AN ELEMENTARY PROOF OF THE SQUARE SUMMABILITY OF THE DISCRETE HILBERT TRANSFORM

Loukas Grafakos

Washington University

We would like to give an elementary proof of Hilbert's inequality

$$\left(\sum_{j\in\mathbb{Z}}\Big|\sum_{\substack{n\in\mathbb{Z}\\ n\neq j}}\frac{a_n}{j-n}\Big|^2\right)^{1/2}\le\pi\left(\sum_{n\in\mathbb{Z}}|a_n|^2\right)^{1/2}, \tag{1}$$

where the a_n's are real and square summable, and also prove that π cannot be replaced by any smaller number.

Hilbert first proved a weaker version of inequality (1), where π was replaced by a larger constant. The original proof used trigonometric series and first appeared in Weyl's [W] doctoral dissertation in 1908. Three years later, Schur [S] obtained a proof of (1), showing that π was the best possible constant. In his proof, he used a version of what we today refer to as Schur's Lemma. This proof can be found in the book [HLP]. Many other proofs and generalizations have been given since then.

The purpose of this note is to give an elementary proof of inequality (1). The proof uses convergence of sequences; remarkably, only one inequality $2ab\le a^2+b^2$; and the identity

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n^2}=\frac{\pi^2}{3}\ .$$

Before we present the proof, we would like to clarify a point about (1). If (a_n) is square summable, it isn't automatic that the left hand-side of (1) converges. Part of the inequality is the statement that the left hand-side of (1) is finite whenever the right hand-side is.

Assume first that (a_n) is compactly supported, i.e. $a_n=0$ except for finitely many n. We show below that the left hand-side of (1) is finite and we prove the required inequality

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for such sequences. Expand the square of the left hand-side of (1). All indices m, n, j below run from $-\infty$ to ∞ unless there is some restriction stated. We obtain

$$\sum_{j}\sum_{n\neq j}\sum_{m\neq j} a_m a_n \frac{1}{(j-n)(j-m)}$$

$$=\sum_{n}\sum_{m} a_m a_n \sum_{j\neq n,m} \frac{1}{(j-n)(j-m)} \ . \tag{2}$$

Two out of the three sums above are over finite sets of indices and the interchange of summations is justified. The sum over all $m = n$ in (2) is clearly equal to

$$\sum_{n} a_n^2 \sum_{j\neq n} \frac{1}{(j-n)^2} = \frac{\pi^2}{3}\sum_{n} a_n^2. \tag{3}$$

Assume below that $m \neq n$. We calculate the sum over j in (2). We have

$$\sum_{j\neq m,n} \frac{1}{(j-n)(j-m)} = \frac{1}{m-n}\sum_{j\neq m,n}\left(\frac{1}{j-m}-\frac{1}{j-n}\right)$$

$$=\frac{1}{m-n}\lim_{k\to\infty}\sum_{\substack{j\neq m,n\\ |j|\leq k}}\left(\frac{1}{j-m}-\frac{1}{j-n}\right)$$

$$=\frac{1}{m-n}\lim_{k\to\infty}\left[\left(\sum_{\substack{j\neq m\\ |j|\leq k}}\frac{1}{j-m}\right)-\frac{1}{n-m}-\left(\sum_{\substack{j\neq n\\ |j|\leq k}}\frac{1}{j-n}\right)+\frac{1}{m-n}\right]$$

$$=\frac{2}{(m-n)^2}+\frac{1}{m-n}\lim_{k\to\infty}\left[\sum_{\substack{j\neq m\\ |j|\leq k}}\frac{1}{j-m}-\sum_{\substack{j\neq n\\ |j|\leq k}}\frac{1}{j-n}\right]$$

$$=\frac{2}{(m-n)^2}\ , \tag{4}$$

since the expression inside brackets above has limit 0 as $k \to \infty$. Because of (4), the off-diagonal terms in (2) are exactly equal to:

$$\sum_{n}\sum_{m\neq n} a_n a_m \frac{2}{(m-n)^2} \ . \tag{5}$$

Using the inequality $2a_m a_n \leq a_n^2 + a_m^2$ we bound (5) by:

$$\sum_{n}\sum_{m\neq n}\frac{a_n^2}{(m-n)^2}+\sum_{m}\sum_{n\neq m}\frac{a_m^2}{(m-n)^2}=\frac{\pi^2}{3}\sum_{n}a_n^2+\frac{\pi^2}{3}\sum_{m}a_m^2=\frac{2\pi^2}{3}\sum_{n}a_n^2. \tag{6}$$

Combining (3) and and the estimate (6) for (5), we obtain inequality (1) for compactly supported sequences. A simple limiting argument gives (1) for general square summable sequences.

We now turn to the the fact that π is the best possible constant. We define b_N to be (5) divided by $\sum_n a_n^2$, where (a_n) is the sequence 1 for $|n| \le N$ and 0 otherwise. Estimate (6) shows that $b_N \le \frac{2\pi^2}{3}$. A simple calculation gives

$$b_N \ge \frac{4N}{2N+1}\left[\sum_{k=1}^{N+1}\frac{1}{k^2}\right] + \frac{4(N-1)}{2N+1}\left[\frac{\frac{1}{1^2}+\left(\frac{1}{1^2}+\frac{1}{2^2}\right)+\cdots+\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(N-1)^2}\right)}{N-1}\right].$$

Applying the squeeze law, we obtain that b_N tends to $\frac{2\pi^2}{3}$ as $N \to \infty$. Using (3) and (5), we obtain that for this choice of (a_n), the ratio of the left-hand side of (1) and $(\sum_n a_n^2)^{1/2}$ converges to π as $N \to \infty$. This proves that π is the best possible constant in (1), *Q.E.D.*

Inequality (1) is known to be strict if (a_n) is nonzero. To see this for compactly supported nonzero sequences, observe that (6) is a strict bound for (5) since for some m and n, $2a_n a_m < a_n^2 + a_m^2$. For general sequences, a further argument is needed since the passage to the limit will destroy the strict inequality. See [HLP] for details on this.

To the best of my knowledge, the determination of the best constant for the l^p inequality, $1 < p \neq 2 < \infty$ remains unresolved. Pichorides [P] solves this problem for the corresponding continuous operator.

We end this note by asking a question: Is there a constant C such that for all square summable sequences (a_n) and all bounded sequences (λ_n), the following inequality holds?

$$\left(\sum_{j\in\mathbb{Z}}\Big|\sum_{\substack{n\in\mathbb{Z}\\ n\neq j}}\lambda_{j+n}\frac{a_n}{j-n}\Big|^2\right)^{1/2} \le C\sup_j|\lambda_j|\left(\sum_{n\in\mathbb{Z}}|a_n|^2\right)^{1/2}. \tag{7}$$

If $\lambda_n = 1$ for all n, then (7) reduces to (1) and one can take $C \ge \pi$.

References

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Department of Mathematics, Washington University, Campus Box 1146, One Brookings Drive, St Louis, MO 63130-4899

E-mail address: grafakos@math.wustl.edu