# THE HÖRMANDER MULTIPLIER THEOREM, II: THE BILINEAR LOCAL $L^2$ CASE

LOUKAS GRAFAKOS, DANQING HE, AND PETR HONZÍK

ABSTRACT. We use wavelets of tensor product type to obtain the boundedness of bilinear multiplier operators on $\mathbb{R}^n \times \mathbb{R}^n$ associated with Hörmander multipliers on $\mathbb{R}^{2n}$ with minimal smoothness. We focus on the local $L^2$ case and we obtain boundedness under the minimal smoothness assumption of $n/2$ derivatives. We also provide counterexamples to obtain necessary conditions for all sets of indices.

## 1. INTRODUCTION

An $m$-linear $(p_1, \dots, p_m, p)$ multiplier $\sigma(\xi_1, \dots, \xi_m)$ is a function on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ such that the corresponding $m$-linear operator

$$T_\sigma(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} \sigma(\xi_1, \dots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} d\xi_1 \cdots d\xi_m,$$

initially defined on $m$-tuples of Schwartz functions, has a bounded extension from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for appropriate $p_1, \dots, p_m, p$.

It is known from the work in [2] for $p > 1$ and [12], [11] for $p \le 1$, that the classical Mihlin condition on $\sigma$ in $\mathbb{R}^{mn}$ yields boundedness for $T_\sigma$ from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots p_m \le \infty$, $1/m < p = (1/p_1 + \cdots + 1/p_m)^{-1} < \infty$. The Mihlin condition in this setting is usually referred to as the Coifman-Meyer condition and the associated multipliers bear the same names as well. The Coifman-Meyer condition cannot be weakened to the Marcinkiewicz condition, as the latter fails in the multilinear setting; see [8]. Related multilinear multiplier theorems with mixed smoothness (but not necessarily minimal) can be found in [14], [15], [7].

A natural question on Hörmander type multipliers is how the minimal smoothness $s$ interplays with the range of $p$'s on which boundedness is expected. In the linear case, this question was studied in [1], [16], and [6]. Let $L^r_s(\mathbb{R}^n)$ be the Sobolev space consisting of all functions $h$ such that $(I - \Delta)^{s/2}(h) \in L^r(\mathbb{R}^n)$, where $\Delta$ is the Laplacian. In the first paper of this series [6], we showed that the conditions $|1/2 - 1/p| < s/n$ and $rs > n$ imply $L^p(\mathbb{R}^n)$ boundedness for $1 < p < \infty$ for $T_\sigma$ in the linear case $m = 1$,

The first author was supported by the Simons Foundation and the University of Missouri Research Board and Council. The third author was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education.

*Mathematics Subject Classification:* Primary 42B15. Secondary 42B25.

*Keywords and phases:* Bilinear multipliers, Hörmander multipliers, wavelets, multilinear operators.

when the multiplier $\sigma$ lies in the Sobolev space $L^r_s(\mathbb{R}^n)$ uniformly over all annuli. This minimal smoothness problem in the bilinear setting was first studied in [17] and later in [14] and [9]. These references contain necessary conditions on $s$ when the multiplier in the Sobolev space $L^r_s$ with $r=2$; other values of $r$ were considered in [10].

Our goal here is to pursue the analogous bilinear question. In this paper we focus on the boundedness of $T_\sigma$ in the local $L^2$ case, i.e., the situation where $1 \le p_1, p_2 \le 2$ and $1 \le p = 1/(1/p_1+1/p_2) \le 2$ under minimal smoothness conditions on $s$. It turns out that to express our result in an optimal fashion, we need to work with $r>2$. We also work with the case $L^2 \times L^2 \to L^1$ as boundedness in the remaining local $L^2$ indices follows by duality and interpolation. We achieve our goal via new technique to study boundedness for bilinear operators based on tensor product wavelet decomposition developed in [5].

The main result of this paper is the following theorem.

**Theorem 1.** *Suppose $\widehat{\psi} \in \mathcal{C}^\infty_0(\mathbb{R}^{2n})$ is positive and supported in the annulus*

$$\{(\xi,\eta) : 1/2 \le |(\xi,\eta)| \le 2\}$$

*such that $\sum_{j\in\mathbb{Z}} \widehat{\psi}_j(\xi,\eta) = \sum_j \widehat{\psi}(2^{-j}(\xi,\eta)) = 1$ for all $(\xi,\eta)\neq 0$. Let $1<r<\infty$, $s > \max\{n/2, 2n/r\}$, and suppose there is a constant $A$ such that*

$$\sup_j \|\sigma(2^j\cdot)\widehat{\psi}\|_{L^r_s(\mathbb{R}^{2n})} \le A < \infty. \tag{1}$$

*Then there is a constant $C = C(n,\Psi)$ such that the bilinear operator*

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta,$$

*initially defined on Schwartz functions $f$ and $g$, satisfies*

$$\|T_\sigma(f,g)\|_{L^1(\mathbb{R}^n)} \le CA\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{2}$$

The optimality of (1) in the preceding theorem is contained in the following result.

**Theorem 2.** *Suppose that for $0<p_1,\dots,p_m<\infty$, $p=(1/p_1+\cdots+1/p_m)^{-1}$, we have*

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le C\sup_{j\in\mathbb{Z}}\|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{mn})} \tag{3}$$

*for all bounded functions $\sigma$ for which $\sup_{j\in\mathbb{Z}}\|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{mn})}<\infty$ (for some fixed $r,s>0$). Then we must necessarily have $s \ge \max\{(m-1)n/2, mn/r\}$.*

Finally, we have another set of necessary conditions for the boundedness of $m$-linear multipliers. The sufficiency of these conditions is shown in the third paper of this series.

**Theorem 3.** *Suppose there exists a constant $C$ such that* (3) *holds for all $\sigma$ such that the right hand side is finite. Then we must necessarily have*

$$\frac{1}{p}-\frac{1}{2} \le \frac{s}{n}+\sum_{i\in I}\Big(\frac{1}{p_i}-\frac{1}{2}\Big),$$

*where $I$ is an arbitrary subset of $\{1,2,\dots,\ m\}$ which may also be empty (in which case the sum is supposed to be zero).*

## 2. Preliminaries

We utilize wavelets with compact supports. Their existence is due to Daubechies [3] and their construction is contained in Meyer's book [13] and Daubechies' book [4]. For our purposes we need product type smooth wavelets with compact supports; the construction of such objects we use here can be found in Triebel [18, Proposition 1.53].

**Lemma 4.** *For any fixed $k \in \mathbb{N}$ there exist real compactly supported functions $\psi_F, \psi_M \in \mathcal{C}^k(\mathbb{R})$, the class of functions with continuous derivatives of order up to k, which satisfy that $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$ and $\int_{\mathbb{R}} x^\alpha \psi_M(x)dx = 0$ for $0 \le \alpha \le k$, such that, if $\Psi^G$ is defined by*

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1)\cdots\psi_{G_{2n}}(x_{2n})$$

*for $G = (G_1,\dots,G_{2n})$ in the set*

$$\mathcal{I} := \Big\{(G_1,\dots,G_{2n}):\ G_i \in \{F,M\}\Big\},$$

*then the family of functions*

$$\bigcup_{\vec{\mu}\in\mathbb{Z}^{2n}} \left[\Big\{\Psi^{(F,\dots,F)}(\vec{x}-\vec{\mu})\Big\} \cup \bigcup_{\lambda=0}^{\infty}\Big\{2^{\lambda n}\Psi^G(2^\lambda\vec{x}-\vec{\mu}):\ G\in\mathcal{I}\setminus\{(F,\dots,F)\}\Big\}\right]$$

*forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1,\dots,x_{2n})$.*

In order to prove our results, we use the wavelet characterization of Sobolev spaces, following Triebel's book [18]. Let us fix the smoothness $s$, for our purposes we always have $s \le n+1$, since we are seeking for the minimal smoothness. Also, we only work with spaces with the integrability index $r > 1$. Take $\varphi$ as a smooth function defined on $\mathbb{R}^{2n}$ such that $\widehat{\varphi}$ is supported in the unit annulus such that $\sum_{j=0}^\infty \widehat{\varphi}_j = 1$, where $\widehat{\varphi}_j = \widehat{\varphi}(2^{-j}\cdot)$ for $j \ge 1$ and $\widehat{\varphi}_0 = \sum_{k\le 0}\widehat{\varphi}(2^{-k}\cdot)$. Then for a distribution $f \in \mathcal{S}'(\mathbb{R}^{2n})$ we define the $F^s_{r,q}$ norm as follows:

$$\|f|F^s_{r,q}(\mathbb{R}^{2n})\| = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j\widehat{f})^\vee(\cdot)|^q\Big)^{1/q}\Big\|_{L^r(\mathbb{R}^{2n})}.$$

We then pick wavelets with smoothness and cancellation degrees $k = 6n$. This number suffices for the purposes of the following lemma.

**Lemma 5** ([18, Theorem 1.64])**.** *Let $0 < r < \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}$, and for $\lambda \in \mathbb{N}$ and $\vec{\mu} \in \mathbb{N}^{2n}$ let $\chi_{\lambda\vec{\mu}}$ be the characteristic function of the cube $Q_{\lambda\vec{\mu}}$ centered at $2^{-\lambda}\vec{\mu}$ with length $2^{1-\lambda}$. For a sequence $\gamma = \{\gamma^{\lambda,G}_{\vec{\mu}}\}$ define the norm*

$$\|\gamma|f^s_{r,q}\| = \Big\|\Big(\sum_{\lambda,G,\vec{\mu}} 2^{\lambda sq}|\gamma^{\lambda,G}_{\vec{\mu}}\chi_{\lambda\vec{\mu}}(\cdot)|^q\Big)^{1/q}\Big\|_{L^r(\mathbb{R}^{2n})}.$$

*Let* $\mathbb{N} \ni k > \max\{s, \frac{4n}{\min(r,q)} + n - s\}$. *Let* $\Psi_{\vec{\mu}}^{\lambda,G}$ *be the* $2n$*-dimensional Daubechies wavelets with smoothness* $k$ *according Lemma 4. Let* $f \in \mathcal{S}'(\mathbb{R}^{2n})$. *Then* $f \in F_{r,q}^{s}(\mathbb{R}^{2n})$ *if and only if it can be represented as*

$$f = \sum_{\lambda,G,\vec{\mu}} \gamma_{\vec{\mu}}^{\lambda,G} 2^{-\lambda n} \Psi_{\vec{\mu}}^{\lambda,G}$$

*with* $\|\gamma|f_{rq}^{s}\| < \infty$ *with unconditional convergence in* $\mathcal{S}'(\mathbb{R}^n)$. *Furthermore this representation is unique,*

$$\gamma_{\vec{\mu}}^{\lambda,G} = 2^{\lambda n}\langle f, \Psi_{\vec{\mu}}^{\lambda,G}\rangle,$$

*and*

$$I : f \to \left\{2^{\lambda n}\langle f, \Psi_{\vec{\mu}}^{\lambda,G}\rangle\right\}$$

*is an isomorphic map of* $F_{r,q}^{s}(\mathbb{R}^{2n})$ *onto* $f_{r,q}^{s}$.

In particular, the Sobolev space $L_s^r(\mathbb{R}^{2n})$ coincides with $F_{r,2}^{s}(\mathbb{R}^{2n})$. In the proof of our results, we use for fixed $\lambda$ the following estimate:

$$\left\| \Big( \sum_{G,\vec{\mu}} |\langle \sigma, \Psi_{\vec{\mu}}^{\lambda,G}\rangle \Psi_{\vec{\mu}}^{\lambda,G}|^2 \Big)^{1/2} \right\|_{L^r} \le C\|\sigma\|_{L_s^r} 2^{-s\lambda}. \tag{4}$$

To verify this, by Lemma 5, we have

$$\left\| \sum_{G,\vec{\mu}} 2^{\lambda s} |\gamma_{\vec{\mu}}^{\lambda,G} \chi_{Q_{\lambda,\vec{\mu}}}| \right\|_{L^r} \le C\|\sigma\|_{L_s^r},$$

with $\gamma_{\vec{\mu}}^{\lambda,G} = 2^{\lambda n}\langle \sigma, \Psi_{\vec{\mu}}^{\lambda,G}\rangle$. Notice that $2^{-\lambda n}\Psi_{\vec{\mu}}^{\lambda,G}$ are $L^\infty$ normalized wavelets, and there exists an absolute constant $B$ such that the support of $\Psi_{\vec{\mu}}^{\lambda,G}$ is always contained in $\cup_{|\vec{v}|\le B} Q_{\lambda,\vec{\mu}+\vec{v}}$. This then implies (4).

## 3. The main lemma

Let $Q$ denote the cube $[-2,2]^{2n}$ in $\mathbb{R}^{2n}$, and consider a Sobolev space $L_s^r(Q)$ as the Sobolev space of distributions supported in $Q$ which are in $L_s^r(\mathbb{R}^{2n})$.

**Lemma 6.** *For* $r \in (1,\infty)$ *let* $s > \max(n/2, 2n/r)$ *and suppose* $\sigma \in L_s^r(Q)$. *Then* $\sigma$ *is a bilinear multiplier bounded from* $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$.

*Proof.* The important inequality is the one for a single generation of wavelets (with $\lambda$ fixed). For a fixed $\lambda$, by the uniform compact supports of the elements in the basis, we can classify the wavelets into finitely many subclasses such that the supports of the elements in each subclass are pairwise disjoint. We denote by $D_{\lambda,\kappa}$ such a subclass and the related symbol

$$\sigma_{\lambda,\kappa} = \sum_{\omega \in D_{\lambda,\kappa}} a_\omega \omega,$$

where $a_\omega = \langle \sigma, \omega \rangle$. The $\omega$'s are $L^2$ normalized, but we change the normalization to $L^r$, i.e. we consider $\tilde{\omega} = \omega / \|\omega\|_{L^r}$ and $b_\omega = a_\omega \|\omega\|_{L^r}$. We have

$$\sigma_{\lambda,\kappa} = \sum_{\omega \in D_{\lambda,\kappa}} b_\omega \tilde{\omega}$$

and from the Sobolev smoothness and the fact that the supports of the wavelets do not overlap, with the aid of (4) we obtain

$$\begin{aligned} B = \Big( \sum_{\omega \in D_{\lambda,\kappa}} |b_\omega|^r \Big)^{1/r} &= \Big( \sum_{\omega} \int \Big( |a_\omega \omega|^2 \Big)^{r/2} dx \Big)^{1/r} \\ &\le \Big\| \Big( \sum_{\omega} |a_\omega \omega|^2 \Big)^{1/2} \Big\|_{L^r} \\ &\le C \|\sigma\|_{L^r_s} 2^{-s\lambda}. \end{aligned}$$

Now, each $\omega$ in $D_{\lambda,\kappa}$ is of the form $\omega = \omega_k \omega_l$ with $\vec{\mu} = (k,l)$, where $k$ and $l$ both range over index sets $U_1$ and $U_2$ of cardinality at most $C 2^{\lambda n}$. Moreover we denote by $b_{kl}$ the coefficient $b_\omega$, and we have

$$\sigma_{\lambda,\kappa} = \sum_{k \in U_1} \tilde{\omega}_k \sum_{l \in U_2} b_{kl} \tilde{\omega}_l.$$

Set $\tau_{\max}$ to be the positive number such that $2n\lambda/r \le \tau_{\max} < 1 + 2n\lambda/r$. For a non-negative number $\tau < 2n\lambda/r = \tau_{\max}$ and a positive constant (depending on $\tau$) $K = 2^{\tau r/2}$ we introduce the following decomposition: We define the level set according to $b$ as

$$D^\tau_{\lambda,\kappa} = \{ \omega \in D_{\lambda,\kappa} : B 2^{-\tau} < |b_\omega| \le B 2^{-\tau+1} \},$$

when $\tau < \tau_{\max}$. We also define the set

$$D^{\tau_{\max}}_{\lambda,\kappa} = \{ \omega \in D_{\lambda,\kappa} : |b_\omega| \le B 2^{-\tau_{\max}+1} \}.$$

We now take the part with heavy columns

$$D^{\tau,1}_{\lambda,\kappa} = \{ \omega_k \omega_l \in D^\tau_{\lambda,\kappa} : \operatorname{card}\{ s : \omega_k \omega_s \in D^\tau_{\lambda,\kappa} \} \ge K \},$$

and the remainder

$$D^{\tau,2}_{\lambda,\kappa} = D^\tau_{\lambda,\kappa} \setminus D^{\tau,1}_{\lambda,\kappa}.$$

We also use the following notations for the index sets: $U_1^{\tau,1}$ is the set of $k$'s such that $\omega_k \omega_l$ in $D^{\tau,1}_{\lambda,\kappa}$, and for each $k \in U_1^{\tau,1}$ we denote $U^{\tau,1}_{2,k}$ the set of corresponding second indices $l$'s such that $\omega_k \omega_l \in D^{\tau,1}_{\lambda,\kappa}$, whose cardinality is at least $K$. We also denote

$$\sigma^{\tau,1}_{\lambda,\kappa} = \sum_{k \in U_1^{\tau,1}} \tilde{\omega}_k \sum_{l \in U^{\tau,1}_{2,k}} b_{kl} \tilde{\omega}_l,$$

thus summing over the wavelets in the set $D^{\tau,1}_{\lambda,\kappa}$. The symbol $\sigma^{\tau,2}_{\lambda,\kappa}$ is then defined by summation over $D^{\tau,2}_{\lambda,\kappa}$.

We first treat the part $\sigma^{\tau,1}_{\lambda,\kappa}$. Denote $\gamma = \text{card}\, U_1^{\tau,1}$. For $\tau < \tau_{\max}$ the $\ell^r$-norm of the part of the sequence $\{b_{kl}\}$ indexed by the set $D^{\tau,1}_{\lambda,\kappa}$ is comparable to

$$C\Bigg(\sum_{k\in U_1^{\tau,1}}\sum_{l\in U_{2,k}^{\tau,1}}(B2^{-\tau})^r\Bigg)^{1/r}$$

which is at least as big as $C(\gamma K(B2^{-\tau})^r)^{1/r}$. However this $\ell^r$-norm is smaller than $B$, therefore we get $\gamma \le C2^{\tau r}/K = C2^{\tau r/2}$. For $\tau = \tau_{\max}$ we trivially have that $\gamma \le C2^{n\lambda} = C2^{\tau_{\max} r/2}$.

For $f,g \in \mathcal{S}$ we estimate the multiplier norm of $\sigma^{\tau,1}_{\lambda,\kappa}$ as follows:

$$\begin{aligned}\|\mathcal{F}^{-1}(\sigma^{\tau,1}_{\lambda,\kappa}\widehat{f}\widehat{g})\|_{L^1} &\le \sum_{k\in U_1^{\tau,1}}\|\widehat{f}\tilde{\omega}_k\|_{L^2}\|\sum_{l\in U_{2,k}^{\tau,1}} b_{kl}\tilde{\omega}_l\widehat{g}\|_{L^2}\\ &\le C\sum_{k\in U_1^{\tau,1}}\|\widehat{f}\tilde{\omega}_k\|_{L^2}\sup_l|b_{kl}|2^{\lambda n/r}\|g\|_{L^2}\\ &\le C2^{\lambda n/r}\|g\|_{L^2}\Bigg(\sum_k\sup_l|b_{kl}|^2\Bigg)^{1/2}\Bigg(\sum_k\|\widehat{f}\tilde{\omega}_k\|_{L^2}^2\Bigg)^{1/2}.\end{aligned}$$

In view of orthogonality and of the fact that $\|\tilde{\omega}_k\|_{L^\infty} \approx 2^{\lambda n/r}$ we obtain the inequality

$$\Bigg(\sum_k\|\widehat{f}\tilde{\omega}_k\|_{L^2}^2\Bigg)^{1/2} \le C2^{\frac{\lambda n}{r}}\|f\|_{L^2}.$$

By the definition of $U_1^{\tau,1}$ we have also that

$$\Bigg(\sum_k\sup_l|b_{kl}|^2\Bigg)^{1/2} \le B2^{-\tau}\gamma^{\frac{1}{2}}.$$

Collecting these estimates, we deduce

$$\|\mathcal{F}^{-1}(\sigma^{\tau,1}_{\lambda,\kappa}\widehat{f}\,\widehat{g})\|_{L^1} \le C\|\sigma\|_{L^r_s}\gamma^{\frac{1}{2}}2^{\lambda(\frac{2n}{r}-s)}2^{-\tau}\|f\|_{L^2}\|g\|_{L^2}. \tag{5}$$

The set $D^{\tau,2}_{\lambda,\kappa}$ has the property that in each column there are at most $K$ elements. Let us denote by $V^2$ the index set of all second indices such that $\tilde{\omega}_k\tilde{\omega}_l \in D^{\tau,2}_{\lambda,\kappa}$, and for each $l \in V^2$ set $V^{1,l}$ the corresponding sets of first indices. Thus

$$D^{\tau,2}_{\lambda,\kappa} = \{\omega_k\omega_l : l \in V^2, k \in V^{1,l}\}.$$

We then have

$$\|\mathcal{F}^{-1}(\sigma_{\lambda,\kappa}^{\tau,2}\widehat{f}\,\widehat{g})\|_{L^1} \le \sum_{l\in V^2}\Big\|\sum_{k\in V^{1,l}} b_{kl}\tilde{\omega}_k\widehat{f}\Big\|_{L^2}\|\tilde{\omega}_l\widehat{g}\|_{L^2}$$

$$\le \Big(\sum_{l\in V^2}\Big\|\sum_{k\in V^{1,l}} b_{kl}\tilde{\omega}_k\widehat{f}\Big\|_{L^2}^2\Big)^{1/2}\Big(\sum_{l\in V^2}\|\tilde{\omega}_l\widehat{g}\|_{L^2}^2\Big)^{1/2}.$$

We need to estimate

$$\sum_{l\in V^2}\Big\|\sum_{k\in V^{1,l}} b_{kl}\tilde{\omega}_k\widehat{f}\Big\|_{L^2}^2 \le C\int_Q\sum_{l\in V^2}\sum_{k\in V^{1,l}} B^2 2^{-2\tau}|\tilde{\omega}_k|^2|\widehat{f}(\xi_1)|^2 d\xi_1$$

$$\le CK2^{\frac{2n\lambda}{r}}B^2 2^{-2\tau}\|f\|_{L^2}^2,$$

since, by the disjointness of the supports of $\tilde{\omega}_k$, $\sum_k|\tilde{\omega}_k|^2 \le C2^{2n\lambda/r}$, and the cardinality of $V^2$ is controlled by $K$.

Returning to our estimate, and using orthogonality, we obtain

$$\|\mathcal{F}^{-1}(\sigma_{\lambda,\kappa}^{\tau,2}\widehat{f}\,\widehat{g})\|_{L^1} \le C\|\sigma\|_{L_s^r}K^{\frac{1}{2}}2^{-s\lambda}2^{-\tau}2^{\frac{2\lambda n}{r}}\|f\|_{L^2}\|g\|_{L^2}. \tag{6}$$

For any $\tau \le \tau_{\max}$ the two inequalities (5) and (6) are the same due to $\gamma \le C2^{\tau r}/K = C2^{\tau r/2}$. Therefore, we have

$$\|\mathcal{F}^{-1}(\sigma_{\lambda,\kappa}^{\tau}\widehat{f}\,\widehat{g})\|_{L^1} \le C\|\sigma\|_{L_s^r}2^{(\frac{r}{4}-1)\tau}2^{\lambda(\frac{2n}{r}-s)}\|f\|_{L^2}\|g\|_{L^2}. \tag{7}$$

The right hand side has a negative exponent in $\lambda$ since $s > 2n/r$.

The behavior in $\tau$ depends on $r$. For $1 < r < 4$ it is a geometric series in $\tau$ and hence summing over $0 \le \tau \le \tau_{\max}$ and $\lambda \ge 0$ is finite. However, if $r \ge 4$, we need to use the following observation:

$$\sum_{\tau=0}^{\tau_{\max}} 2^{(\frac{r}{4}-1)\tau} \le C\tau_{\max}2^{(\frac{r}{4}-1)\tau_{\max}} \le C\big(\tfrac{2n\lambda}{r}\big)2^{(\frac{r}{4}-1)\frac{2n\lambda}{r}}. \tag{8}$$

Therefore, by summing over $\tau$ in (7) we obtain

$$\sum_{\tau=0}^{\tau_{\max}}\|\mathcal{F}^{-1}(\sigma_{\lambda,\kappa}^{\tau}\widehat{f}\,\widehat{g})\|_{L^1} \le C\|\sigma\|_{L_s^r}\big(\tfrac{2n\lambda}{r}\big)2^{(\frac{r}{4}-1)(1+\frac{2n\lambda}{r})}2^{\lambda(\frac{2n}{r}-s)}\|f\|_{L^2}\|g\|_{L^2}.$$

Since $(2n\lambda/r)2^{(r/4-1)2n\lambda/r}2^{\lambda(2n/r-s)} = (2n\lambda/r)2^{\lambda(n/2-s)}$, these estimates form a summable series in $\lambda$ only if $s > n/2$.

We have $1 \le \kappa \le C_n$ and $\sigma = \sum_{\lambda=0}^{\infty}\sum_\kappa \sigma_{\lambda,\kappa}$. Therefore for $s$ and $r$ related as in $s > \max(2n/r, n/2)$ we have convergent series, and we obtain the result by summation in $\tau$ first and then in $\lambda$. □

*Remark* 1. We see from the proof (or by an easy dilation argument) that the condition $Q$ is $[-2,2]^n$ is not essential and the statement keeps valid when $Q$ is any fixed compact set.

*Remark* 2. In the case $r<4$, the summation in (8) is finite even if $\tau_{\max}=\infty$, which means that the restriction that $\sigma$ is compactly supported is unnecessary when $r\in(1,4)$.

## 4. The proof of Theorem 1

*Proof.* We use an idea developed in [5], where we consider off-diagonal and diagonal cases separately. For the former we use the Hardy-Littlewood maximal function and a "square" function, and for the latter we use use Lemma 6 in Section 3.

We introduce notations needed to study these cases appropriately. We define $\sigma_j(\xi,\eta)=\sigma(\xi,\eta)\widehat{\psi}(2^{-j}(\xi,\eta))$ and write $m_j(\xi,\eta)=\sigma_j(2^j(\xi,\eta))$. We note that all $m_j$ are supported in the unit annulus, the dyadic annulus centered at zero with radius comparable to 1, and $\|m_j\|_{L^r_s}\le A$ uniformly in $j$ by assumption (1).

By the discussion in the previous section, for each $m_j$ we have the decomposition $m_j(\xi,\eta)=\sum_\kappa\sum_\lambda\sum_{k,l}b_{k,l}\tilde{\omega}_k(\xi)\tilde{\omega}_l(\eta)=\sum_\lambda m_{j,\lambda}$ with $\tilde{\omega}_k\approx 2^{\lambda n/r}$ and $(\sum_{k,l}|b_{k,l}|^r)^{1/r}\le CA2^{-\lambda s}$. Assume that both $\Psi_F$ and $\Psi_M$ are supported in $B(0,N)$ for some large fixed number $N$. We define the off-diagonal parts

$$m^2_{j,\lambda}(\xi,\eta)=\sum_\kappa\sum_{|l|\le 2\sqrt{n}N}\sum_k b_{k,l}\tilde{\omega}_k(\xi)\tilde{\omega}_l(\eta) \tag{9}$$

and

$$m^3_{j,\lambda}(\xi,\eta)=\sum_\kappa\sum_{|k|\le 2\sqrt{n}N}\sum_l b_{k,l}\tilde{\omega}_k(\xi)\tilde{\omega}_l(\eta),$$

then the remainder in the $\lambda$ level is $m^1_{j,\lambda}(\xi,\eta)=[m_{j,\lambda}-m^2_{j,\lambda}-m^3_{j,\lambda}](\xi,\eta)$ with each wavelet involved away from the axes. Notice that since $|\eta|$ is small, we have that $\frac{1}{2}\le|\xi|\le 2$ for large $\lambda$. Moreover for $i=1,2,3$, we define $m^i_j=\sum_\lambda m^i_{j,\lambda}$, $\sigma^i_j=m^i_j(2^{-j}\cdot)$, $\sigma^i=\sum_j\sigma^i_j$. Notice that $\sigma$ is equal to the sum $\sigma^1+\sigma^2+\sigma^3$.

*(i) The Off-diagonal Cases*

We consider the off-diagonal cases $m^2_{j,\lambda}$ and $m^3_{j,\lambda}$ first. By symmetry, it suffices to consider

$$T_{m^2_{j,\lambda}}(f,g)(x)=\int_{\mathbb{R}^{2n}}m^2_{j,\lambda}(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix(\xi+\eta)}d\xi d\eta.$$

By the definition $\tilde{\omega}_l=2^{\lambda n/2}\Psi(2^\lambda x-l)/\|\omega_l\|_{L^r}$, we have $|(\tilde{\omega}_l\widehat{g})^\vee(x)|\le C2^{\lambda n/r}M(g)(x)$, where $M(g)(x)$ is the Hardy-Littlewood maximal function. Recall the boundedness of $b_{k,l}$ and $\tilde{\omega}_k$, we therefore have

$$|(\sum_k b_{k,l}\tilde{\omega}_k\widehat{f})^\vee|\le 2^{\lambda(n/r-s)}|(m\widehat{f}\chi_{1/2\le|\xi|\le 2})^\vee|$$

with $\|m\|_{L^\infty}\le C$, where the summation over $k$ runs through allowed $k$'s in (9). In view of the finiteness of $N$ and the number of $\kappa$'s, we finally obtain a pointwise control

$$|T_{m^2_{j,\lambda}}(f,g)(x)|\le C2^{(2n/r-s)\lambda}|T_m(f')(x)|M(g)(x),$$

where $\widehat{f'} = \widehat{f}\chi_{1/2\le|\xi|\le 2}$.

Observe that

$$T_{\sigma_j^2}(f,g)(x) = 2^{jn}T_{m_j^2}(f_j,g_j)(2^jx)$$

with $\widehat{f_j}(\xi) = 2^{jn/2}\widehat{f}(2^j\xi)\chi_{1/2\le|\xi|\le 2}$ and $\widehat{g}_j(\xi) = 2^{jn/2}\widehat{g}(2^j\xi)$. Note that we did not define $f_j$ and $g_j$ in similar ways. By a standard argument using the square function characterization of the Hardy space $H^1$, we control $\|T_{\sigma^2}(f,g)\|_{L^1}$ by

$$\begin{aligned}\Big\|\Big(\sum_j |T_{\sigma_j^2}(f,g)|^2\Big)^{1/2}\Big\|_{L^1} &= \Big\|\Big(\sum_j |2^{jn}T_{m_j^2}(f_j,g_j)(2^j\cdot)|^2\Big)^{1/2}\Big\|_{L^1}\\ &\le \sum_\lambda 2^{(2n/r-s)\lambda}\|g\|_{L^2}\Big(\int\sum_j|\widehat{f_j}(\xi)|^2d\xi\Big)^{1/2}.\end{aligned}$$

Because of the definition of $\widehat{f_j}$, we see that

$$\int\sum_j|\widehat{f_j}(\xi)|^2d\xi = \int\sum_j|\widehat{f}(\xi)|^2\chi_{2^{j-1}\le|\xi|\le 2^{j+1}}d\xi \le C\|f\|_{L^2}^2.$$

The exponential decay in $\lambda$ given by the condition $rs>2n$ then concludes the proof of the off-diagonal cases.

*(ii) The Diagonal Case*

This case is relatively simple by an argument similar to the diagonal part in [5], because we have dealt with the key ingredient in Lemma 6. We give a brief proof here for completeness. By dilation we have that

$$\|T_{\sigma^1}(f,g)(x)\|_{L^1} \le \|\sum_j\sum_\lambda T_{\sigma^1_{j,\lambda}}(f,g)\|_{L^1} \le \sum_\lambda\sum_j \|2^{jn}T_{m^1_{j,\lambda}}(f_j,g_j)(2^j\cdot)\|_{L^1},$$

where $\widehat{f_j}(\xi) = 2^{jn/2}\widehat{f}(2^{jn}\xi)\chi_{C2^{-\lambda}\le|\xi|\le 2}(\xi)$ because in the support of $m^1_{j,\lambda}$ we have $C2^{-\lambda}\le|\xi|\le 2$, and $g_j$ is defined similarly. For the last line we apply Lemma 6 and obtain, when $r\ge 4$, the estimate

$$\sum_\lambda C^{\frac{2n\lambda}{r}}2^{\lambda(n/2-s)}\sum_j\|\widehat{f_j}\|_{L^2}\|\widehat{g}_j\|_{L^2} \le \sum_\lambda C^{\frac{2n\lambda}{r}}2^{\lambda(n/2-s)}\Big(\sum_j\|\widehat{f_j}\|_{L^2}^2\Big)^{1/2}\Big(\sum_j\|\widehat{g}_j\|_{L^2}^2\Big)^{1/2}.$$

And when $r<4$, we have a similar control

$$\sum_\lambda C2^{\lambda(2n/r-s)}\sum_j\|\widehat{f_j}\|_{L^2}\|\widehat{g}_j\|_{L^2} \le \sum_\lambda C2^{\lambda(2n/r-s)}\Big(\sum_j\|\widehat{f_j}\|_{L^2}^2\Big)^{1/2}\Big(\sum_j\|\widehat{g}_j\|_{L^2}^2\Big)^{1/2}.$$

Observe that

$$\sum_j\|\widehat{f_j}\|_{L^2}^2 = \int|\widehat{f}(\xi)|^2\sum_j\chi_{2^{-\lambda-j}\le|\xi|\le 2^{1-j}}(\xi)d\xi \le C\lambda\|f\|_{L^2}^2,$$

so in either case with the restriction $s>\max\{n/2, 2n/r\}$ the sum over $\lambda$ is controlled by $\|f\|_{L^2}\|g\|_{L^2}$. Thus we conclude the proof of the diagonal case and of Theorem 1. $\square$

## 5. NECESSARY CONDITIONS

For a bounded function $\sigma$, let $T_\sigma$ be the $m$-linear multiplier operator with symbol $\sigma$. In this section we obtain examples for $m$-linear multiplier operators that impose restrictions on the indices and the smoothness in order to have

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le C\sup_{j\in\mathbb{Z}}\|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{mn})}. \tag{10}$$

These conditions show in particular that the restriction on $s$ in Theorem 1 is necessary.

We first prove Theorem 2 via two counterexamples; these are contained in Proposition 7 and Proposition 9, respectively.

**Proposition 7.** *Under the hypothesis of Theorem 2 we must have $s \ge (m-1)n/2$.*

*Proof.* We use the bilinear case with dimension one to demonstrate the idea first. Then we easily extend the argument to higher dimensions.

We fix a Schwartz function $\varphi$ with $\hat{\varphi}$ supported in $[-1/100, 1/100]$. Let $\{a_j(t)\}_j$ be a sequence of Rademacher functions indexed by positive integers, and for $N > 1$ define

$$\widehat{f_N}(\xi_1) = \sum_{j=1}^N a_j(t_1)\hat{\varphi}(N\xi_1 - j), \qquad \widehat{g_N}(\xi_2) = \sum_{k=1}^N a_k(t_2)\hat{\varphi}(N\xi_2 - k).$$

Let $\phi$ be a smooth function $\phi$ supported in $[-\frac{1}{10}, \frac{1}{10}]$ assuming value 1 in $[-\frac{1}{20}, \frac{1}{20}]$. We construct the multiplier $\sigma_N$ of the bilinear operator $T_N$ as follows,

$$\sigma_N = \sum_{j=1}^N\sum_{k=1}^N a_j(t_1)a_k(t_2)a_{j+k}(t_3)c_{j+k}\phi(N\xi_1 - j)\phi(N\xi_2 - k), \tag{11}$$

where $c_l = 1$ when $9N/10 \le l \le 11N/10$ and 0 elsewhere. Hence

$$\begin{aligned} T_N(f_N, g_N)(x) &= \sum_{j=1}^N\sum_{k=1}^N a_{j+k}(t_3)c_{j+k}\tfrac{1}{N^2}\varphi(x/N)\varphi(x/N)e^{2\pi i x(j+k)/N} \\ &= \sum_{l=2}^{2N}\sum_{k=s_l}^{S_l} a_l(t_3)c_l\tfrac{1}{N^2}\varphi(x/N)\varphi(x/N)e^{2\pi i x l/N}, \end{aligned}$$

where $s_l = \max(1, l-N)$ and $S_l = \min(N, l-1)$. We estimate $\|f_N\|_{L^{p_1}(\mathbb{R})}$, $\|g_N\|_{L^{p_2}(\mathbb{R})}$, $\|\sigma_N\|_{L^r_s(\mathbb{R}^2)}$ and $\|T_N(f_N, g_N)\|_{L^p(\mathbb{R})}$.

First we prove that $\|f_N\|_{L^{p_1}(\mathbb{R})} \approx N^{1-\frac{p_1}{2}}$. By Khinchine's inequality we have

$$\begin{aligned} \int_0^1 \|f_N\|_{L^{p_1}}^{p_1} dt_1 &= \int_{\mathbb{R}}\int_0^1 \Big|\sum_{j=1}^N a_j(t_1)\frac{\varphi(x/N)}{N}e^{2\pi i x j/N}\Big|^{p_1} dt_1 dx \\ &\approx \int_{\mathbb{R}}\Big(\sum_{j=1}^N\Big|\frac{\varphi(x/N)}{N}\Big|^2\Big)^{p_1/2} dx \end{aligned}$$

$$
\begin{aligned}
&\approx N^{-p_1/2}\int_{\mathbb{R}}\big|\varphi(x/N)\big|^{p_1}dx\\
&\approx N^{1-\frac{p_1}{2}}.
\end{aligned}
$$

Hence $\|f_N\|_{L^{p_1}(\mathbb{R}\times[0,1],\ dxdt)} \approx N^{\frac{1}{p_1}-\frac{1}{2}}$. Similarly $\|g_N\|_{L^{p_1}(\mathbb{R}\times[0,1],\ dxdt)} \approx N^{\frac{1}{p_2}-\frac{1}{2}}$. The same idea gives that

$$
\begin{aligned}
\int_0^1 \|T_N(f_N,g_N)\|_{L^p}^p dt_3 &\approx \int_{\mathbb{R}}\Big(\sum_{l=2}^{2N}\big|c_l(S_l-s_l)\tfrac{1}{N^2}\varphi^2(x/N)e^{2\pi ixl/N}\big|^2\Big)^{p/2}dx\\
&\approx \int_{\mathbb{R}}\Big(\sum_{l=9N/10}^{11N/10}(S_l-s_l)^2\Big)^{p/2}\tfrac{1}{N^{2p}}|\varphi(x/N)|^{2p}dx\\
&\approx N^{\frac{3p}{2}-2p}\int_{\mathbb{R}}|\varphi(x/N)|^{2p}dx\\
&\approx N^{1-\frac{p}{2}}.
\end{aligned}
$$

In other words we showed that $\|T_N(f_N,g_N)\|_{L^p(\mathbb{R}\times[0,1],\ dxdt)} \approx N^{\frac{1}{p}-\frac{1}{2}}$.

As for $\sigma_N$, we have the following result whose proof can be found in [6, Lemma 4.2].

**Lemma 8.** *For the multiplier $\sigma_N$ defined in* (11) *and any $s\in(0,1)$, there exists a constant $C_s$ such that*

$$
\|\sigma_N\|_{L^r_s(\mathbb{R}^2)} \le C_s N^s. \tag{12}
$$

Apply (3) to $f_N$, $g_N$ and $T_N$ defined above and integrate with respect to $t_1$, $t_2$ and $t_3$ on both sides, we have

$$
\Big(\int_0^1\int_0^1\int_0^1 \|T_N(f_N,g_N)\|_{L^p}^p dt_3dt_1dt_2\Big)^{1/p} \le C_sN^s\Big(\int_0^1\|f_N\|_{L^{p_1}}^p dt_1\int_0^1\|g_N\|_{L^{p_2}}^p dt_2\Big)^{1/p},
$$

which combining the estimates obtained on $f_N$, $g_N$ and $T_N(f_N,g_N)$ above implies

$$
N^{\frac{1}{p}-\frac{1}{2}} \le C_sN^sN^{\frac{1}{p_1}-\frac{1}{2}}N^{\frac{1}{p_2}-\frac{1}{2}},
$$

so we automatically have $N^{1/2}\le C_sN^s$, which is true when $N$ goes to $\infty$ only if $s\ge 1/2$.

We now discuss the case $m\ge 2$ and $n=1$. We use for $1\le k\le m$

$$
\widehat{f_k}(\xi_k) = \sum_{j=1}^N a_j(t_k)\widehat{\varphi}(N\xi_k - j),
$$

and

$$
\sigma_N = \sum_{j_1=1}^N\cdots\sum_{j_m=1}^N a_{j_1}(t_1)\cdots a_{j_m}(t_m)a_{j_1+\cdots+j_m}(t_{m+1})c_{j_1+\cdots+j_m}\prod_{k=1}^m \phi(N\xi_k - j_k).
$$

By an argument similar to the case $m=2$ and $n=1$, we have

$$\|f_k\|_{L^{p_k}(\mathbb{R}\times[0,1],\ dxdt)} \approx N^{\frac{1}{p_k}-\frac{1}{2}},$$

$\|\sigma_N\|_{L^r_s} \le CN^s$ and

$$\|T(f_1,\dots,f_m)\|_{L^p(\mathbb{R})} \approx N^{\frac{1}{p}-\frac{1}{2}}, \tag{13}$$

hence we obtain that $s \ge (m-1)/2$.

For the higher dimensional cases, we define

$$F_k(x_1,\dots,x_n) = \prod_{\tau=1}^{n} f_k(x_\tau),$$

and $\sigma(\xi_1,\dots,\xi_n) = \prod_{\tau=1}^n \sigma_N(\xi_\tau)$, then $\|F_k\|_{L^{p_k}} \approx N^{n(\frac{1}{p_k}-\frac{1}{2})}$, $\|\sigma\|_{L^r_s} \le CN^s$, and

$$\|T(F_1,\dots,F_m)\| \approx N^{n(\frac{1}{p}-\frac{1}{2})}.$$

We therefore obtain the restriction $s \ge (m-1)n/2$. $\square$

**Proposition 9.** *Under the hypothesis of Theorem 2 we must have $s \ge mn/r$.*

*Proof.* Let $\varphi$ and $\phi$ be as in Proposition 7. Define $\widehat{f_j}(\xi_j) = \widehat{\varphi}(N(\xi_j - a))$ with $|a| = 1$, and $\sigma(\xi,\dots,\xi_m) = \prod_{j=1}^m \phi(N(\xi_j - a))$, then a direct calculation gives $\|f_j\|_{L^{p_j}(\mathbb{R}^n)} \approx N^{-n+n/p_j}$ and $\|\sigma\|_{L^r_s(\mathbb{R}^{mn})} \le CN^sN^{-mn/r}$. Moreover,

$$T_\sigma(f_1,\dots,f_m)(x) = N^{-mn}(\varphi(x/N)e^{2\pi i x\cdot a})^m.$$

We can therefore obtain that $\|T_\sigma(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n)} \approx N^{-mn+n/p}CN^sN^{-mn/r}$. Then we come to the inequality $N^{-mn+n/p} \le CN^sN^{-mn/r}\prod_j N^{-n+n/p_j}$, which forces $s - mn/r \ge 0$ by letting $N$ go to infinity. $\square$

Next, we obtain from (10) the restrictions for the indices $p_j$ claimed in Theorem 3.

*Proof of Theorem 3.* By symmetry it suffices to consider the case $I = \{1,2,\ \dots,\ k\}$ with $k \in \{0,1,\dots,m\}$ and the explanation $I = \emptyset$ when $k = 0$. Define for $\xi \in \mathbb{R}$

$$\widehat{f_N}(\xi) = \sum_{j=-N}^{N} \widehat{\varphi}(N\xi - j)a_j(t), \qquad \widehat{g_N}(\xi) = \sum_{j=-N}^{N} \widehat{\varphi}(N\xi - j),$$

and

$$\begin{aligned}&\sigma_N(\xi_1,\dots,\xi_m)\\ &= \sum_{j_1=-N}^{N} \cdots \sum_{j_m=-N}^{N} a_{j_1+\cdots+j_m}(t)c_{j_1+\cdots+j_m}a_{j_1}(t_1)\cdots a_{j_k}(t_k)\phi(N\xi_1 - j_1)\cdots\phi(N\xi_m - j_m).\end{aligned}$$

The idea is that in this setting if we take the first $k$ functions as $f_N$ and the remaining as $g_N$, we have

$$
\begin{aligned}
&T_{\sigma_N}(\overbrace{f_N,\dots,f_N}^{k\text{ terms}},\overbrace{g_N,\dots,g_N}^{m-k\text{ terms}})(x)\\
&=\sum_{j_1=-N}^{N}\cdots\sum_{j_m=-N}^{N} a_{j_1+\cdots+j_m}(t)c_{j_1+\cdots+j_m}N^{-m}[\varphi(x/N)]^m e^{2\pi i x(j_1+\cdots+j_m)/N}.
\end{aligned}
$$

This expression is independent of $k$ and by (13) we know

$$\|T_{\sigma_N}(f_N,\dots,f_N,g_N,\dots,g_N)\|_{L^p} \approx N^{1/p-1/2}.$$

Previous calculations show also $\|f_N\|_{L^{p_i}} \approx C_{p_i}N^{1/p_i-1/2}$ and $\|\sigma_N\|_{L^r_s} \le CN^s$. Lemma 4.3 in [6] gives that $\|g_N\|_{L^{p_i}} \le C_{p_i}$ for $p_i \in (1,\infty]$. Consequently, we have

$$N^{\frac{1}{p}-\frac{1}{2}} \le CN^{\sum_{i=1}^{k}(\frac{1}{p_i}-\frac{1}{2})}N^s$$

and this verifies our conclusion when $n=1$.

For the higher dimensional case, we just use the tensor products and $\sigma$ similar to what we have in Proposition 7, and thus conclude the proof. □

Notice that when $k=m$, Theorem 3 coincides with Proposition 7.

## REFERENCES


[1] A. P. Calderón, A. Torchinsky, *Parabolic maximal functions associated with a distribution,* II. Adv. in Math. **24** (1977), no. 1, 101–171.

[2] R. Coifman, Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 3, xi, 177–202.

[3] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.

[4] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[5] L. Grafakos, D. He, P. Honzík, *Rough bilinear singular integrals*, submitted.

[6] L. Grafakos, D. He, P. Honzík, H. V. Nguyen, *The Hörmander multiplier theorem, I: the linear case*, submitted.

[7] L. Grafakos, D. He, H. V. Nguyen, L. Yan, *Multilinear Multiplier Theorems and Applications*, submitted.

[8] L. Grafakos, N. Kalton, *The Marcinkiewicz multiplier condition for bilinear operators*, Studia Math. **146** (2001), no. 2, 115–156.

[9] L. Grafakos, A. Miyachi, N. Tomita, *On multilinear Fourier multipliers of limited smoothness*, Can. J. Math. **65** (2013), no. 2, 299–330.

[10] L. Grafakos, Z. Si, *The Hörmander multiplier theorem for multilinear operators*, J. Reine Angew. Math. **668** (2012), 133–147.

[11] L. Grafakos, R. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164.

[12] C. Kenig, E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), no. 1, 1–15.

[13] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Advanced Mathematics, **37**, Cambridge University Press, Cambridge, 1992.

[14] A. Miyachi, N. Tomita, *Minimal smoothness conditions for bilinear Fourier multipliers*, Rev. Mat. Iberoam. **29** (2013), no. 2, 495–530.

[15] C. Muscalu, J. Pipher, T. Tao, C. Thiele, *Bi-parameter paraproducts*, Acta Math. **193** (2004), no. 2, 269–296.

[16] A. Seeger, *Estimates near $L^1$ for Fourier multipliers and maximal functions*, Arch. Math. (Basel) **53** (1989), no. 2, 188–193.

[17] N. Tomita, *A Hörmander type multiplier theorem for multilinear operators*, J. Funct. Anal. **259** (2010), no. 8, 2028–2044.

[18] H. Triebel, *Theory of function spaces. III, vol. 100 of Monographs in Mathematics*, 2006.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA MO 65211, USA
*E-mail address*: grafakosl@missouri.edu

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN (ZHONGSHAN) UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA
*E-mail address*: hedanqing@mail.sysu.edu.cn

FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY IN PRAGUE, KE KARLOVU 3, 121 16 PRAHA 2, CZECH REPUBLIC
*E-mail address*: honzik@gmail.com