HARDY SPACE ESTIMATES FOR MULTILINEAR OPERATORS, I

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ABSTRACT. In this article, we study bilinear operators given by inner products of finite vectors of Calderón-Zygmund operators. We find that necessary and sufficient condition for these operators to map products of Hardy spaces into Hardy spaces is to have a certain number of moments vanishing and under these assumptions we prove a Hölder-type inequality in the H^p space context.

0. Introduction

Probably, the most important example of a multilinear operator is the determinant of the Jacobian of a map $F : \mathbf{R}^n \to \mathbf{R}^n$. In two dimensions the determinant of the Jacobian of a map (f, g) from $\mathbf{R}^2$ to $\mathbf{R}^2$ is the bilinear map

$$J(f,g) = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}$$

which has very interesting mapping properties. A theorem of P.L. Lions, says that J maps the product of Sobolev spaces $L^2_1 \times L^2_1$ into the Hardy space H^1. This theorem has been extended for L^p Sobolev spaces by [CMLS] as follows: The determinant of the Jacobian maps $L^p_1 \times L^q_1$ into H^γ as long as $1 \geq \gamma > \frac{2}{3}$, $p, q > 1$ and $p^{-1} + q^{-1} = \gamma^{-1}$. The spaces $H^\gamma(\mathbf{R}^n)$, $\gamma \leq 1$ are the usual real variable Hardy spaces as defined in [S] or [SW]. The result of [CMLS] is false when $\gamma = 2/3$ and leads naturally to the following question: Why can't the Jacobian-determinant map into some H^γ space for $\gamma \leq \frac{2}{3}$?

In these articles we prove that a more general class of bilinear operators map into H^r for arbitrarily small $r > 0$ *only when* they have a certain number of moments vanishing. The determinant of the Jacobian has always integral zero but it does not have higher moments vanishing and this is the reason it cannot map into H^γ for $\gamma \leq \frac{2}{3}$. Other bilinear operators

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on $\mathbf{R}^n$ have higher moments vanishing and they map into H^r for $r < \frac{n}{n+1}$. (The index $\frac{2}{3}$ corresponds to the case $n = 2$.)

A good example of an operator with integral and first moments vanishing in $\mathbf{R}^2$ is the determinant of the Hessian of a map $(f, g) : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$H(f,g) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 g}{\partial y \partial x} - \frac{\partial^2 f}{\partial y \partial x}\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 g}{\partial x^2}.$$

By introducing the Riesz transforms, $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$ the Jacobian-determinant J and Hessian-determinant H can be studied through the bilinear maps $\tilde{J}$ and $\tilde{H}$ given by

$$\begin{aligned}\tilde{J}(f,g) &= (R_1 f)(R_2 g) - (R_2 f)(R_1 g)\\ \tilde{H}(f,g) &= (R_1^2 f)(R_2^2 g) - 2(R_1 R_2 f)(R_2 R_1 g) + (R_2^2 f)(R_1^2 g)\end{aligned}$$

which are of the form

$$\sum_{j=1}^{N}(T_j^1 f)(T_j^2 g) \tag{0.1}$$

for some Calderón-Zygmund operators $\{T_j^1\}, \{T_j^2\}$.

In part II of this work, we prove that $\tilde{H}$ maps $L^p \times L^q$ into H^r for $p, q > 1$, $1 \geq r > 1/2$ and $r^{-1} = p^{-1} + q^{-1}$. We conclude that H maps functions with Laplacean in $L^p \times L^q$ into H^r for the same p, q, r as above. $r = 1/2$ is a natural lower bound in this case since the assumptions $p, q > 1$ imply that $r = (p^{-1} + q^{-1})^{-1} > 1/2$.

The question investigated in this article is under what conditions can we have boundedness into H^r for $r \leq 1/2$. Since $r = (p^{-1} + q^{-1})^{-1}$ we must have $p \leq 1$ or $q \leq 1$ and obviously the L^p spaces are not a suitable starting point. If we replace the L^p spaces with H^p for $p \leq 1$ however, we get boundedness into H^r for arbitrarily small r. We treat general bilinear operators of the form (0.1) and we assume that these operators have for a given $r > 0$ a required number of moments vanishing to map into H^r.

1. Preliminaries

We are given two families of tempered distributions $\{K_i^1\}_{i=1}^N, \{K_i^2\}_{i=1}^N$, homogeneous of degree 0 and we are assuming that:

1) The Fourier transforms of $\{K_i^1\}, \{K_i^2\}$ are bounded functions.
2) $\{K_i^1\}, \{K_i^2\}$ are sufficiently smooth away from the origin and $|\frac{\partial^\gamma}{\partial x^\gamma} K_i^j| \leq C|x|^{-n-|\gamma|}$ for all sufficiently large γ $\quad (j = 0$ or $1)$

3) For all sufficiently large multi indices α and γ the partial derivatives of $\{K_i^1\}$ and $\{K_i^2\}$ satisfy:

$$\left|\frac{\partial^\gamma}{\partial y^\gamma}\frac{\partial^\alpha}{\partial x^\alpha}\Big(K_i^j(x-y)-K_i^j(x)\Big)\right| \le A\frac{|y|}{|x|^{n+|\alpha|+|\gamma|+1}} \quad \text{for} \quad |x|>2|y| \quad (j=0 \text{ or } 1)$$

We call T_i^1 the operator given by convoluting with K_i^1 and T_i^2 the operator given with convolution with K_i^2. Theorem 12 in [FS] says that the operators $\{T_i^1\},\{T_i^2\}$ map $H^p \to H^p$. The main result in this article is the H^p boundedness of the bilinear product

$$B(f,g)=\sum_i (T_i^1 f)(T_i^2 g)$$

where f,g lie in suitable Hardy spaces. We have the following theorem.

Theorem. *Suppose $\{T_i^1\},\{T_i^2\}$, $i=1,2,\dots,N$ are Calderón-Zygmund operators on $\mathbf{R}^n$ as above. Fix $p,q\le 1$ and let*

$$B(f,g)=\sum_{i=1}^N (T_i^1 f)(T_i^2 g).$$

Assume that for some $k\ge 0$ integer, for all multi indices $|\alpha|\le k$ and for all f H^p-atoms and g H^q-atoms the moments

$$\int x^\alpha B(f,g)(x)dx=0, \quad |\alpha|\le k.$$

Then B can be extended to a bounded operator from $H^p\times H^q \to H^r$ where $\dfrac{n}{n+k+1}<r\le\dfrac{n}{n+k}$ and $\dfrac{1}{r}=\dfrac{1}{p}+\dfrac{1}{q}$.

Remarks.

1. Note that if f is an H^p-atom and g is an H^q-atom the integral defining the moments of $B(f,g)$ is well defined, for $(T_i f)(x)$ is L^p for all $p>1$ and decays like $|x|^{-[n(\frac{1}{p}-1)]-n-1}$ as $|x|\to\infty$. Therefore, the functions $x^\alpha(T_i^1 f)(x)(T_i^2 g)(x)$, $|\alpha|\le k$ are L^1 around 0 and decay like

$$|x|^{-[n(\frac{1}{p}-1)]-[n(\frac{1}{q}-1)]-2n-2+|\alpha|} \quad \text{as} \quad |x|\to\infty$$

The exponent above is less than $-n$ as long as $|\alpha|\le k$ and $r\le\frac{n}{n+k}$. The operator $B(f,g)$ is certainly well defined when f and g are finite sums of atoms and the assumptions make sense for the range of p, q and r as above. The conclusion is that $B(f, g)$ maps $H^p \times H^q \to H^r$ as long as it has r moments vanishing.

2. Our theorem is not vacuous if we assume that $k \geq n$. Otherwise, $k \leq n-1$ implies that $\frac{n}{n+k+1} \geq \frac{1}{2}$ and thus $r > \frac{1}{2}$ which is impossible since $r = (\frac{1}{p} + \frac{1}{q})^{-1} \leq \frac{1}{2}$.

Before we begin the proof of our theorem we state and prove a lemma that will be the main tool in the treatment of all the terms that will appear in the decomposition of $B(f, g)$ except the main term.

Suppose ϕ is a smooth function and f an H^p distribution ($p \leq 1$). We are interested in computing the value of the constant C_ϕ in the following inequality

$$\left| \int f(x)\phi(x)dx \right| \leq C_\phi$$

where, by $\int f(x)\phi(x)dx$ we denote the action of the distribution f on the test function ϕ. We have the following

Lemma 1. *Let ϕ be sufficiently smooth. For any $f \in H^p$, $p \leq 1$ there exists an L^p function f^+ with $\|f^+\|_{L^p} \leqq C_p \|f\|_{H^p}$ such that*

$$\left| \int f(x)\phi(x)dx \right| \leq N_{x_0}(\phi) f^+(x_0), \;\; \textit{all } x_0 \in \mathbf{R}^n$$

where $N_{x_0}(\phi) = \sum_{s=0}^{N} \int |x - x_0|^{2s} |\Delta^s \phi|(x)dx$ for some N depending on n and p only.

Proof. We call $N_{x_0}(\phi)$ the "norm" of ϕ.

To prove the lemma we use the atomic decomposition of f. Assume first that $f = a_{Q_0}$ is an atom supported in the unit cube Q_0 of $\mathbf{R}^n$. We will show that

$$\left| \int a_{Q_0}(x)\phi(x)dx \right| \leq N_{x_0}(\phi) h(x_0) \tag{1.1}$$

where $h(x_0) = c(3 + |x_0|)^{-2s}$ and $s \in \mathbf{Z}^+$ chosen such that $2s > \frac{n}{p}$. We are assuming that a_{Q_0} has at least $2s - n$ moments vanishing.

To verify (1.1) we consider two cases

Case 1. $x_0 \in 3Q_0$ (triple of Q_0). Then

$$\left| \int a_{Q_0}(x)\phi(x)dx \right| \leq C \int |\phi| dx \leq h(x_0) N_{x_0}(\phi).$$

Case 2. $x_0 \notin 3Q_0$.

We choose a $\zeta(x)$ in $C^\infty(\mathbf{R}^n)$ function, supported in $|x| < 1$ and we call $\zeta_{x_0}(x) = \zeta\left(\frac{2(x-x_0)}{|x_0|}\right)$. Then $\zeta_{x_0}(x)$ is supported in $|x - x_0| \leq \frac{|x_0|}{2}$. Note that for $s > n/2p$, $\Delta^{-s} a_{Q_0}$ is defined because a_{Q_0} has enough vanishing moments. We have

$$\int a_{Q_0} \phi dx = \int (\Delta^{-s} a_{Q_0})(\Delta^s \phi) dx$$

$$= \int \zeta_{x_0}(\Delta^{-s} a_{Q_0})(\Delta^s \phi) dx + \int (1 - \zeta_{x_0})(\Delta^{-s} a_{Q_0})(\Delta^s \phi)(x)\, dx = I + II.$$

Term II is the easiest to treat. First we claim that $\|\Delta^{-s} a_{Q_0}\|_{L^\infty} \leq C$. To check this, write

$$\Delta^{-s} a_{Q_0}(x) = C \int_{Q_0} |x - y|^{-n+2s} a_{Q_0}(y) dy.$$

Use the fact that $a_{Q_0}(y)$ has at least $-n+2s > 0$ moments vanishing to subtract a suitable polynomial $\sum\limits_{|\alpha|<-n+2s} C_\alpha |x|^{-n+2s-|\alpha|} y^\alpha$ from $|x-y|^{-n+2s}$ so that

$$\left| |x-y|^{-n+2s} - \sum_{|\alpha|<-n+2s} C_\alpha |x|^{-n+2s-|\alpha|} y^\alpha \right| \leq C|y|^{-n+2s}$$

and thus get $|\Delta^{-s} a_{Q_0}(x)| \leq C \int\limits_{Q_0} |y|^{-n+2s} dy \leq C$. Then we estimate term II by

$$II \leq \int_{|x-x_0| \geq \frac{|x_0|}{2}} \|\Delta^{-s} a_{Q_0}\|_{L^\infty} |\Delta^s \phi| dx$$

$$\leq C|x_0|^{-2s} \int_{|x-x_0| \geq \frac{|x_0|}{2}} |x - x_0|^{2s} |\Delta^s \phi(x)| dx \leq h(x_0)\ N_{x_0}(\phi).$$

Term I can be handled with another integration by parts.

$$= \int \Delta^s (\zeta_{x_0} \Delta^{-s} a_{Q_0}) \phi dx$$

$$= \int \left(\sum_{j=0}^{2s-1} c_j (D_x^{2s-j} \zeta_{x_0}) D_x^j (\Delta^{-s} a_{Q_0}) + \zeta_{x_0} a_{Q_0} \right) \phi\ dx$$

where $D_x^j F$ is a sum of derivatives of F in x of total order j. Since a_{Q_0} and ζ_{x_0} have disjoint supports their product is zero.

Finally, we claim that $D_x^j(\Delta^{-s}a_{Q_0})(x)$ decays like $|x_0|^{-j}$ as $|x_0| \to \infty$ for $|x-x_0| \le \frac{|x_0|}{2}$. We can see this by the following argument. We have

$$\Delta^{-s}a_{Q_0}(x) = \int |x-y|^{-n+2s}a_{Q_0}(y)dy$$

and therefore $$D_x^j(\Delta^{-s}a_{Q_0})(x) = \int D_x^j(|x-y|^{-n+2s})a_{Q_0}(y)dy.$$

Note that the function $|x-y|^{-n+2s}$ is smooth near x since $y \in Q_0$ and $|x| \sim |x_0| \notin 3Q_0$. Using the fact that a_{Q_0} has $-n+2s$ moments vanishing we can subtract the Taylor polynomial of degree $-n+2s$ of $D_x^j(|x-y|^{-n-2s})$ at x to get

$$D_x^j(\Delta^{-s}a_{Q_0}(x) = \int \left\{ D_x^j(|x-y|^{-n+2s}) - \sum_{|\alpha| \le -n+2s} \frac{\partial^\alpha}{\partial x^\alpha}(D_x^j|x|^{-n+2s})\frac{y^\alpha}{\alpha!} \right\} a_{Q_0}(y)dy$$

The expression inside the curly brackets above decays like $|x-y|^{-n+2s-j-(-n+2s)} \sim |x|^{-j}$ as $|x| \to \infty$. Since $|x-x_0| \le \frac{|x_0|}{2}$ we have $|x| \sim |x_0|$ and thus we get the required estimate

$$|D_x^j(\Delta^{-s}a_{Q_0})(x)| \le C|x_0|^{-j} \text{ as } |x_0| \to \infty.$$

Clearly, we also have $|D_x^{2s-j}\zeta_{x_0}| \le C|x_0|^{-2s+j}$.

Summing on j we obtain

$$\left| \sum_{j=0}^{2s-1} c_j(\Delta^{2s-j}\zeta_{x_0})D_x^j(\Delta^{-s}a_{Q_0}) \right| \le C|x_0|^{-2s}$$

and thus

$$I \le C|x_0|^{-2s}\int |\phi|dx \le \zeta(x_0)N_{x_0}(\phi).$$

Putting estimates I and II together we get the desired conclusion for Case 2. Our lemma is now proved in the case where $f = a_{Q_0}$ = an atom supported in the unit cube centered at the origin. To obtain an estimate for a general atom a_Q we use translations and dilations. Observe that the norm N_{x_0} satisfies the following properties:

$$N_{x_0}\left(\frac{1}{t^n}\phi\left(\frac{\cdot}{t}\right)\right) = N_{\frac{x_0}{t}}(\phi)$$
$$N_{x_0}(\phi(\cdot + y)) = N_{x_0+y}(\phi).$$

For a general cube Q, the properties above and (1.1) give

$$\left| \int a_Q(x)\phi(x)dx \right| \le |Q|^{-\frac{1}{p}}N_{x_0}(\phi)h\left(\frac{x_0}{|Q|^{1/n}}\right).$$

Summing over all Q we obtain

$$\left| \int f(x)\phi(x)dx \right| \leq N_{x_0}(\phi) f^+(x_0)$$

where $f^+(x_0) = \sum \lambda_Q |Q|^{-\frac{1}{p}} h\left(\frac{x_0}{|Q|^{1/n}}\right)$. We easily check that $\|f^+\|_{L^p} \leq C\|f\|_{H^p}$,

$$\begin{aligned}
\left(\int f^+(x_0)^p dx_0\right)^{\frac{1}{p}} &= \left(\int \left(\sum \lambda_Q |Q|^{-\frac{1}{p}} h\left(\frac{x_0}{|Q|^{1/n}}\right)\right)^p dx_0\right)^{\frac{1}{p}} \\
&\leq \left(\sum \lambda_Q^p \int |Q|^{-1} h^p \left(\frac{x_0}{|Q|^{1/n}}\right) dx_0\right)^{\frac{1}{p}} \\
&\leq \|h\|_{L^p} \left(\sum \lambda_Q^p\right)^{\frac{1}{p}} \leq C_p \|f\|_{H^p}.
\end{aligned}$$

Our lemma is now proved.

2. Begining of the Proof

Fix p, q, r as in the statement of the theorem. Fix a smooth compactly supported function $\phi \geq 0$ in $\mathbf{R}^n$ and define $\phi_{t,x_0}(x) = \frac{1}{t^n}\phi\left(\frac{x_0 - x}{t}\right)$ where $x_0 \in \mathbf{R}^n$ fixed. We will show that

$$\sup_{t>0} \left| \int \phi_{t,x_0}(x) B(f,g)(x) dx \right| \in L^r$$

for f, g finite sums of H^p and H^q atoms respectively.

Without loss of generality we may assume that support of $\phi \subset \{x : |x| < 1\}$. Fix a smooth cutoff $\eta(x)$ supported in $|x| < 4$ such that $\eta \equiv 1$ on $|x| < 2$. Call for simplicity $\eta_0(x) = \eta\left(\frac{x_0 - x}{t}\right)$, $\eta_1(x) = 1 - \eta_0(x)$. The reader should remember the dependence of η_0 and η_1 on t. We write $B(f,g) = B_1 + B_2 + B_3 + B_4$ where

$$\begin{aligned}
B_1 &= B(\eta_0 f, \eta_0 g) \\
B_2 &= B(\eta_1 f, g) \\
B_3 &= B(f, \eta_1 g) \\
B_4 &= -B(\eta_1 f, \eta_1 g)
\end{aligned}$$

We treat term B_4 first.

$$
\begin{aligned}
-\int \phi_{t,x_0} B_4 dx = \\
&+\sum \int \phi_{t,x_0}(x)(T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0))(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))dx \\
&+\sum_i T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x)(T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0))dx \\
&+\sum_i T_i^1(\eta_1 f)(x_0) \int \phi_{t,x_0}(x)(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))dx \\
&+\sum_i T_i^1(\eta_1 f)(x_0) T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x)dx
\end{aligned}
\tag{2.1}
$$

We claim that for any fixed x such that $|x - x_0| \leq t$ we have that,

$$
\sup_{t>0} |T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0)| \leq C f^+(x_0)
$$

where $f^+(x_0)$ is an L^p function of x_0 with $\|f^+\|_{L^p} \leq C\|f\|_{H^p}$.

We verify this last assertion. Fix an i and write

$$
T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0) = \int (1 - \eta(\tfrac{x_0 - y}{t}))(K_i^1(x - y) - K_i^1(x_0 - y))f(y)dy
$$

Call $\Phi(y) = (1 - \eta(\frac{x_0 - y}{t}))(K_i^1(x - y) - K_i^1(x_0 - y))$. Φ is a smooth function of y and by Lemma 1 we get

$$
|T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0))| \leq N_{x_0}(\Phi) f^+(x_0).
$$

It suffices to prove that $\sup_{t>0} |N_{x_0}(\Phi)| \leq C$ to verify our claim.

By the basic estimates for $\{K_i^1\}$ we get that

$$
\left| \Delta_y^\alpha K_i^1(x - y) - \Delta_y^\alpha K_i^1(x_0 - y) \right| \leq \frac{C|x - x_0|}{|x - y|^{n+2|\alpha|}}
$$

Since $|x - y| \sim |x_0 - y|$ and $|x - x_0| \leq t$, we get

$$
\sup_{t>0} |N_{x_0}(\Phi)| \leq C \sup_{t>0} \sum_{s=0}^{N} \int_{|y - x_0| > 2t} |y - x_0|^{2s} \frac{|x - x_0|}{|x - y|^{n+2s+1}} dy \leq C.
$$

We denote by

$$(T_i^1)_*f(x_0) = \sup_{t>0}\left|\int (1-\eta(\tfrac{x_0-y}{t}))K_i^1(x_0-y)f(y)dy\right|$$

the smoothly truncated maximal singular integral of f. By [FS] we get that $\|(T_i^1)_*f\|_{H^p} \leq C\|f\|_{H^p}$. We now use (2.1) to estimate B_4 as follows:

$$\begin{aligned}|\int \phi_{t,x_0}B_4dx| \leqq &\sum_i f^+(x_0)g^+(x_0)\int \phi_{t,x_0}dx\ + \\ &\sum_i f^+(x_0)(T_i^2)_*g(x_0)\int \phi_{t,x_0}dx\ + \\ &\sum_i (T_i^1)_*f(x_0)g^+(x_0)\int \phi_{t,x_0}dx\ + \\ &\sum_i (T_i^1)_*f(x_0)(T_i^2)_*g(x_0)\int \phi_{t,x_0}dx\end{aligned}$$

Since the right hand side above is independent of t

$$\begin{aligned}&\sup_{t>0}|\int \phi_{t,x_0}B_4dx| \leq \ (2.2) = \\ &\sum_i f^+(x_0)g^+(x_0) + (T_i^1)_*f(x_0)g^+(x_0) + f^+(x_0)(T_i^2)_*g(x_0) + (T_i^1)_*f(x_0)(T_i^2)_*g(x_0).\end{aligned}$$

We raise (2.2) to the power r and we integrate with respect to x_0. We then apply Hölder's inequality to the right hand side with exponents p/r and q/r. We finally get that

$$\begin{aligned}&\|\sup_{t>0}|\int \phi_{t,x_0}B_4dx|\|_{L^r} \\ \leqq &\sum_i \|f^+\|_{L^p}\|g^+\|_{L^q} + \|(T_i^1)_*f\|_{L^p}\|g^+\|_{L^q} + \|f^+\|_{L^p}\|(T_i^2)_*g\|_{L^q} + \|(T_i^1)_*f\|_{L^p}\|(T_i^2)_*g\|_{L^q} \\ \leqq &C\|f\|_{H^p}\|g\|_{H^q}\end{aligned}$$

and this is the required estimate for term B_4. We now estimate term B_3. We have

$$\int \phi_{t,x_0}B_3dx = \int \phi_{t,x_0}B(f,\eta_1 g)dx = I_1 + I_2 \quad \text{where}$$

$$I_1 = \sum_i \int \phi_{t,x_0}(x) T_i^1(f)(x)(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))dx$$

$$I_2 = \sum_i T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x) T_i^1(f)(x)dx.$$

Let's start with term I_1. Fix an i and call $\Phi(x) = \phi_{t,x_0}(x)(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))$. Φ is a smooth function of x. We claim that $N_{x_0}(\Phi) \leq Cg^+(x_0)$. To prove the claim, we first estimate

$$\begin{aligned}
&\left| \frac{\partial^\beta}{\partial x^\beta}(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) \right| \\
=&\left| \int \eta_1(y) \left(\frac{\partial^\beta}{\partial x^\beta} \left(K_i^2(x-y) - K_i^2(x_0 - y) \right) \right) g(y)dy \right| \\
=&\left| \int \Psi(y) g(y) dy \right| \leq N_{x_0}(\Psi) g^+(x_0).
\end{aligned}$$

where we set $\Psi(y) = \eta_1(y) \left(\frac{\partial^\beta}{\partial x^\beta} \left(K_i^2(x-y) - K_i^2(x_0-y) \right) \right)$. An easy calculation using the basic estimates for $\{K_i^2\}$, shows that for $|x - x_0| \leq t$,

$$N_{x_0}(\Psi) \leq \sum_{s=0}^N \int\limits_{|y-x_0|>2t} |y - x_0|^{2s} \left| \Delta^s \left(\eta_1(y) \frac{\partial^\beta}{\partial x^\beta} (K_i^2(x-y) - K_i^2(x_0 - y)) \right) \right| dy \leq Ct^{-|\beta|}$$

Therefore

$$\left| \frac{\partial^\beta}{\partial x^\beta} \left(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0) \right) \right| \leq Ct^{-|\beta|} g^+(x_0).$$

Now,

$$\begin{aligned}
N_{x_0}(\Phi) &\leq \sum_{s=0}^N \int |x - x_0|^{2s} |\Delta^s (\phi_{t,x_0}(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))| dx \\
&\leq C \sum_{s=0}^N \int_{|x-x_0|<t} |x - x_0|^{2s} \sum_{j=0}^{2s} |D_x^{2s-j} \phi_{t,x_0}| |D_x^j (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))| dx
\end{aligned}$$

where by $D_x^j F$ we denote any derivative of F in x of total order j.

The above can be dominated by

$$C \sum_{s=0}^N \int_{|x-x_0|<t} |x - x_0|^{2s} \sum_{j=0}^{2s} t^{-n-(2s-j)} t^{-j} dx \;\; g^+(x_0) \leq Cg^+(x_0)$$

This estimate finishes the proof of the claim.

An application of Lemma 1 gives that

$$|I_1| \le \sum_i N_{x_0}(\Phi)(T_i^1 f)^+(x_0) \le C \sum_i g^+(x_0)(T_i^1 f)^+(x_0)$$

Since the right hand side doesn't depend on t, the $\sup_{t>0}|I_1|$ satisfies the same estimate Hölder's inequality will give that

$$\begin{aligned}\|\sup_{t>0}|I_1|\|_{L^r} &\le C\sum_i \|g^+\|_{L^q}\|(T_i^1 f)^+\|_{L^p}\\ &\le C\sum_i \|g\|_{H^q}\|T_i^1 f\|_{H^p}\\ &\le C\|f\|_{H^p}\|g\|_{H^q}\end{aligned}$$

We now continue with term I_2. Term I_2 satisfies the estimate

$$|I_2| \le C\sum_i (T_i^2)_* g(x_0)(T_i^1 f)^*(x_0)$$

where by $(T_i^1 f)^*$ we denote some smooth maximal function of $T_i f$ and the same argument as before will give that

$$\|\sup_{t>0}|I_2|\|_{L^r} \le C\|g\|_{H^q}\|f\|_{H^p}$$

This estimate concludes the treatment of term B_3. We deal similarly with term B_2.

3. The Main Term

We are now left with term B_1 which is the main term of the bilinear operator B. We start by introducing some notation. For $0 \le \delta \le 1$, we denote by Λ_δ the Lipschitz space of all bounded functions f on $\mathbf{R}^n$ with

$$\sup_{x\in\mathbf{R}^n}\sup_{h\in\mathbf{R}^n}|h|^{-\delta}|f(x+h)-f(x)| = \|f\|_{\Lambda_\delta} < +\infty$$

For $m \in \mathbf{Z}$, $m \ge 1$ and $0 \le \delta \le 1$, we denote by Λ_δ^m the space of all bounded functions f on $\mathbf{R}^n$ whose partials of order m exist and are in Λ_δ and whose partials of order up to $m-1$ are bounded. The $\| \quad \|_{\Lambda_\delta^m}$ norm of a function is defined as the sum of the $\| \quad \|_{\Lambda_\delta}$ norms of its partials of order m. It is easy to see that functions in Λ_δ^m satisfy

$$|f(x+h) - \sum_{|\alpha|\le m}\frac{\partial^\alpha f}{\partial x^\alpha}(x)\frac{h^\alpha}{\alpha!}| \le \frac{\|f\|_{\Lambda_\delta^m}}{m!}|h|^{m+\delta} \quad \text{for all} \quad x\in\mathbf{R}^n\ ,\ h\in\mathbf{R}^n.$$

For $m, \ell \in \mathbf{Z}^+ \cup \{0\}$ and $0 \leq \gamma, \delta \leq 1$ we define the space $\Lambda^{m,\ell}_{\gamma,\delta}$ of all bounded functions $b(y,z)$ on $\mathbf{R}^n \times \mathbf{R}^n$ which are in Λ^m_γ as functions of y and in Λ^ℓ_δ as functions of z and satisfy the condition:

$$\begin{aligned} \Bigg| b(y+h,z+k) - \sum_{|\alpha|\leq m} \frac{\partial^\alpha}{\partial y^\alpha} b(y,z+k)\frac{h^\alpha}{\alpha!} - \sum_{|\beta|\leq \ell} \frac{\partial^\beta}{\partial z^\beta} b(y+h,z)\frac{k^\beta}{\beta!} \\ + \sum_{|\alpha|\leq m}\sum_{|\beta|\leq \ell} \frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^\beta}{\partial z^\beta} b(y,z)\frac{h^\alpha}{\alpha!}\frac{k^\beta}{\beta!}\Bigg| \leq C|h|^{m+\gamma}|k|^{\ell+\delta} \end{aligned} \tag{3.1}$$

for all $y, z, k, h \in \mathbf{R}^n$.

This double Lipschitz condition above is equivalent to either one of the statements below

$$|h|^{-m-\gamma}\left(b(y+h,z) - \sum_{|\alpha|\leq m} \frac{\partial^\alpha}{\partial y^\alpha} b(y,z)\frac{h^\alpha}{\alpha!}\right) \qquad \text{is in } \Lambda^\ell_\delta \text{ as a function of } z \tag{3.2}$$

$$|k|^{-\ell-\delta}\left(b(y,z+k) - \sum_{|\beta|\leq \ell} \frac{\partial^\beta}{\partial z^\beta} b(y,z)\frac{k^\beta}{\beta!}\right) \qquad \text{is in } \Lambda^m_\gamma \text{ as a function of } y \tag{3.3}$$

for all $y, z, h, k \in \mathbf{R}^n$, $|h|, |k| \leq C$.

We will now state and prove the main lemma needed to estimate term B_1.

Lemma 2. *Suppose that $p, q \leq 1$ and let $m = [n(\frac{1}{p} - 1)]$, $\ell = [n(\frac{1}{q} - 1)]$. Suppose that the compactly supported function $b(y,z)$ is in $\Lambda^{m,\ell}_{\gamma,\delta}$. For $x_0 \in \mathbf{R}^n$ let*

$$S(f,g)(x_0) = \sup_{t>0}\left| \iint f(y)g(z)\tfrac{1}{t^{2n}} b\left(\tfrac{x_0-y}{t}, \tfrac{x_0-z}{t}\right) dydz\right|$$

Then $S(f,g)$ maps $H^p \times H^q \to L^r$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ as long as

$$1 \geq \gamma > n(\frac{1}{p} - 1) - [n(\frac{1}{p} - 1)] \quad \textit{and}$$
$$1 \geq \delta > n(\frac{1}{q} - 1) - [n(\frac{1}{q} - 1)].$$

Proof. Let $a_Q(y)$ be an H^p-atom and $b_R(z)$ be an H^q-atom. We will first estimate $S(a_Q, b_R)(x_0)$. Let y_Q be the center of the center of the cube Q and z_R the center of the cube R. We have the following four basic estimates:

Case 1: $x_0 \in 3Q$ and $x_0 \in 3R$, then

$$S(a_Q, b_R)(x_0) \leq C|Q|^{-\frac{1}{p}}\chi_{3Q}(x_0)|R|^{-\frac{1}{q}}\chi_{3R}(x_0).$$

Case 2: $x_0 \in 3Q$ and $x_0 \notin 3R$, then

$$S(a_Q, b_R)(x_0) \leqq C|Q|^{-\frac{1}{p}}\chi_{3Q}(x_0)\frac{|R|^{-\frac{1}{q}+1+\frac{\ell+\delta}{n}}}{\mathrm{dist}(x_0, R)^{n+\ell+\delta}}.$$

Case 3: $x_0 \notin 3Q$ and $x_0 \in 3R$, then

$$S(a_Q, b_R)(x_0) \leq C\frac{|Q|^{-\frac{1}{p}+1+\frac{m+\gamma}{n}}}{\mathrm{dist}(x_0, Q)^{n+m+\gamma}}|R|^{-\frac{1}{q}}\chi_{3R}(x_0).$$

Case 4: $x_0 \notin 3Q$ and $x_0 \notin 3R$, then

$$S(a_Q, b_R)(x_0) \leq C\frac{|Q|^{-\frac{1}{p}+1+\frac{m+\gamma}{n}}}{\mathrm{dist}(x_0, Q)^{n+m+\gamma}}\frac{|R|^{-\frac{1}{q}+1+\frac{\ell+\delta}{n}}}{\mathrm{dist}(x_0, R)^{n+\ell+\delta}}$$

We indicate how to prove the basic estimates.

In Case 1 we just use the L^∞ bounds for atoms.

In Case 2 we need the following estimate which can be easily proved by integrating (3.3) over the y-support of b.

(3.4) $$\int |b(y, z+k) - \sum_{|\beta|\leq \ell} \frac{\partial^\beta b}{\partial z^\beta}(y, z)\frac{k^\beta}{\beta!}|dy \leq C|k|^{\ell+\delta} \text{for all } z, k \text{ in } \mathbf{R}^n.$$

Using the fact that b_R has moments up to order ℓ vanishing we have

$$\iint a_Q(y)b_R(z)\tfrac{1}{t^{2n}}b\left(\tfrac{x_0-y}{t}, \tfrac{x_0-z}{t}\right)dydz = (3.5)$$

$$\int a_Q(y)\int b_R(z)\tfrac{1}{t^n}\left(b\left(\tfrac{x_0-y}{t}, \tfrac{x_0-z}{t}\right) - \sum_{|\beta|\leq\ell}\frac{\partial^\beta b}{\partial z^\beta}\left(\tfrac{x_0-y}{t}, \tfrac{x_0-z_R}{t}\right)\tfrac{1}{\beta!}\left(\tfrac{z_R-z}{t}\right)^\beta\right)\frac{dz}{t^n}dy$$

By (3.4) the above can be estimated by

$$C|Q|^{-\frac{1}{p}}\chi_{3Q}(x_0)|R|^{-\frac{1}{q}}\tfrac{1}{t^n}\int_R\left|\tfrac{z-z_R}{t}\right|^{\ell+\delta}dz \leq C|Q|^{-\frac{1}{p}}\chi_{3Q}(x_0)\frac{|R|^{-\frac{1}{q}+1+\frac{\ell+\delta}{n}}}{t^{n+\ell+\delta}}$$

Note that (3.5) is nonzero if $t \geq \operatorname{dist}((x_0,x_0),Q\times R) \sim \operatorname{dist}(x_0,R)$, since $x_0\in 3Q$ and $x_0\notin 3R$. The required estimate for $S(a_Q,b_R)$ in Case 2 follows. Case 3 is similar.

To get the estimate in Case 4 we use the double Lipschitz estimate for b. We have

$$\begin{aligned}
(3.6) = &\iint a_Q(y)b_R(z)\tfrac{1}{t^{2n}}b\left(\tfrac{x_0-y}{t},\tfrac{x_0-z}{t}\right)dydz = \\
&\iint a_Q(y)b_R(z)\tfrac{1}{t^{2n}}\Big\{b\left(\tfrac{x_0-y}{t},\tfrac{x_0-z}{t}\right)-\sum_{|\alpha|\leq m}\frac{\partial^\alpha b}{\partial y^\alpha}\left(\tfrac{x_0-y_Q}{t},\tfrac{x_0-z}{t}\right)\tfrac{1}{\alpha!}\left(\tfrac{y_Q-y}{t}\right)^\alpha \\
&-\sum_{|\beta|\leq\ell}\frac{\partial^\beta b}{\partial z^\beta}\left(\tfrac{x_0-y}{t},\tfrac{x_0-z_R}{t}\right)\tfrac{1}{\beta!}\left(\tfrac{z_R-z}{t}\right)^\beta \\
&+\sum_{|\alpha|\leq m}\sum_{|\beta|\leq\ell}\frac{\partial^\alpha\partial^\beta b}{\partial y^\alpha\partial z^\beta}\left(\tfrac{x_0-y_Q}{t},\tfrac{x_0-z_R}{t}\right)\tfrac{1}{\alpha!}\left(\tfrac{y_Q-y}{t}\right)^\alpha\tfrac{1}{\beta!}\left(\tfrac{z_R-z}{t}\right)^\beta\Big\}dydz
\end{aligned}$$

Apply (3.1) to bound the above by

$$\begin{aligned}
&C|Q|^{-\frac{1}{p}}|R|^{-\frac{1}{q}}\tfrac{1}{t^{2n}}\int_Q\int_R\left|\tfrac{y-y_Q}{t}\right|^{m+\gamma}\left|\tfrac{z-z_R}{t}\right|^{\ell+\delta}dydz \\
\leq &Ct^{-2n-m-\ell-\gamma-\delta}|Q|^{-\frac{1}{p}+1+\frac{m+\gamma}{n}}|R|^{-\frac{1}{q}+1+\frac{\ell+\delta}{n}}
\end{aligned}$$

Note that (3.6) is nonzero as long as $t\geq \operatorname{dist}((x_0,x_0),Q\times R)\sim \operatorname{dist}(x_0,Q)+\operatorname{dist}(x_0,R)$, because $x_0\notin 3Q$ and $x_0\notin 3R$.

Since $\operatorname{dist}((x_0,x_0),Q\times R)^{-2n-m-\gamma-\ell-\delta} \leqq C\operatorname{dist}(x_0,Q)^{-n-m-\gamma}\operatorname{dist}(x_0,R)^{-n-\ell-\delta}$, the required estimate for (3.6) follows immediately.

We have now proved our basic estimates and we continue with the proof of the lemma.

Let $f\in H^p, g\in H^q$ be finite sums of atoms $f=\sum\lambda_Q a_Q$, $g=\sum\mu_R b_R$ where $\lambda_Q>0$, $\mu_R>0$ and

$$(\sum\lambda_Q^p)^{\frac{1}{p}}\approx\|f\|_{H^p},\ (\sum\mu_R^q)^{\frac{1}{q}}\approx\|g\|_{H^q}$$

Bound $S(f,g)(x_0)\leq\sum_Q\sum_R\lambda_Q\mu_R S(a_Q,b_R)(x_0)$ by $\Sigma_1(x_0)+\Sigma_2(x_0)+\Sigma_3(x_0)+\Sigma_4(x_0)$ where

$$\Sigma_j(x_0)=\sum_Q\sum_R\lambda_Q\mu_R S(a_Q,b_R)(x_0)$$

and the sum above is taken over all Q and R related to x_0 as in case j above, $1 \le j \le 4$.

We will show that each Σ_j is in $L^r(dx_0)$ with L^r quasinorm bounded by $C\|f\|_{H^p}\|g\|_{H^q}$. Then we can sum on j, $1 \le j \le 4$. Use Hölder's inequality with exponents p/r, q/r to get

$$
\begin{aligned}
&\left(\int \Sigma_1(x_0)^r dx_0\right)^{\frac{1}{r}} \\
\le& C\left(\int \left(\sum \lambda_Q |Q|^{-\frac{1}{p}} \chi_{3Q}(x_0)\right)^p dx_0\right)^{\frac{1}{p}} \left(\int \left(\sum \mu_R |R|^{-\frac{1}{q}} \chi_{3R}(x_0)\right)^q dx_0\right)^{-\frac{1}{q}} \\
\le& C\left(\int_{3Q} \sum \lambda_Q^p |Q|^{-1} dx_0\right)^{\frac{1}{p}} \left(\int_{3R} \sum \mu_R^q |R|^{-1} dx_0\right)^{\frac{1}{q}} \quad \text{(since } p, q \le 1) \\
&\le C\left(\sum \lambda_Q^p\right)^{\frac{1}{p}} \left(\sum \mu_R^q\right)^{\frac{1}{q}} \le C\|f\|_{H^p}\|g\|_{H^q}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left(\int \Sigma_4(x_0)^r dx_0\right)^{\frac{1}{r}} \\
\le& C\left(\int\limits_{\mathrm{dist}(x_0,Q)\ge C|Q|^{\frac{1}{n}}} \left(\sum \lambda_Q \frac{|Q|^{-\frac{1}{p}+1+\frac{m+\gamma}{n}}}{\mathrm{dist}(x_0,Q)^{n+m+\gamma}}\right)^p dx_0\right)^{\frac{1}{p}} \\
&\qquad \left(\int\limits_{\mathrm{dist}(x_0,R)\ge C|R|^{\frac{1}{n}}} \left(\sum \mu_R \frac{|R|^{-\frac{1}{q}+1+\frac{\ell+\delta}{n}}}{\mathrm{dist}(x_0,R)^{n+\ell+\delta}}\right)^q dx_0\right)^{\frac{1}{q}} \\
\le& C\left(\int\limits_{\mathrm{dist}(x_0,Q)\ge C|Q|^{\frac{1}{n}}} \sum \lambda_Q^p \frac{|Q|^{-1+p+\frac{(m+\gamma)p}{n}}}{\mathrm{dist}(x_0,Q)^{(n+m+\gamma)p}} dx_0\right)^{\frac{1}{p}} \\
&\qquad \left(\int\limits_{\mathrm{dist}(x_0,R)\ge C|R|^{\frac{1}{n}}} \sum \mu_R^q \frac{|R|^{-1+q+\frac{(\ell+\delta)q}{n}}}{\mathrm{dist}(x_0,R)^{(n+\ell+\delta)q}} dx_0\right)^{\frac{1}{q}} \\
\le& C\left(\sum \lambda_Q^p\right)^{\frac{1}{p}} \left(\sum \mu_R^q\right)^{\frac{1}{q}} \le C\|f\|_{H^p}\|g\|_{H^q}
\end{aligned}
$$

where we used the fact that $(n+m+\gamma)p > n$ and

$$\int_{\operatorname{dist}(x_0,Q)\geq C|Q|^{\frac{1}{n}}} \operatorname{dist}(x_0,Q)^{-(n+m+\gamma)p} dx_0 \sim |Q|^{\frac{1}{n}(n-(n+m+\gamma)p)} \sim |Q|^{1-p-\frac{(m+\gamma)p}{n}}.$$

The same way we can prove that

$$\left(\int \Sigma_2(x_0)^r dx_0\right)^{\frac{1}{r}} \leq C\|f\|_{H^p}\|g\|_{H^p} \text{ and}$$
$$\left(\int \Sigma_3(x_0)^r dx_0\right)^{\frac{1}{r}} \leq C\|f\|_{H^p}\|g\|_{H^q}$$

using a combination of the estimates above. Finally we approximate the general $f \in H^p$ and $g \in H^q$ by finite sums of atoms to finish the proof of the lemma.

We now continue the proof of our theorem by estimatng term B_1. We have

$$\int \phi_{t,x_0} B_1 dx = \iint f(y)g(z)\tfrac{1}{t^{2n}} b\left(\tfrac{x_0-y}{t}, \tfrac{x_0-z}{t}\right) dydz$$

where we set $b(y,z) = \sum_i \eta(y)\eta(z) \int \phi(\sigma)K_i^1(y-\sigma)K_i^2(z-\sigma)d\sigma$. To apply Lemma 2 we need to prove that $b(y,z) \in \Lambda_{\gamma,\delta}^{m,\ell}$ where m,γ,ℓ,δ as in Lemma 2.

Note that the assumption that $B(f,g)$ has moments up to order k vanishing gives that the kernel of $x^\alpha B(f,g)(x)$ is identically zero for all $|\alpha| \leq k$, i.e.

$$\sum_i \int x^\alpha K_i^1(y-x)K_i^2(z-x)dx = 0 \text{ for all } y,z \in \mathbf{R}^n.$$

We can therefore write

$$b(y,z) = \eta(x)\eta(y)\sum_i \int \left(\phi(\sigma) - \sum_{|\alpha|\leq k} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)\frac{(\sigma-y)^\alpha}{\alpha!}\right) K_i^1(y-\sigma)K_i^2(z-\sigma)d\sigma.$$

The fact that $b(y,z) \in \Lambda_{\gamma,\delta}^{m,\ell}$ where m,ℓ,γ,δ as in Lemma 2, will be a consequence of the following two lemmas.

Lemma 3. *Let G be a function on $\mathbf{R}^n$ of class Λ_1^s and let $a(y,\sigma) = G(y-\sigma)$.*
(a) If m,ℓ are non-negative integers such that $m+\ell = s$ and $\gamma,\delta > 0$ such that $\gamma+\delta = 1$, then $a(y,\sigma)$ is of class $\Lambda_{\gamma,\delta}^{m,\ell}$.
(b) If m,ℓ are integers such that $m+\ell = s-1$ then $a(y,\sigma)$ is of class $\Lambda_{1,1}^{m,\ell}$.

Lemma 4. *Let K be a convolution Calderón-Zygmund kernel on $\mathbf{R}^n$. If $a(y,\sigma) \in \Lambda^{m,\ell}_{\gamma,\delta}$ then*

$$b(y,z) = \int a(y,\sigma)K(z-\sigma)d\sigma \in \Lambda^{m,\ell}_{\gamma,\delta}$$

Proofs. We denote by $\partial^\alpha G$ the partial derivative of G of order $\alpha = (\alpha_1, \ldots, \alpha_n)$. To prove Lemma 3 (a), by (3.2), it suffices to show that the function

$$\begin{aligned} F(\sigma) =& \frac{1}{|h|^{m+\gamma}} \left(a(y+h,\sigma) - \sum_{|\alpha| \le m} \frac{\partial^\alpha a}{\partial y^\alpha}(y,\sigma) \frac{h^\alpha}{\alpha!} \right) \\ =& \frac{1}{|h|^{m+\gamma}} \left(G(y+h-\sigma) - \sum_{|\alpha| \le m} \partial^\alpha G(y-\sigma) \frac{h^\alpha}{\alpha!} \right) \end{aligned}$$

is in Λ^ℓ_δ in the variable σ uniformly in $y \in \mathbf{R}^n$. This will be a consequence of the following two observations:

(1) $F(\sigma)$ is in Λ^ℓ_0 with norm $\le C|h|^{1-\gamma}$.
(2) $F(\sigma)$ is in Λ^ℓ_1 with norm $\le C|h|^{-\gamma}$.

Interpolation will then give that $F(\sigma)$ is in Λ^ℓ_δ with norm $\le C$. (Recall $\gamma + \delta = 1$.)

Both observations follow from Taylor's theorem. For some $\xi_{y,h}$ between y and $y+h$ we have that $F(\sigma) = |h|^{-m-\gamma} \left(\sum_{|\alpha|=m} (\partial^\alpha G(\xi_{y,h} - \sigma) - \partial^\alpha G(y-\sigma)) \frac{h^\alpha}{\alpha!} \right)$ and therefore for a fixed β with $|\beta| = \ell$ we have

$$\partial^\beta F(\sigma) = |h|^{-m-\gamma} \left(\sum_{|\alpha|=m} (\partial^{\alpha+\beta} G(\xi_{y,h} - \sigma) - \partial^{\alpha+\beta} G(y-\sigma)) \frac{h^\alpha}{\alpha!} \right)$$

Since $|\alpha+\beta| = m+\ell = s$ and since $\partial^\sigma G$ is in Λ_1 if $|\sigma| = s$ it follows that $\partial^{\alpha+\beta} G$ is in Λ_1 and thus $(\partial^\beta F)(\sigma)$ is in $\Lambda_0 = L^\infty$ with norm $\le C|h|^{-\gamma} \sum_{|\alpha|=s} \|\partial^\alpha G\|_{\Lambda_1} |\xi_{y,h} - y| \le C|h|^{1-\gamma}$. Also, from the translation invariance of the Lipschitz norms it follows that $\partial^{\alpha+\beta} G(\xi_{y,h} - \sigma)$ and $\partial^{\alpha+\beta} G(y-\sigma)$ are in Λ_1 in σ, and therefore the function $\partial^\beta F(\sigma)$ is in Λ_1 in σ with norm $\le C|h|^{-\gamma}$. We proved that the arbitrary partial derivative $\partial^\beta F$ of F of order ℓ is in Λ_0 with norm $\le C|h|^{1-\gamma}$ and in Λ_1 with norm $\le C|h|^{-\gamma}$. This concludes the proofs of the observations. Note that both norm estimates are independent of y.

Part (b) of lemma 3 follows by a similar argument. We need to show that the function

$$\begin{aligned} F(\sigma) =& \frac{1}{|h|^{m+1}} \left(a(y+h,\sigma) - \sum_{|\alpha|\le m} \frac{\partial^\alpha a}{\partial y^\alpha}(y,\sigma)\frac{h^\alpha}{\alpha!} \right) \\ =& \frac{1}{|h|^{m+1}} \left(G(y+h-\sigma) - \sum_{|\alpha|\le m} \partial^\alpha G(y-\sigma)\frac{h^\alpha}{\alpha!} \right) \end{aligned}$$

is of class Λ_1^ℓ in the variable σ uniformly in $y \in \mathbf{R}^n$. For some $\xi'_{y,h}$ between y and $y+h$, we have that $F(\sigma) = \frac{|h|^{-m-1}}{(m+1)!} \sum_{|\alpha|=m+1} \partial^\alpha G(\xi'_{y,h}-\sigma)\frac{h^\alpha}{\alpha!}$. Fix a multiindex β with $|\beta| = \ell$. Clearly

$$\partial^\beta F(\sigma) = \frac{|h|^{-m-1}}{(m+1)!} \sum_{|\alpha|=m+1} \partial^{\alpha+\beta} G(\xi'_{y,h}-\sigma)\frac{h^\alpha}{\alpha!}.$$

Since in this case $|\alpha+\beta| = m+1+\ell = s$ and the partials of G of order s are in Λ_1 it follows that $\partial^\beta F$ is in Λ_1 with norm independent of y. Therefore F is in Λ_1^ℓ with norm independent of y and this concludes the proof of Lemma 3.

We now indicate how to prove Lemma 4. It is a well known fact that convolution Calderón-Zygmund operators map the Lipschitz spaces Λ_γ^m into themselves. If $a(y,\sigma) \in \Lambda_{\gamma,\delta}^{m,\ell}$ then by (3.2) the function $|h|^{-m-\gamma}\{a(y+h,\sigma) - \sum\limits_{|\alpha|\le m} \frac{\partial^\alpha a}{\partial y^\alpha}(y,\sigma)\frac{h^\alpha}{\alpha!}\}$ is in Λ_δ^ℓ in the variable σ. Convolution with K in σ will give that

$$|h|^{-m-\gamma}\left\{ b(y+h,z) - \sum_{|\alpha|\le m} \frac{\partial^\alpha b}{\partial y^\alpha}(y,z)\frac{h^\alpha}{\alpha!} \right\} \in \Lambda_\delta^\ell \quad \text{in } z$$

and by (3.2) again we get that $b(y,z) \in \Lambda_{\gamma,\delta}^{m,\ell}$. This finishes the proof of Lemma 4.

To conclude the proof of our theorem, it suffices to check that the functions

$$a_i(y,\sigma) = \left\{ \phi(\sigma) - \sum_{|\alpha|\le k} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)\frac{(\sigma-y)^\alpha}{\alpha!} \right\} K_i^1(y-\sigma)$$

are in $\Lambda_{\gamma,\delta}^{m,\ell}$. Then Lemma 4 will give that $b(y,z)$ is in $\Lambda_{\gamma,\delta}^{m,\ell}$ also.

We write $a_i(y,\sigma) = \sum\limits_{|\alpha|=k+1} \phi^{(\alpha)}(\xi_{\sigma,y})\frac{(\sigma-y)^\alpha}{\alpha!} K_i^1(y-\sigma)$. Fix i and α with $|\alpha| = k+1$. It suffices to show that $\phi^{(\alpha)}(\xi_{\sigma,y})(\sigma-y)^\alpha K_i^1(y-\sigma) \in \Lambda_{\gamma,\delta}^{m,\ell}$. Since $\phi^{(\alpha)}(\xi_{\sigma,y})$ is a smooth function of y and σ it is enough to show that $(\sigma - y)^\alpha K_i^1(y-\sigma) \in \Lambda_{\gamma,\delta}^{m,\ell}$ for suitable m, ℓ, γ, δ.

As a function of the variable x, the function $x^\alpha K_i^1(x)$ is smooth everywhere except possibly at zero and it behaves like $|x|^{k+1}|x|^{-n} = |x|_1^{k-n}$ as $|x| \to 0$. Therefore, $x^\alpha K_i^1(x) \in \Lambda^{k-n+1}$. We are going to apply Lemma 3 with $G(x) = x^\alpha K_i^1(x)$.

Let $m = [n(\frac{1}{p} - 1)]$, $\ell = [n(\frac{1}{q} - 1)]$. Since $\frac{n+k}{n} \le \frac{1}{r} < \frac{n+k+1}{n}$ it follows that $m + \ell$ is either $k - n$ or $k - n - 1$. If $m + \ell = k - n$ just pick $\gamma > n(\frac{1}{p} - 1) - m$ and $\delta > n(\frac{1}{q} - 1) - \ell$ such that $\gamma + \delta = 1$ (this is possible since $1 > n(\frac{1}{r} - 2) - m - \ell$) and apply part (a) of Lemma 3 with $s = k - n$. If $m + \ell = k - n - 1 = s - 1$, apply part (b) of Lemma 3.

It follows that $(\sigma - y)^\alpha K_i^1(y - \sigma)$ is in $\Lambda_{\gamma,\delta}^{m,\ell}$ where γ and δ satisfy the hypotheses of Lemma 1. The same is true for $a_i(y, \sigma)$ and therefore for $b(y, z)$ by Lemma 4.

4. Applications and examples

The vanishing moments properties of the bilinear operators $B(f, g) = \sum_j (T_j^1 f)(T_j^2 g)$ can be written in terms of relations involving multipliers. Let m_j^1 and m_j^2 be the multipliers corresponding to the Calderón-Zygmund operators T_j^1 and T_j^2. $B(f, g)$ has mean value zero if and only if $\widehat{B(f,g)}(0) = 0$, i.e.

$$\sum_j (\widehat{T_j^1 f} * \widehat{T_j^2 g})(0).$$

This is equal to

$$\sum_j \int m_j^1(-\xi)\hat{f}(-\xi) m_j^2(\xi)\hat{g}(\xi) d\xi = 0$$

and since f and g are arbitrary the above is equivalent to the statement

$$\sum_j m_j^1(-\xi) m_j^2(\xi) = 0 \qquad \text{for all} \quad \xi \neq 0$$

Similar reasoning shows that B has two moments zero if and only if the following identities hold:

$$\sum_j m_j^1(-\xi) m_j^2(\xi) = 0 \qquad \text{for all} \quad \xi \neq 0$$

$$\sum_j m_j^1(-\xi)(\frac{\partial}{\partial \xi_i} m_j^2)(\xi) = 0 \qquad \text{for all} \quad \xi \neq 0 \ , \ i = 1, 2, \dots n.$$

The second identity can be replaced by

$$\sum_j \frac{\partial}{\partial \xi_i} m_j^1(-\xi) m_j^2(\xi) = 0$$

in view of the first identity and the product rule.

Generalizing the above, we get that B has all moments of order up to and including k vanishing if and only if

$$\sum_j m_j^1(-\xi)\left(\frac{\partial^m}{\partial \xi_i^m} m_j^2\right)(\xi) = 0 \tag{4.1}$$

holds for all $\xi \neq 0$, $i = 1, 2, \ldots n$, $m = 0, 1, \ldots, k$. The identities above give us an easy way to decide whether a bilinear operator has vanishing moments. For example, using (4.1), it is trivial to check that the bilinear operator $\tilde{J}(f,g) = R_1 f R_2 g - R_2 f R_1 g$ has integral zero. To include an example, we check that the bilinear operator $\tilde{H}(f,g) = (R_1^2 f)(R_2^2 g) - 2(R_1R_2 f)(R_2 R_1 g) + (R_2^2 f)(R_1^2 g)$ has vanishing first moments. We calculate (4.1) when $m = 1$ and $i = 1$. Let $T_1^1 = R_1^2$, $T_1^2 = R_2^2$, $T_2^1 = -2R_1R_2$, $T_2^2 = R_2R_1$, $T_3^1 = R_2^2$, $T_3^2 = R_1^2$ and let $m_1^1(\xi) = -\xi_1^2/|\xi|^2$, $m_1^2(\xi) = -\xi_2^2/|\xi|^2$, $m_2^1(\xi) = 2\xi_1\xi_2/|\xi|^2$, $m_2^2(\xi) = -\xi_1\xi_2/|\xi|^2$, $m_3^1(\xi) = -\xi_2^2/|\xi|^2$, $m_3^2(\xi) = -\xi_1^2/|\xi|^2$ be the corresponding multipliers. Then

$$\sum_{j=1}^3 m_j^1(-\xi)\frac{\partial}{\partial \xi_1} m_j^2(\xi) = (-\frac{\xi_1^2}{|\xi|^2})(\frac{2\xi_1\xi_2^2}{|\xi|^4}) + (\frac{2\xi_1\xi_2}{|\xi|^2})(\frac{-\xi_2^3+\xi_1^2\xi_2}{|\xi|^4}) + (-\frac{\xi_2^2}{|\xi|^2})(-\frac{2\xi_1\xi_2^2}{|\xi|^4}) = 0.$$

An example of an operator with two vanishing moments is given on $\mathbf{R}^1$ by the bilinear map $B(f,g) = fg - (Cf)(Cg) + (Sf)(Sg)$, where the operators Sf and Cf are defined on the Fourier side by

$$\widehat{Sf}(\xi) = \sin(\log|\xi|)\hat{f}(\xi) \qquad \text{and} \qquad \widehat{Cf}(\xi) = \cos(\log|\xi|)\hat{f}(\xi).$$

One can easily check using (4.1) that B has integral and first moments zero and hence by our theorem it maps $H^p \times H^q \to H^r$ for $p, q > 1$ and $1/2 \geq r = (p^{-1} + q^{-1})^{-1} > 1/3$.

Examples of bilinear operators with moments of all orders vanishing are given on $\mathbf{R}^1$ by the maps

$$D_1(f,g) = fg - (Hf)(Hg)$$
$$D_2(f,g) = f(Hg) + (Hf)g$$

where H is the usual Hilbert transform. It follows from our theorem that D_1 and D_2 map $H^p \times H^q \to H^r$ for all $p, q \leq 1$ and r their harmonic mean. D_1 and D_2 are the real and imaginary parts of the holomorphic function $(f + iHf)(g + iHg)$ and they can also be studied through complex analysis.

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