HARDY SPACE ESTIMATES FOR MULTILINEAR OPERATORS, II

Loukas Grafakos

Yale University

Abstract. We continue the study of multilinear operators given by products of finite vectors of Calderón-Zygmund operators. We determine the set of all $r \leq 1$ for which these operators map products of Lebesgue spaces $L^p(R^n)$ into the Hardy spaces $H^r(R^n)$. At the endpoint case $r = n/n+m+1$, where m is the highest vanishing moment of the multilinear operator, we prove a weak type result.

0. Introduction

A well known by now theorem of P.L. Lions says that the determinant of the Jacobian of a function from $\mathbb{R}^n \to \mathbb{R}^n$ maps the product of Sobolev spaces $L_1^n \times \cdots \times L_1^n$ into the Hardy space H^1. Coifman, Lions, Meyer and Semmes, [CLMS], went below H^1 by showing that for $p, q > 1$, the Jacobian-determinant maps $L_1^p(\mathbb{R}^2) \times L_1^q(\mathbb{R}^2)$ into $H^r(\mathbb{R}^2)$, where $r^{-1} = p^{-1} + q^{-1}$, as long as $r > 2/3$. Their result can be generalized to give the n-dimensional version that the determinant of the Jacobian maps $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_n}(\mathbb{R}^n)$ into $H^r(\mathbb{R}^n)$, as long as the harmonic mean r of the p_j's is strictly greater than $n/n+1$. In this work we prove a positive result in the endpoint case $r = n/n+1$. We treat more general multilinear operators with vanishing integral since our methods show that this is the only assumption needed. We also study the case of multilinear operators with higher moments vanishing. The number of vanishing moments is related to the lowest r for which these operators map products of Lebesgue spaces into H^r. If such an operator has all moments of order $\leq m$ vanishing, then it maps products of Lebesgue spaces into H^r for $r > n/n+m+1$. Also, a weak type estimate holds in the endpoint case $r = n/n+m+1$ and no boundedness result holds for $r < n/n+m+1$.

1. Statements of results

Throughout this article, N and K will denote fixed integers ≥ 2. We are given a matrix of convolution Calderón-Zygmund kernels $\{K_i^j\}_{i=1,j=1}^{N\ \ K}$ on $\mathbb{R}^n$. We call T_i^j the associated Calderón-Zygmund operator. We denote by $L(f_1,\dots,f_K)$ the K-linear operator

$$L(f_1,\dots,f_K)=\sum_{i=1}^{N}(T_i^1 f_1)\dots(T_i^K f_K). \tag{1.1}$$

originally defined for smooth compactly supported functions $f_1,\dots,f_K$. For $p\le 1$, we denote by H^p the usual real variable Hardy space as defined in [S] or [FST], i.e. the set of all distributions f on $\mathbb{R}^n$ for which the maximal function $\sup_{t>0}|\phi_t * f(x)|$ is in L^p, where $\phi_t(x)=\frac{1}{t^n}\phi(\frac{x}{t^n})$ and ϕ is smooth, nonzero and compactly supported. We also denote by $H^{p,\infty}$ the weak H^p as defined in [FRS] (or [FSO] in the case $p=1$), i.e. the set of all f in $\mathbb{R}^n$ for which the maximal function $\sup_{t>0}|\phi_t * f(x)|$ is in weak L^p. The weak L^p (quasi)norm of this maximal function is by definition the $\|\ \ \|_{H^{p,\infty}}$ (quasi)norm of f.

Our first result concerns the general multilinear operators L of the type above and it presents very clearly the method that will be used in this article. Note however, that there is an unpleasant restriction about the exponents that will be lifted later.

Theorem I . *Assume that for all $(f_1,\dots,f_K)\in(C_0^\infty)^K$, the K-linear operator L satisfies:*

$$\int L(f_1,\dots,f_K)\,dx=0.$$

Suppose that $p_1,\dots,p_K>1$ are given and let $r=(p_1^{-1}+\dots+p_K^{-1})^{-1}$ be their harmonic mean. Assume that the harmonic mean of any proper subset of the p_j's is greater than 1. Then

1) If $r>1$, L maps $L^{p_1}\times\dots\times L^{p_K}\to L^r$.

2) If $1\ge r>n/n+1$, L maps $L^{p_1}\times\dots\times L^{p_K}\to H^r$.

3) If $r=n/n+1$, L maps $L^{p_1}\times\dots\times L^{p_K}\to H^{r,\infty}$.

Next, we treat the case of multilinear operators with vanishing higher moments. The significance of the number of vanishing moments is that it gives the lowest r for which such operators map into H^r. We also get rid of the assumption that the harmonic mean of any subset of the p_j's is always greater than 1. We are assuming however, that the K-linear operators L that have a special form.

When $K=2$, we consider operators L of the general form (1.1), i.e. inner products of two vectors of Calderón-Zygmund operators. For $K\ge 3$, we consider operators built inductively as follows:

We are assuming that for any j there exist $\Lambda_i^j=\Lambda_i^j(f_1,\dots,f_{j-1},f_{j+1},\dots,f_K)$ $(K-1)$-linear operators already defined by the induction hypothesis with the same number of vanishing moments, such that

$$L(f_1,\ldots,f_K)=\sum_{i=1}^{M}T_i^j(f_j)\,\Lambda_i^j(f_1,\ldots,f_{j-1},f_{j+1},\ldots,f_K) \tag{1.2}$$

Condition (1.2) essentially says that the multilinear operators L look like determinants of matrices. They are built by induction starting from arbitrary bilinear operators as the ones in theorem I (when $K=2$) and at each stage they look like sums of products of multilinear operators of one smaller degree multiplied by a Calderón-Zygmund operator. These sums have a certain degree of symmetry because it follows from a repeated application of (1.2) that for each $j_1,\ldots,j_l$, there exist $(K-l)$-linear operators $\Lambda_i^{j_1,\ldots,j_l}$ with the same number of vanishing moments such that

$$L(f_1,\ldots,f_K)=\sum_i (T_i^{j_1}f_{j_1})\ldots(T_i^{j_l}f_{j_l})\,\Lambda_i^{j_1,\ldots,j_l}(\text{remaining } f_j\text{'s}).$$

In most applications we have in mind, the multilinear operators have this form, for example determinants of matrices.

In the case of bilinear operators, $K=2$, there are no additional assumptions about the operators L and this is why we state and prove this case separately. Also, this case is going to serve as the first step of an inductive argument that will be used later.

Theorem IIa. *Assume that for some m, $0\le m\le n-1$ and for all $f,g\in C_0^\infty(\mathbb{R}^n)$ the bilinear operator $B(f,g)=\sum_{i=1}^N (T_i^1 f)(T_i^2 g)$ satisfies:*

$$\int x^\alpha B(f,g)\,dx=0 \qquad \textit{for all multiindices}\quad \alpha \quad \textit{with}\quad |\alpha|\le m.$$

Suppose that $p,q>1$ are arbitrary and let $r=(p^{-1}+q^{-1})^{-1}$ be their harmonic mean. Then
1) If $r>1$, B maps $L^p\times L^q\to L^r$.
2) If $1\ge r>n/n+m+1$, B maps $L^p\times L^q\to H^r$.
3) If $r=n/n+m+1$, B maps $L^p\times L^q\to H^{r,\infty}$.

Next, we generalize theorem IIa for K-linear operators of the form (1.2) and for these type of operators we don't have any additional assumption about the p_j's

Theorem IIb. *Assume that for some m, $0\le m\le n(K-1)-1$ and for all $f_j\in C_0^\infty(\mathbb{R}^n)$ the K-linear operator $L(f_1,\ldots,f_K)$ has the form (1.1), where each Λ_i^j satisfy*

$$\int x^\alpha \Lambda_i^j\,dx=0 \qquad \textit{for all multiindices}\quad \alpha \quad \textit{with}\quad |\alpha|\le m.$$

Suppose that $p_1,\dots,p_K>1$ *are arbitrary and let* $r=(\sum_k p_k^{-1})^{-1}$ *be their harmonic mean. Then*

1) *If* $r>1$, L *maps* $L^{p_1}\times\cdots\times L^{p_K}\to L^r$.

2) *If* $1\geq r>n/n+m+1$, L *maps* $L^{p_1}\times\cdots\times L^{p_K}\to H^r$.

3) *If* $r=n/n+m+1$, L *maps* $L^{p_1}\times\cdots\times L^{p_K}\to H^{r,\infty}$.

Remarks:

a. The assumption $m\leq n(K-1)-1$ is necessary in theorem II, since otherwise $r=n/n+m+1<1/K$ which would contradict that $p_j>1$.

b. The hypothesis that the harmonic mean of any subset of the p_j's is greater than 1 seems to be necessary in conclusions 2) and 3) of theorem I. It is obviously not needed in conclusion 1) of theorem I and it is always automatically satisfied when $r=1$ or when $K=2$. This condition imposes an upper bound on the degree K of multilinearity of the K-linear operator L. For, let $p_j=p>1$ and let $r<1$ be the harmonic mean of the p_j's. Then $Kr=p$. The assumption on the harmonic mean of any subset of the p_j's gives $p/(K-1)>1$. We conclude that $K<1/(1-r)$ which is a restriction on the size of K. Note, however, that when $r=1$ there is no upper bound on K nor any restriction about the exponents and our theorem implies for example, that any K-linear operator as above with mean value zero maps $L^{p_1}\times\cdots\times L^{p_K}\to H^1$ when $\sum p_j^{-1}=1$.

c. The vanishing integral hypothesis for L in theorem I can be relaxed to the milder condition that *for all* f_1 smooth with compact support and *for some* $f_2,\dots,f_K$ in the corresponding Lebesgue spaces the integrals $\int L(f_1,f_2,\dots,f_K)\,dx$ vanish. Then conclusion 2) of theorem I will be that the operator $g\to L(g,f_2,\dots,f_K)$ maps L^{p_1} to H^r with norm no larger than a constant times the product of the L^{p_j} norms of the f_j's, $j=2,\dots,K$. Conclusion 3) of theorem I will be similar.

2. Proof of theorem I

We fix $p_1,\dots,p_K>1$ and we let r be their harmonic mean. Clearly only the case $r\leq 1$ is interesting because the case $r>1$ is just Hölder's inequality together with the L^p boundedness of Calderón - Zygmund operators. Fix a smooth compactly supported function ϕ in $\mathbb{R}^n$, an $x_0\in\mathbb{R}^n$ and define $\phi_{t,x_0}(x)=\frac{1}{t^n}\phi(\frac{x-x_0}{t})$. Without loss of generality we may assume that ϕ is supported in $|x|\leq 1$. We need to show that $\sup_{t>0}|\int\phi_{t,x_0}L(f_1,\dots,f_K)\,dx|$ is in L^r when $r>n/n+1$ and in $L^{r,\infty}$ when $r=n/n+1$. We also fix a smooth cutoff $\eta(x)$ such that $\eta\equiv 1$ on $|x|<2$ and supported in $|x|<4$. We call for simplicity $\eta_0(x)=\eta(\frac{x_0-x}{t})$ and $\eta_1(x)=1-\eta_0(x)$. The reader should remember the dependence of η_0,η_1 on t. We now decompose $L(f_1,\dots,f_K) = L_0 + L_1 + \cdots + L_{K+1}$, where

$$\begin{aligned}
L_0 &= L(\eta_0 f_1, \eta_0 f_2, \dots, \eta_0 f_K)\\
L_1 &= \sum_{j=1}^{K} L(f_1,\dots,\eta_1 f_j,\dots,f_K)\\
L_2 &= -\sum_{\substack{1\le j,l\le K\\ j<l}} L(f_1,\dots,\eta_1 f_j,\dots,\eta_1 f_l,\dots,f_K)\\
&\quad \text{etc}\\
L_{K+1} &= (-1)^K L(\eta_1 f_1, \eta_1 f_2, \dots, \eta_1 f_K)
\end{aligned}$$

In each L_u above exactly u functions among the f_j's are multiplied by η_1 and the remaining are left intact. To get this decomposition of L we expand $L(\eta_0 f_1,\dots,\eta_0 f_K) = L(f_1 - \eta_1 f_1,\dots,f_K - \eta_K f_K)$ and then we solve for $L(f_1,\dots,f_K)$.

Note that for any fixed i,k and any x such that $|x-x_0|\le t$ we have:

$$\begin{aligned}
&\sup_{t>0} |T_i^k(\eta_1 f)(x) - T_i^k(\eta_1)(x_0)|\\
\le &\sup_{t>0} \left| \int \left(K_i^k(x-y) - K_i^k(x_0-y)\right)\eta_1(y) f(y)\ dy \right|\\
\le &C\sup_{t>0} \int_{|y-x_0|\ge t} |x-x_0|\,|y-x_0|^{-n-1} |f(y)|\ dy \le C|f|^*(x_0)
\end{aligned}$$

where by $g^*(x_0)$ we denote the Hardy-Littlewood maximal function of g at the point x_0. We also use the notation $(T_i^j)_*$ for the maximal truncated operator of T_i^j. The term L_0 is the main term term of the decomposition and is treated last. We begin with term L_1. We write it as

$$\sum_{j=1}^{K} L(f_1,\dots,(\eta_1 f_j)(x) - (\eta_1 f_j)(x_0),\dots,f_K) + \sum_{j=1}^{K} L\left(f_1,\dots,(\eta_1 f_j)(x_0),\dots,f_K\right).$$

We then have:

$$\begin{aligned}
&\sup_{t>0}\left|\int \phi_{t,x_0} L_1\ dx\right|\\
\le &\sum_{j=1}^{K}\sum_{i=1}^{N} \sup_{t>0} \int |\phi_{t,x_0}| \prod_{\substack{1\le k\le K\\ k\ne j}} |T_i^k f_k| \ \Big(|T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0)| + |T_i^j(\eta_1 f_j)(x_0)|\Big)\ dx\\
\le &C\sum_{j=1}^{K}\sum_{i=1}^{N} \Big(\prod_{\substack{1\le k\le K\\ k\ne j}} |T_i^k f_k|)^*(x_0)\ \ [|f_j^*(x_0)| + (T_i^j)_* f_j(x_0)] = (2.1)
\end{aligned}$$

Define σ_j by $\sigma_j^{-1}+p_j^{-1}=r^{-1}$. By Hölder's inequality the L^r norm in x_0 of (2.1) is bounded by

$$
\begin{aligned}
&C\sum_{j=1}^{K}\sum_{i=1}^{N}[\||f_j|^*\|_{L^{p_j}}+\|(T_i^j f_j)_*\|_{L^{p_j}}] \quad \|(\prod_{\substack{1\le k\le K\\ k\neq j}}|T_i^k f_k|)^*\|_{L^{\sigma_j}}\\
\le &C\sum_{j=1}^{K}\sum_{i=1}^{N}\|f_j\|_{L^{p_j}} \quad \|\prod_{\substack{1\le k\le K\\ k\neq j}}|T_i^k f_k|\|_{L^{\sigma_j}}\\
\le &C\sum_{j=1}^{K}\sum_{i=1}^{N}\|f_j\|_{L^{p_j}} \prod_{\substack{1\le k\le K\\ k\neq j}}\|T_i^k f_k\|_{L^{p_k}}\\
\le &C\sum_{j=1}^{K}\|f_j\|_{L^{p_j}} \prod_{\substack{1\le k\le K\\ k\neq j}}\|f_k\|_{L^{p_k}} = C\prod_{k=1}^{K}\|f_k\|_{L^{p_k}}
\end{aligned}
$$

We conclude that the L^r (quasi)norm in x_0 of $\sup_{t>0}|\int\phi_{t,x_0}L_1\,dx|$ is bounded by $C\prod_{k=1}^{K}\|f_k\|_{L^{p_k}}$ and that the measure of the set $\{x_0 : \sup_{t>0}|\int\phi_{t,x_0}L_1\,dx| > \lambda\}$ is bounded by $C\lambda^{-r}\prod_{k=1}^{K}\|f_k\|_{L^{p_k}}^r$.

Term L_2 is treated similarly. First write $L_2=L_{21}+L_{22}+L_{23}+L_{24}$ where

$$
\begin{aligned}
L_{21}&=-\sum_{\substack{1\le j,l\le K\\ j<l}}L(f_1,\dots,(\eta_1 f_j)(x)-(\eta_1 f_j)(x_0),\dots,(\eta_1 f_l)(x)-(\eta_1 f_l)(x_0),\dots,f_K)\\
L_{22}&=-\sum_{\substack{1\le j,l\le K\\ j<l}}L(f_1,\dots,(\eta_1 f_j)(x_0),\dots,(\eta_1 f_l)(x)-(\eta_1 f_l)(x_0),\dots,f_K)\\
L_{23}&=-\sum_{\substack{1\le j,l\le K\\ j<l}}L(f_1,\dots,(\eta_1 f_j)(x)-(\eta_1 f_j)(x_0),\dots,(\eta_1 f_l)(x_0),\dots,f_K)\\
L_{24}&=-\sum_{\substack{1\le j,l\le K\\ j<l}}L(f_1,\dots,(\eta_1 f_j)(x_0),\dots,(\eta_1 f_l)(x_0),\dots,f_K)
\end{aligned}
$$

Same reasoning as before will show that any term L_{2u}, $u=1,2,3,4$ satisfies the following estimate:

$$\sup_{t>0}\left|\int\phi_{t,x_0}L_{2u}\,dx\right|\le C\sum_{\substack{1\le j,l\le K\\ j<l}}\sum_{i=1}^{K}(\prod_{\substack{1\le k\le K\\ k\neq j,l}}|T_i^k f_k|)^*(x_0)[(C_j f_j)(x_0)\,(C_l f_l)(x_0)]$$

where each $C_j f_j$ is either $|f_j|^*$ or $(T_i^j)_* f$ and therefore $\|C_j f_j\|_{L^{p_j}} \le C\|f_j\|_{L^{p_j}}$. Now define σ_{jl} by $\sigma_{jl}^{-1} + p_j^{-1} + p_l^{-1} = 1$. Hölder's inequality gives that for each $u = 1, 2, 3, 4$

$$\begin{aligned}
\|\sup_{t>0} |\int \phi_{t,x_0} L_{2u}\, dx|\|_{L^r} \le & C \sum_{\substack{1\le j,l\le K \\ j<l}} \sum_{i=1}^{K} \|(\prod_{\substack{1\le k\le K \\ k\ne j,l}} |T_i^k f_k|)^*\|_{L^{\sigma_{jl}}} [\|C_j f_j\|_{L^{p_j}}\, \|C_l f_l\|_{L^{p_l}}] \\
\le & C \sum_{\substack{1\le j,l\le K \\ j<l}} \sum_{i=1}^{K} \| \prod_{\substack{1\le k\le K \\ k\ne j,l}} |T_i^k f_k|\|_{L^{\sigma_{jl}}} [\|f_j\|_{L^{p_j}}\, \|f_l\|_{L^{p_l}}] \\
\le & C \sum_{\substack{1\le j,l\le K \\ j<l}} \sum_{i=1}^{K} \prod_{\substack{1\le k\le K \\ k\ne j,l}} \|T_i^k f_k\|_{L^{p_k}} [\|f_j\|_{L^{p_j}}\, \|f_l\|_{L^{p_l}}] \\
\le & C \sum_{\substack{1\le j,l\le K \\ j<l}} \prod_{\substack{1\le k\le K \\ k\ne j,l}} \|f_k\|_{L^{p_k}} [\|f_j\|_{L^{p_j}}\, \|f_l\|_{L^{p_l}}] \\
\le & C \prod_{1\le k\le K} \|f_k\|_{L^{p_k}}
\end{aligned}$$

We conclude that the L^r (quasi)norm in x_0 of $\sup_{t>0} |\int \phi_{t,x_0} L_2\, dx|$ is bounded by $C \prod_{k=1}^K \|f_k\|_{L^{p_k}}$ and that the measure of the set $\{x_0 : \sup_{t>0} |\int \phi_{t,x_0} L_2\, dx| > \lambda\}$ is bounded by $C\lambda^{-r} \prod_{k=1}^K \|f_k\|_{L^{p_k}}^r$.

We treat terms $L_3, L_4, \ldots, L_{K+1}$ in a similar way. In particular, we write term L_{K+1} as a sum of 2^K terms of the form $A = L(g_1, \ldots, g_K)$ where each g_j is either $(\eta_1 f_j)(x) - (\eta_1 f_j)(x_0)$ or $(\eta_1 f_j)(x_0)$. Same reasoning as before will show that the maximal function of L_{K+1} satisfies

$$\sup_{t>0} |\int \phi_{t,x_0} A\, dx| \le C \sum_{i=1}^{N} (C_i^1 f_1)(x_0) \ldots (C_i^K f_K)(x_0)$$

where each $C_i^j f_j$ is $|f_j|^* + (T_i^j)_* f_j$. Hölder's inequality gives that $\|\sup_{t>0} |\int \phi_{t,x_0} A dx|\|_{L^r}$ is bounded by $C \prod \|f_k\|_{L^{p_k}}$. Exactly the same estimate as above holds for the maximal function of L_{K+1} and the weak type estimates follow from Chebychev's inequality.

We are now left with term L_0. This is where we are going to use the assumption that L has mean value zero. We will show that for some $1 < s_j < p_j$ we have

(2.2) When $r > n/n+1$ $$\sup_{t>0} |\int \phi_{t,x_0} L_0\, dx| \le C \prod_{k=1}^{K} ((|f_k|^{s_k})^*(x_0))^{1/s_k}$$

(2.3) When $r = n/n+1$ $$\sup_{t>0} |\int \phi_{t,x_0} L_0\, dx| \le C \prod_{k=1}^{K} ((|f_k|^{p_k})^*(x_0))^{1/p_k}$$

Let's now indicate how (2.2) and (2.3) imply assertions 2) and 3) of theorem I. To get assertion 2) observe that when $r > n/n+1$

$$\|\sup_{t>0}|\int \phi_{t,x_0} L_0\ dx|\|_{L^r} \leq C\| \prod_{k=1}^{K} ((|f_k|^{s_k})^*(x_0))^{1/s_k}\ \|_{L^r}$$
$$\leq C \prod_{k=1}^{K} \|\ ((|f_k|^{s_k})^*(x_0))^{1/s_k}\ \|_{L^{p_k}} \leq C \prod_{k=1}^{K} \|f_k\|_{L^{p_k}}$$

where we used above that $p_k/s_k > 1$. We denote by $|A|$ the measure of the set A. To derive conclusion 3) of theorem I , let $\epsilon_0 = \lambda/C, \epsilon_{K+1} = 1$ and $\epsilon_1, \ldots, \epsilon_K > 0$ be arbitrary. It follows from (2.3) that

$$|\{x_0 : \sup_{t>0}|\int \phi_{t,x_0} L_0\ dx| > \lambda\}| \leq \sum_{j=1}^{K} |\{x_0 : (|f_j|^{p_j})^*(x_0) > (\frac{\epsilon_{j-1}}{\epsilon_j})^{p_j}\}|$$

By the weak type (1,1) result for the Hardy-Littlewood maximal function we get that the above is bounded by $C \sum_{j=1}^{K} (\frac{\epsilon_{j-1}}{\epsilon_j})^{-p_j} \int |f_j|^{p_j}\ dx$. This expression minimizes in $\epsilon_1, \ldots, \epsilon_K > 0$ when all the terms that appear in the sum are equal. This happens when

$$\frac{\epsilon_{j-1}}{\epsilon_j} = \frac{\|f_j\|_{L^{p_j}}^{p_j} (\lambda/C)^r}{\prod \|f_j\|_{L^{p_j}}^r} \qquad \text{for all } j = 2, 3, \ldots, K.$$

With this choice of ϵ_j's we get the weak type estimate

$$|\{x_0 : \sup_{t>0}|\int \phi_{t,x_0} L_0\ dx| > \lambda\}| \leq C\lambda^{-r} \prod \|f_j\|_{L^{p_j}}^r.$$

It remains to prove (2.2) and (2.3). We denote by T^* the adjoint operator of T and by $[\phi_{t,x_0}\ ,\ (T_i^1)^*]$ the commutator of ϕ_{t,x_0} and $(T_i^1)^*$. Since ϕ is a Lipschitz function of order 1,it follows that

$$|[\phi_{t,x_0}\ ,\ (T_i^1)^*](f)| \leq |\int K_i^1(x-y)(\phi_{t,x_0}(x) - \phi_{t,x_0}(y))f(y)\ dy| \leq \frac{C}{t^{n+1}} \int \frac{|f(y)|}{|x-y|^{n-1}}\ dy$$

and by the Hardy-Littlewood-Sobolev fractional integral theorem we get

(2.4) $$\|[\phi_{t,x_0}\ ,\ (T_i^1)^*](f)\|_{L^\sigma} \leq Ct^{-n-1}\|f\|_{L^\tau} \qquad \text{when} \qquad 1/\tau - 1/\sigma = 1/n.$$

Since $L(f_1, \dots, f_K)$ has integral zero *for all* sufficiently smooth functions f_1, the identity $\sum_{i=1}^N (T_i^1)^*(\prod_{k=2}^K T_i^k(\eta_0 f_k)) \equiv 0$ justifies the third equality below. We have

$$\begin{aligned}
\int \phi_{t,x_0} L_0 \; dx =& \sum_{i=1}^N \int \phi_{t,x_0} T_i^1(\eta_0 f_1) \dots T_i^K(\eta_0 f_K) \; dx \\
=& \sum_{i=1}^N \int \eta_0 f_1 \;\; (T_i^1)^* \left(\phi_{t,x_0} \prod_{k=2}^K T_i^k(\eta_0 f_k) \right) \; dx \\
=& \sum_{i=1}^N \int \eta_0 f_1 \;\; \left((T_i^1)^*(\phi_{t,x_0} \prod_{k=2}^K T_i^k(\eta_0 f_k)) - \phi_{t,x_0} (T_i^1)^*(\prod_{k=2}^K T_i^k(\eta_0 f_k)) \right) dx \\
=& \sum_{i=1}^N \int \eta_0 f_1 \;\; [\phi_{t,x_0} \, , \, (T_i^1)^*](F_i) \; dx = (2.5)
\end{aligned}$$

where $F_i = \prod_{k=2}^K T_i^k(\eta_0 f_k)$. Apply first Hölder's inequality with exponents p_1 and $p_1^{'} = p_1/(p_1 - 1)$ and then (2.4) with $\sigma = p_1^{'}$ and $\tau = s = (p_2^{-1} + \cdots + p_K^{-1})^{-1}$ the harmonic mean of $p_2, \dots, p_K$. This is where we use the assumption that $s > 1$. We get

$$\begin{aligned}
|(2.5)| \leq& \|\eta_0 f_1\|_{L^{p_1}} \sum_{i=1}^N \|[\phi_{t,x_0} \, , \, (T_i^1)^*](F_i)\|_{L^{p_1'}} \\
\leq& C t^{-n-1} \|\eta_0 f_1\|_{L^{p_1}} \sum_{i=1}^N \|F_i\|_{L^s} \\
\leq& C t^{-n-1} \|\eta_0 f_1\|_{L^{p_1}} \sum_{i=1}^N \|T_i^2(\eta_0 f_2)\|_{L^{p_2}} \dots \|T_i^K(\eta_0 f_K)\|_{L^{p_K}} \\
\leq& C t^{-n-1} \|\eta_0 f_1\|_{L^{p_1}} \, \|\eta_0 f_2\|_{L^{p_2}} \dots \|\eta_0 f_K\|_{L^{p_K}} \\
\leq& C t^{-n-1} \prod_{j=1}^K ((|f_j|^{p_j})^*(x_0))^{1/p_j} \;\; t^{n/p_j} = C \prod_{j=1}^K ((|f_j|^{p_j})^*(x_0))^{1/p_j}
\end{aligned}$$

This establishes (2.3). To prove (2.2) observe that the assumption $r > n/n+1$ gives $s^{-1} - (p_1^{'})^{-1} = p_2^{-1} + \cdots + p_K^{-1} - (p_1^{'})^{-1} = r^{-1} - 1 < n^{-1}$. Therefore for a suitable selection of $s_j < p_j$ we can make the expression $(s_2)^{-1} + \cdots + (s_K)^{-1} - ({s_1}^{'})^{-1}$ equal to n^{-1}. Then the same argument as before will give that

$$|(2.5)| \leq C t^{-n-1} \prod_{j=1}^K ((|f_j|^{s_j})^*(x_0))^{1/s_j} \, t^{n/s_j} = C \prod_{j=1}^K ((|f_j|^{s_j})^*(x_0))^{1/s_j}$$

The exponent of t above is zero because of the choice of the s_j's . Taking the supremum over all $t > 0$ we obtain (2.2). The proof of theorem I is now complete.

3. Proof of theorem IIa

Clearly, we only need to do the case $r \le 1$. Fix a ϕ and η as in Theorem I and split the bilinear operator $B(f,g)$ as the sum of $B_0 + B_{11} + B_{12} + B_3$ where

$$\begin{aligned} B_0 &= B(\eta_0 f, \eta_0 g) \\ B_{11} &= B(f, \eta_1 g) \\ B_{12} &= B(\eta_1 f, g) \\ B_2 &= -B(\eta_1 f, \eta_1 g) \end{aligned}$$

The arguments presented in theorem I will give the required estimates for the terms B_{11}, B_{12} and B_3. (Note the mean value zero assumption was only used in the treatment of term L_0.) It remains to get the required etimates for term B_0 which is the main term of the decomposition. We have

$$\int \phi_{t,x_0} B_0 \, dx = \int \int f(y) g(z) b_t(y,z) \; dydz$$

where $b_t(y,z) = \eta_0(y)\eta_0(z) \int \sum_{i=1}^N K_i^1(x-y) K_i^2(x-z) \phi_{t,x_0} \, dx$. The following lemma, whose proof we postpone until the end of this section describes the behavior of $b_t(y,z)$.

Lemma 1. *$b_t(y,z)$ is a smooth function off the diagonal $y = z$ and satisfies the following estimate $|b_t(y,z)| \le C t^{-n-m-1} |y-z|^{m+1-n} \eta(\frac{x_0 - y}{t}) \eta(\frac{x_0 - z}{t})$ for $|y-z|$ small.*

Assuming the lemma we estimate $|\int \phi_{t,x_0} B_0 \, dx|$ by

$$(3.1) \qquad C t^{-n-m-1} \int_{|y-x_0| \le 2t} \int_{|z-x_0| \le 2t} |f(y)| |g(z)| |y-z|^{-n+m+1} \; dydz$$

We denote by I_{m+1} the potential of order $m+1$, i.e. convolution with the kernel $|x|^{-n+m+1}$ in $\mathbb{R}^n$. Assume first that $m + 1 < n$.

In the case $r > n/n+m+1$ select $p_1 < p$ and $q_1 < q$ such that $1/q_1 - 1/p_1' = (m+1)/n$. This is always possible since the expression $1/q - 1/p' = 1/r - 1$ is assumption strictly less than $(m+1)/n$. Hölder's inequality together with the Hardy-Littlewood-Sobolev theorem on fractional integrals give that

$$\begin{aligned} |(3.1)| \le & C t^{-n-m-1} \| f \chi_{|y-x_0| \le 2t} \|_{L^{p_1}} \| I_{m+1}(g \chi_{|z-x_0| \le 2t}) \|_{L^{p_1'}} \\ \le & C t^{-n-m-1} \| f \chi_{|y-x_0| \le 2t} \|_{L^{p_1}} \| g \chi_{|z-x_0| \le 2t} \|_{L^{q_1}} \\ \le & C t^{-n-m-1} t^{n/p_1 + n/q_1} \left((|f|^{p_1})^*(x_0) \right)^{1/p_1} \left((|g|^{q_1})^*(x_0) \right)^{1/q_1} \end{aligned}$$

where by χ_A we denote the characteristic function of the set A. By the choice of p_1 and q_1, the exponent of t above is equal to zero and we conclude that

$$(3.2) \quad \text{if } r > n/n+m+1 \quad \sup_{t>0} \left| \int \phi_{t,x_0} B_0 \, dx \right| \le C \left((|f|^{p_1})^*(x_0) \right)^{1/p_1} \left((|g|^{q_1})^*(x_0) \right)^{1/q_1}.$$

In the case $r = n/n+m+1$ simply repeat the argument above with $p = p_1$ and $q = q_1$. We get

$$(3.3) \quad \text{if } r = n/n+m+1 \quad \sup_{t>0} \left| \int \phi_{t,x_0} B_0 \, dx \right| \le C \left((|f|^{p})^*(x_0) \right)^{1/p} \left((|g|^{q})^*(x_0) \right)^{1/q}.$$

Conclusions 2) and 3) of theorem IIa follow as in theorem I. In fact (3.2) and (3.3) are repetitions of (2.2) and (2.3) in section 2.

When $m+1 = n$ only the case $r > n/n+m+1 = 1/2$ can occur. Then (3.2) follows from (3.1) directly from Hölder's inequality.

It remains to prove Lemma 1. We have that $b_t(y,z) = \frac{1}{t^{2n}} b(\frac{x_0 - y}{t}, \frac{x_0 - z}{t})$ where

$$b(y,z) = \eta(y)\eta(z) \int \sum_{i=1}^{N} K_i^1(y-\sigma) K_i^2(z-\sigma) \phi(\sigma) \, d\sigma.$$

The estimate for b_t in Lemma 1 is then equivalent to the estimate

$$|b(y,z)| \le C|y-z|^{m+1-n} \eta(y)\eta(z).$$

The vanishing moments assumptions for $B(f,g)$ are equivalent to the conditions

$$\int \sum_i K_i^1(y-\sigma) K_i^2(z-\sigma) \sigma^\alpha \, d\sigma = 0 \qquad \text{for all} \qquad |\alpha| \le m.$$

We can therefore write $b(y,z) = \eta(y)\eta(z) d(y,z)$ where

$$d(y,z) = \int \sum_i K_i^1(y-\sigma) K_i^2(z-\sigma) [\phi(\sigma) - \sum_{|\alpha| \le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)(\sigma - y)^\alpha] \, d\sigma$$

It will suffice to show that for $|y|, |z| \le 4$, we have $|d(y,z)| \le |y-z|^{m+1-n}$. Fix a smooth function $\zeta(\sigma)$ on $\mathbb{R}^n$, equal to 1 on $|\sigma| \le 16$ and supported in $|\sigma| \le 32$. Split $d(y,z) = I_1 + I_2$ where

$$I_1 = \int \sum_i K_i^1(\sigma) K_i^2((z-y)-\sigma) [\phi(\sigma + y) - \sum_{|\alpha| \le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y) \sigma^\alpha] \zeta(\sigma) \, d\sigma$$

$$I_2 = \int \sum_i K_i^1(\sigma) K_i^2((z-y)-\sigma) [\phi(\sigma + y) - \sum_{|\alpha| \le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y) \sigma^\alpha] (1 - \zeta(\sigma)) \, d\sigma$$

Then $\phi(\sigma + y) = 0$ in the integral I_2 and $|\sigma - (z-y)| \sim |\sigma|$. It follows that

$$\begin{aligned}
|I_2| &\le C \sum_i \int_{|\sigma|\ge 16} |\sigma|^{-n} |\sigma - (z-y)|^{-n} \sum_{|\alpha|\le m} |\frac{\partial^\alpha \phi}{\partial y^\alpha}(y)||\sigma|^\alpha |1-\zeta(\sigma)|\, d\sigma \\
&\le C \int_{|\sigma|\ge 16} |\sigma|^{-n}|\sigma|^{-n}|\sigma^m|\, d\sigma \\
&\le C \le C|y-z|^{m+1-n}
\end{aligned}$$

since $|y-z|$ is small and $m+1-n \le 0$.

Finally we treat term I_1. First fix an i and a y and let

$$a_y(\sigma) = K_i^1(\sigma)\,\zeta(\sigma)\left(\phi(\sigma+y) - \sum_{|\alpha|\le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)\sigma^\alpha\right).$$

a_y is a smooth function away from zero, has compact support and behaves like $|\sigma|^{m+1-n}$ as $|\sigma| \to 0$. To see this last assertion use the mean value theorem to write $a_y(\sigma)$ as $K_i^1(\sigma)\sum_{|\alpha|=m+1} \frac{\partial^\alpha \phi}{\partial y^\alpha}(\xi_{y,y+\sigma})\sigma^\alpha$ for some $\xi_{y,y+\sigma}$ between y and $y+\sigma$. It follows that $a_y(\sigma) \sim |\sigma|^{-n+m+1}$ as $\sigma \to 0$.

Also note that $I_1 = \sum_i (T_i^2 a_y)(z-y)$. The required estimate for I_1 will then follow from the following lemma

Lemma 2. *Let $\psi(\sigma)$ be a compactly supported smooth function except at the origin such that $\psi(\sigma) \sim B_0|\sigma|^l$ as $|\sigma| \to 0$, for some $-n < l < 0$ and B_0 constant. Then for any Calderón-Zygmund operator T, there is a constant C such that the estimate $|T(\psi)(w)| \le CB_0|w|^l$ holds as $|w| \to 0$.*

The following short proof of Lemma 2 was suggested to me by Peter Jones. Fix a smooth compactly supported function θ equal to 1 on half of its support so that $|x|^l \sim \sum_{j\ge 0} 2^{-jl}\theta(2^j x)$ as $|x| \to 0$. Then $T(\psi)(w) \sim \sum_{j\ge0} B_0 2^{-jl} T(\theta)(2^j w)$. $T(\theta)(w)$ is bounded near zero and has rapid decay as $|w| \to \infty$. Therefore the terms in the sum giving $T(\psi)$ with $2^j \le \frac{C}{|w|}$ contribute $\le CB_0 \sum_{j \le \log \frac{C}{|w|}} 2^{-jl} \le CB_0 \le CB_0|w|^l$ while the terms with with $2^j \ge \frac{C}{|w|}$ contribute $C_N B_0 \sum_{j\ge \log\frac{C}{|w|}} 2^{-jl}(2^j|w|)^{-N} \le CB_0|w|^l$ as $|w| \to 0$. This finishes the proof of Lemma 2.

We conclude that $(T_i^2 a_y)(z-y)$ is $O(|y-z|^{m+1-n})$ as $|y-z| \to 0$. Since $I_1 = \sum_i (T_i^2 a_y)(z-y)$ it follows that I_1 satisfies the required estimate $I_1 \le C|y-z|^{m+1-n}$. The proof of Lemma 1 and hence of theorem IIa are now complete.

5. Proof of theorem IIb

We will now combine some ideas from theorems I and IIa to prove theorem IIb. Again we only need to do the case $r \le 1$. Fix ϕ and η as before and split the K-linear operator $L = L_0 + L_1 + \cdots + L_{K+1}$ as in theorem I. Already the treatment of term L_1 presents some differences. First of all for a fixed j define s_j by $s_j^{-1} + p_j^{-1} = r^{-1}$. L_1 is the sum of K terms of the form

$$L(f_1, \dots, (\eta_1 f_j)(x) - (\eta_1 f_j)(x_0), \dots, f_K) + L\left(f_1, \dots, (\eta_1 f_j)(x_0), \dots, f_K\right).$$

By Lemma 1 in [CG], for any F in H^p and ψ sufficiently smooth we have that $|\int F\psi\, dx| \le F^+(x_0)N_{x_0}(\psi)$ for any x_0 where $N_{x_0}(\psi)$ is the norm of ψ as defined in [CG] and F^+ is an L^p function with $\|F^+\|_{L^p} \le C\|F\|_{H^p}$. A computation after Lemma 1 shows that if $\psi(x) = T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0)$, then $\sup_{t>0} N_{x_0}(\psi) \le C|f_j|^*(x_0)$. An application of this fact with $\Lambda_i^j = F$ gives that the maximal function of L_2 satisfies

$$\begin{aligned}
&\sup_{t>0} \left| \int \phi_{t,x_0} L_2 \, dx \right| \\
\le &\sum_{j=1}^{K} \sum_{i=1}^{M} \sup_{t>0} \left| \int \phi_{t,x_0} \left(T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0) + T_i^j(\eta_1 f_j)(x_0) \right) \Lambda_i^j \, dx \right| \\
\le &C \sum_{j=1}^{K} \sum_{i=1}^{M} |f_j|^*(x_0)(\Lambda_i^j)^+(x_0) + (T_i^j)_* f_j(x_0)(\Lambda_i^j)^*(x_0) = (4.1)
\end{aligned}$$

If $s_j > 1$, the argument in theorem I applies. Suppose then that $s_j \le 1$. We can assume by induction that $\|\Lambda_i^j\|_{H^{s_j}} = \|(\Lambda_i^j)^*\|_{L^{s_j}} \le C \prod_{k \ne j} \|f_k\|_{L^p}$. By Hölder's inequality, the L^r norm in x_0 of (4.1) is bounded by

$$\begin{aligned}
&C \sum_{j=1}^{K} \sum_{i=1}^{M} \||f_j|^*\|_{L^{p_j}} \|(\Lambda_i^j)^+\|_{L^{s_j}} + \|(T_i^j)_* f_j\|_{L^{p_j}} \|(\Lambda_i^j)^*\|_{L^{s_j}} \\
\le &C \sum_{j=1}^{K} \sum_{i=1}^{M} \|f_j\|_{L^{p_j}} \prod_{\substack{1 \le k \le K \\ k \ne j}} \|f_k\|_{L^{p_k}} \\
= &C \prod_{k=1}^{K} \|f_k\|_{L^{p_k}}
\end{aligned}$$

Term L_2 can be treated similarly. A simple computation shows that the N_{x_0} norm of the function $\psi = (T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0))(T_i^l(\eta_1 f_l)(x) - T_i^l(\eta_1 f_l)(x_0))$ satisfies

$\sup_{t>0} N_{x_0}(\psi) \leq C|f_j|^*(x_0)|f_l|^*(x_0)$. As in theorem I, we write term L_2 as $L_2 = L_{21} + L_{22} + L_{23} + L_{24}$. Since the Λ_i^j satisfy property (1.1), for each j, l we can write $L = \sum_{i=1}^{R}(T_i^j f_j)(T_i^l f_l)\Lambda_i^{j,l}$, where each $\Lambda_i^{j,l}$ is a $(K-2)$-linear operator of $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{l-1}, f_{l+1}, \ldots, f_K$. Another application of Lemma 1 in [CG] will give that the maximal function of any term L_{2u}, $u = 1, 2, 3, 4$ satisfies the following estimate:

$$\sup_{t>0} \left|\int \phi_{t,x_0} L_{2u}\, dx\right| \leq C \sum_{\substack{1\leq j,l\leq K \\ j<l}} \sum_{i=1}^{M} (C_j f_j)(x_0)\,(C_l f_l)(x_0)[(\Lambda_i^{j,l})^*(x_0) + (\Lambda_i^{j,l})^+(x_0)] = (4.2)$$

where $C_j f_j = |f_j|^* + (T_i^j)_* f$ and therefore $\|C_j f_j\|_{L^{p_j}} \leq C\|f_j\|_{L^{p_j}}$. Now define s_{jl} by $s_{jl}^{-1} + p_j^{-1} + p_l^{-1} = 1$. If $s_{jl} > 1$, the argument in theorem I establishes the result. If $s_{jl} \leq 1$, we can assume by induction that $\|\Lambda_i^{j,l}\|_{H^{s_{jl}}} = \|(\Lambda_i^{j,l})^*\|_{L^{s_{jl}}} \leq C\prod_{k\neq j,l}\|f_k\|_{L^{p_k}}$. By Hölder's inequality, the L^r norm in x_0 of (4.2) is bounded by

$$\begin{aligned}
&\|\sup_{t>0}\left|\int \phi_{t,x_0} L_{2u}\, dx\right|\|_{L^r} \\
\leq & C \sum_{\substack{1\leq j,l\leq K \\ j<l}} \sum_{i=1}^{M} \|C_j f_j\|_{L^{p_j}}\,\|C_l f_l\|_{L^{p_l}}\,[\|(\Lambda_i^{j,l})^*\|_{L^{s_{jl}}} + \|(\Lambda_i^{j,l})^+\|_{L^{s_{jl}}}] \\
\leq & C \sum_{\substack{1\leq j,l\leq K \\ j<l}} \sum_{i=1}^{M} [\|f_j\|_{L^{p_j}}\,\|f_l\|_{L^{p_l}}] \prod_{\substack{1\leq k\leq K \\ k\neq j,l}} \|f_k\|_{L^{p_k}} \\
\leq & C \prod_{1\leq k\leq K} \|f_k\|_{L^{p_k}}
\end{aligned}$$

We conclude that the L^r (quasi)norm of the maximal function of L_2 is bounded by a constant multiple of the product of the L^{p_k} norms of the f_k's.

Clearly, this procedure can go on for terms $L_3, \ldots, L_{K-1}$. In the case of term L_{K-1}, the operators $\Lambda_i^{j_1,\ldots,j_{K-1}}$ are bilinear operators as those in theorem IIa. Finally, as in theorem I, the terms L_K and L_{K+1} satisfy the following estimate:

$$\sup_{t>0}\left|\int \phi_{t,x_0} L_K\, dx\right| + \sup_{t>0}\left|\int \phi_{t,x_0} L_{K+1}\, dx\right| \leq C\sum_i (C_i^1 f_1)(x_0)\ldots(C_i^K f_K)(x_0).$$

It follows that the L^r (quasi)norms of the maximal functions of L_K and L_{K+1} are bounded by constant multiples of the product of the $\|f_k\|_{L^{p_k}}$'s.

We are now left with the main term of the decomposition, L_0. We have

$$\int \phi_{t,x_0} L_0 \, dx = \int \cdots \int \prod_{k=1}^{K} f_k(y_k) b_t(y_1, \ldots, y_K) \; dy_K \ldots dy_1 \tag{4.3}$$

where $b_t(y_1, \ldots, y_K) = \prod_{k=1}^{K} \eta_0(y_k) \int \sum_{i=1}^{N} \prod_{j=1}^{K} K_i^j(y_j - x)\phi_{t,x_0} \, dx$.

We first do the case $r = n/n + m + 1$. We find $q_2 > p_2, \ldots, q_K > p_K$ such that $\sum_{k=2}^{K} \frac{1}{p_k} - \frac{1}{q_k} = \frac{m+1}{n}$. Let $\delta_k = \frac{n}{m+1}(\frac{1}{p_k} - \frac{1}{q_k})$. The δ_k's are positive numbers and their sum is 1. The following lemma describes the behavior of $b_t(y_1, \ldots, y_K)$.

Lemma 3. *$b_t(y_1, \ldots, y_K))$ is a smooth function off the planes $y_i = y_j$ and the satisfies the estimate $|b_t(y_1, \ldots, y_K)| \leq Ct^{-n-m-1} \prod_{k=1}^{K} \eta_0(y_k) \prod_{j=2}^{K} |y_1 - y_j|^{-n+\delta_j(m+1)}$.*

Assuming the lemma, we prove the theorem as follows. Hölder's inequality with exponents $p_1^{-1} + q_2^{-1} + \cdots + q_K^{-1} = 1$ together with (4.3) give the following

$$|\int \phi_{t,x_0} L_0 \, dx| \leq Ct^{-n-m-1} \|f_1 \chi_{|x_0 - y_1| \leq 2t}\|_{L^{p_1}} \prod_{j=2}^{K} \|I_{(m+1)\delta_j}(f_j \chi_{|x_0 - y_j| \leq 2t})\|_{L^{q_j}}$$

where by I_k we denote convolution with $|x|^{-n+k}$ on $\mathbb{R}^n$. By the Hardy-Littlewood-Sobolev fractional integral theorem we get that

$$\begin{aligned}
&\sup_{t>0} |\int \phi_{t,x_0} L_0 \, dx| \\
\leq & C \sup_{t>0} t^{-n-m-1} \|f_1 \chi_{|x_0 - y_1| \leq 2t}\|_{L^{p_1}} \prod_{j=2}^{K} \|f_j \chi_{|x_0 - y_j| \leq 2t}\|_{L^{p_j}} \\
\leq & C \sup_{t>0} t^{-n-m-1} \prod_{j=1}^{K} ((|f_j|^{p_j})^*(x_0))^{1/p_j} \, t^{n/p_j} \\
= & C \prod_{j=1}^{K} ((|f_j|^{p_j})^*(x_0))^{1/p_j}
\end{aligned}$$

since we are assuming that $r = n/n + m + 1$. This estimate is the equivalent of (2.3) in theorem I and the required weak type result follows as in theorem I.

In the case $r > n/n+m+1$, select $s_j < p_j$ such that $\sum_{j=1}^{K} s_j^{-1} = n/n+m+1$. By the previous result, we get that

$$\sup_{t>0} \int \phi_{t,x_0} L_0 \, dx \leq C \prod_{j=1}^{K} ((|f_j|^{s_j})^*(x_0))^{1/s_j}$$

and this estimate is the equivalent of (2.2) in theorem I. The required result follows as before.

It remains to prove Lemma 3. Note that $b_t(y_1, \ldots, y_K) = \frac{1}{t^{Kn}} b(\frac{x_0 - y_1}{t}, \ldots, \frac{x_0 - y_K}{t})$ where

$$b(y_1, \ldots, y_K) = \prod_{k=1}^{K} \eta(y_k) \int \sum_{i=1}^{N} K_i^1(y_1 - \sigma) \ldots K_i^K(y_K - \sigma)\phi(\sigma)\ d\sigma.$$

The estimate for b_t in Lemma 3 is equivalent to the following estimate

$$|b(y_1, \ldots, y_K)| \leq C \prod_{k=1}^{K} \eta(y_k) \prod_{j=2}^{K} |y_1 - y_j|^{-n+\delta_j(m+1)}.$$

The vanishing moments assumptions for L are equivalent to the conditions

$$\int \sum_i \prod_j K_i^j(y_j - \sigma)\sigma^\alpha\ d\sigma = 0 \qquad \text{for all} \qquad |\alpha| \leq m.$$

We can therefore write $b(y_1, \ldots, y_K) = \prod_k \eta(y_k)\ d(y_1, \ldots, y_K)$ where

$$d(y_1, \ldots, y_K) = \int \sum_i \prod_j K_i^j(y_j - \sigma)[\phi(\sigma) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha \phi}{\partial y_1^\alpha}(y_1)(\sigma - y_1)^\alpha]\ d\sigma$$

All we need to show is that for $|y_2|, \ldots, |y_K| \leq 4$, we have that $|d(y_1, \ldots, y_K)| \leq \prod_{j=2}^{K} |y_1 - y_j|^{-n+\delta_j(m+1)}$. Fix a smooth function $\zeta(\sigma)$ on $\mathbb{R}^n$, equal to 1 on $|\sigma| \leq 16$ and supported in $|\sigma| \leq 32$. Split $d(y_1, \ldots, y_K) = I_1 + I_2$ where

$$I_1 = \int \sum_i K_i^1(\sigma) \prod_{j=2}^{K} K_i^j((y_j - y_1) - \sigma)[\phi(\sigma + y_1) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha \phi}{\partial x_1^\alpha}(y_1)\sigma^\alpha]\zeta(\sigma)\ d\sigma$$

$$I_2 = \int \sum_i K_i^1(\sigma) \prod_{j=2}^{K} K_i^j((y_j - y_1) - \sigma)[\phi(\sigma + y_1) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha \phi}{\partial y_1^\alpha}(y_1)\sigma^\alpha](1 - \zeta(\sigma))\ d\sigma$$

Then $\phi(\sigma + y_1) = 0$ in the integral I_2 and $|\sigma - (y_j - y_1)| \sim |\sigma|$ for all $j \geq 2$. It follows that

$$\begin{aligned}
|I_2| &\leq C \sum_i \int_{|\sigma| \geq 16} |\sigma|^{-n} \prod_{j=2}^{K} |\sigma - (y_j - y_1)|^{-n} \sum_{|\alpha| \leq m} |\frac{\partial^\alpha \phi}{\partial y_1^\alpha}(y_1)||\sigma|^\alpha |1 - \zeta(\sigma)|\ d\sigma \\
&\leq C \int_{|\sigma| \geq 16} |\sigma|^{-n} |\sigma|^{-(K-1)n} |\sigma|^m\ d\sigma \\
&\leq C \leq C \prod_{j=2}^{K} |y_1 - y_j|^{-n+\delta_j(m+1)}
\end{aligned}$$

since $|y_j - y_1| \leq 8$ and the numbers $-n + \delta_j(m+1)$ are negative.

We now treat term I_1. First fix an i and $y_1, y_2, \ldots, y_K$ such that $y_j \neq y_1$ for $j \neq 1$ and let

$$a_{y_1}(\sigma) = K_i^1(\sigma) \prod_{j=3}^{K} K_i^j((y_j - y_1) - \sigma)\, \zeta(\sigma) \left(\phi(\sigma + y_1) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha \phi}{\partial y_1^\alpha}(y_1) \sigma^\alpha \right).$$

a_{y_1} is a smooth function away from zero except possibly at the points $y_j - y_1$, it has compact support and behaves like $B_0|\sigma|^{m+1-n}$ as $|\sigma| \to 0$, where the constant $B_0 \sim |y_3 - y_1|^{-n} \cdots |y_K - y_1|^{-n}$. It follows that $a_y(\sigma) \sim B_0|\sigma|^{-n+m+1}$ as $\sigma \to 0$.

Also note that $I_1 = \sum_i (T_i^2 a_{y_1})(y_2 - y_1)$. It follows from Lemma 2 that

$$|I_1| \leq \sum_i |(T_i^2 a_{y_1})(y_2 - y_1)| \leq C|y_2 - y_1|^{-n+m+1} \prod_{j=3}^{K} |y_j - y_1|^{-n}.$$

The above is also true when y_2 is replaced by y_s, $s = 3, \ldots, K$. We get that

$$|I_1| \leq C|y_s - y_1|^{-n+m+1} \prod_{\substack{3 \leq j \leq K \\ j \neq s}} |y_j - y_1|^{-n} \qquad \text{for } s = 2, \ldots, K. \tag{4.4}$$

We raise (4.4) to the power δ_s, $s = 2, \ldots, K$ and we multiply all the resulting inequalities. We get the desired conclusion for $|I_1|$ and hence for $b(y_1, \ldots, y_K)$.

6. Examples and final remarks

In this section, we discuss examples of operators that satisfy the hypotheses of theorems I and II. The determinant of the Jacobian of a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ gives rise to the following bilinear operator

$$\tilde{J}(f, g) = (R_1 f)(R_2 g) - (R_2 f)(R_1 g)$$

where R_j are the usual Riesz transforms. $\tilde{J}$ has always integral 0 but it doesn't have any other vanishing moments. It can be easily checked that for $p, q > 1$ with $1/p + 1/q = 3/2$, $\tilde{J}$ *never* maps $L^p \times L^q$ to $H^{2/3}$ and therefore our weak type result is sharp. In general, if a bilinear operator has all moments up to and including order m vanishing and one moment of order $m+1$ nonzero, it doesn't map $L^p \times L^q$ to $H^{n/n+m+1}$ when $1/p + 1/q = (n+m+1)/n$. Hence, the number of vanishing moments of the bilinear operator gives the lowest r for which the operator maps products of Lebesgue spaces into H^r. The same is true for more general multilinear operators.

We now discuss probably the most important example that satisfies the hypotheses of our theorem II, a bilinear map that involves sums of products of derivatives of order 2 and is the analogue of the determinant of the Jacobian. The Hessian of a map $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is the spacial $2 \times 2 \times 2$ matrix which has on the top the 2×2 matrix:

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

and on the bottom the 2×2 matrix:

$$\begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{pmatrix}$$

We denote by $H(f, g)$ the determinant of the $2 \times 2 \times 2$ Hessian matrix above defined as follows:

$$H(f, g) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 g}{\partial y \partial x} - \frac{\partial^2 f}{\partial y \partial x} \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 g}{\partial x^2}.$$

After formally replacing the partial derivatives of F with the corresponding Riesz transforms we get the following bilinear operator

$$\tilde{H}(f, g) = (R_1^2 f)(R_2^2 g) - (R_1 R_2 f)(R_2 R_1 g) - (R_2 R_1 f)(R_1 R_2 g) + (R_2^2 f)(R_1^2 g).$$

It was shown in the last section of [CG], that $\tilde{H}$ has integral and first moments zero. By theorem IIa it follows that $\tilde{H}$ maps $L^p \times L^q$ to H^r for all $p, q > 1$, where r is their harmonic mean. It follows that the determinant of the Hessian H, of a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ maps pairs of functions with Laplacean in $L^p \times L^q$ into H^r for all $p, q > 1$, where r is their harmonic mean. This result generalizes the corresponding theorem about the Jacobian in the case of second order derivatives and has analogues in higher dimensions.

We now discuss generalizations of $\tilde{H}$ in $\mathbb{R}^n$. Let $F = (F_1, \ldots, F_n)$ be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Form the $n \times \cdots \times n$ matrix M by stacking the $n \times n$ matrices $\left(\frac{\partial^2 F_k}{\partial x_i \partial x_j} \right)_{i,j}$, $k = 1, 2, \ldots, n$ on the top of each other. We call M the n-dimensional Hessian of F. The determinant of this matrix is defined by induction on n as the sum of its n^2 minor determinants suitably signed. After formally replacing the $\frac{\partial}{\partial x_j}$ derivative of F by the j^{th} Riesz transform, the n-dimensional determinant of this matrix gives rise to an n-linear operator $\tilde{H}^n$ with vanishing integral and first moments. When $n = 2$ the resulting bilinear operator $\tilde{H}^2$ is the operator $\tilde{H}$ defined above. When $n = 3$ the resulting trilinear operator $\tilde{H}^3(f, g, h)$ is the sum of the nine terms $(-1)^{i+j}(R_i R_j h)\tilde{H}_{ij}(f, g)$ where each $\tilde{H}_{ij}$ corresponds to the determinant of a Hessian of a $2 \times 2 \times 2$ minor. It is therefore clear that that $\tilde{H}^3$ satisfies the hypotheses of theorem IIb. Our result then says that that for $p, q > 1$, $\tilde{H}^3$ maps $L^p \times L^q$ into H^r when the harmonic mean r of p and q is $> 3/5$. In general the operator $\tilde{H}^n$ maps $L^p \times L^q$

into H^r for $1 \geq r > n/(n+2)$ since it follows by induction that $\tilde{H}^n$ has integral and first moments zero. ($k = 1$.)

It is conceivable that determinants of matrices of higher order derivatives of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ give rise to multilinear operators with higher moments vanishing but these cases are not investigated in this article. Examples of bilinear operators with moments of all orders vanishing in one dimension are $D_1(f,g) = fg - (Hf)(Hg)$ and $D_2(f,g) = f(Hg) + (Hf)g$, where H is the usual Hilbert transform. D_1 and D_2 are the real and imaginary parts of holomorphic functions and their mapping properties are well understood. More generally, examples of K-linear operators with all moments vanishing are given by the real and imaginary parts of $\prod_{k=1}^{K}(f_k + iHf_k)$.

References

[CG] R.R. Coifman and L. Grafakos, *Hardy space estimates for multilinear operators, I*, Revista Mat. Iberoamericana **8** (1992).

[CLMS] R.R. Coifman, P.L. Lions, Y. Meyer and S. Semmes, *Compacité par compensation et espaces de Hardy*, Comptes Rendus de la Academie de Sciences, to appear.

[CRW] R.R.Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Annals of Mathematics, **103** (1976), 611-635.

[FRS] C. Fefferman, N. M. Riviere and Y. Sagher, it Interpolation between H^p spaces: The real method, Transactions of the AMS **191** (1974) 75-81.

[FSO] R. Fefferman and F. Soria, *The space Weak H^1*, Studia Mathematica, **85** (1987), 1-16.

[FST] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **229** (1972), 137-193.

[CM][1] R.R. Coifman and Y. Meyer, *Non linear harmonic analysis, operator theory and PDE*, Beijing Lectures in harmonic analysis, Annals of Mathematical Studies, Prince
ton Univ. Press, Princeton 1986.

[CM][2] R.R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Transactions of the AMS, **212** (1975), 315-331.

[S] E. M. Stein, *Singular Integrals and Differentiability Properties of functions*, Princeton Univ. Press, Princeton 1970.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton 1971.

Department of Mathematics, Yale University, Box 2155 Yale Station, New Haven, CT 06520-2155

Current address: Loukas Grafakos Department of Mathematics, Washington University in St Louis, 1 Brooking Drive, St Louis, MO 63130-4899