \varepsilon \sum_{k=-N}^{M} |x_0|\, \chi_{E_k} \le \varepsilon |x_0|,$$

and

$$\Big| \sum_{k=-N}^{M} \lambda_k \varepsilon^{(1-\theta)k} x_k \Big| \le \varepsilon \sum_{k=-N}^{M} |x_1| \chi_{E_k} \le \varepsilon |x_1|.$$

Consequently, for $j = 0, 1$, we have

$$\sup_{|\lambda_k| \le 1} \Big\| \sum_{k=-N}^{M} \lambda_k \varepsilon^{(j-\theta)k} x_k \Big\|_{X_j} \le \varepsilon. \tag{7}$$

Let f be a function $f\colon \overline{S} \to X_0 \cap X_1$ defined by

$$f(z) = \sum_{k=-N}^{M} \varepsilon^{k(z-\theta)} x_k, \qquad z \in \overline{S}.$$

Since $|\varepsilon^{it}| = 1$ for every $t \in \mathbb{R}$, thus combining with (7) yields

$$\sup_{t \in \mathbb{R}} \|f(j+it)\|_{X_j} \le \varepsilon.$$

Since $f(\theta) = \sum_{k=-N}^{M} x_k = x$, $x \in (X_0, X_1)_\theta$ with

$$\|x\|_{(X_0,X_1)_\theta} \le \varepsilon$$

and this completes the proof since $\varepsilon > 1$ was arbitrary. $\square$

Let X be a quasi-Banach space on a measure space. Following the Banach case, we say that an element $x \in X$ has order continuous quasi-norm if for any sequence (x_n) of measurable functions such that $0 \le x_n \le |x|$, and $x_n \to |x|$ a.e., we have $\|x_n - |x|\|_X \to 0$. A quasi-Banach lattice X is said to be order continuous if every $x \in X$ has order continuous quasi-norm.

The following corollary is an immediate consequence of Lemma 3.1 combined with the proof of Theorem 1.14 in [15, pp. 244-245].

Corollary 3.1. *Let* (X_0, X_1) *be a couple of complex quasi-Banach lattices on a measure space* (Ω, Σ, μ)*. If* $x \in X_0 \cap X_1$ *has an order continuous norm in* $X_0^{1-\theta} X_1^{\theta}$*, then for every* $0 < \theta < 1$*,*

$$\|x\|_{(X_0,X_1)_\theta} \leq \|x\|_{X_0^{1-\theta} X_1^{\theta}}.$$

As a consequence of the above corollary and Lemma 2.2, we obtain a variant of Calderón's theorem for quasi-Banach lattices.

Theorem 3.1. *Let* (X_0, X_1) *be a couple of complex quasi-Banach lattices on a measure space with nontrivial lattice convexity constants. If the space* $X_0^{1-\theta} X_1^{\theta}$ *has order continuous quasi-norm, then*

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^{\theta}$$

up to equivalences of norms (*isometrically, provided that lattice convexity constants are equal to* 1)*. In particular this holds if at least one of the spaces* X_0 *or* X_1 *is order continuous.*

Proof. The first part follows by application of Lemma 3.1 and Corollary 3.1. Now if X_0 or X_1 is order continuous, then it is easy to see that $X_0^{1-\theta} X_1^{\theta}$ is also order continuous. □

We remark that Theorem 1.1 combined with the preceding relationships between spaces $(X_0, X_1)_\theta$ and $X_0^{1-\theta} X_1^{\theta}$ generated by couples of complex quasi-Banach lattices give multilinear theorems for analytic families of multilinear operators on quasi-Banach lattices of type $X_0^{1-\theta} X_1^{\theta}$. As a special case we obtain the following interpolation theorem for mulitilinear operators:

Theorem 3.2. *For each* $1 \leq i \leq m$*, let* (X_{0i}, X_{1i}) *be complex quasi-Banach function lattices and let* Y_j *be complex* p_j*-convex maximal quasi-Banach function lattices with* p_j*-convexity constants equal* 1 *for* $j = 0, 1$*. Suppose that either* X_{0i} *or* X_{1i} *is order continuous for each* $1 \leq i \leq m$*. Let* T *be a multilinear operator defined on* $(X_{01} + X_{11}) \times \cdots \times (X_{0m} + X_{1m})$ *and taking values in* $Y_0 + Y_1$ *such that* $T\colon X_{i1} \times \cdots \times X_{im} \to Y_i$ *is bounded with quasi-norm* M_i *for* $i = 0, 1$*. Then for* $0 < \theta < 1$*,* $T\colon (X_{01})^{1-\theta}(X_{11})^{\theta} \times \cdots \times (X_{0m})^{1-\theta}(X_{1m})^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$ *is bounded with the quasi-norm*

$$\|T\| \leq M_0^{1-\theta}\, M_1^{\theta}.$$

We conclude this section by noting that in the case of probability measures and under the assumption of separability, the following interpolation theorem for operators was proved by Kalton [13, Theorem 2.2] and was applied to study a problem in uniqueness of structure in quasi-Banach lattices. The proof presented in [13] is completely different than ours and uses a deep theorem by Nikishin and the theory of Hardy H_p-spaces on the unit disc.

Theorem 3.3. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measures spaces. Let* X_i*,* $i = 0, 1$*, be complex* p_i*-convex quasi-Banach lattices on* $(\Omega_1, \Sigma_1, \mu_1)$ *and let* Y_i *be complex* p_i*-convex maximal quasi-Banach lattices on* $(\Omega_2, \Sigma_2, \mu_2)$ *with* p_i*-convexity constants equal* 1*. Suppose that either* X_0 *or* X_1 *is order continuous. Let* $T\colon X_0 + X_1 \to L^0(\mu_2)$ *be a continuous operator such that* $T(X_0) \subset Y_0$ *and* $T(X_1) \subset Y_1$*. Then for* $0 < \theta < 1$*,* T *maps* $X_\theta = X_0^{1-\theta} X_1^\theta$ *into* $Y_\theta = Y_0^{1-\theta} Y_1^\theta$ *and*

$$\|T\|_{X_\theta \to Y_\theta} \leq \|T\|_{X_0 \to Y_0}^{1-\theta} \, \|T\|_{X_1 \to Y_1}^{\theta}.$$

4 Applications to Lebesgue, Lorentz, and Hardy spaces

In this section we discuss applications of our results in the context of classical quasi-Banach spaces which appear in many areas of analysis. We first provide some results concerning the descriptions of the Calderón spaces $X_0^{1-\theta} X_1^\theta$, where X_0 and X_1 are r.i. quasi-Banach spaces and we provide applications to Lorentz spaces. In these descriptions we estimate the constants of equivalence of quasi-norms.

We first introduce some definitions. We recall that a quasi-Banach function lattice X on (Ω, Σ, μ) is said to be a *rearrangement invariant space* (r.i. for short) if for every $f \in L^0(\mu)$ and $g \in X$ with $\mu_f = \mu_g$, we have $f \in X$ and $\|f\|_X = \|g\|_X$. Here μ_f denotes the distribution function of $|f|$ with respect to μ, i.e., $\mu_f(s) = \mu(\{\omega \in \Omega;\, |f(\omega)| > s\})$, $s \geq 0$. The decreasing rearrangement f^* of f with respect to μ is defined by $f^*(t) = \inf\{s > 0;\, \mu_f(s) \leq t\}$, $t \geq 0$. In what follows the space of all $f \in L^0(\mu)$ such that $f^*(+\infty) := \lim_{t\to\infty} f^*(t) = 0$ is denoted by Λ_0

Let (Ω, Σ, μ) be a nonatomic measure space and let $I = (0, \mu(\Omega))$ be equipped with the Lebesgue measure λ. If E is a quasi-Banach lattice on (I, λ), then we define

$$E^{(*)} = E^{(*)}(\mu) := \big\{f \in L^0(\mu);\, f^* \in E\big\}.$$

It is easily checked that if the dilation operator $D_2 f(t) = f(t/2)$, for all $t \in I$, is bounded on the cone of non-negative decreasing elements in E, then $E^{(*)}$, equipped with the quasi-norm given by $\|f\|_{E^{(*)}} := \|f^*\|_E$ for all $f \in E^{(*)}$, is a r.i. quasi-Banach space on (Ω, Σ, μ).

An r.i. space X on a nonatomic measure space (Ω, Σ, μ) is said to be generated by a quasi-Banach lattice E on $I = (0, \mu(\Omega))$ provided the following conditions are satisfied: $f \in X$ if and only if $f \in E^{(*)}$ and $\|f\|_X = \|f\|_{E^{(*)}}$.

The *Lorentz space* $L_{p,q} := L_{p,q}(\Omega)$ on a measure space (Ω, Σ, μ), $0 < p < \infty, 0 < q \leq \infty$ consists of all $f \in L^0(\mu)$ such that the following quasi-norm is finite

$$\|f\|_{L_{p,q}} := \begin{cases} \Big(\displaystyle\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \frac{dt}{t}\Big)^{1/q} & \text{if } 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if}\, q = \infty\,. \end{cases}$$

Notice that $L_{p,q}$ is generated by the weighted quasi-Banach lattice $L^q(w)$, where $w(t) = t^{1/p-1/q}$ for all $\in I$ and $0 < p, q < \infty$ and $w(t) = t^{1/p}$ in the case when $0 < p < \infty$ and $q = \infty$.

We now apply the results of the previous sections. We begin by noticing that Theorem 3.1 contains the following well-known fact: If $0 < p_0,\ p_1 \leq \infty$ and (L^{p_0}, L^{p_1}) is a couple on any measure space, then

$$[L^{p_0}, L^{p_1}]_\theta = L^{p_\theta}$$

isometrically, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ for all $0 < \theta < 1$; in fact simple calculations show that

$$(L^{p_0})^{1-\theta}(L^{p_1})^\theta = L^{p_\theta}$$

with equality of norms. Since L^p is p-convex and L^∞ is 1-convex with the convexity constants equal 1, the result follows Theorem 3.1 if $p_0 \neq p_1$. If $p_0 = p_1 = \infty$, the statement is obvious.

We state the following technical result which seems of independent interest.

The following Lemma is surely well-known to specialists, but we include a proof.

Lemma 4.1. *Let (X_0, X_1) be a couple of r.i. quasi-Banach spaces on a nonatomic measure space (Ω, Σ, μ). Assume that X_j $(j = 0, 1)$ is generated by a quasi-Banach lattice E_j on (I, λ). Then the following hold:*

(i) $X_0^{1-\theta}X_1^\theta \hookrightarrow \big(E_0^{1-\theta}E_1^\theta\big)^{(*)}$ *and the inclusion map* id *has norm*

$$\|\mathrm{id}\| \leq C_0^{1-\theta}C_1^\theta,$$

where $C_j = \sup\big\{\|D_2 g\|_{E_j};\ \|g\|_{E_j} \leq 1, \quad g \text{ is a nonnegative decreasing function}\big\}$.

(ii) *Let P be a positive linear operator such that $P(E_j) \subset E_j$ for $j = 0, 1$. Assume that Pf is a nonincreasing function on I for every $0 \leq f \in E_0 + E_1$ and there exists $C > 0$ such that $g \leq CPg$ for any nonegative nonincreasing function $g \in E_0 + E_1$. If $\big(E_0^{1-\theta}E_1^\theta\big)^{(*)} \subset \Lambda_0$, then $\big(E_0^{1-\theta}E_1^\theta\big)^{(*)} \hookrightarrow X_0^{1-\theta}X_1^\theta$ and the inclusion map* id *has norm*

$$\|\mathrm{id}\| \leq C\|P\|_{E_0}^{1-\theta}\|P\|_{E_1}^\theta.$$

Proof. Let $|f| \leq |f_0|^{1-\theta}|f_1|^\theta$ μ-a.e. with $f_j \in X_j$ $(j = 0, 1)$. Then we have

$$f^*(t) \leq f_0^*(t/2)^{1-\theta}f_1^*(t/2)^\theta = (D_2 f_0^*(t))^{1-\theta}(D_2 f_1^*(t))^\theta, \quad t \in I.$$

Since $\|D_2 f_j^*\|_{E_j} \leq C_j\|f_j\|_{X_j}$ for $j = 0, 1$, $f^* \in E_0^{1-\theta}E_1^\theta$ with

$$\|f^*\|_{E_\theta} \leq C_0^{1-\theta}C_1^\theta\|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^\theta$$

and so the required estimate follows.

(ii). We will use the following known property $S(f_0^{1-\theta} f_1^{\theta}) \le S(f_0)^{1-\theta} S(f_1)^{\theta}$, $f_0, f_1 \in \mathcal{X}$, valid for positive operators $S\colon \mathcal{X} \to L^0(\mu)$ defined on solid linear subspace $\mathcal{X}$ of $L^0(\mu)$.

Let $f \in \big(E_0^{1-\theta} E_1^{\theta}\big)^{(*)}$. Then $f^* \le g_0^{1-\theta} g_1^{\theta}$ with $0 \le g_j \in E_j$ $(j = 0, 1)$. Applying the preceding property yields

$$f^* \le C P f^* \le C (P g_0)^{1-\theta} (P g_1)^{\theta}.$$

Since $f^*(+\infty) = 0$, the Ryff Theorem (see [1, Chapter 2, Corollary 7.6]) implies that there exists a measure preserving map $\sigma\colon \Omega \to I$ such that $|f| = f^* \circ \sigma$ μ-a.e. on the support of f. Thus if we put $h_j = P(g_j) \circ \sigma$ for $j = 0, 1$ we deduce that

$$|f| \le C\, h_0^{1-\theta} h_1^{\theta}, \quad \mu\text{-a.e.}$$

To conclude observe that $h_j \in X_j$ by $h_j^* = P(g_j)^* = P(g_j)$ (since $P(g_j)$ is a nonincreasing function on I) for $j = 0, 1$. Since P is positive and $P\colon E_j \to E_j$, P is bounded and so

$$\|h_j\|_{X_j} = \|h_j^*\|_{E_j} \le \|P\|_{E_j} \|g_j\|_{E_j}.$$

Combining these facts, we see that $f \in X_0^{1-\theta} X_1^{\theta}$ with

$$\|f\|_{X_\theta} \le K \|g_0\|_{E_0}^{1-\theta} \|g_1\|_{E_1}^{\theta},$$

where $K = C\|P\|_{E_0}^{1-\theta} \|P\|_{E_1}^{\theta}$. Since $f^* \in \big(E_0^{1-\theta} E_1^{\theta}\big)^{(*)}$ and $f^* \le g_0^{1-\theta} g_1^{\theta}$ with arbitrary $g_0 \in E_0$ and $g_1 \in E_1$, the proof is complete. □

Remark 4.1. *It is easily seen that examples of positive operators which satisfy the hypotheses of the Lemma* 4.1(*ii*) *are operators* $P_r = D_2 \circ Q_r$, $0 < r < \infty$ *where for every* $0 < r < \infty$ *is given by*

$$Q_r f(t) := \Big(\int_t^{\infty} \frac{f(s)^r}{s}\, ds \Big)^{1/r}.$$

To apply the Lemma 4.1 to the Lorentz spaces $\Lambda_{p,q}$ we recall the following result (see, e.g., [16, Theorem 2]): If $1 \le p \le \infty$, $1/p + 1/p' = 1$ and u and v are weighted functions on $(0, \infty)$, then there exists $C > 0$ such that

$$\Big(\int_0^{\infty} \Big| u(t) \int_t^{\infty} f(s)\, ds \Big|^p dt \Big)^{1/p} \le C \Big(\int_0^{\infty} |f(t) v(t)|^p\, dt \Big)^{1/p}, \quad f \in L_p(v),$$

if and only if

$$B = \sup_{r>0} \Big(\int_0^r u(t)^p\, dt \Big)^{1/p} \Big(\int_r^{\infty} v(t)^{-p'}\, dt \Big)^{1/p'} < \infty.$$

In addition $B \le C \le p^{1/p} (p')^{1/p'} B$.

We single out the following specific case of the well-known Hardy inequality for the Hardy operator Q given by

$$Qf(t) = \int_t^\infty \frac{f(s)}{s}\, ds, \quad t > 0,$$

which states that if $1 < p < \infty$ and $1 \le q < \infty$, then for all $f \in L^q(w)$, where $w(t) = t^{1/p-1/q}$ for all $t > 0$ we have

$$\Big(\int_0^\infty (t^{1/p}|Qf(t)|)^q \,\frac{dt}{t}\Big)^{1/q} \le p\Big(\int_0^\infty (t^{1/p}|f(t)|)^q \,\frac{dt}{t}\Big)^{1/q}.$$

For simplicity we discuss applications only to the classical Lorentz spaces on infinite nonatomic measure spaces.

Corollary 4.1. *Let $0 < p_j, q_j < \infty$ and let L_{p_j,q_j} for $j = 0,1$ be Lorentz spaces on an infinite nonatomic measure space (Ω, Σ, μ). Then for $0 < \theta < 1$ the quasi-norm of*

$$X_\theta := (L_{p_0,q_0})^{1-\theta}(L_{p_1,q_1})^\theta$$

is equivalent to that of $L_{p,q}$, where $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Moreover for all $f \in X_\theta$ we have

$$2^{-1/p}\,\|f\|_{L_{p,q}} \le \|f\|_{X_\theta} \le \frac{2^{1/p}}{(\log 2)^s}\, s^s (p_0^{1-\theta} p_1^\theta)^s\,\|f\|_{L_{p,q}},$$

where $s = 1$ whenever $1 < p_0, p_1 < \infty$ and $1 \le q_0, q_1 < \infty$ and $s > \max\{1/p_0, 1/q_0, 1/p_1, 1/q_1\}$ otherwise.

Proof. Clearly L_{p_j,q_j} is generated by $E_j := L^{q_j}(w_j)$, where $w_j(t) = t^{1/p_j-1/q_j}$ for $t > 0$. Since $E_0^{1-\theta}E_1^\theta = L^q(w)$ with equality of norms where $w(s) = w_0(s)^{1-\theta}w_1(t)^\theta = t^{1/p-1/q}$ for all $t > 0$,

$$\big(E_0^{1-\theta}E_1^\theta\big)^{(*)} = L_{p,q} \tag{8}$$

with equality of quasi-norms. Thus the left hand of the required inequality follows from Lemma 4.1(i) by $\|D_2\|_{E_j} \le 2^{1/p_j}$ for $j = 0,1$.

To conclude we apply Lemma 4.1(ii). We first assume that $1 < p_j < \infty$ and $1 \le q_j < \infty$ for $j = 0,1$. Now observe that for any nonnegative and nonincreasing function $g \in L^1_{loc}(\lambda)$ we have

$$g(t) \le \frac{1}{\log 2}\int_{t/2}^t \frac{g(s)}{s}\, d\lambda \le \frac{1}{\log 2}\, D_2 \circ Qg(t), \quad t > 0.$$

Since $\|D_2\|_{E_j} \le 2^{1/p_j}$, the aforementioned Hardy inequality shows that $P := D_2 \circ Q\colon E_j \to E_j$ for $j = 0,1$ and

$$\|P\|_{E_j} \le \frac{2^{1/p_j} p_j}{\log 2}.$$

This in combination with (8) shows by $L_{p,q} \subset \Lambda_0$ that an operator P satisfies all hypotheses in (ii) of Lemma 4.1 and so

$$\big(E_0^{1-\theta} E_1^{\theta}\big)^{(*)} \hookrightarrow (L_{p_0,q_0})^{1-\theta} (L_{p_1,q_1})^{\theta},$$

where the inclusion map id has norm

$$\|\mathrm{id}\| \leq \frac{2^{1/p}}{\log 2}\, p_0^{1-\theta} p_1^{\theta}$$

and consequently the required right hand estimate of quasi-norms follows by (8).

We now consider the general case. We use the easily-verified fact that for any couple (X_0, X_1) of quasi-Banach lattices on a measure space and every $s > 0$,

$$(X_0^{1-\theta} X_1^{\theta})^s = (X_0^s)^{1-\theta} (X_1^s)^{\theta}, \quad \theta \in (0,1).$$

with equality of quasi-norms. In particular, for any $s > \max_{j=0,1}\{1/p_j, 1/q_j\}$ (note that s can be taken to be equal to 1 when $1 < p_0, p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$) this yields that

$$X_\theta^s := ((L_{p_0,q_0})^{1-\theta} (L_{p_1,q_1})^{\theta})^s = ((L_{p_0,q_0})^s)^{1-\theta} ((L_{p_1,q_1})^s)^{\theta} = (L_{sp_0,sq_0})^{1-\theta} (L_{sp_1,sq_1})^{\theta}$$

with equality of quasi-norms.

The second part of the proof gives $(X_\theta)^s = L_{sp,sq}$ with equivalence of quasi-norms. As an immediate consequence, we obtain (by $(X^s)^{1/s} = X$ with equality of quasi-norms) the required equality of considered spaces. Furthermore, we have

$$\|f\|_{(X_\theta)^s} \leq \frac{C_s}{\log 2}\, \|f\|_{(L_{p,q})^s}, \quad f \in (L_{p,q})^s,$$

where $C_s = 2^{1/ps} s p_0^{1-\theta} p_1^{\theta}$.

Combining we obtain $X_\theta = L_{p,q}$ with

$$\|f\|_{X_\theta} \leq \frac{2^{1/p}}{(\log 2)^s}\, s^s (p_0^{1-\theta} p_1^{\theta})^s\, \|f\|_{L_{p,q}}, \quad f \in L_{p,q}.$$

and so this completes the proof. □

Next, we have the following result on interpolation of analytic multilinear operators on products of Lorentz spaces.

Theorem 4.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces. For $1 \leq i \leq m$, fix $0 < q_0, q_1, q_{0i}, q_{1i} < \infty$, $0 < r_0, r_1, r_{0i}, r_{1i} \leq \infty$ and for $0 < \theta < 1$, define q, r, q_i, r_i by setting*

$$\frac{1}{q_i} = \frac{1-\theta}{q_{0i}} + \frac{\theta}{q_{1i}}, \quad \frac{1}{r_i} = \frac{1-\theta}{r_{0i}} + \frac{\theta}{r_{1i}}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Assume that $\mathcal{X}_i$ is a dense linear subspace of $L_{q_{0i},r_{0i}}(\Omega_1)\cap L_{q_{1i},r_{1i}}(\Omega_1)$ and that $\{T_z\}_{z\in\overline{S}}$ is an admissible analytic family of multilinear operators $T_z\colon \mathcal{X}_1\times\cdots\times\mathcal{X}_m \to L_{q_0,r_0}(\Omega_2)\cap L_{q_1,r_1}(\Omega_2)$. Suppose that for every $(h_1,...,h_m)\in\mathcal{X}_1\times\cdots\times\mathcal{X}_m$, $t\in\mathbb{R}$ and $j=0,1$, we have

$$\|T_{j+it}(h_1,...,h_m)\|_{L_{q_j,r_j}(\Omega_2)} \le K_j(t)\|h_1\|_{L_{q_{j1},r_{j1}}(\Omega_1)}\cdots\|h_m\|_{L_{q_{jm},r_{jm}}(\Omega_1)}$$

where K_j are Lebesgue measurable functions such that $K_j\in L^{p_j}(P_j(\theta,\cdot)\,dt)$ for all $\theta\in(0,1)$, where p_j is chosen so that $0<p_j<q_j$ and $p_j\le r_j$ for each $j=0,1$.

Then for all $(f_1,...,f_m)\in\mathcal{X}_1\times\cdots\times\mathcal{X}_m$, $0<\theta<1$, and $s\in\mathbb{R}$ we have

$$\|T_{\theta+is}(f_1,...,f_m)\|_{(L_{q_0,r_0},L_{q_1,r_1})_\theta} \le \left(\frac{q_0}{q_0-p_0}\right)^{\frac{1-\theta}{p_0}}\left(\frac{q_1}{q_1-p_1}\right)^{\frac{\theta}{p_1}} K_\theta(s)\prod_{i=1}^m \|f_i\|_{(L_{q_0,r_0},L_{q_1,r_1})_\theta},$$

where

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta,t)\log K_0(t+s)\,dt + \int_{\mathbb{R}} P_1(\theta,t)\log K_1(t+s)\,dt.$$

If in addition the measures spaces are infinite and nonatomic, then for all $(f_1,...,f_m)$ in $\mathcal{X}_1\times\cdots\times\mathcal{X}_m$, and $s\in\mathbb{R}$, and $0<\theta<1$ we have

$$\|T_{\theta+is}(f_1,...,f_m)\|_{L_{q,r}(\Omega_2)} \le C\left(\frac{q_0}{q_0-p_0}\right)^{\frac{1-\theta}{p_0}}\left(\frac{q_1}{q_1-p_1}\right)^{\frac{\theta}{p_1}} K_\theta(s)\prod_{i=1}^m \|f_i\|_{L_{q_i,r_i}(\Omega_1)},$$

where

$$C = 2^{\frac{1}{q}+\sum_{i=1}^m\frac{1}{q_i}}\left(\frac{u\,q_0^{1-\theta}q_1^{\theta}}{\log 2}\right)^u$$

with $u=1$ if $1<q_0,q_1<\infty$ and $1\le r_0,r_1\le\infty$, while $u>\max\{1/q_0,1/q_1,1/r_0,1/r_1\}$ otherwise.

Proof. We apply Theorem 1.1. At first we notice that for any $0<p_0<\infty$ we have

$$\|f\|_{L_{q_0,r_0}} = \left(\big\||f|^{p_0}\big\|_{L_{\frac{q_0}{p_0},\frac{r_0}{p_0}}}\right)^{1/p_0} = \|f\|_{\left(L_{\frac{q_0}{p_0},\frac{r_0}{p_0}}\right)^{p_0}},\qquad f\in L_{q_0,r_0}$$

and the space $E_0 = L_{\frac{q_0}{p_0},\frac{r_0}{p_0}}$ is normable as long as $p_0<q_0$ and $p_0\le r_0$. In fact, when $r_0<\infty$, the following norm on E_0

$$|||g|||_{E_0} = \left(1-\frac{p_0}{q_0}\right)\left(\int_0^\infty\left[t^{\frac{q_0}{r_0}}\sup_{\substack{B\subset\Omega_2\\ \mu_2(B)\ge\min\{t,\mu(\Omega_2)\}}}\frac{1}{\max\{t,\mu_2(B)\}}\int_B|g|\,d\mu_2\right]^{\frac{r_0}{p_0}}dt\right)^{\frac{p_0}{r_0}}$$

satisfies

$$|||h|||_{E_0} \le \|h\|_{L_{\frac{q_0}{p_0},\frac{r_0}{p_0}}} \le \frac{q_0}{q_0-p_0}\,|||h|||_{E_0}$$

(see [8, Exercise 1.4.3]) and hence (taking $h=|f|^{p_0}$) we have

$$\left(\frac{q_0}{q_0-p_0}\right)^{-\frac{1}{p_0}}\|f\|_{L_{q_0,r_0}} \le |||f|||_{E_0^{p_0}} \le \|f\|_{L_{q_0,r_0}}. \tag{9}$$

When $r_0 = \infty$, we define

$$|||g|||_{E_0} = \Big(1 - \frac{p_0}{q_0}\Big) \sup_{0<\mu_2(B)<\infty} \frac{1}{\mu_2(B)^{1-\frac{p_0}{q_0}}} \int_B |g|\, d\mu_2\,.$$

By [8, Exercise 1.1.12], this is a norm on $L_{\frac{q_0}{p_0},\infty}$ that satisfies

$$|||g|||_{E_0} \le \|g\|_{L_{\frac{q_0}{p_0},\infty}} \le \frac{q_0}{q_0 - p_0}\, |||g|||_{E_0},$$

and thus (9) also holds in this case. This implies that $M^{(p_0)}(L_{q_0,r_0}) \le \big(\frac{q_0}{q_0-p_0}\big)^{1/p_0}$ and analogously $M^{(p_1)}(L_{q_1,r_1}) \le \big(\frac{q_1}{q_1-p_1}\big)^{1/p_1}$. Hence, the desired inequality

$$\|T_{\theta+is}(f_1,...,f_m)\|_{(L_{q_0,r_0},L_{q_1,r_1})_\theta} \le \Big(\frac{q_0}{q_0-p_0}\Big)^{\frac{1-\theta}{p_0}} \Big(\frac{q_1}{q_1-p_1}\Big)^{\frac{\theta}{p_1}} K_\theta(s) \prod_{i=1}^m \|f_i\|_{(L_{q_0,r_0},L_{q_1,r_1})_\theta},$$

is a consequence of Theorem 1.1. The inequality with $L_{q,r}$ in place of $(L_{q_0,r_0}, L_{q_1,r_1})_\theta$ and L_{q_i,r_i} in place of $(L_{q_{0i},r_{0i}}, L_{q_{1i},r_{1i}})_\theta$ follows from Corollary 4.1. □

Next, we discuss an extension motivated by applications to Hardy spaces.

Under the hypotheses of Theorem 1.1, suppose additionally that: There is an operator $\mathcal{M}$ defined on a linear subspace of $\widetilde{L}^0(\Omega,\Sigma,\mu)$ and taking values in $\widetilde{L}^0(\Omega,\Sigma,\mu)$ such that:

(i) For $j = 0$ and $j = 1$ the function $(t,x) \mapsto \mathcal{M}(h(j+it,\cdot))(\omega)$, $(t,\omega) \in \mathbb{R}\times\Omega$ is $\mathcal{L}\times\Sigma$-measurable for any function $h\colon \partial S\times\Omega \to \mathbb{C}$ such that $\omega \mapsto h(j+it,\omega)$ is Σ-measurable for almost all $t \in \mathbb{R}$.

(ii) $\mathcal{M}(\lambda h)(\omega) = |\lambda|\mathcal{M}(h)(\omega)$ for all $\lambda \in \mathbb{C}$.

(iii) For every function h as in above there is an exceptional set $E_h \in \Sigma$ with $\mu(E_h) = 0$ such that for $j \in \{0,1\}$

$$\mathcal{M}\Big(\int_{-\infty}^{\infty} h(t,\cdot)P_j(\theta,t)\,dt\Big)(\omega) \le \int_{-\infty}^{\infty} \mathcal{M}(h(t,\cdot))(\omega)P_j(\theta,t)\,dt$$

for all $z \in \mathbb{C}$, all $\theta \in (0,1)$, and all $\omega \notin E_h$. Moreover, $E_{\psi h} = E_h$ for every analytic function ψ on S which is bounded on $\overline{S}$.

An example of this situation arises when $\Omega = \mathbb{R}^n$, μ is Lebesgue measure, and

$$\mathcal{M}(h)(x) = \sup_{\delta>0} |\phi_\delta * h(x)| \tag{10}$$

where ϕ is a Schwartz function on $\mathbb{R}^n$ with nonvanishing integral. Under assumptions (i), (ii), (iii) Cwikel and Sagher [6] (page 981 estimate (5)) show that

$$\log\,[M(F_\theta)(x)] \le \int_{-\infty}^{\infty} \log[\mathcal{M}(F_{it})(x)]\,P_0(\theta,t)\,dt + \int_{-\infty}^{\infty} \log[\mathcal{M}(F_{1+it})(x)]\,P_1(\theta,t)\,dt \tag{11}$$

whenever F_z is an analytic function on S which is continuous and bounded on $\overline{S}$ and which is of admissible growth. In our case, we take F_z as defined in (5). But in this case, (11) serves as a substitute for (6) and thus the proof of (3) for T_z also works for $\mathcal{M} \circ T_z$. We obtain the following result:

Theorem 4.2. *For each $1 \le i \le m$, let $\overline{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of complex maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j-convex for $j = 0, 1$. Assume that $\mathcal{X}_i$ is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \le i \le m$, and that $\{T_z\}_{z \in \overline{S}}$ is an admissible analytic family of multilinear operators $T_z \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to Y_0 \cap Y_1$. Assume that $\mathcal{M}$ is defined on the range of T_z, takes values in $L^0(\Omega, \Sigma, \mu)$, and satisfies (i), (ii), (iii). Suppose that for every $(x_1, ..., x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$,*

$$\|\mathcal{M}(T_{j+it}(x_1, ..., x_m))\|_{Y_j} \le K_j(t) \|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}},$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot)\, dt)$ for all $\theta \in (0,1)$.

Then for all $(x_1, ..., x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $s \in \mathbb{R}$, and $0 < \theta < 1$ we have

$$\|\mathcal{M}(T_{\theta+is}(x_1, ..., x_m))\|_{Y_0^{1-\theta} Y_1^{\theta}} \le \left(M^{(p_0)}(Y_0)\right)^{1-\theta} \left(M^{(p_1)}(Y_1)\right)^{\theta} K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s)\, dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s)\, dt.$$

The preceding theorem has an important application to interpolation of multilinear operators that take values in Hardy spaces.

Example 4.1. *A particular case of Theorem 4.2 arises when $Y_0 = L^{p_0}$, $Y_1 = L^{p_1}$, in which case $Y_0^{1-\theta} Y_1^{\theta} = L^p$, where $1/p = (1-\theta)/p_0 + \theta/p_1$. If $\mathcal{M}$ is given by (10), then $\|\mathcal{M}(h)\|_{L^p} = \|h\|_{H^p}$, where H^p is the classical Hardy space of Fefferman and Stein.*

In this case, estimates of the form

$$\|T_{j+it}(x_1, ..., x_m)\|_{H^{p_j}} \le K_j(t) \|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}$$

for admissible analytic families T_z when $j = 0, 1$ imply the intermediate estimates

$$\|T_{\theta+s}(x_1, ..., x_m)\|_{H^p} \le K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta}$$

for $0 < p_0, p_1 < \infty$, $s \in \mathbb{R}$, and $0 < \theta < 1$. Analogous estimates hold for the Hardy-Lorentz spaces $H^{q,r}$ where estimates of the form

$$\|T_{j+it}(x_1, ..., x_m)\|_{H^{q_j, r_j}} \le K_j(t) \|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}$$

for admissible analytic families T_z when $j = 0, 1$ imply

$$\|T_{\theta+is}(x_1, ..., x_m)\|_{H^{q,r}} \le C\, K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta}$$

with

$$C = 2^{\frac{1}{q}} \left(\frac{u\, q_0^{1-\theta} q_1^{\theta}}{\log 2}\right)^u \left(\frac{q_0}{q_0 - p_0}\right)^{\frac{1-\theta}{p_0}} \left(\frac{q_1}{q_1 - p_1}\right)^{\frac{\theta}{p_1}},$$

where $0 < p_j < q_j < \infty$, $p_j \le r_j \le \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$ while $u = 1$ if $1 < q_0, q_1 < \infty$ and $1 \le r_0, r_1 \le \infty$ and $u > \max\{1/q_0, 1/q_1, 1/r_0, 1/r_1\}$ otherwise.

5 An application to the bilinear Bochner-Riesz operators

Stein's [19] motivation to study analytic families of operators might have been the study of the Bochner-Riesz operators

$$B^\delta(f)(x) := \int_{|\xi|\le 1} \left(1 - |\xi|^2\right)^\delta \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

in which the "smoothness" variable δ affects the degree p of integrability of $B^\delta(f)$ on $L^p(\mathbb{R}^n)$. Here f is a Schwartz function on $\mathbb{R}^n$ and $\widehat{f}$ is its Fourier transform defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \qquad \xi \in \mathbb{R}^n.$$

Using interpolation for analytic families of operators, Stein showed that whenever $\delta > (n-1)|1/p - 1/2|$, then B^δ maps $L^p(\mathbb{R}^n)$ to itself for $1 \le p \le \infty$.

Recent interest in bilinear operator has led to the consideration of the *bilinear Bochner-Riesz operators*. For Schwartz functions f, g on $\mathbb{R}^n$ these are defined as

$$S^\delta(f,g)(x) := \iint_{|\xi|^2+|\eta|^2\le 1} \left(1 - |\xi|^2 - |\eta|^2\right)^\delta \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta, \tag{12}$$

where the integral is over $\mathbb{R}^{2n}$. Several boundedness results concerning these means have recently been obtained in [2]. Among them we state two:

- For any $\delta > 0$, there is an estimate

$$\|S^\delta(f,g)\|_{L^1} \le C_\delta\, \|f\|_{L^2} \|g\|_{L^2}. \tag{13}$$

- For any $\delta > n - 1/2$, there is an estimate

$$\|S^\delta(f,g)\|_{L^{1/2}} \le C'_\delta\, \|f\|_{L^1} \|g\|_{L^1}. \tag{14}$$

Both estimates hold for all functions f, g in the corresponding spaces with constants C_δ, C'_δ only depending on δ and the dimension n.

We make some remarks about the extension of these estimates to the case where δ is replaced by $z = \delta + it$, where δ, t are real.

The bilinear Bochner-Riesz means S^z can also be written in the form

$$S^z(f,g)(x) = \int\int K_z(x-y_1, x-y_2) f(y_1) g(y_2) dy_1\, dy_2$$

for some kernel K_z. A well-known calculation (see [8, Appendix B.5]) shows that the kernel of $S^{\delta+it}$ is

$$K_{\delta+it}(x_1,x_2) = \frac{\Gamma(\delta+1+it)}{\pi^{\delta+it}} \frac{J_{\delta+it+n}(2\pi|x|)}{|x|^{\delta+it+n}}, \qquad x=(x_1,x_2)\,.$$

Consider the case where $\delta > n-1/2$. Then using known asymptotics for Bessel functions ([8] Appendix B.8) we have that this kernel satisfies an estimate of the form:

$$|K_{\delta+it}(x_1,x_2)| \le \frac{C(n+\delta+it)}{(1+|x|)^{\delta+n+1/2}}.$$

where $C(n+\delta+it)$ is a constant that satisfies

$$C(n+\delta+it) \le C_{n+\delta} e^{B\,|t|^2}$$

for some $B>0$. Then for $\delta > n-1/2$, we have

$$|K_{\delta+it}(x_1,x_2)| \le C_{n+\delta}\, e^{B|t|^2} \frac{1}{(1+|x_1|)^{n+\epsilon}} \frac{1}{(1+|x_2|)^{n+\epsilon}},$$

with $\epsilon = \frac{1}{2}(\delta - n - 1/2)$. It follows that the bilinear operator $S^{\delta+it}$ is bounded by a product of two linear operators, each of which has a good integrable kernel. It follows that $S^{\delta+it}$ is bounded from $L^1\times L^1$ to $L^{1/2}$ with constant $K_1(t) \le C'_{n+\delta} e^{B|t|^2}$ whenever $\delta > n-1/2$. This yields an extension of (14) for complex values of δ.

To obtain the analogous extension of (13), we slightly modify the proof of Theorem 4.7 in [2]. This theorem claims that

$$\big\|S^z\big\|_{L^2\times L^2\to L^1} \le C' \sup_{u\in[-1,1]} \|(1-u^2-|\cdot|^2)^z_+\|_{W^{1+\alpha,1}(\mathbb{R})}, \tag{15}$$

where $W^{1+\alpha,1}$ is the Sobolev space of functions with $1+\alpha$ "derivatives" in L^1 with $\alpha>0$. The norm on the Sobolev space $W^{s,p}$ is defined via $\|v\|_{W^{s,p}} = \|(I-\Delta)^{s/2}v\|_{L^p}$ and is comparable to the norm $\|v\|_{L^p} + \|v\|_{\dot W^{s,p}}$, where $\|v\|_{\dot W^{s,p}} = \|(-\Delta)^{s/2}v\|_{L^p}$. In (15) the constant C' depends only on the dimension and α. To estimate the Sobolev norm in (15) when $z=\delta+it$, $\delta>0$, we need Lemma 4.4. in [2] which claims that

$$\big\|(1-|\cdot|^2)^z_+\big\|_{W^{s,q}(\mathbb{R}^n)} \le C\, e^{c|\mathrm{Im}\, z|^2} \tag{16}$$

when $0 < s < \operatorname{Re} z + \frac{1}{q}$, $1 \le q < \infty$ with constants c, C depending only on n, q, s. Fixing $z = \delta + it$, $\delta > 0$, we pick $0 < \alpha < \delta$ and we write:

$$\begin{aligned}
\big\|(1-u^2-|\cdot|^2)_+^{\delta+it}\big\|_{\dot W^{1+\alpha,1}(\mathbb{R})} &= (1-u^2)^{\delta}\,\big\|(1-|\tfrac{\cdot}{(1-u^2)^{1/2}}|^2)^{\delta+it})\big\|_{\dot W^{1+\alpha,1}(\mathbb{R})}\\
&= (1-u^2)^{\delta-\alpha}\big\|(1-|\cdot|^2)^{\delta+it}\big\|_{\dot W^{1+\alpha,1}(\mathbb{R})}\\
&\le C(1-u^2)^{\delta-\alpha}\,e^{c|t|^2} \le C\,e^{c|t|^2},
\end{aligned}$$

since $\delta-\alpha > 0$ and then $(1-u^2)^{\delta-\alpha} \le 1$ for $u \in [-1,1]$. It follows that $S^{\delta+it}$ is bounded from $L^2 \times L^2$ to L^1 with constant $K_0(t) \le C'_{n+\delta} e^{c|t|^2}$ whenever $\delta > 0$. This yields an extension of (13) for complex values of δ.

In [2], intermediate results for S^δ are obtained rather indirectly via bilinear real interpolation applied to operators appearing in a decomposition of S^δ. Here we provide intermediate estimates via a direct proof based on complex interpolation for analytic families of bilinear operators.

Theorem 5.1. *Let $1 < p < 2$. For any $\lambda > (2n-1)(1/p-1/2)$, S^λ maps $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to $L^{p/2}(\mathbb{R}^n)$.*

Proof. To apply Theorem 1.1 we set $X_{01} = X_{02} = L^2$, $X_{11} = X_{12} = L^1$, $Y_0 = L^1$, $Y_1 = L^{1/2}$, $\mathcal{X}_i$ is the space of Schwartz functions on $\mathbb{R}^n$, which is dense in L^1 and L^2.

We fix $\delta > 0$ and we consider the bilinear analytic family $\{T_z\}_{z\in\overline{S}}$, where $T_z := S^{(n-\frac{1}{2})z+\delta}$ for all $z \in \overline{S}$. (Recall that $S = (0,1)\times\mathbb{R}$ is the unit strip). We claim that this family is admissible. Indeed, for f, g Schwartz functions we have

$$T_z(f,g)(x) = \iint_{|\xi|^2+|\eta|^2\le 1} \big(1-|\xi|^2-|\eta|^2\big)^{(n-\frac{1}{2})z+\delta}\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta, \qquad x \in \mathbb{R}^n,$$

and the map $z \mapsto T_z(f,g)$ is analytic in S, continuous and bounded on $\overline{S}$, and jointly measurable in (t,x) when $z = it$ or $z = 1+it$. Moreover, for all $x \in \mathbb{R}^n$ we have

$$\sup_{z\in\overline{S}} \frac{\log|T_z(f,g)(x)|}{e^{\alpha|\operatorname{Im} z|}} < \infty$$

with $\alpha = 0 < \pi$ when f, g are Schwartz functions; in fact $|T_z(f,g)(x)| \le \|\widehat{f}\|_{L^1}\|\widehat{g}\|_{L^1}$.

Based on the preceding discussion, we have that when $\operatorname{Re} z = 0$, T_z maps $L^2 \times L^2$ to L^1 with constant $K_0(t) \le C_{n,\delta} e^{c\,|t|^2}$ for some $C_{n,\delta}, c > 0$. We also have that when $\operatorname{Re} z = 1$, T_z maps $L^1 \times L^1$ to $L^{1/2}$ with constant $K_1(t) \le C'_{n,\delta} e^{B\,|t|^2}$ for some $C'_{n,\delta}$, $B > 0$. We notice that for these functions $K_i(t)$ we have that the constant $K(\theta, 1, 1/2)$ in (4) is finite; in this case $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{1}$, hence $\theta = 2(\frac{1}{p} - \frac{1}{2})$. An application of Theorem 1.1 yields that S^λ maps $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to $L^{p/2}(\mathbb{R}^n)$ when $\lambda = 2(n-\frac{1}{2})\,(\frac{1}{p}-\frac{1}{2}) + \delta > (2n-1)(\frac{1}{p}-\frac{1}{2})$. □

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