MAXIMAL OPERATOR AND WEIGHTED NORM INEQUALITIES FOR MULTILINEAR SINGULAR INTEGRALS

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Abstract. The maximal operator associated with multilinear Calderón-Zygmund singular integrals is introduced and shown to be bounded on product of Lebesgue spaces. Moreover weighted norm inequalities are obtained for this maximal operator as well as for the corresponding singular integrals.

1. Introduction

The analysis of multilinear singular integrals has much of its origins in several works by Coifman and Meyer in the 70's; see for example [3]. More recently, in [5] and [6], an updated systematic treatment of multilinear singular integral operators of Calderón-Zygmund type was presented in light of some new developments. See also [7] and the references therein for a detailed description of previous work in the subject. In this article we prove the boundedness of a maximal operator associated to multilinear singular integrals and we use it to obtain multilinear weighted norm inequalities.

We will consider multilinear operators T initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T: \ \mathcal{S}(\mathbf{R}^n) \times \cdots \times \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n).$$

Every such operator is associated with a distributional kernel on $(\mathbf{R}^n)^{m+1}$. We will assume that this distributional kernel coincides with a function K defined away from the diagonal $y_0 = y_1 = y_2 = \cdots = y_m$ in $(\mathbf{R}^n)^{m+1}$ which satisfies the size estimate

$$|K(y_0, y_1, \ldots, y_m)| \le \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}} \tag{1}$$

and, for some $\varepsilon > 0$, the regularity condition

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \le \frac{A|y_j - y_j'|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}, \tag{2}$$

1991 *Mathematics Subject Classification.* Primary 42B20, 42B25. Secondary 46B70, 47B38.

Key words and phrases. Calderón-Zygmund theory, multilinear operators, maximal singular integrals, weighted norm inequalities.

Grafakos' research partially supported by the University of Missouri Research Board and by the NSF under grant DMS 9623120.

Torres' research partially supported by a General Research Fund of the University of Kansas and by the NSF under grants DMS 9696267 and DMS 0070514.

whenever $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2} \max_{0\le k \le m} |y_j - y_k|$. Kernels K satisfying (1) and (2) will be called of class m-$CZK(A,\varepsilon)$. The association between T and K is expressed via the representation

$$T(f_1,\dots,f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x,y_1,\dots,y_m) f_1(y_1)\dots f_m(y_m)\, dy_1\dots dy_m, \tag{3}$$

whenever $f_1,\dots,f_m$ are C^∞ functions with compact support and $x \notin \cap_{j=1}^m \text{supp } f_j$.

By homogeneity considerations, given exponents $1 \le q_1,\dots,q_m < \infty$ and a multilinear operator T associated with a kernel in m-$CZK(A,\varepsilon)$, it is meaningful to consider boundedness properties of the form

$$T : L^{q_1} \times \dots \times L^{q_m} \to L^q,$$

only when

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{1}{q}. \tag{4}$$

It was shown in [6] that the boundedness of these general multilinear operators T on just one such product of Lebesgue spaces implies the boundedness on all other products of Lebesgue spaces with exponents $1 < q_j \le \infty$ satisfying (4) with $q < \infty$. A simple limiting argument then shows that the integral representation (3) still holds for L^{q_j} functions as long as $x \notin \cap_{j=1}^m \text{supp } f_j$. Moreover, there are endpoint weak-type estimates when some of the exponents q_j are equal to one. In particular,

$$T : L^1 \times \dots \times L^1 \to L^{1/m,\infty}. \tag{5}$$

For translation invariant operators similar results were obtained in [8].

When all the above continuity properties hold, we say that T is an m-linear Calderón-Zygmund operator. Necessary and sufficient conditions for boundedness of operators with kernels in m-$CZK(A,\varepsilon)$ can be described in the form of multilinear T1-Theorems, [1] and [6] .

In this article we study the maximal truncated operator

$$T_*(f_1,\dots,f_m)(x) = \sup_{\delta>0} |T_\delta(f_1,\dots,f_m)(x)|,$$

where, using the notation $\vec{y} = (y_1,\dots,y_m)$ and $d\vec{y} = dy_1\dots dy_m$, we set

$$T_\delta(f_1,\dots,f_m)(x) = \int_{|x-y_1|^2+\dots+|x-y_m|^2>\delta^2} K(x,y_1,\dots,y_m) f_1(y_1)\dots f_m(y_m)\, d\vec{y}.$$

We note that if $f_j \in L^{q_j}(\mathbf{R}^n)$ with $1 \le q_j \le \infty$, then $T_\delta(f_1,\dots,f_m)$ is given by an absolutely convergent integral and thus is well defined. Indeed, if $(y_1,\dots,y_m)$ satisfies $|x-y_1|^2 + \dots + |x-y_m|^2 > \delta^2$, then for some j, say $j = m$, we have $|x - y_j| = |x - y_m| > \delta/\sqrt{n}$. Then, using Hölder's inequality in each variable at a time, we obtain

$$\begin{aligned}
&|T_\delta(f_1,\dots,f_m)(x)|\\
\le& C_1(n)\|f_1\|_{L^{q_1}}\int_{|x-y_m|>\frac{\delta}{\sqrt{n}}}\int_{(\mathbf{R}^n)^{m-2}}\frac{|f_2(y_2)|\dots|f_m(y_m)|dy_2\dots dy_{m-1}}{(|x-y_2|+\dots+|x-y_m|)^{mn-\frac{n}{q_1'}}}\,dy_m\\
\le& \qquad\dots\\
\le& C_{m-1}(n)\|f_1\|_{L^{q_1}}\dots\|f_{m-1}\|_{L^{q_{m-1}}}\int_{|x-y_m|>\frac{\delta}{\sqrt{n}}}\frac{|f_m(y_m)|dy_m}{|x-y_m|^{mn-(\frac{n}{q_1'}+\dots+\frac{n}{q_{m-1}'})}}\\
\le& C_m(n)\|f_1\|_{L^{q_1}}\dots\|f_m\|_{L^{q_m}}\frac{1}{\delta^{mn-(\frac{n}{q_1'}+\dots+\frac{n}{q_m'})}}<\infty.
\end{aligned}$$

($q'=q/(q-1)$ here denotes the dual index of q.) Thus $T_*(f_1,\dots,f_m)(x)$ is also pointwise well-defined when $f_j\in L^{q_j}(\mathbf{R}^n)$ with $1\le q_j\le\infty$. In Theorem 1 below we prove a pointwise estimate for T_* when the f_j's lie in suitable Lebesgue spaces.

An immediate consequence of the boundedness of T_* is that if T is given by a principal value integral of the form

$$(6)\quad T(f_1,\dots,f_m)(x)=\lim_{\delta\to0}\int_{|x-y_1|^2+\dots+|x-y_m|^2>\delta^2}K(x,y_1,\dots,y_m)f_1(y_1)\dots f_m(y_m)\,d\vec{y}$$

when the functions f_j are in the Schwartz class, then the integrals in (6) converge a.e. for all f_j in $L^{q_j}(\mathbf{R}^n)$. We refer again to [6] where several examples of such operators are given.

The A^∞ estimate for T_* obtained in Theorem 2, gives weighted norm inequalities analogous to those in [2] for linear operators.

The authors announced some of the results proved here at the recent 6th International Conference on Harmonic Analysis and Partial Differential Equations held at El Escorial, Spain. The authors would like to take this opportunity to thank their colleagues in Spain for their hospitality during that conference. They would also like to thank Carlos Pérez for some useful comments.

2. Cotlar's Inequality for Multilinear Singular Integrals

The Hardy-Littlewood maximal function with respect to balls on $\mathbf{R}^n$ will be denoted by M. We will also use the notation $\vec{f}=(f_1,\dots,f_m)$ whenever it is convenient. For a given $x\in\mathbf{R}^n$ we will denote by $S_\delta(x)$ the cube $\{\vec{y}:\ \sup_{1\le j\le m}|x-y_j|\le\delta\}$. Throughout this paper we will let W be the norm of T in (5). Recall that A is the constant that appears in the size and smoothness estimates (1) and (2) of the kernel K associated with T.

Theorem 1. *Let T be an m-linear Calderón-Zygmund operator. Then, for all $\eta>0$, there exists a constant $C_\eta=C_\eta(n,m)<\infty$ such that for all $\vec{f}$ in any product of $L^{q_j}(\mathbf{R}^n)$ spaces, with $1\le q_j<\infty$, the following inequality holds for all x in $\mathbf{R}^n$*

$$(7)\qquad T_*(\vec{f})(x)\le C_\eta\bigg((M(|T(\vec{f})|^\eta)(x))^{1/\eta}+(A+W)\prod_{j=1}^m Mf_j(x)\bigg).$$

Proof. It is clear that is enough to prove the theorem for η arbitrarily small, so we provide an argument for $0 < \eta < 1/m$. Fix x in $\mathbf{R}^n$. Let $U_\delta = \{\vec{y} \in S_\delta(x) : |x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2\}$. It is easy to see that

$$\sup_{\delta>0} \left| \int_{U_\delta} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, d\vec{y} \right| \le CA \prod_{j=1}^{m} Mf_j(x), \tag{8}$$

so it suffices to show (7) with $T_*(\vec{f})(x)$ replaced by

$$\widetilde{T}_*(\vec{f})(x) = \sup_{\delta>0} |\widetilde{T}_\delta(f_1, \dots, f_m)(x)|, \tag{9}$$

where

$$\widetilde{T}_\delta(f_1, \dots, f_m)(x) = \int_{\vec{y} \notin S_\delta(x)} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, d\vec{y}.$$

Fix $\delta > 0$ and let $B(x, \delta/2)$ be the ball of center x and radius $\delta/2$. Note that, since $\vec{f}$ is in a product of Lebesgue spaces and T is a Calderón-Zygmund operator, $T(\vec{f})$ is in some L^p space and hence it is finite almost everywhere. Moreover, using linearity and (3), we have for $z \in B(x, \delta/2)$

$$\widetilde{T}_\delta(\vec{f})(z) = T(\vec{f})(z) - T(\vec{f_0})(z), \tag{10}$$

where $\vec{f_0} = (f_1\chi_{B(x,\delta)}, \dots, f_m\chi_{B(x,\delta)})$. Also, using (2), we obtain

$$|\widetilde{T}_\delta(\vec{f})(x) - \widetilde{T}_\delta(\vec{f})(z)| \le \int_{\vec{y} \notin S_\delta(x)} \frac{A|x-z|^\varepsilon \prod_{j=1}^m |f_j(y_j)|}{(|x-y_1| + \cdots + |x-y_m|)^{nm+\varepsilon}}\, d\vec{y}. \tag{11}$$

Now, the right hand side of (11) can be written as a sum of integrals over sets $R_{j_1,\dots,j_l}$ in $(\mathbf{R}^n)^m$ for some $\{j_1, \dots, j_l\} \subsetneqq \{1, \dots, m\}$ so that for $\vec{y} = (y_1, \dots, y_m) \in R_{j_1,\dots,j_l}$ we have $|x - y_j| \le \delta$ if and only if $j \in \{j_1, \dots, j_l\}$. Then $l < m$ and it follows that

$$\begin{aligned}
&\int_{\vec{y} \in R_{j_1,\dots,j_l}} \frac{A|x-z|^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{nm+\varepsilon}} \prod_{j=1}^m |f_j(y_j)|\, d\vec{y} \\
\le& A\delta^\varepsilon \prod_{j\in\{j_1,\dots,j_l\}} \int_{|x-y_j|\le\delta} |f_j(y_j)|\, dy_j \prod_{j\notin\{j_1,\dots,j_l\}} \int_{|x-y_j|>\delta} \frac{|f_j(y_j)|}{|x-y_j|^{\frac{nm+\varepsilon}{m-l}}}\, dy_j \\
\le& CA \prod_{j\in\{j_1,\dots,j_l\}} Mf_j(x) \prod_{j\notin\{j_1,\dots,j_l\}} \delta^{\frac{n+\varepsilon}{m-l}} \int_{|x-y_j|>\delta} \frac{|f_j(y_j)|}{|x-y_j|^{\frac{nm+\varepsilon}{m-l}}}\, dy_j \\
\le& CA \prod_{j=1}^m Mf_j(x).
\end{aligned}$$

Using (10) and (11), we obtain for z in $B(x, \delta/2)$

$$|\widetilde{T}_\delta(\vec{f})(x)| \le CA \prod_{j=1}^m Mf_j(x) + |T(\vec{f})(z) - T(\vec{f_0})(z)|. \tag{12}$$

Fix now $0 < \eta < 1/m$. Raising (12) to the power η, integrating over $z \in B = B(x, \delta/2)$, and dividing by $|B|$ we obtain

$$|\widetilde{T}_\delta(\vec{f})(x)|^\eta \leq \big(CA \prod_{j=1}^m Mf_j(x)\big)^\eta + M(|T(\vec{f})|^\eta)(x) + \frac{1}{|B|}\int_B |T(\vec{f_0})(z)|^\eta \, dz. \tag{13}$$

We estimate the last term in (13) as follows

$$\int_B |T(\vec{f_0})(z)|^\eta \, dz = m\eta \int_0^\infty \lambda^{m\eta-1} |\{z \in B : |T(\vec{f_0})(z)|^{1/m} > \lambda\}| \, d\lambda$$

$$\leq m\eta \int_0^\infty \lambda^{m\eta-1} \min\left(|B|, \frac{W^{1/m}}{\lambda} (\prod_{j=1}^m \|f_j \chi_{B(x,\delta)}\|_{L^1})^{1/m}\right) d\lambda.$$

Letting

$$R = W^{1/m} (\prod_{j=1}^m \|f_j \chi_{B(x,\delta)}\|_{L^1})^{1/m},$$

we get

$$\int_B |T(\vec{f_0})(z)|^\eta \, dz \leq m\eta \int_0^{R/|B|} \lambda^{m\eta-1}|B| \, d\lambda + m\eta \int_{R/|B|}^\infty \lambda^{m\eta-2} R \, d\lambda \leq C_\eta R^{m\eta} |B|^{1-m\eta},$$

where we have used that $m\eta < 1$. Finally

$$\frac{1}{|B|}\int_B |T(\vec{f_0})(z)|^\eta \, dz \leq C_\eta W^\eta |B|^{-m\eta} (\prod_{j=1}^m \|f_j \chi_{B(x,\delta)}\|_{L^1})^\eta \leq C_\eta W^\eta \left(\prod_{j=1}^m Mf_j(x)\right)^\eta,$$

and if we insert this estimate in (13) and raise to the power $1/\eta$ we obtain (7). □

Remark 1. We note that if T satisfies any strong type estimate for some $q_j > 1$ with norm $\|T\|$ then $W \leq C(n, m, q_j)(A + \|T\|)$. See [6].

We also note that a particular case of Theorem 1 for $\eta = 1/m$ can be obtained with rather different arguments which are of interest in their own but are not needed in this article. For the linear case $m = 1$ see, for example, the book [4]. The point here is to obtain the estimate for η sufficiently small so that the full range of $q's$ in the next corollary can be achieved.

Corollary 1. *Let T be an m-linear Calderón-Zygmund operator. Then, for all exponents $q_1, \dots, q_m$ and q satisfying (4), we have*

$$T_* : L^{q_1} \times \cdots \times L^{q_m} \to L^q$$

when $1 < q_1, \dots, q_m \leq \infty$ and $q < \infty$. We also have

$$T_* : L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}$$

when at least one q_j is equal to one. Moreover, in either case the norm of T_* is controlled by a constant multiple of $A + W$.

Proof. The strong estimates follow directly from (7) with any $\eta \le 1/m$ and the boundedness properties of T (see [6]) and M. For the weak estimates we just observe, for instance if $q = 1/m$, that by picking $\eta < 1/m$, we have

$$\|M(|T(\vec{f})|^\eta)^{1/\eta}\|_{L^{1/m,\infty}} = \|M(|T(\vec{f})|^\eta)\|^{1/\eta}_{L^{1/(m\eta),\infty}}$$
$$\le C\||T(\vec{f})|^\eta\|^{1/\eta}_{L^{1/(m\eta),\infty}} = C\|T(\vec{f})\|_{L^{1/m,\infty}},$$

because M maps $L^{p,\infty}$ into itself for all $1 < p < \infty$. □

3. Weighted Norm Inequalities

For simplicity in the proofs, in this section we use the uncentered Hardy-Littlewood maximal function with respect to cubes in $\mathbf{R}^n$ which we denote by M_c. Recall that a weight w is in the class A_∞ if and only if there exist c, $\theta > 0$ such that for every cube Q and every measurable set $E \subset Q$,

$$\frac{w(E)}{w(Q)} \le c\left(\frac{|E|}{|Q|}\right)^\theta, \tag{14}$$

where, for a measurable set F, $w(F) = \int_F w(x)\,dx$.

Recall the modified maximal truncated singular integral $\widetilde{T}_*$ defined in (9).

Theorem 2. *Let T be a m-linear Calderón-Zygmund operator and let W be the least bound in (5). Let $\vec{f}$ be in any product of $L^{q_j}(\mathbf{R}^n)$ spaces, with $1 \le q_j < \infty$. Also let $w \in A_\infty$ and θ be as in (14). Then there exists a positive constant C such that for all $\alpha > 0$ and all $\gamma > 0$ sufficiently small we have*

$$w\left(\{\widetilde{T}_*(\vec{f}) > 2^{m+1}\alpha\}\cap\{\prod_{j=1}^m M_c f_j \le \gamma\alpha\}\right) \le C(A+W)^{\frac{\theta}{m}}\gamma^{\frac{\theta}{m}} w\left(\{\widetilde{T}_*(\vec{f}) > \alpha\}\right). \tag{15}$$

Proof. Write

$$\Omega = \{x : \widetilde{T}_*(\vec{f})(x) > \alpha\} = \cup_s Q_s,$$

where Q_s are Whitney cubes. In view of (14), it suffices to show that for all Whitney cubes Q_s we have the estimate

$$|Q_s \cap \{\widetilde{T}_*(\vec{f}) > 2^{m+1}\alpha\} \cap \{\prod_{j=1}^m M_c f_j \le \gamma\alpha\}| \le C(A+W)^{1/m}\gamma^{1/m}|Q_s|, \tag{16}$$

where W is the bound for T in the weak estimate (5).

For each Whitney cube Q_s fix a large multiple of it Q_s^* and a point y_s in ${}^c\Omega \cap Q_s^*$ with the property that

$$\max_{z\in Q_s}|y_s - z| \le \frac{1}{2}\operatorname{dist}(y_s, {}^c(Q_s^*)). \tag{17}$$

In order to prove (16) for a given cube Q_s we may assume that there exists a point ξ_s in Q_s such that

$$M_c f_1(\xi_s) \ldots M_c f_m(\xi_s) \le \gamma\alpha,$$

otherwise there is nothing to prove.

Given $\vec{f} = (f_1, \ldots, f_m)$, define $f_j^0 = f_j \chi_{Q_s^*}$ and $f_j^\infty = f - f_j^0$ for $j = 1, \ldots, m$. The set

$$Q_s \cap \{\widetilde{T}_*(\vec{f}) > 2^{m+1}\alpha\} \cap \{\prod_{j=1}^m M_c f_j \leq \gamma\alpha\}$$

is contained in the union of 2^m sets of the form

$$Q_s \cap \{\widetilde{T}_*(f_1^{r_1}, \ldots, f_m^{r_m}) > 2\alpha\} \cap \{\prod_{j=1}^m M_c f_j \leq \gamma\alpha\}, \tag{18}$$

where $r_j \in \{0, \infty\}$ for all $1 \leq j \leq m$. First we estimate the measure of the set corresponding to $r_1 = \cdots = r_m = 0$. We have

$$\begin{aligned} &|Q_s \cap \{\widetilde{T}_*(f_1^0, \ldots, f_m^0)(x) > 2\alpha\} \cap \{\prod_{j=1}^m M_c f_j(x) \leq \gamma\alpha\}| \\ \leq & \frac{C(A+W)^{1/m}}{\alpha^{1/m}} \Big(\int_{\mathbf{R}^n} |f_1^0(t_1)| dt_1 \ldots \int_{\mathbf{R}^n} |f_m^0(t_m)| dt_m \Big)^{1/m} \\ \leq & \frac{C(A+W)^{1/m}}{\alpha^{1/m}} \Big(\frac{1}{|Q_s^*|} \int_{Q_s^*} |f_1(t_1)| dt_1 \ldots \frac{1}{|Q_s^*|} \int_{Q_s^*} |f_m(t_m)| dt_m \Big)^{1/m} |Q_s| \\ \leq & \frac{C(A+W)^{1/m}}{\alpha^{1/m}} \big(\prod_{j=1}^m M_c f_j(\xi_s) \big)^{1/m} |Q_s| \leq C(A+W)^{1/m} \gamma^{1/m} |Q_s|, \end{aligned} \tag{19}$$

where we have used that $\widetilde{T}_*$ maps $L^1 \times \cdots \times L^1$ into weak $L^{1/m}$ with bound at most $C(A+W)$, a consequence of Corollary 1.

Next, we will show that all the remaining sets are empty if γ is chosen to be small. When this is established, combining (19) with (14) and summing over all Whitney cubes Q_s yields (15). Consider first the case where exactly l of the r_j are ∞ for some $1 \leq l < m$. We give the arguments for one of these cases. The rest are similar and can be easily obtained from the argument below by permuting the indices. We have

$$\begin{aligned} &\Big| \int_{\vec{y} \notin S_\delta(x)} K(x, \vec{y}) f_1^\infty(y_1) \ldots f_l^\infty(y_l) f_{l+1}^0(y_{l+1}) \ldots f_m^0(y_m) \, d\vec{y} \Big| \\ \leq & CA \prod_{j=l+1}^m \int_{Q_s^*} |f_j(y_j)| \, dy_j \prod_{k=1}^l \int_{{}^c(Q_s^*)} \frac{|f_k(y_k)|}{|x - y_k|^{mn/l}} \, dy_k \\ \leq & CA \prod_{j=l+1}^m M_c f_j(\xi_s) |Q_s|^{m-l} \prod_{k=1}^l \int_{{}^c(Q_s^*)} \frac{|f_k(y_k)|}{|\xi_s - y_k|^{mn/l}} \, dy_k \\ \leq & CA \prod_{j=1}^m M_c f_j(\xi_s) \leq CA\gamma\alpha, \end{aligned}$$

where we have used that $m > l$. By picking γ small enough, we can make the set in (18) empty when $r_1 = \cdots = r_l = \infty$ and $r_{l+1} = \cdots = r_m = 0$. Likewise with all the remaining sets where at least one r_j is infinity. We are now left with the set in (18) where all the r_j's are equal to infinity, that is, the set

$$Q_s \cap \{\widetilde{T}_*(f_1^\infty, \dots, f_m^\infty) > 2\alpha\} \cap \{\prod_{j=1}^m M_c f_j(x) \le \gamma\alpha\}. \tag{20}$$

Set $\vec{f}^\infty = (f_1^\infty, \dots, f_m^\infty)$. We claim that for $x \in Q_s$ we have

$$|\widetilde{T}_\delta(\vec{f}^\infty)(x) - \widetilde{T}_\delta(\vec{f}^\infty)(y_s)| \le CA \prod_{j=1}^m M_c f_j(\xi_s). \tag{21}$$

We have

$$\begin{aligned}
&\widetilde{T}_\delta(\vec{f}^\infty)(x) - \widetilde{T}_\delta(\vec{f}^\infty)(y_s)\\
=&\int_{\vec{y}\notin S_\delta(x)} K(x,\vec{y}) f_1^\infty(y_1)\dots f_m^\infty(y_m)\,d\vec{y} - \int_{\vec{y}\notin S_\delta(y_s)} K(y_s,\vec{y}) f_1^\infty(y_1)\dots f_m^\infty(y_m)\,d\vec{y}\\
=&I - II
\end{aligned}$$

where

$$I = \int_{\left({}^c S_\delta(x)\cap S_\delta(y_s)\right)\cup\left({}^c S_\delta(y_s)\cap S_\delta(x)\right)} K(x,\vec{y}) f_1^\infty(y_1)\dots f_m^\infty(y_m)\,d\vec{y}$$

$$II = \int_{\vec{y}\notin S_\delta(y_s)} \big[K(x,\vec{y}) - K(y_s,\vec{y})\big] f_1^\infty(y_1)\dots f_m^\infty(y_m)\,d\vec{y}$$

Since $|x - y_s| \le \frac{1}{2}\max_{1\le j\le n}|x - y_j|$ when $y_j \notin Q_s^*$, applying (2) we obtain

$$\begin{aligned}
|II| \le& \int_{(\mathbf{R}^n)^m} \frac{A|x-y_s|^\varepsilon}{(|x-y_1| + \dots + |x-y_m|)^{nm+\varepsilon}} \prod_{j=1}^m |f_j^\infty(y_j)|\,d\vec{y}\\
\le& CA|Q_s|^{\varepsilon/n} \prod_{j=1}^m \int_{{}^c(Q_s^*)} \frac{|f(y_j)|}{|x-y_j|^{\frac{n+\varepsilon}{m}}}\,dy_j \le CA\prod_{j=1}^m M_c f_j(\xi_s).
\end{aligned}$$

As far as I is concerned, we consider two cases:
(a) If $\vec{y}$ belongs to ${}^c S_\delta(x) \cap S_\delta(y_s)$, then we have

$$|y_1 - y_s|^2 + \dots + |y_m - y_s|^2 \le n\delta^2, \qquad |y_1 - x|^2 + \dots + |y_m - x|^2 \ge \delta^2\,.$$

In this case, $|y_j - x|^2 \ge \frac{1}{2}|y_j - x|^2 + \frac{1}{4}\ell(Q_s)^2$ and summing over $j = 1, \dots, m$ yields

$$|y_1 - x| + \dots + |y_m - x| \ge c\,(\delta + \ell(Q_s))\,,$$

where $\ell(Q_s)$ is the length of the cube Q_s. Under the assumptions in case (a), for a given point $\xi_2 \in Q_s$ we have

$$|y_j - \xi_s| \le |y_j - y_s| + |y_s - \xi_s| \le \sqrt{n}\,\delta + c\,\ell(Q_s)$$

and so the integral I in case (a) can be estimated by

$$\frac{A}{(c\,(\delta + \ell(Q_s)))^{mn}} \prod_{j=1}^m \int_{|y_j - \xi_s| \le c'(\delta + \ell(Q_s))} |f_j(y_j)|\,dy_j \le CA\prod_{j=1}^m M_c f_j(\xi_s).$$

(b) If $\vec{y} \in {}^c S_\delta(y_s) \cap S_\delta(x)$, then we have

$$|y_1 - x|^2 + \cdots + |y_m - x|^2 \le n\delta^2, \qquad |y_1 - y_s|^2 + \cdots + |y_m - y_s|^2 \ge \delta^2 \,.$$

But in this case,

$$|y_j - x| \ge \frac{1}{2}|y_j - y_s| - |y_s - x| + \frac{1}{2}|y_j - y_s| \ge \frac{1}{2}|y_j - y_s| \ge \frac{1}{4}|y_j - y_s| + \ell(Q_s)$$

using the definition of y_s in the second inequality. Squaring and summing over j yields

$$|y_1 - x| + \cdots + |y_m - x| \ge c\,(\delta + \ell(Q_s))\,.$$

Under the assumptions in case (b), for a given point $\xi_2 \in Q_s$ we have

$$|y_j - \xi_s| \le |y_j - x| + |x - \xi_s| \le \sqrt{n}\,\delta + c\,\ell(Q_s)$$

and so the integral I in case (b) can be estimated by

$$\frac{A}{(c\,(\delta + \ell(Q_s)))^{mn}} \prod_{j=1}^{m} \int_{|y_j - \xi_s| \le c'(\delta + \ell(Q_s))} |f_j(y_j)|\,dy_j \le CA \prod_{j=1}^{m} M_c f_j(\xi_s).$$

as in the case (a). This proves (21). We also claim that for all $\delta > 0$

$$|\widetilde{T}_\delta(\vec{f}^{\infty})(y_s)| \le \widetilde{T}_*(\vec{f}\,)(y_s) + CA \prod_{j=1}^{m} M_c f_j(\xi_s). \tag{22}$$

Assuming (22) momentarily, observe that (21) and (22) imply

$$|\widetilde{T}_*(\vec{f}^{\infty})(x)| \le \widetilde{T}_*(\vec{f}\,)(y_s) + CA \prod_{j=1}^{m} M_c f_j(\xi_s) \le \alpha + CA\gamma\alpha \le 2\alpha,$$

if γ is small enough because y_s is in ${}^c\Omega$. For these γ's the set (20) is then empty.

It suffices therefore to prove (22). Let

$$d_1 = \mathrm{dist}(y_s, {}^c(Q_s^*)) \qquad \text{and} \qquad d_2 = \max_{z \in \partial({}^c(Q_s^*))} |y_s - z|.$$

Note that $d_1 \approx d_2 \approx |Q_s|^{1/n}$. For $\delta \ge d_2$, (22) follows immediately because $Q_s^* \subset S_{d_2}(y_s)$ and f_j^∞ agrees with f_j in the complement of Q_s^*. On the other hand, for $\delta < d_2$ we have that

$$\widetilde{T}_\delta(\vec{f}^{\infty})(y_s) = \widetilde{T}_{\max(\delta,d_1)}(\vec{f}^{\infty}\,)(y_s)$$

and hence

$$\widetilde{T}_\delta(\vec{f}^{\infty})(y_s) \le \widetilde{T}_*(\vec{f}\,)(y_s) + |\widetilde{T}_{\max(\delta,d_1)}(\vec{f}^{\infty})(y_s) - \widetilde{T}_{d_2}(\vec{f}^{\infty})(y_s)|, \tag{23}$$

since $\widetilde{T}_{d_2}(\vec{f}^{\infty})(y_s) = \widetilde{T}_{d_2}(\vec{f})(y_s)$. To prove (22) it suffices to show that the second term on the right of inequality (23) is controlled by $CA\prod_{j=1}^m M_c f_j(\xi_s)$. We have

$$\begin{aligned}
&|\widetilde{T}_{\max(\delta,d_1)}(\vec{f}^{\infty})(y_s) - \widetilde{T}_{d_2}(\vec{f}^{\infty})(y_s)| \\
&\le \int_{\vec{t}\in S_{d_2}(y_s)-S_{\max(\delta,d_1)}(y_s)} \frac{A\prod_{j=1}^m |f_j^{\infty}(t_j)|}{(|y_s-t_1|+\cdots+|y_s-t_m|)^{nm}}\,d\vec{t} \\
&\le \int_{\vec{t}\in S_{2d_2}(\xi_s)-S_{d_1/2}(\xi_s)} \frac{CA\prod_{j=1}^m |f_j(t_j)|}{(|\xi_s-t_1|+\cdots+|\xi_s-t_m|)^{nm}}\,d\vec{t} \qquad \text{by (17)} \\
&\le \sum_{k=1}^m \int_{\frac12 d_1<|\xi_s-t_k|\le 2\sqrt{n}d_2} \frac{CA\prod_{j=1}^m |f_j(t_j)|}{(|\xi_s-t_1|+\cdots+|\xi_s-t_m|)^{nm}}\,d\vec{t}.
\end{aligned}$$

We estimate the term with $k=m$; the other are analogous. We have

$$\begin{aligned}
&A\int_{(\mathbf{R}^n)^{m-1}}\int_{\frac12 d_1<|\xi_s-t_m|\le 2\sqrt{n}d_2} \frac{\prod_{j=1}^m |f_j(t_j)|dt_m dt_{m-1}\dots dt_1}{(|\xi_s-t_1|+\cdots+|\xi_s-t_m|)^{nm}} \\
&\le CAM_c f_1(\xi_s)\int_{(\mathbf{R}^n)^{m-2}}\int_{\frac12 d_1<|\xi_s-t_m|\le 2\sqrt{n}d_2} \frac{\prod_{j=2}^m |f_j(t_j)|dt_m dt_{m-1}\dots dt_2}{(|\xi_s-t_2|+\cdots+|\xi_s-t_m|)^{n(m-1)}} \\
&\le \qquad \dots \\
&\le CA\prod_{j=1}^{m-1} M_c f_j(\xi_s)\int_{\frac12 d_1<|\xi_s-t_m|\le 2\sqrt{n}d_2} \frac{|f_m(t_m)|}{|\xi_s-t_m|^n}\,dt_m \le CA\prod_{j=1}^m M_c f_j(\xi_s).
\end{aligned}$$

This proves (22) and the proof of the theorem is complete. □

Corollary 2. *Let $1\le p_1,\dots,p_m<\infty$, and p be such that $1/p_1+\cdots+1/p_m=1/p$, and $w\in A_\infty$. Let T be an m-linear Calderón-Zygmund operator. Then there is a $C_{p,n}<\infty$ so that for all $\vec{f}=(f_1,\dots,f_m)$ satisfying $\|T_*(\vec{f})\|_{L^p(w)}<\infty$ we have*

$$\|T_*(\vec{f})\|_{L^p(w)} \le C_{p,n}(A+W)\prod_{j=1}^m \|M_c f_j\|_{L^{p_j}(w)}. \tag{24}$$

Moreover, if $p_0=\min(p_1,\dots,p_m)>1$, and $w\in A_{p_0}$, then

$$\|T_*(\vec{f})\|_{L^p(w)} \le C_{p,n}(A+W)\prod_{j=1}^m \|f_j\|_{L^{p_j}(w)}. \tag{25}$$

Proof. The first part of the corollary with T_* replaced by $\widetilde{T}_*$ follows from Theorem 2 and standard estimates using distribution functions. For this we need the assumption that $\|T_*(\vec{f})\|_{L^p(w)}<\infty$. Estimate (24) then also follows for T_* which is controlled by $\widetilde{T}_*$ and M_c. For (25), just observe that $A_{p_0}\subset A_{p_j}$ and M_c is bounded on $L^{p_j}(w)$ when $w\in A_{p_j}$. □

Remark 2. The hypothesis $\|T_*(\vec{f})\|_{L^p(w)}<\infty$ is always satisfied if each component in $\vec{f}$ is a bounded function with compact support and w is in A_{p_0}, $p_0>1$ as above. In fact, in this case, $T_*(\vec{f})(x) \sim |x|^{-nm}$ near infinity and thus $T_*(\vec{f})$ is in $L^p(w)$ outside a compact set since $mp \geq p_0$ and $w(x)|x|^{-np_0}$ is integrable at infinity. Moreover inside a compact set w^q is integrable for some $q > 1$, and thus $|T_*(\vec{f})|^p \in L^{q'}$ as it easily follows from $T_* : L^{mpq'} \times \cdots \times L^{mpq'} \to L^{pq'}$, a consequence of Corollary 1.

We can extend the weighted norm inequalities above to a Calderón-Zygmund operator T itself. To do so we first need the following simple lemma.

Lemma 1. *Let T be an m-linear pointwise multiplier operator of the form*

$$T(f_1, \ldots, f_m)(x) = b(x)f_1(x) \ldots f_m(x),$$

where b is a measurable function. If T maps $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for some $1 < p_1, \ldots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/p$, then b is in L^∞ with norm at most a multiple of the norm of T.

Proof. We proceed by induction on m. In the linear case the statement is well-known. Assume then that the result is true for $(m-1)$-linear pointwise multipliers. Suppose that $T(f_1, \ldots, f_m) = bf_1 \ldots f_m$ maps $L^{p_1} \times \cdots \times L^{p_m}$ into L^p, for some $1 < p_1, \ldots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/p$. Then, since T agrees with its m-transposes, duality and interpolation gives that T is bounded on all product of Lebesgue spaces with $1 < q_1, \ldots, q_m < \infty$ and $1/q_1 + \cdots + 1/q_m = 1/q$. See e.g. [6] for details. In particular, T maps $L^{2(m-1)} \times \cdots \times L^{2(m-1)} \times L^2$ into L^1. It follows that $T_{m-1}(f_1, \ldots, f_{m-1}) = bf_1 \ldots f_{m-1}$ is an $(m-1)$-linear pointwise multiplier that maps $L^{2(m-1)} \times \cdots \times L^{2(m-1)}$ into L^2. The induction hypothesis gives that b is bounded and the claimed estimate for $\|b\|_{L^\infty}$ follows. □

Corollary 3. *Let T be an m-linear Calderón-Zygmund operator. Fix exponents $1 < p_1, \ldots, p_m < \infty$, and p such that $1/p_1 + \cdots + 1/p_m = 1/p$, and let w be a weight in A_∞. Then, there is a constant $C_{p,n} < \infty$ so that for all $\vec{f} = (f_1, \ldots, f_m)$ with each f_j bounded and compactly supported we have*

$$\|T(\vec{f})\|_{L^p(w)} \leq C_{p,n}(A + W) \prod_{j=1}^{m} \|M_c f_j\|_{L^{p_j}(w)}. \tag{26}$$

Moreover, if $w \in A_{p_0}$, with $p_0 = \min(p_1, \ldots, p_m)$, then

$$\|T(\vec{f})\|_{L^p(w)} \leq C_{p,n}(A + W) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w)} \tag{27}$$

and, in particular, T extends as a bounded operator from $L^{p_1}(w) \times \cdots \times L^{p_m}(w)$ into $L^p(w)$

Proof. We will control T by T_*. First we observe that since T_* is bounded on products of L^{p_j} spaces, then the truncated singular integrals T_δ are uniformly bounded and thus there is a subsequence T_{δ_j} which converges weakly in L^p to a limit T_0. Next, we claim that the given T differs from T_0 by a pointwise multiplier. That is, for all f_j bounded and with compact support we have,

$$T(f_1, \ldots, f_m) - T_0(f_1, \ldots, f_m) = bf_1 \ldots f_m,$$

where b is a well-defined measurable function. We just sketch the proof of this claim following the arguments for the analogous linear case in [9] p. 34. We set

$$\Delta(g_1, \dots g_m) = T(g_1, \dots, g_m) - T_0(g_1, \dots, g_m)$$

and we observe that for all $g_j \in L^{q_j}$ we have

$$\Delta(g_1, \dots g_m)(x) = 0, \tag{28}$$

whenever $x \notin \cap_{j=1}^m \mathrm{supp} g_j$. To see (28), note that if δ is smaller than the distance from x to $\cap_{j=1}^m \mathrm{supp} g_j$, then $T = T_\delta$. Next we observe that for all $g_j \in L^{q_j}$ and all cubes Q_j we have

$$\Delta(\chi_{Q_1} g_1, \dots \chi_{Q_m} g_m) = \chi_{Q_1} \dots \chi_{Q_m} \Delta(g_1, \dots, g_m). \tag{29}$$

Indeed, if $x \notin \cap_{j=1}^m Q_j$, then both terms in (29) are zero by (28). If $x \in \cap_{j=1}^m Q_j$, then we write each $\chi_{Q_j} g_j$ as $g_j - \chi_{^cQ_j} g_j$ and we use multilinearity and (28) to prove (29).

Once we know (29) we use linearity and density to obtain that

$$\Delta(f_1 g_1, \dots f_m g_m) = f_1 \dots f_m \Delta(g_1, \dots, g_m)$$

for all $g_j \in L^{q_j}$ and f_j in L^∞ with compact support. We now take $O_r = B(0, r)$. For $x \in O_r$ (29) gives

$$\Delta(\chi_{O_r}, \dots, \chi_{O_r}) = \Delta(\chi_{O_r} \chi_{O_{r+1}}, \dots, \chi_{O_r} \chi_{O_{r+1}}) = \chi_{O_{r+1}} \Delta(\chi_{O_{r+1}}, \dots, \chi_{O_{r+1}})$$

and this identity implies that the function

$$b(x) = \Delta(\chi_{O_r}, \dots, \chi_{O_r})(x), \qquad \text{when } x \in O_r$$

is well defined on $\mathbf{R}^n$. Now take f_j compactly supported and bounded. Then pick an $r > 0$ so that $\cup_{j=1}^m \mathrm{supp} f_j \subset B(0, r)$. Then

$$\Delta(f_1, \dots, f_m) = \Delta(\chi_{O_r} f_1, \dots, \chi_{O_r} f_m) = b f_1 \dots f_m.$$

Finally, since both T and T_0 are bounded, it follows from Lemma 1 that b is in L^∞. Then,

$$|T(\vec{f})| \le |T_0(\vec{f})| + \|b\|_{L^\infty} |f_1 \dots f_m| \le T_*(\vec{f}) + \|b\|_{L^\infty} |f_1 \dots f_m|,$$

and all the estimates for T follow from the corresponding ones for T_* once we observe that

$$\|b\|_{L^\infty} \le \|T - T_0\| \le \|T\| + \|T_*\| \le C(A + W),$$

where $\|.\|$ denotes the operator norm in the unweighted Lebesgue spaces. □

Remark 3. For $\vec{f}$ as in the corollary, we clearly also have the estimate

$$\|T(\vec{f})\|_{L^{1/m,\infty}(w)} \le C_{m,n}(A + W) \prod_{j=1}^m \|M_c f_j\|_{L^{1,\infty}(w)} \tag{30}$$

and, in particular, if $w \in A_1$, then

$$T : L^1(w) \times \dots \times L^1(w) \to L^{1/m,\infty}(w), \tag{31}$$

since the Hardy-Littlewood maximal function is of weak-type (1,1) if and only if w is in A_1.

Remark 4. Using Corollary 3 we can now improve on Remark 2. Let $p_0 > 1$ and $w \in A_{p_0}$. Then for all $\vec{f}$ in a product of L^{p_j} spaces with $1 < p_1, \dots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$, we have $\|T_*(\vec{f})\|_{L^p(w)} < \infty$. In fact, we can use Cotlar's inequality (7) to control T_* pointwise. Taking $\eta = 1/m$ in (7) and using that $pm \geq p_0 > 1$ we obtain

$$\begin{aligned}
\|T_*(\vec{f})\|_{L^p(w)} \leq & C\left(\|M(|T(\vec{f})|^{1/m})\|^m_{L^{pm}(w)} + \|\prod_{j=1}^m Mf_j\|_{L^p(w)} \right) \\
\leq & C\left(\||T(\vec{f})|^{1/m}\|^m_{L^{pm}(w)} + \prod_{j=1}^m \|Mf_j\|_{L^{p_j}(w)} \right) \\
\leq & C\left(\|T(\vec{f})\|_{L^p(w)} + \prod_{j=1}^m \|Mf_j\|_{L^{p_j}(w)} \right) \\
\leq & C \prod_{j=1}^m \|Mf_j\|_{L^{p_j}(w)} < \infty.
\end{aligned}$$

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