ON MULTILINEAR FRACTIONAL INTEGRALS

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Abstract. In $\mathbf{R}^n$, we prove $L^{p_1}\times\cdots\times L^{p_K}$ boundedness for the multilinear fractional integrals $I_\alpha(f_1,\dots,f_K)(x)=\int f_1(x-\theta_1 y)\dots f_K(x-\theta_K y)|y|^{\alpha-n}\,dy$ where the θ_j's are nonzero and distinct. We also prove multilinear versions of two inequalities about fractional integrals and a multilinear Lebesgue differentiation theorem.

1. Introduction. Although it is not known whether the bi(sub)linear maximal function

$$M(f,g)(x)=\sup_{N>0}\frac{1}{2N}\int_{-N}^{N}|f(x+t)g(x-t)|\,dt$$

or the bilinear Hilbert transform

$$H(f,g)(x)=\text{p.v.}\int f(x+t)g(x-t)\,\frac{dt}{t}$$

map $L^p(\mathbf{R}^1)\times L^{p'}(\mathbf{R}^1)\to L^1(\mathbf{R}^1)$ boundedness into L^1 for the correspoding multilinear fractional integrals can be obtained.

Throughout this note, K will denote an integer ≥ 2 and θ_j, $j=1,\dots,K$ will be fixed, distinct and nonzero real numbers. We are going to be working in $\mathbf{R}^n$ and α will be a fixed real number number stricly between 0 and n. We denote by $\mathbf{f}$ the K-tuple $(f_1,\dots,f_K)$ and by I_α the K-linear fractional integral operator defined as follows:

$$I_\alpha(\mathbf{f})(x)=\int f_1(x-\theta_1 y)\dots f_K(x-\theta_K y)|y|^{\alpha-n}\,dy.$$

When $K=1$ the operators I_α are the usual fractional integrals as studied in [ST]. We also denote by $M(\mathbf{f})$ the K-sublinear maximal function

$$M(\mathbf{f})(x)=\sup_{N>0}(\Omega_n N^n)^{-1}\int|f_1(x-\theta_1 y)|\dots|f_K(x-\theta_K y)|dy$$

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where Ω_n is the volume of the unit ball in $\mathbf{R}^n$. It is trivial to check that for any positive $p_1, \dots, p_K$ with harmonic mean $s > 1$, M maps $L^{p_1} \times \cdots \times L^{p_K}$ into L^s. If we denote by f^* the Hardy-Littlewood maximal function of f, then $M(\mathbf{f})$ is dominated by the product $C_{\theta_k}((f_1^{p_1/s})^*)^{s/p_1} \dots ((f_1^{p_K/s})^*)^{s/p_K}$ and hence its boundedness follows from Hölder's inequality and the L^s boundedness of the Hardy-Littlewood maximal function. This argument breaks down when $s = 1$ but a slight modification of it gives that M maps into weak L^1 in this endpoint case. It is conceivable however that M map into L^1 since it carries K-tuples of compactly supported functions into compactly supported functions. This problem remains unresolved. The $L^p \times L^q \to L^r$ boundedness of the bilinear Hilbert transform $H(f, g)$ is more subtle and it remains unresolved even in the case $r > 1$.

In this note, we study the easier problem of the multilinear fractional integrals. Our first result concerns the $L^{p_1} \times \cdots \times L^{p_K} \to L^r$ boundedness of I_α for $r \geq 1$.

Theorem 1. *Let s be the harmonic mean of $p_1, \dots, p_K > 1$ and let r be such that the identity $1/r + \alpha/n = 1/s$ holds. Then I_α maps $L^{p_1} \times \cdots \times L^{p_K}$ into L^r for $n/(n+\alpha) \leq s < n/\alpha$ (equivalently $1 \leq r < \infty$).*

Note that in the case $K = 1$, the corresponding range of s is the smaller interval $1 < s < n/\alpha$ (equivalently $n/(n-\alpha) < r < \infty$).

When $K = 1$, the following theorem has been proved by Hirschman [HI] for periodic functions and by Hedberg [HE] for positive functions.

Theorem 2. *Let p_j be positive real numbers and let $s > 1$ be their harmonic mean. Then for $q, r > 1$ and $0 < \theta < 1$, the following inequality is true:*

$$\|I_{\alpha\theta}(\mathbf{f})\|_{L^r} \leq C \|I_\alpha(\mathbf{f})\|_{L^q}^\theta \prod_k \|f_k\|_{L^{p_k}}^{1-\theta} \qquad \text{where} \qquad \frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s}.$$

In the endpoint case $s = n/\alpha$, Trudinger [T] for $\alpha = 1$, and Strichartz [STR] for other α proved exponential integrability of I_α when $K = 1$. Hedberg [HE] gave a simpler proof of theorem 3 below when $K = 1$.

By ω_{n-1} we denote the area of the unit sphere $\mathbf{S}^{n-1}$. The factor L in the exponent below is a normalizing factor and should be there by homogeneity.

Theorem 3. *Let $s = n/\alpha$ be the harmonic mean of $p_1, \dots, p_K > 1$. Let B be a ball of radius R in $\mathbf{R}^n$ and let $f_j \in L^{p_j}(B)$ be supported in the ball B. Then for any $\gamma < 1$, there exists a constant $C_0(\gamma)$ depending only on n, on α, on the θ_j's and on γ, such that the following inequality is true:*

$$(1.1) \qquad \int_B e^{\frac{n}{\omega_{n-1}}\gamma\left(\frac{L\, I_\alpha(f_1,\dots,f_K)}{\|f_1\|_{L^{p_1}}\cdots\|f_K\|_{L^{p_K}}}\right)^{n/(n-\alpha)}} dx \leq C_0(\gamma) R^n \qquad \text{where} \qquad L = \prod_k |\theta_k|^{n/p_k}.$$

All the comments in this paragraph refer to the case $K = 1$. [HMT] (for $\alpha = 1$) and later Adams [A] (for all α) showed that inequality (1.1) cannot hold if $\gamma > 1$. Moser [M] showed exponential integrability of $n\omega_{n-1}^{1/n-1}\big(|\phi(x)|/\|\nabla\phi\|_{L^n}\big)^{n/n-1}$ suggesting that theorem 3 be true in the endpoint case $\gamma = 1$. (Use formula (18) page 125 in [ST] to show that Moser's result follows from an improved theorem 3 with $\gamma = 1$.) In fact, Adams [A] proved inequality (1.1) in the endpoint case $\gamma = 1$ and also deduced the sharp constants for Moser's exponential inequality for higher order derivatives. Chang and Marshall [CM] proved a similar sharp exponential inequality concerning the Dirichlet integral.

2. Proof of theorem 1. We denote by $|B|$ the measure of the set B and by χ_A the characteristic function of the set A. We also use the notation $s^{'} = s/(s-1)$ for $s \geq 1$.

We consider first the case $s \geq 1$. We will show that I_α maps $L^{p_1} \times \cdots \times L^{p_K} \to L^{r,\infty}$. The required result when $s > 1$ is going to follow from an application of the Marcinkiewitz interpolation theorem. Without loss of generality we can assume that $f_j \geq 0$ and that $\|f_j\|_{L_{p_j}} = 1$. Fix a $\lambda > 0$ and define $\mu > 0$ by $L^{-1}(\frac{\omega_{n-1}}{(\alpha-n)s^{'}+n})^{1/s^{'}}\mu^{-n/r} = \frac{\lambda}{2}$ where ω_{n-1} and L are as in theorem 3. Hölder's inequality and our choice of μ give that

$$
\begin{aligned}
I_\alpha^\infty(\mathbf{f})(x) &= \int_{|y|>\mu} f_1(x-\theta_1 y)\ldots f_K(x-\theta_K y)|y|^{\alpha-n}\,dy \\
&\leq \|\prod f_k(x-\theta_k y)\|_{L^s(y)}\||y|^{\alpha-n}\chi_{|y|>\mu}\|_{L^{s'}} \\
&\leq \prod \|f_k(x-\theta_k y)\|_{L^{p_k}(y)}\big(\tfrac{\omega_{n-1}}{(\alpha-n)s^{'}+n}\big)^{1/s^{'}}\mu^{n(\alpha/n-1+1/s^{'})} = \lambda/2 \qquad (2.1)
\end{aligned}
$$

Let $I_\alpha^0(\mathbf{f})(x) = \int_{|y|\leq\mu} f_1(x-\theta_1 y)\ldots f_K(x-\theta_K y)|y|^{\alpha-n}\,dy$. We compute its L^s norm:

$$
\begin{aligned}
\|I_\alpha^0(\mathbf{f})\|_{L^s} &\leq \left\|\left(\int \big(\prod f_k\big)^s |y|^{\alpha-n}\chi_{|y|\leq\mu}dy\right)^{1/s}\left(\int 1^{s^{'}}\ |y|^{\alpha-n}\chi_{|y|\leq\mu}dy\right)^{1/s^{'}}\right\|_{L^s} \\
&\leq C\mu^{\alpha/s^{'}}\left(\iint \big(\prod f_k\big)^s |y|^{\alpha-n}\chi_{|y|\leq\mu}dxdy\right)^{1/s} \\
&\leq C\mu^{\alpha/s^{'}}\big(\prod\|f_k\|_{L^{p_k}}^s\int_{|y|\leq\mu}|y|^{\alpha-n}dy\big)^{1/s} = C\mu^{\alpha/s^{'}}\mu^{\alpha/s} = C\mu^\alpha \qquad (2.2)
\end{aligned}
$$

By (2.1) the set $\{x : I_\alpha^\infty(\mathbf{f})(x) > \lambda/2\}$ is empty. This fact together with Chebychev's inequality and (2.2) gives us the following inequality:

$$
|\{x : I_\alpha(\mathbf{f})(x) > \lambda\}| \leq |\{x : I_\alpha^0(\mathbf{f})(x) > \lambda/2\}| \leq 2^s\lambda^{-s}\|I_\alpha^0\|_{L^s}^s \leq C\lambda^{-s}\mu^{s\alpha} = C_{\theta_k}\lambda^{-r}
$$

which is the required weak type estimate for I_α.

We now do the case $n/(n+\alpha) \le s \le 1$. The corresponding range of r's is $1 \le r \le n/(n-\alpha)$. Assume that $K = 2$ and that $p_1 \ge p_2 > 1$. Also assume that $r = 1$ first. Since $s < n/\alpha$ we must have that $p_2 < n/\alpha$. We get

$$\begin{aligned}
\|I_\alpha(f_1, f_2)\|_{L^1} &= \iint f_1(x-\theta_1 y) f_2(x-\theta_2 y)|y|^{\alpha-n} dxdy \\
&= \int f_1(x) \int f_2(x-(\theta_2-\theta_1)y)|y|^{\alpha-n} dy\, dx \\
&= |\theta_2-\theta_1|^{-\alpha} \int f_1(x) I_\alpha(f_2)(x) dx \le C_{\theta_1,\theta_2} \|f_1\|_{L^{p_1}} \|I_\alpha(f_2)\|_{L^{p_1'}}. \qquad (2.3)
\end{aligned}$$

Since $1 < p_2 < n/\alpha$, we can apply theorem 1, Ch. V on fractional integrals in [ST] to bound (2.3) by $C_{\theta_2,\theta_1}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}$. The case for a general $r > 1$ follows by interpolating between the endpoint case $r = 1$ and the case r close to ∞. Suppose now that the theorem is true for $K-1$, $K \ge 3$. We will show that it true for K. Again we first do the case $r = 1$. We may assume without loss of generality that $p_1 \ge \cdots \ge p_K > 1$.

$$\begin{aligned}
\|I_\alpha(\mathbf{f})\|_{L^1} &= \iint f_1(x-\theta_1 y) \dots f_K(x-\theta_K y)|y|^{\alpha-n} dxdy \\
&= \int f_1(x) \int f_2(x-(\theta_2-\theta_1)y) \dots f_K(x-(\theta_K-\theta_1)y)|y|^{\alpha-n} dy\, dx \\
&= \prod_{k\ne 1} |\theta_k-\theta_1|^{-\alpha} \int f_1(x) I_\alpha(f_2,\dots,f_K)(x) dx \le C_{\theta_k} \|f_1\|_{L^{p_1}} \|I_\alpha(f_2,\dots,f_K)\|_{L^{p_1'}} \qquad (2.4)
\end{aligned}$$

Define s_1 by $1/s_1 = 1/s - 1/p_1$. Since $r = 1$, we have that $1/p_1' + \alpha/n = 1/s_1$. We can apply the induction hypothesis only if we have that $n/(n+\alpha) \le s_1 < n/\alpha$. This inequality follows from the identity $1 + \alpha/n = 1/s$ which relates s and $r = 1$. From our induction hypothesis we get that (2.4) is bounded by $C_{\theta_j} \prod \|f_k\|_{L^{p_k}}$ The case $r \ge 1$ follows by interpolation.

3. Proof of theorem 2.

As in the proof of theorem 1, fix $f_j \ge 0$ such that $\|f_j\|_{L^{p_j}} = 1$. Like [HE], split

$$\begin{aligned}
&I_{\alpha\theta}(\mathbf{f})(x) = \int_{|y|<\delta} \prod f_k(x-\theta_k y)|y|^{\alpha\theta-n} dy + \int_{|y|\ge\delta} \prod f_k(x-\theta_k y)|y|^{\alpha\theta-n} dy \le \\
&\sum_{m=1}^{\infty} \int_{|y|\sim\delta 2^{-m}} \prod f_k(x-\theta_k y)|y|^{\alpha\theta-n} dy + \int_{|y|\ge\delta} \prod f_k(x-\theta_k y)|y|^{\alpha-n}|y|^{(\theta-1)\alpha} dy \le \\
&\sum_{m=1}^{\infty} (\delta 2^{-m})^{\alpha\theta} \int_{|y|\sim\delta 2^{-m}} \prod f_k(x-\theta_k y)|y|^{-n} dy + \delta^{(\theta-1)\alpha} \int_{|y|\ge\delta} \prod f_k(x-\theta_k y)|y|^{\alpha-n} dy
\end{aligned}$$

$$\leq C\delta^{\alpha(\theta-\epsilon)}M(\mathbf{f})(x)+\delta^{\alpha(\theta-1)}I_\alpha(\mathbf{f})(x).$$

Now choose $\delta=(I_\alpha(\mathbf{f})(x)/M(\mathbf{f})(x))^{1/\alpha}$ to get $I_{\alpha\theta}(\mathbf{f})(x)\leq C(I_\alpha(\mathbf{f})(x))^\theta\,(M(\mathbf{f})(x))^{1-\theta}$. Hölder's inequality with exponents $1/r=1/(\frac{s}{1-\theta})+1/(\frac{q}{\theta})$ will give $\|I_{\alpha\theta}(\mathbf{f})\|_{L^r}\leq C\|I_\alpha^\theta(\mathbf{f})\|_{L^{q/\theta}}\,\|M(\mathbf{f})\|_{L^{s/(1-\theta)}}=C\|I_\alpha(\mathbf{f})\|_{L^q}^\theta\|M(\mathbf{f})\|_{L^s}^{1-\theta}\leq C\|I_\alpha(\mathbf{f})\|_{L^q}^\theta$ by the boundedness of the maximal function M on L^q. This concludes the proof of theorem 2.

4. Proof of theorem 3. A simple dilation argument shows that if we know theorem 3 for a specific value of $R=R_0$ with a constant $C_0^{'}(\gamma)$ on the right hand side of (1.1), then we also know it for all other values of R with constant $C_0^{'}(\gamma)(R/R_0)^n$. We select $R_0=1/P$ where $P=2\min|\theta_k|^{-1}$ and we will assume that the radius of B is R_0. Furthermore, we can assume that the f_j's satisfy $f_j\geq 0$ and $\|f_j\|_{L^{p_j}}=1$. Now fix $x\in B$. The same argument as in theorem 2 with $\theta=1$ gives that

$$I_\alpha(\mathbf{f})(x)\leq C\delta^\alpha M(\mathbf{f})(x)+\int_{|y|\geq\delta}\prod f_k(x-\theta_k y)|y|^{\alpha-n}dy. \tag{4.1}$$

Since all f_k are supported in the ball B and $x\in B$ the integral in (4.1) is over the set $\{y:\delta\leq|y|\leq PR_0=1\}$. Hölder's inequality with exponents $p_1,\ldots,p_K$ and $\frac{n}{n-\alpha}$ gives

$$\begin{aligned}&\int_{\delta\leq|y|\leq 1}\prod f_k(x-\theta_k y)|y|^{\alpha-n}dy\leq\\ &\prod\|f_k(x-\theta_k y)\|_{L^{p_k}(y)}\Big(\int_{\delta\leq|y|\leq 1}|y|^{-n}dy\Big)^{(n-\alpha)/n}=L^{-1}\big(\omega_{n-1}\ln\tfrac{1}{\delta}\big)^{(n-\alpha)/n}.\end{aligned}\tag{4.2}$$

Combining (4.1) and (4.2) we get:

$$I_\alpha(\mathbf{f})(x)\leq C\delta^\alpha M(\mathbf{f})(x)+L^{-1}\left(\tfrac{\omega_{n-1}}{n}\ln\big(\tfrac{1}{\delta}\big)^n\right)^{(n-\alpha)/n}. \tag{4.3}$$

The choice $\delta=1$ gives $I_\alpha(\mathbf{f})(x)\leq CM(\mathbf{f})(x)$ for all $x\in B$ and therefore the selection $\delta=\delta(x)=\epsilon\big(I_\alpha(\mathbf{f})(x)(CM(\mathbf{f})(x))^{-1}\big)^{1/\alpha}$ will satisfy $\delta\leq 1$ for all $\epsilon\leq 1$. (4.3) now implies

$$I_\alpha(\mathbf{f})(x)\leq\epsilon^\alpha I_\alpha(\mathbf{f})(x)+L^{-1}\left(\tfrac{\omega_{n-1}}{n}\ln\big(\tfrac{(CM(\mathbf{f})(x))^{n/\alpha}}{\epsilon^n I_\alpha(\mathbf{f})(x)^{n/\alpha}}\big)\right)^{(n-\alpha)/n}.$$

Algebraic manipulation of the above gives:

$$\tfrac{n}{\omega_{n-1}}\gamma\big(LI_\alpha(\mathbf{f})(x)\big)^{n/(n-\alpha)}\leq\ln\big(\tfrac{(CM(\mathbf{f})(x))^{n/\alpha}}{\epsilon^n I_\alpha(\mathbf{f})(x)^{n/\alpha}}\big) \tag{4.4}$$

where we set $\gamma = (1-\epsilon^\alpha)^{n/(n-\alpha)}$. We exponentiate (4.4) and we integrate over the set $B_1 = \{x \in B : I_\alpha(\mathbf{f})(x) \geq 1\}$ to obtain

$$\int_{B_1} e^{\frac{n}{\omega_{n-1}}\gamma\left(LI_\alpha(\mathbf{f})(x)\right)^{n/(n-\alpha)}} dx \leq \frac{1}{\epsilon^n}\int_{B_1} \frac{(CM(\mathbf{f})(x))^{n/\alpha}}{I_\alpha(\mathbf{f})(x)^{n/\alpha}} dx \leq \frac{C_1}{\epsilon^n}\int M(\mathbf{f})(x)^{n/\alpha} dx \leq \frac{C_2}{\epsilon^n}.$$

The last inequality follows from the boundedness of the maximal function of $\mathbf{f}$ on $L^{n/\alpha}$. The integral of the same exponential over the set $B_2 = B - B_1$ is estimated trivially by

$$\int_{B_2} e^{\frac{n}{\omega_{n-1}}\gamma\left(LI_\alpha(\mathbf{f})(x)\right)^{n/(n-\alpha)}} dx \leq e^{\frac{n}{\omega n-1}L^{n/(n-\alpha)}}|B_2| \leq C_3\Omega_n R_0^n = C_4.$$

Adding the integrals above over B_1 and B_2 we obtain the required inequality with a constant $C_0^{'}(\gamma) = \max(C_2, C_4)(1+(1-\gamma^{(n-\alpha)/n})^{-n/\alpha})$. The constant $C_0(\gamma)$ in the statement of theorem 3 is then $C_0^{'}(\gamma)R_0^{-n} = C_0^{'}(\gamma)P^n$.

We obtain the following

Corollary. *Let B, f_k, p_k, and s as in theorem 3. Then $I_\alpha(f_1,\ldots,f_K)$ is in $L^q(B)$ for every $q>0$. In fact the following inequality is true:*

$$\|I_\alpha(f_1,\ldots,f_K)\|_{L^q(B)} \leq C\prod_k \|f_k\|_{L^{p_k}}$$

for some constant C depending only on q on n on α and on the θ_j's.

The corollary follows since exponential integrability of I_α implies integrability to any power q. (Here γ is fixed <1.)

5. A multilinear differentiation theorem.

We end this note by proving the following multilinear Lebesgue differentiation theorem. Let $f_j \in L^{p_j}(\mathbf{R}^n)$ and suppose that their harmonic mean is $s \geq 1$. Then

$$\lim_{\epsilon\to 0} T_\epsilon(\mathbf{f})(x) = \lim_{\epsilon\to 0}\frac{1}{\Omega_n\epsilon^n}\int_{|y|\leq\epsilon} f_1(x-\theta_1 y)\ldots f_K(x-\theta_K y)dy = f_1(x)\ldots f_K(x) \quad \text{a.e.}$$

The case $s=1$ is a consequence of the weak type inequality $|\{x \in \mathbf{R}^n : M(\mathbf{f})(x) > \lambda\}| \leq \frac{C}{\lambda}\|f_1\|_{L^{p_1}}\ldots\|f_1\|_{L^{p_K}}$ which is easily obtained from $|\{x \in \mathbf{R}^n : M(\mathbf{f})(x) > \lambda\}| \leq \sum_{j=1}^K |\{x \in \mathbf{R}^n : (f_j)^*(x) > (\epsilon_{j-1}/\epsilon_j)^{p_j}\}| \leq C\sum_{j=1}^K(\epsilon_{j-1}/\epsilon_j)^{-p_j}\|f_j\|_{L^{p_j}}$ after minimizing over all $\epsilon_1,\ldots,\epsilon_K > 0$. (Take $\epsilon_0 = \lambda$.) The standard argument presented in [SWE], page 61, will prove that the sequence $\{T_\epsilon(\mathbf{f})(x)\}_{\epsilon>0}$ is Cauchy for almost all x and therefore it converges. Since for continuous $f_1,\ldots,f_K$ it converges to the value of their product at the point $x \in \mathbf{R}^n$, to deduce the general case it will suffice to show that $\{T_\epsilon(\mathbf{f})\}_{\epsilon>0}$ converges to the product of the f_j's in the L^s norm as $\epsilon \to 0$. (Then some subsequence will converge to the product a.e.) Denoting by $(\tau_y f)(x) = f(x-y)$ the translation of f by $-y$, we get

$$\|T_\epsilon(\mathbf{f}) - f_1 \dots f_K\|_{L^s} \leq \frac{1}{\Omega_n \epsilon^n} \int_{|y|\leq\epsilon} \| \prod_j \tau_{\theta_j y} f_j - \prod_j f_j \|_{L^s} dy$$

$$\leq \frac{1}{\Omega_n \epsilon^n} \int_{|y|\leq\epsilon} \sum_{j=1}^{K} \|\tau_{\theta_j y} f_j - f_j\|_{L^{p_j}} \prod_{k\neq j} \|f_j\|_{L^{p_k}} dy \to 0$$

as $|y| \to 0$ since the last integrand is a continous function of y which vanishes at the origin. The last inequality above follows by adding and subtracting $2K-2$ suitable terms and applying Hölder's inequality K times.

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