MULTILINEAR INTERPOLATION BETWEEN ADJOINT OPERATORS

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Abstract. Multilinear interpolation is a powerful tool used in obtaining strong type boundedness for a variety of operators assuming only a finite set of restricted weak-type estimates. A typical situation occurs when one knows that a multilinear operator satisfies a weak L^q estimate for a single index q (which may be less than one) and that all the adjoints of the multilinear operator are of similar nature, and thus they also satisfy the same weak L^q estimate. Under this assumption, in this expository note we give a general multilinear interpolation theorem which allows one to obtain strong type boundedness for the operator (and all of its adjoints) for a large set of exponents. The key point in the applications we discuss is that the interpolation theorem can handle the case $q \leq 1$. When $q > 1$, weak L^q has a predual, and such strong type boundedness can be easily obtained by duality and multilinear interpolation (c.f. [1], [5], [7], [12], [14]).

1. Multilinear operators

We begin by setting up some notation for multilinear operators. Let $m \geq 1$ be an integer. We suppose that for $0 \leq j \leq m$, (X_j, μ_j) are measure spaces endowed with positive measures μ_j. We assume that T is an m-linear operator of the form

$$T(f_1, \dots, f_m)(x_0) := \int \dots \int K(x_0, \dots, x_m) \prod_{i=1}^{m} f_i(x_i)\ d\mu_i(x_i)$$

where K is a complex-valued locally integrable function on $X_0 \times \ldots \times X_m$ and f_j are simple functions on X_j. We shall make the technical assumption that K is bounded and is supported on a product set $Y_0 \times \ldots \times Y_m$ where each $Y_j \subseteq X_j$ has finite measure. Of course, most interesting operators (e.g. multilinear singular integral operators) do not obey this condition, but in practice one can truncate and/or mollify the kernel of a singular integral to obey this condition, apply the multilinear interpolation theorem to the truncated operator, and use a standard limiting argument to recover estimates for the untruncated operator.

One can rewrite T more symmetrically as an $m+1$-linear form Λ defined by

$$\Lambda(f_0, f_1, \dots, f_m) := \int \dots \int K(x_0, \dots, x_m) \prod_{i=0}^{m} f_i(x_i)\ d\mu_i(x_i).$$

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One can then define the m adjoints T^{*j} of T for $0 \leq j \leq m$ by duality as

$$\int f_j(x_j) T^{*j}(f_1, \dots, f_{j-1}, f_0, f_{j+1}, \dots, f_m)(x_j)\ d\mu_j(x_j) := \Lambda(f_0, f_1, \dots, f_m).$$

Observe that $T = T^{*0}$.

We are interested in the mapping properties of T from the product of spaces $L^{p_1}(X_1, \mu_1) \times \ldots \times L^{p_m}(X_m, \mu_m)$ into $L^{p_0}(X_0, \mu_0)$ for various exponents p_j, and more generally for the adjoints T^{*j} of T. Actually, it will be more convenient to work with the $(m+1)$-linear form Λ, and with the tuple of reciprocals $(1/p_0', 1/p_1, \dots, 1/p_m)$ instead of the exponents p_j directly. (Here we adopt the usual convention that p' is defined by $1/p' + 1/p := 1$ even when $0 < p < 1$; this notation is taken from Hardy, Littlewood and Pólya.)

Recall the definition of the weak Lebesgue space $L^{p,\infty}(X_i, \mu_i)$ for $0 < p < \infty$ by

$$\|f\|_{L^{p,\infty}(X_i,\mu_i)} := \sup_{\lambda > 0} \lambda \mu_i(\{x_i \in X_i : |f(x_i)| \geq \lambda\})^{1/p}.$$

We also define $L^{\infty,\infty} = L^\infty$. If $1 < p < \infty$, we define the restricted Lebesgue space $L^{p,1}(X_i, \mu_i)$ by duality as

$$\|f\|_{L^{p,1}(X_i,\mu_i)} := \sup\{\Big|\int f(x_i) g(x_i)\ d\mu_i(x_i)\Big| : g \in L^{p,\infty}(X_i, \mu_i), \|g\|_{L^{p,\infty}(X_i,\mu_i)} \leq 1\}.$$

We also define $L^{1,1} = L^1$. This definition is equivalent to the other standard definitions of $L^{p,1}(X_i, \mu_i)$ up to a constant depending on p.

Definition 1. *Define a* tuple *to be a collection of $m+1$ numbers $\alpha = (\alpha_0, \dots, \alpha_m)$ such that $-\infty < \alpha_i \leq 1$ for all $0 \leq i \leq m$, such that such that $\alpha_0 + \ldots + \alpha_m = 1$, and such that at most one of the α_i is non-positive. If for all $j \in \{0, 1, 2, \dots, m\}$ we have $0 < \alpha_j < 1$, we say that the tuple α is* good*. Otherwise there is exactly one a_i such that $a_i \leq 0$ and we say that the tuple α is* bad*. The smallest number j_0 for which the $\min_{0 \leq j \leq m} \alpha_j$ is attained for a tuple α is called the* bad index of the tuple.

If α is a good tuple and $B > 0$, we say that Λ is *of strong-type α with bound B* if we have the multilinear form estimate

$$|\Lambda(f_0, \dots, f_m)| \leq B \prod_{i=0}^m \|f_i\|_{L^{1/\alpha_i}(X_i,\mu_i)}$$

for all simple functions $f_0, \dots, f_m$. By duality, this is equivalent to the multilinear operator estimate

$$\|T(f_1, \dots, f_m)\|_{L^{1/(1-\alpha_0)}(X_0,\mu_0)} \leq B \prod_{i=1}^m \|f_i\|_{L^{1/\alpha_i}(X_i,\mu_i)}$$

or more generally

$$\|T^{*j}(f_1, f_{j-1}, f_0, f_{j+1}, \dots, f_m)\|_{L^{1/(1-\alpha_j)}(X_j,\mu_j)} \leq B \prod_{\substack{0 \leq i \leq m \\ i \neq j}} \|f_i\|_{L^{1/\alpha_i}(X_i,\mu_i)}$$

for $0 \leq j \leq m$.

If α is a tuple with bad index j, we say that Λ is *of restricted weak-type α with bound B* if we have the estimate

$$\|T^{*j}(f_1, \dots, f_{j-1}, f_0, f_{j+1}, \dots, f_m)\|_{L^{1/(1-\alpha_j),\infty}(X_j,\mu_j)} \leq B \prod_{\substack{0\leq i\leq m \\ i\neq j}} \|f_i\|_{L^{1/\alpha_i,1}(X_i,\mu_i)}$$

for all simple functions f_i. In view of duality, if α is a good index, then the choice of the index j above is irrelevant.

2. The interpolation theorem

We have the following interpolation theorem for restricted weak-type estimates, inspired by [12]:

Theorem 1. *Let $\alpha^{(1)}, \dots, \alpha^{(N)}$ be tuples for some $N > 1$, and let α be a good tuple such that $\alpha = \theta_1\alpha^{(1)} + \ldots + \theta_N\alpha^{(N)}$, where $0 \leq \theta_s \leq 1$ for all $1 \leq s \leq N$ and $\theta_1 + \ldots + \theta_N = 1$.*

Suppose that Λ is of restricted weak-type $\alpha^{(s)}$ with bound $B_s > 0$ for all $1 \leq s \leq N$. Then Λ is of restricted weak-type α with bound $C\prod_{s=1}^N B_s^{\theta_s}$, where $C > 0$ is a constant depending on $\alpha^{(1)}, \dots, \alpha^{(N)}, \theta_1, \dots, \theta_N$.

Proof. Since α is a good tuple, it suffices by duality to prove the multilinear form estimate

$$|\Lambda(f_0, \dots, f_m)| \leq C(\prod_{s=1}^N B_s^{\theta_s}) \prod_{i=0}^m \|f_i\|_{L^{1/\alpha_i,1}(X_i,\mu_i)}.$$

We will let the constant C vary from line to line. For $1 < p < \infty$, the unit ball of $L^{p,1}(X_i, \mu_i)$ is contained in a constant multiple of the convex hull of the normalized characteristic functions $\mu_i(E)^{1/p}\chi_E$ (see e.g. [13]) it suffices to prove the above estimate for characteristic functions:

$$|\Lambda(\chi_{E_0}, \dots, \chi_{E_m})| \leq C(\prod_{s=1}^N B_s^{\theta_s}) \prod_{i=0}^m \mu_i(E_i)^{\alpha_i}.$$

We may of course assume that all the E_i have positive finite measure. Let A be the best constant such that

$$|\Lambda(\chi_{E_0}, \dots, \chi_{E_m})| \leq A(\prod_{s=1}^N B_s^{\theta_s}) \prod_{i=0}^m \mu_i(E_i)^{\alpha_i} \tag{1}$$

for all such E_j; by our technical assumption on the kernel K we see that A is finite. Our task is to show that $A \leq C$.

Let $\varepsilon > 0$ be chosen later. We may find $E_0, \dots, E_m$ of positive finite measure such that

$$|\Lambda(\chi_{E_0}, \dots, \chi_{E_m})| \geq (A - \varepsilon)Q, \tag{2}$$

where we use $0 < Q < \infty$ to denote the quantity

$$Q := (\prod_{s=1}^N B_s^{\theta_s}) \prod_{i=0}^m \mu_i(E_i)^{\alpha_i} = \prod_{s=1}^N (B_s \prod_{i=0}^m \mu_i(E_i)^{\alpha_i^{(s)}})^{\theta_s}.$$

Fix $E_0, \ldots, E_m$. From the definition of Q we see that there exists $1 \leq s_0 \leq N$ such that

$$B_{s_0} \prod_{i=0}^{m} \mu_i(E_i)^{\alpha_i^{(s_0)}} \leq Q. \tag{3}$$

Fix this s_0, and let j be the bad index of $\alpha^{(s_0)}$. Let F be the function

$$F := T^{*j}(\chi_{E_1}, \ldots, \chi_{E_{j-1}}, \chi_{E_0}, \chi_{E_{j+1}}, \ldots, \chi_{E_m}).$$

Since Λ is of restricted weak-type $\alpha^{(s_0)}$ with bound B_{s_0}, we have from (3) that

$$\|F\|_{L^{1/(1-\alpha_j^{(s_0)}),\infty}(X_j,\mu_j)} \leq B_{s_0} \prod_{\substack{0 \leq i \leq m \\ i \neq j}} \mu_i(E_i)^{\alpha_i^{(s_0)}} \leq Q \mu_j(E_j)^{-\alpha_j^{(s_0)}}. \tag{4}$$

In particular if we define the set

$$E_j' := \{x_j \in E_j : |F(x_j)| \geq 2^{1-\alpha_j^{(s_0)}} Q \mu_j(E_j)^{-1}\} \tag{5}$$

then (4) implies that

$$\mu_j(E_j') \leq \tfrac{1}{2} \mu_j(E_j). \tag{6}$$

By construction of E_j' we have $|\int \chi_{E_j \setminus E_j'}(x_j) F(x_j)\ d\mu_j(x_j)| \leq 2^{1-\alpha_j^{(s_0)}} Q$, or equivalently that

$$|\Lambda(\chi_{E_0}, \ldots, \chi_{E_{j-1}}, \chi_{E_j \setminus E_j'}, \chi_{E_{j+1}}, \ldots, \chi_{E_m})| \leq CQ.$$

On the other hand, from (1) and (6) we have

$$|\Lambda(\chi_{E_0}, \ldots, \chi_{E_{j-1}}, \chi_{E_j'}, \chi_{E_{j+1}}, \ldots, \chi_{E_m})| \leq 2^{-\alpha_j} AQ.$$

Adding the two estimates and using (2) we obtain $CQ + 2^{-\alpha_j} AQ \leq (A - \varepsilon) Q$. Since α is good, we have $\alpha_j > 0$. The claim $A < C$ then follows by choosing ε sufficiently small. $\square$

From the multilinear Marcinkiewicz interpolation theorem (see e.g. Theorem 4.6 of [5]) we can obtain strong-type estimates at a good tuple α if we know restricted weak-type estimates for all tuples in a neighborhood of α. From this and the previous theorem we obtain

Corollary 1. *Let $\alpha^{(1)}, \ldots, \alpha^{(N)}$ be tuples for some $N > 1$, and let α be a good tuple in the interior of the convex hull of $\alpha^{(1)}, \ldots, \alpha^{(N)}$. Suppose that Λ is of restricted weak-type $\alpha^{(s)}$ with bound $B > 0$ for all $1 \leq s \leq N$. Then Λ is of strong-type α with bound CB, where $C > 0$ is a constant depending on $\alpha, \alpha^{(1)}, \ldots, \alpha^{(N)}$.*

By interpolating this result with the restricted weak-type estimates on the individual T^{*j}, one can obtain some strong-type estimates for T^{*j} mapping onto spaces $L^p(X_j, \mu_j)$ where p is possibly less than or equal to 1. By duality one can thus get some estimates where some of the functions are in L^∞. However it is still an open question whether one can get the entire interior of the convex hull of $\alpha^{(1)}, \dots, \alpha^{(N)}$ this way[1].

3. Applications

We now pass to three applications. The first application is to re-prove an old result of Wolff [15]: if T is a linear operator such that T and its adjoint T^* both map L^1 to $L^{1,\infty}$, then T is bounded on L^p for all $1 < p < \infty$ (assuming that T can be approximated by truncated kernels as mentioned in the introduction). Indeed, in this case Λ is of restricted weak-type $(1,0)$ and $(0,1)$, and hence of strong-type $(1/p, 1/p')$ for all $1 < p < \infty$ by Corollary 1.

The next application involves the multilinear Calderón-Zygmund singular integral operators on $\mathbf{R}^n \times \dots \times \mathbf{R}^n = (\mathbf{R}^n)^m$ defined by

$$T(f_1, \dots, f_m)(x_0) := \lim_{\varepsilon \to 0} \int \cdots \int_{\sum_{j,k} |x_k - x_j| \geq \varepsilon} K(x_0, x_1, \dots, x_m) f_1(x_1) \dots f_m(x_m)\, dx_1 \dots dx_m,$$

where $|K(\vec{x})| \leq C(\sum_{j,k=0}^m |x_k - x_j|)^{-nm}$, $|\nabla K(\vec{x})| \leq C(\sum_{j,k=0}^m |x_k - x_j|)^{-nm-1}$, and $\vec{x} = (x_0, x_1, \dots, x_m)$. These integrals have been extensively studied by Coifman and Meyer [2],[3],[4] and recently by Grafakos and Torres [6]. It was shown in [6] and also by Kenig and Stein [8] (who considered the case $n = 1$, $m = 2$) that if such operators map $L^{q_1} \times \dots \times L^{q_m}$ into $L^{q,\infty}$ for only one m-tuple of indices, then they must map $L^1 \times \dots \times L^1$ into $L^{1/m,\infty}$. Since the adjoints of these operators satisfy similar boundedness properties, we see that the corresponding form Λ is of restricted weak-type $(1 - m, 1, \dots, 1)$, and similarly for permutations. It then follows[2] from Corollary 1 that T maps $L^{p_1} \times \dots \times L^{p_m}$ into L^p for all m-tuples of indices with[3] $1 < p_j < \infty$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$ and $p > 1$. The condition $p > 1$ can be removed by further interpolation with the $L^1 \times \ldots \times L^1 \to L^{1/m}$ estimate. This argument simplifies the interpolation proof used in [6].

Our third application involves the bilinear Hilbert transform $H_{\alpha,\beta}$ defined by

$$H_{\alpha,\beta}(f,g)(x) = \lim_{\varepsilon \to 0} \int_{|t| \geq \varepsilon} f(x - \alpha t) g(x - \beta t) \frac{dt}{t}, \quad x \in \mathbf{R}. \tag{7}$$

The proof of boundedness of $H_{\alpha,\beta}$ from $L^2 \times L^2$ into $L^{1,\infty}$ (for example see [9]) is technically simpler than that of $L^{p_1} \times L^{p_2}$ into L^p when $2 < p_1, p_2, p' < \infty$ given in Lacey and Thiele [10]. Since the adjoints of the operators $H_{\alpha,\beta}$ are $H^{*1}_{\alpha,\beta} = H_{-\alpha,\beta-\alpha}$

[1] In [12] this was achieved, but only after strengthening the hypothesis of restricted weak-type to that of "positive type". Essentially, this requires the set E'_j defined in (5) to be stable if one replaces the characteristic functions χ_{E_i} with arbitrary bounded functions on E_i.

[2] Strictly speaking, we have to first fix ε, and truncate the kernel K to a compact set, before applying the Theorem, and then take limits at the end. We leave the details of this standard argument to the reader. A similar approximation technique can be applied for the bilinear Hilbert transform below.

[3] The convex hull of the permutations of $(1-m, 1, \dots, 1)$ is the tetrahedron of points $(x_0, \dots, x_m)$ with $x_0 + \ldots + x_m = 1$ and $x_i \leq 1$ for all $0 \leq i \leq m$, so in particular the points $(1/p_1, \dots, 1/p_m)$ described above fall into this category.

and $H^{*2}_{\alpha,\beta} = H_{\alpha-\beta,-\beta}$ which are "essentially" the same operators, we can use the single estimate $L^2 \times L^2 \to L^{1,\infty}$ for all of these operators to obtain the results in [10], since the corresponding form Λ is then of restricted weak-type $(0, 1/2, 1/2)$, $(1/2, 0, 1/2)$, and $(1/2, 1/2, 0)$. (See also the similar argument in [12]).

The operator in (7) is in fact bounded in the larger range $1 < p_1, p_2 < \infty$, $p > 2/3$ and similarly for adjoints, see [11]. The interpolation theorem given here allows for a slight simplification in the arguments in that paper (cf. [12]), although one cannot deduce all these estimates solely from the $L^2 \times L^2 \to L^{1,\infty}$ estimate.

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