ESTIMATES FOR MAXIMAL SINGULAR INTEGRALS

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Abstract. It is shown that maximal truncations of nonconvolution L^2-bounded singular integral operators with kernels satisfying Hörmander's condition are weak type $(1,1)$ and L^p bounded for $1 < p < \infty$. Under stronger smoothness conditions, such estimates can be obtained using a generalization of Cotlar's inequality. This inequality is not applicable here and the point of this article is to treat the boundedness of such maximal singular integral operators in an alternative way.

1. Introduction

Consider a function $k(x)$ on $\mathbf{R}^n \setminus \{0\}$ which satisfies

$$\begin{aligned} &\sup_{R>0} \int_{R\le|x|\le 2R} |k(x)|\,dx < \infty \\ (1)\qquad &\sup_{y\in\mathbf{R}^n\setminus\{0\}} \int_{|x|\ge 2|y|} |k(x-y)-k(x)|\,dx < \infty \\ \text{and}\qquad &\sup_{0<R_1<R_2<\infty} \left| \int_{R_1\le|x|\le R_2} k(x)\,dx \right| < \infty\,. \end{aligned}$$

It is a classical result of Benedek, Calderón, and Panzone [1] that any linear operator T given by convolution with a tempered distribution W in $\mathcal{S}'(\mathbf{R}^n)$ which coincides with k on $\mathbf{R}^n \setminus \{0\}$ extends to a bounded operator on $L^2(\mathbf{R}^n)$. By the standard theory, such an operator T must be of weak type $(1,1)$ and also $L^p(\mathbf{R}^n)$ bounded for $1 < p < \infty$. Moreover, it was shown by Riviere [5] that the maximal operator

$$T^*(f)(x) = \sup_{\varepsilon>0} |T(f\chi_{|x-\cdot|\ge\varepsilon})(x)|$$

is also bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ and is of weak type $(1,1)$.

The purpose of this note is to extend Riviere's theorem to the nonconvolution setting, although the analogous L^2 boundedness is still an open question under the general kernel conditions given below.

Suppose that $K(x,y)$ is a complex valued function on $\mathbf{R}^n \times \mathbf{R}^n \setminus D$, where $D = \{(x,x) : x \in \mathbf{R}^n\}$ is the diagonal of $\mathbf{R}^{2n}$. We assume that K satisfies the size

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condition

$$\sup_{R>0} \int_{R\le|x-y|\le 2R} |K(x,y)|\,dy = A_1 < \infty \tag{2}$$

and the smoothness estimate

$$\sup_{\substack{y,z\in\mathbf{R}^n\\ y\neq z}} \int_{|x-y|\ge 2|y-z|} |K(x,y)-K(x,z)|\,dx \le A_2 < \infty, \tag{3}$$

often referred to as Hörmander's condition. It follows from the equivalence of the $T1$ theorem given in [6] (Chapter VII, Section 3.4, Theorem 4) that the condition

$$\sup_{\substack{x_0\in\mathbf{R}^n\\ \varepsilon,N>0}} \left(\frac{1}{N^n}\int_{|x-x_0|<N} \left|\int_{\varepsilon<|x-y|\le N} K(x,y)\,dy\right|^2 dx\right)^{\frac{1}{2}} = A_3 < \infty. \tag{4}$$

is necessary for the L^2 boundedness of an operator with kernel K that satisfies (2). Therefore condition (4) plays the role of the third condition in (1), but we will not assume that K satisfies condition (4) here.

We denote by $K^*(x,y) = K(y,x)$ the transpose kernel of $K(x,y)$ and we let $(2)^*$, $(3)^*$, and $(4)^*$ be conditions (2), (3), and (4) respectively with K^* in the place of K.

It is still an open question whether the six conditions (2), (3), (4), $(2)^*$, $(3)^*$, and $(4)^*$ imply that a continuous linear operator T from $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$ whose Schwartz kernel coincides with K on $\mathbf{R}^n\times\mathbf{R}^n\setminus D$ admits a bounded extension from $L^2(\mathbf{R}^n)$ into itself. (This is known under less stringent conditions on K, see [3]). Such an operator T is related to the kernel K in the following way: If f is a Schwartz function on $\mathbf{R}^n$ whose support is not all of $\mathbf{R}^n$, then

$$T(f)(x) = \int_{\mathbf{R}^n} K(x,y)f(y)\,dy\,, \qquad \text{whenever } x\in\mathbf{R}^n\setminus \operatorname{supp} f\,. \tag{5}$$

Assuming however that T, as defined in (5), admits an extension which is L^2 bounded, we will obtain the boundedness of the corresponding maximal singular integral operator using only conditions (2) and (3) on K and K^*. By the classical theory such a T admits an extension (also denoted by T) which is of weak type $(1,1)$ and bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$ with norms

$$\|T\|_{L^1\to L^{1,\infty}} \le c_n(A_1+A_2+\|T\|_{L^2\to L^2}) \tag{6}$$

$$\|T\|_{L^p\to L^p} \le c_n \max(p,(p-1)^{-1})(A_1+A_2+\|T\|_{L^2\to L^2})\,, \tag{7}$$

where c_n depends only on the dimension. $L^{1,\infty}$ denotes here the space weak L^1.

It is not easy to define the maximal singular integral operator corresponding to T on L^p under the general conditions (2) and (3) on K. Indeed, the problem is that the integral

$$\int_{|x-y|\ge\varepsilon} K(x,y)f(y)\,dy$$

may not converge absolutely, even for f in $L^\infty(\mathbf{R}^n)$. Moreover, even the doubly truncated integral

$$\int_{\varepsilon \le |x-y| \le N} K(x,y) f(y)\, dy$$

may not converge absolutely for $f \in L^p(\mathbf{R}^n)$ if $p < \infty$. To define things properly, for $0 < \varepsilon < N \le \infty$ we set

$$K^{\varepsilon,N}(x,y) = K(x,y)\chi_{\varepsilon \le |x-y| < N}$$

and we introduce linear operators

$$T^{\varepsilon,N}(f)(x) = \int_{\mathbf{R}^n} K^{\varepsilon,N}(x,y) f(y)\, dy$$

for f in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$. Assuming that K and K^* satisfy conditions (2) and (3) and that the operator T, as defined in (5), is L^2 bounded with norm

$$\|T\|_{L^2 \to L^2} = \sup_{\substack{f \in \mathcal{S}(\mathbf{R}^n) \\ f \neq 0}} \frac{\|T(f)\|_{L^2}}{\|f\|_{L^2}} = B < \infty\,,$$

it follows that from [6] (Proposition 1, Chapter I, 7.1 Appendix) and the subsequent note, that the operators $T^{\varepsilon,N}$ are also L^2 bounded uniformly in ε and N with norm at most a dimensional multiple of the quantity $A_1 + B$, i.e.

$$\sup_{0<N<\infty} \sup_{0<\varepsilon<N} \|T^{\varepsilon,N}\|_{L^2 \to L^2} \le C_n(A_1 + B)\,.$$

(This is shown for the truncated operators $T^{\varepsilon,\infty}$, but note that $T^{\varepsilon,N} = T^{\varepsilon,\infty} - T^{N,\infty}$.) It follows then by standard theory that the truncated operators $T^{\varepsilon,N}$ also admit extensions (also denoted by $T^{\varepsilon,N}$) which are of weak type $(1,1)$ and bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ with norms

$$\|T^{\varepsilon,N}\|_{L^1 \to L^{1,\infty}} \le c_n(A_1 + A_2 + B)$$
$$\|T^{\varepsilon,N}\|_{L^p \to L^p} \le c_n \max(p, (p-1)^{-1})(A_1 + A_2 + B)\,,$$

for some dimensional constant c_n. We therefore have an appropriate definition of $T^{\varepsilon,N}(f)$ when f lies in $\bigcup_{1 \le p < \infty} L^p(\mathbf{R}^n)$. For such f we set

$$T^*(f) = \sup_{0<N<\infty} \sup_{0<\varepsilon<N} \left| T^{\varepsilon,N}(f) \right| \tag{8}$$

and we note that in view of the discussion above, the right hand side in (8) is well-defined. T^* is called the maximal singular integral operator associated with T. Note that we defined the maximal singular integral using a double truncation since we would like to be able to realize $T^{\varepsilon,N}(f)$ as a convergent integral if f is a bounded function.

We now state our main result.

Theorem 1. *Suppose that K and K^* satisfy (2), (3), and that the linear operator T associated with K as in (5) has an L^2-bounded extension with norm B. Then there exist dimensional constants C_n, C'_n such that the estimate*

$$\|T^*(f)\|_{L^p(\mathbf{R}^n)} \le C_n(A_1 + A_2 + B)\max(p,(p-1)^{-1})\|f\|_{L^p(\mathbf{R}^n)} \tag{9}$$

is valid for all $1 < p < \infty$ and all f in $L^p(\mathbf{R}^n)$ and

$$\|T^*(f)\|_{L^{1,\infty}(\mathbf{R}^n)} \le C'_n(A_1 + A_2 + B)\|f\|_{L^1(\mathbf{R}^n)} \tag{10}$$

holds for all $f \in L^1(\mathbf{R}^n)$.

It is a classical result (c.f. [4], [6]) that estimates (9) and (10) can be obtained using a generalization of Cotlar's inequality [2], if the smoothness condition (3) is replaced by the more restrictive Lipschitz type condition

$$|K(x,y) - K(z,y)| + |K^*(x,y) - K^*(z,y)| \le A\,\frac{|x-z|^\gamma}{|x-y|^{n+\gamma}} \tag{11}$$

whenever $|x - z| \le \frac{1}{2}|x - y|$. The point of this article is to extend these estimates to rougher kernels which fail to satisfy (11) (and thus Cotlar's inequality), but which satisfy the weaker Hörmander smoothness condition (3). Our approach is based on that of Riviere [5] but presents some extra complications in view of the additional upper truncations of the kernel K.

Corollary 1. *With the same hypotheses as in Theorem 1, if f is compactly supported and of class in $L\log L$, then $T^*(f)$ is integrable over a ball.*

The corollary is an easy consequence of Theorem 1 using the fact that $\|T^*\|_{L^p\to L^p} \le C_n(p-1)^{-1}$ as $p \to 1$ and Yano's [7] extrapolation result. See also Zygmund [8] (4.41).

2. The main decomposition

Fix $\alpha, \gamma > 0$. Recall the Calderón-Zygmund decomposition of an integrable function f on $\mathbf{R}^n$ at height $\alpha\gamma$ which guarantees the existence of functions g and b on $\mathbf{R}^n$ such that

(P1) $f = g + b$.
(P2) $\|g\|_{L^1} \le \|f\|_{L^1}$, $\|g\|_{L^\infty} \le 2^n\alpha\gamma$, thus $\|g\|_{L^p} \le (2^n\alpha\gamma)^{1/p'}\|f\|_{L^1}^{1/p}$ for $1 < p < \infty$.
(P3) $b = \sum_j b_j$ where each b_j is supported in a cube Q_j. Furthermore the cubes Q_k and Q_j have disjoint interiors when $j \ne k$.
(P4) $\int_{Q_j} b_j(x)\,dx = 0$.
(P5) $\|b_j\|_{L^1} \le 2^{n+1}\alpha\gamma|Q_j|$.
(P6) $\sum_j |Q_j| \le (\alpha\gamma)^{-1}\|f\|_{L^1}$.

We will refer to g as the good function and b as the bad function of this decomposition. For a cube Q we will denote by Q^* a cube concentric with Q with sidelength $l(Q_j^*) = 5\sqrt{n}\,l(Q_j)$.

The following lemma is the key ingredient in the proof of Theorem 1.

Lemma 1. *Suppose that f is an integrable function on $\mathbf{R}^n$, $\alpha, \gamma > 0$ and let $f = g+b$ be the Calderón-Zygmund decomposition of f at height $\alpha\gamma$. Let T and K be as in Theorem 1. If $\gamma \le (2^{n+5}A_1)^{-1}$, then we have*

$$\left|\left\{x \in (\cup_j Q_j^*)^c : \ |T^*(b)(x)| > \frac{\alpha}{2}\right\}\right| \le 2^{n+8}A_2 \frac{\|f\|_{L^1}}{\alpha}. \tag{12}$$

We will prove Lemma 1 in section 3. We now prove Theorem 1 using Lemma 1.

We begin with the proof of (9). Fix $1 < p < \infty$ and a function $f \in L^p(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ which we take initially to have compact support. We have

$$\begin{aligned}
T^{\varepsilon,N}(f)(x) &= \int_{\varepsilon \le |x-y| < N} K(x,y) f(y)\, dy = T^{\varepsilon,\infty}(f)(x) - T^{N,\infty}(f)(x) \\
&= \int_{\varepsilon \le |x-y|} K(x,y) f(y)\, dy - \int_{N \le |x-y|} K(x,y) f(y)\, dy \\
&= \int_{\varepsilon \le |x-y|} (K(x,y) - K(z_1,y)) f(y)\, dy + \int_{\varepsilon \le |x-y|} K(z_1,y) f(y)\, dy \\
&\qquad - \int_{N \le |x-y|} (K(x,y) - K(z_2,y)) f(y)\, dy - \int_{N \le |x-y|} K(z_2,y) f(y)\, dy \\
&= \int_{\varepsilon \le |x-y|} (K(x,y) - K(z_1,y)) f(y)\, dy + T(f)(z_1) - T(f\chi_{|x-\cdot|<\varepsilon})(z_1) \\
&\qquad - \int_{N \le |x-y|} (K(x,y) - K(z_2,y)) f(y)\, dy - T(f)(z_2) + T(f\chi_{|x-\cdot|<N})(z_2)
\end{aligned}$$

where z_1 and z_2 are arbitrary points in $\mathbf{R}^n$ that satisfy $|z_1 - x| \le \frac{\varepsilon}{2}$ and $|z_2 - x| \le \frac{N}{2}$. We used that f has compact support in order to be able to write $T^{\varepsilon,\infty}(f)$ and $T^{N,\infty}(f)$ as convergent integrals.

At this point we take absolute values, average over $|z_1 - x| \le \frac{\varepsilon}{2}$ and $|z_2 - x| \le \frac{N}{2}$ and we apply Hölder's inequality in two terms. We obtain the estimate

$$\begin{aligned}
|T^{\varepsilon,N}(f)(x)| \le & \frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1-x|\le\frac{\varepsilon}{2}} \int_{|x-y|\ge\varepsilon} |K(x,y) - K(z_1,y)|\, |f(y)|\, dy\, dz_1 \\
& + \frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1-x|\le\frac{\varepsilon}{2}} |T(f)(z_1)|\, dz_1 \\
& + \left(\frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1-x|\le\frac{\varepsilon}{2}} |T(f\chi_{|x-\cdot|<\varepsilon})(z_1)|^p\, dz_1\right)^{\frac{1}{p}} \\
& + \frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2-x|\le\frac{N}{2}} \int_{|x-y|\ge N} |K(x,y) - K(z_2,y)|\, |f(y)|\, dy\, dz_2 \\
& + \frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2-x|\le\frac{N}{2}} |T(f)(z_2)|\, dz_2 \\
& + \left(\frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2-x|\le\frac{N}{2}} |T(f\chi_{|x-\cdot|<N})(z_2)|^p\, dz_2\right)^{\frac{1}{p}},
\end{aligned}$$

where v_n is the volume of the unit ball in $\mathbf{R}^n$. Applying condition $(3)^*$ and estimate (7) we obtain for f in $L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ with compact support

$$\begin{aligned} |T^{\varepsilon,N}(f)(x)| \le{}& 2A_2\|f\|_{L^\infty} \\ &+\frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n\int_{|z_1-x|\le\frac{\varepsilon}{2}}|T(f)(z_1)|\,dz_1 \\ &+\frac{1}{v_n}\left(\frac{2}{N}\right)^n\int_{|z_2-x|\le\frac{N}{2}}|T(f)(z_2)|\,dz_2 \\ &+c_n(A_2+B)\max(p,(p-1)^{-1})\left(\frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n\int_{|z_1-x|\le\varepsilon}|f(z_1)|^p\,dz_1\right)^{\frac1p} \\ &+c_n(A_2+B)\max(p,(p-1)^{-1})\left(\frac{1}{v_n}\left(\frac{2}{N}\right)^n\int_{|z_2-x|\le N}|f(z_2)|^p\,dz_2\right)^{\frac1p}. \end{aligned} \tag{13}$$

We now use density to remove the compact support condition on f and obtain (13) for all functions f in $L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$. Taking the supremum over all $0<\varepsilon<N$ and over all $N>0$ we deduce that for all $f\in L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ we have the estimate

$$T^*(f)(x)\le 2A_2\|f\|_{L^\infty}+S_p(f)(x), \tag{14}$$

where S_p is the sublinear operator defined by

$$S_p(f)(x)=2M(T(f))(x)+2^{n+1}c_n(A_2+B)\max(p,(p-1)^{-1})M(|f|^p)(x)^{\frac1p},$$

and M is the Hardy-Littlewood maximal operator.

Recalling that M maps L^1 into $L^{1,\infty}$ with bound at most 3^n (and also L^p into $L^{p,\infty}$ with bound at most $2\cdot 3^{n/p}$ for $1<p<\infty$), we conclude that S_p maps $L^p(\mathbf{R}^n)$ into $L^{p,\infty}(\mathbf{R}^n)$ with norm at most

$$\|S_p\|_{L^p\to L^{p,\infty}}\le \widetilde{c}_n(A_2+B)\max(p,(p-1)^{-1}), \tag{15}$$

where $\widetilde{c}_n$ is another dimensional constant.

Now fix $1<p<\infty$ and $f\in L^p(\mathbf{R}^n)$. Let $\alpha>0$. Write $f=f_\alpha+f^\alpha$, where

$$f_\alpha=f\chi_{|f|\le\alpha/(16A_2)} \qquad\text{and}\qquad f^\alpha=f\chi_{|f|>\alpha/(16A_2)}.$$

The function f_α is in $L^\infty\cap L^p$ and f^α is in $L^1\cap L^p$. Moreover, it is easy to see that

$$\|f^\alpha\|_{L^1}\le(16A_2/\alpha)^{p-1}\|f\|_{L^p}^p. \tag{16}$$

Apply the Calderón-Zygmund decomposition to f^α at height $\alpha\gamma$ to write $f^\alpha=g^\alpha+b^\alpha$, where g^α is the good function and b^α is the bad function of this decomposition. We obtain

$$\|g^\alpha\|_{L^p}\le 2^{n/p'}(\alpha\gamma)^{1/p'}\|f^\alpha\|_{L^1}^{1/p}\le 2^{(n+4)/p'}(A_2\gamma)^{1/p'}\|f\|_{L^p}. \tag{17}$$

We now use (14) to get

$$|\{x\in\mathbf{R}^n:\ T^*(f)(x)>\alpha\}|\le b_1+b_2+b_3, \tag{18}$$

where

$$b_1 = \Big|\Big\{x \in \mathbf{R}^n :\ 2A_2\|f_\alpha\|_{L^\infty} + S_p(f_\alpha)(x) > \frac{\alpha}{4}\Big\}\Big|,$$
$$b_2 = \Big|\Big\{x \in \mathbf{R}^n :\ 2A_2\|g^\alpha\|_{L^\infty} + S_p(g^\alpha)(x) > \frac{\alpha}{4}\Big\}\Big|,$$
$$b_3 = \Big|\Big\{x \in \mathbf{R}^n :\ T^*(b^\alpha)(x) > \frac{\alpha}{2}\Big\}\Big|.$$

Observe that $2A_2\|f_\alpha\|_{L^\infty} \le \frac{\alpha}{8}$. Select $\gamma = \frac{1}{2^{n+5}(A_1+A_2)}$. Using (P2) we obtain

$$2A_2\|g^\alpha\|_{L^\infty} \le A_2 2^{n+1}\alpha\gamma \le \alpha 2^{-4} < \frac{\alpha}{8}$$

and therefore

$$b_1 \le \Big|\Big\{x \in \mathbf{R}^n :\ S_p(f_\alpha)(x) > \frac{\alpha}{8}\Big\}\Big|,$$
$$b_2 \le \Big|\Big\{x \in \mathbf{R}^n :\ S_p(g^\alpha)(x) > \frac{\alpha}{8}\Big\}\Big|. \tag{19}$$

Since $\gamma \le (2^{n+5}A_1)^{-1}$, it follows from (12) that

$$b_3 \le |\cup_j Q_j^*| + 2^{n+8}A_2\frac{\|f^\alpha\|_{L^1}}{\alpha} \le \left(\frac{(5\sqrt{n})^n}{\gamma} + 2^{n+8}A_2\right)\frac{\|f^\alpha\|_{L^1}}{\alpha}$$

and using (16) we obtain

$$b_3 \le C_n(A_1+A_2)^p\alpha^{-p}\|f\|_{L^p}^p.$$

Using Chebychev's inequality in (19) and (15) we finally obtain that

$$b_1 + b_2 \le (8/\alpha)^p\,(\widetilde{c}_n)^p(A_1+A_2+B)^p \max(p,(p-1)^{-1})^p\big(\|f\|_{L^p}^p + \|g^\alpha\|_{L^p}^p\big).$$

Combining the estimates for b_1, b_2, and b_3 and using (17) we obtain

$$\|T^*(f)\|_{L^{p,\infty}} \le C_n(A_1+A_2+B)\max(p,(p-1)^{-1})\|f\|_{L^p(\mathbf{R}^n)}\,. \tag{20}$$

The only difference between estimate (20) and the required estimate (9) is that the L^p norm on the left is replaced by the $L^{p,\infty}$ norm. Once (10) is established below, interpolating between $L^1 \to L^{1,\infty}$ and $L^{2p} \to L^{2p,\infty}$ would yield the required estimate (9) for p near 1; also interpolating between $L^{\frac{p+1}{2}} \to L^{\frac{p+1}{2},\infty}$ and $L^{2p} \to L^{2p,\infty}$ would yield (9) for p near ∞; (it may be helpful to use here the value of the Marcinkiewicz interpolation constant calculated in [4] p. 30.)

We now proceed with the proof of (10). Given f in $L^1(\mathbf{R}^n)$ we apply the Calderón-Zygmund decomposition of f at height $\gamma\alpha$ for some $\gamma, \alpha > 0$. We then write $f = g+b$, where $b = \sum_j b_j$ and each b_j is supported in some cube Q_j.

Using (9) we have

$$
\begin{aligned}
&\Big|\Big\{x\in\mathbf{R}^n:\ |T^*(f)(x)|>\alpha\Big\}\Big|\\
\le&\Big|\Big\{x\in\mathbf{R}^n:\ |T^*(g)(x)|>\frac{\alpha}{2}\Big\}\Big|+\Big|\Big\{x\in\mathbf{R}^n:\ |T^*(b)(x)|>\frac{\alpha}{2}\Big\}\Big|\\
\le&\frac{4}{\alpha^2}\|T^*(g)\|_{L^2}^2+\big|\cup_j Q_j^*\big|+\Big|\Big\{x\notin\cup_j Q_j^*:\ |T^*(b)(x)|>\frac{\alpha}{2}\Big\}\Big|\\
\le&\frac{4}{\alpha^2}C_n^2(A_1+A_2+B)^2\|g\|_{L^2}^2+\sum_j|Q_j^*|+\Big|\Big\{x\notin\cup_j Q_j^*:\ |T^*(b)(x)|>\frac{\alpha}{2}\Big\}\Big|\\
\le&\frac{2^{n+2}}{\alpha}\,\gamma\, C_n^2(A_1+A_2+B)^2\|f\|_{L^1}+\frac{(5\sqrt{n})^n}{\alpha\gamma}\|f\|_{L^1}+\Big|\Big\{x\notin\cup_j Q_j^*:\ |T^*(b)(x)|>\frac{\alpha}{2}\Big\}\Big|
\end{aligned}
$$

Choosing $\gamma=(2^{n+5}(A_1+A_2+B))^{-1}$ and using Lemma 1 we obtain the required estimate

$$
|\{x\in\mathbf{R}^n:\ |T^*(f)(x)|>\alpha\}|\le C_n'(A_1+A_2+B)\frac{\|f\|_{L^1}}{\alpha}
$$

with $C_n'=2^{-3}C_n^2+(5\sqrt{n})^n2^{n+5}+2^{n+8}$. This concludes the proof of (10).

It remains to prove Lemma 1; this will be done in the next section.

3. The proof of Lemma 1

We now turn our attention to Lemma 1. The claimed estimate in the lemma will be a consequence of the fact that for $x\in(\cup_j Q_j^*)^c$ we have the key inequality

$$
T^*(b)(x)\le 4E_1(x)+2^{n+2}\alpha\gamma E_2(x)+2^{n+3}\alpha\gamma A_1, \tag{21}
$$

where

$$
\begin{aligned}
E_1(x)&=\sum_j\int_{Q_j}|K(x,y)-K(x,y_j)|\,|b_j(y)|\,dy,\\
E_2(x)&=\sum_j\int_{Q_j}|K(x,y)-K(x,y_j)|\,dy,
\end{aligned}
$$

and y_j is the center of Q_j.

If we had (21), then we could easily derive (12). Indeed, fix $\gamma\le(2^{n+5}A_1)^{-1}$. Then we have $2^{n+3}\alpha\gamma A_1<\dfrac{\alpha}{3}$ and using (21) we obtain

$$
\begin{aligned}
&\Big|\Big\{x\in(\cup_j Q_j^*)^c:\ |T^*(b)(x)|>\frac{\alpha}{2}\Big\}\Big|\\
\le&\Big|\Big\{x\in(\cup_j Q_j^*)^c:\ 4E_1(x)>\frac{\alpha}{12}\Big\}\Big|+\Big|\Big\{x\in(\cup_j Q_j^*)^c:\ 2^{n+2}\alpha\gamma E_2(x)>\frac{\alpha}{12}\Big\}\Big|\\
\le&\frac{48}{\alpha}\int_{(\cup_j Q_j^*)^c}E_1(x)\,dx+2^{n+6}\gamma\int_{(\cup_j Q_j^*)^c}E_2(x)\,dx,
\end{aligned}
\tag{22}
$$

since $\dfrac{\alpha}{2}=\dfrac{\alpha}{3}+\dfrac{\alpha}{12}+\dfrac{\alpha}{12}$. We have

$$
\begin{aligned}
\int_{(\cup_j Q_j^*)^c} E_1(x)\,dx &\le \sum_j \int_{Q_j} |b_j(y)| \int_{(Q_j^*)^c} |K(x,y)-K(x,y_j)|\,dx\,dy \\
&\le \sum_j \int_{Q_j} |b_j(y)| \int_{|x-y_j|\ge 2|y-y_j|} |K(x,y)-K(x,y_j)|\,dx\,dy \\
&\le A_2 \sum_j \int_{Q_j} |b_j(y)|\,dy = A_2 \sum_j \|b_j\|_{L^1} \le A_2 2^{n+1}\|f\|_{L^1},
\end{aligned}
\tag{23}
$$

where we used the fact that if $x\in(Q_j^*)^c$ then $|x-y_j|\ge \dfrac{1}{2}l(Q_j^*)=\dfrac{5}{2}\sqrt{n}\,l(Q_j)$. But since $|y-y_j|\le \dfrac{\sqrt{n}}{2}\,l(Q_j)$ this implies that $|x-y_j|\ge 2|y-y_j|$. Here we used the fact that the diameter of a cube is equal to $\sqrt{n}$ times its sidelength. Likewise we can obtain that

$$
\int_{(\cup_j Q_j^*)^c} E_2(x)\,dx \le A_2 \sum_j |Q_j| \le A_2 \frac{\|f\|_{L^1}}{\alpha\gamma}\,. \tag{24}
$$

Combining (23) and (24) with (22) yields (12).

Therefore the main task in the proof of (12) is to show (21). Recall that $b=\sum_j b_j$ and to estimate $T^*(b)$ it suffices to estimate each $|T^{\varepsilon,N}(b_j)|$ uniformly in ε and N. To achieve this we will use that

$$
|T^{\varepsilon,N}(b_j)| \le |T^{\varepsilon,\infty}(b_j)| + |T^{N,\infty}(b_j)|\,. \tag{25}
$$

We work with $T^{\varepsilon,\infty}$ and we note that $T^{N,\infty}$ can be treated similarly. For fixed $x\notin \cup_j Q_j^*$ and $\varepsilon>0$ we define

$$
\begin{aligned}
J_1(x,\varepsilon) &= \{j:\ \forall y\in Q_j \text{ we have } |x-y|<\varepsilon\},\\
J_2(x,\varepsilon) &= \{j:\ \forall y\in Q_j \text{ we have } |x-y|>\varepsilon\},\\
J_3(x,\varepsilon) &= \{j:\ \exists y\in Q_j \text{ we have } |x-y|=\varepsilon\}.
\end{aligned}
$$

Note that $T^{\varepsilon,\infty}(b_j)(x)=0$ whenever $x\notin\cup_j Q_j^*$ and $j\in J_1(x,\varepsilon)$. Also note that $K^{\varepsilon,\infty}(x,y)=K(x,y)$ whenever $x\notin\cup_j Q_j^*$, $j\in J_2(x,\varepsilon)$ and $y\in Q_j$. Therefore

$$
\sup_{\varepsilon>0}|T^{\varepsilon,\infty}(b)(x)| \le \sup_{\varepsilon>0}\Big|\sum_{j\in J_2(x,\varepsilon)} T(b_j)(x)\Big| + \sup_{\varepsilon>0}\Big|\sum_{j\in J_3(x,\varepsilon)} T(b_j\chi_{|x-\cdot|\ge\varepsilon})(x)\Big| \tag{26}
$$

but since

$$
\sup_{\varepsilon>0}\Big|\sum_{j\in J_2(x,\varepsilon)} T(b_j)(x)\Big| \le \sum_j |T(b_j)(x)| = E_1(x)\,, \tag{27}
$$

it suffices to estimate the second term on the right in (26).

Here we need to make some geometric observations. Fix $\varepsilon > 0$, $x \in (\cup_j Q_j^*)^c$ and also fix a cube Q_j with $j \in J_3(x,\varepsilon)$. Then we have

$$\varepsilon \geq \frac{1}{2}\big(l(Q_j^*) - l(Q_j)\big) = \frac{1}{2}(5\sqrt{n}-1)l(Q_j) \geq 2\sqrt{n}\, l(Q_j)\,. \tag{28}$$

Since $j \in J_3(x,\varepsilon)$ there exists a $y_0 \in Q_j$ with $|x-y_0| = \varepsilon$. Using (28) we obtain that for any $y \in Q_j$ we have

$$\begin{aligned}
&\frac{\varepsilon}{2} \leq \varepsilon - \sqrt{n}\, l(Q_j) \leq |x-y_0| - |y-y_0| \leq |x-y|,\\
&|x-y| \leq |x-y_0| + |y-y_0| \leq \varepsilon + \sqrt{n}\, l(Q_j) \leq \frac{3\varepsilon}{2}.
\end{aligned}$$

We have therefore proved that

$$\bigcup_{j\in J_3(x,\varepsilon)} Q_j \subset B(x, \tfrac{3\varepsilon}{2}) \setminus B(x, \tfrac{\varepsilon}{2})\,.$$

We now let $c_j(\varepsilon) = |Q_j|^{-1} \displaystyle\int_{Q_j} b_j(y)\chi_{|x-y|\geq\varepsilon}(y)\, dy$ and we note that property (P5) of the Calderón-Zygmund decomposition yields the estimate $|c_j(\varepsilon)| \leq 2^{n+1}\alpha\gamma$. Then

$$\begin{aligned}
&\sup_{\varepsilon>0}\bigg|\sum_{j\in J_3(x,\varepsilon)} \int_{Q_j} K(x,y) b_j(y)\chi_{|x-y|\geq\varepsilon}(y)\, dy\bigg|\\
\leq &\sup_{\varepsilon>0}\bigg|\sum_{j\in J_3(x,\varepsilon)} \int_{Q_j} K(x,y)\big(b_j(y)\chi_{|x-y|\geq\varepsilon}(y) - c_j(\varepsilon)\big)\, dy\bigg|\\
&\qquad\qquad + \sup_{\varepsilon>0}\bigg|\sum_{j\in J_3(x,\varepsilon)} c_j(\varepsilon)\int_{Q_j} K(x,y)\, dy\bigg|\\
\leq &\sup_{\varepsilon>0}\bigg|\sum_{j\in J_3(x,\varepsilon)} \int_{Q_j} \big(K(x,y)-K(x,y_j)\big)\big(b_j(y)\chi_{|x-y|\geq\varepsilon}(y) - c_j(\varepsilon)\big)\, dy\bigg|\\
&\qquad\qquad + 2^{n+1}\alpha\gamma \sup_{\varepsilon>0}\int_{B(x,\frac{3\varepsilon}{2})\setminus B(x,\frac{\varepsilon}{2})} |K(x,y)|\, dy\\
\leq &\sum_j \int_{Q_j} \big|K(x,y)-K(x,y_j)\big|\big(|b_j(y)| + 2^{n+1}\alpha\gamma\big)\, dy\\
&\qquad\qquad + 2^{n+1}\alpha\gamma \sup_{\varepsilon>0}\int_{\frac{\varepsilon}{2}\leq|x-y|\leq\frac{3\varepsilon}{2}} |K(x,y)|\, dy\\
\leq &E_1(x) + 2^{n+1}\alpha\gamma E_2(x) + 2^{n+1}\alpha\gamma(2A_1).
\end{aligned}$$

The last estimate above together with (27) and combined with (25) and the analogous estimate for $\sup_{N>0}|T^{N,\infty}(b)(x)|$ (which can be obtained entirely similarly), yields (21). This finishes the proof of Lemma 1 and thus of Theorem 1.

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