APPENDIX

THE MARY WEISS LEMMA

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Abstract. We prove a version of a Lemma due to Mary Weiss on $\mathbb{R}^n$ equipped with the family of dilations $\lambda^\alpha(x_1,\dots,x_n)=(\lambda_1^{\alpha_1}x_1,\dots,\lambda_n^{\alpha_n}x_n)$. We consider both the case of a full order derivative and of fractional derivatives.

If F is a function on $\mathbb{R}^n$ and $|\nabla F|$ in L^p for some $p>n$, it can be shown that for fixed y, the function $F_y(x)=\frac{|F(x)-F(y)|}{|x-y|}$ is in L^p. A Lemma due to Mary Weiss [CaC] says that the supremum of F_y over all $y\in\mathbb{R}^n-\{x\}$ is also in L^p.

We will generalize this Lemma for arbitrary dilations and fractional order differentiation. We first introduce some notation. We let $\alpha=(\alpha_1,\dots,\alpha_n)$ denote a multiindex with real entries such that $1=\alpha_1\le\alpha_2\le\cdots\le\alpha_n$. Define $k_0=\max\{j:\alpha_j=1\}$, and for $z\in\mathbb{R}^n$, let $z'=(z_1,\dots,z_{k_0})$ and $z''=(z_{k_0+1},\dots,z_n)$. We set

$$\nabla_{z'}f=\Big(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_{k_0}}\Big)\qquad\text{and}\tag{0.1}$$

$$\nabla_{z''}f=\Big(\frac{\partial f}{\partial z_{k_0+1}},\dots,\frac{\partial f}{\partial z_n}\Big).\tag{0.2}$$

Let $\|\cdot\|$ denote the unique positive solution ρ of the equation

$$\sum_{j=1}^n\frac{z_j^2}{\rho^{2\alpha_j}}=1.\tag{0.3}$$

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We call $\|\cdot\|$ the nonisotropic norm associated with the multiindex α. Note that $\|\cdot\|$ is homogeneous of degree one with respect to the family of dilations

$$z \to \lambda^{\alpha} z = (\lambda^{\alpha_1} z_1, \ldots, \lambda^{\alpha_n} z_n). \tag{0.4}$$

We define the nonisotropic fractional differentiation operator $\mathbb{D}^{\beta}$ by

$$\widehat{\mathbb{D}^{\beta} f}(\zeta) = \|\zeta\|^{\beta} \hat{f}(\zeta). \tag{0.5}$$

We also define the nonisotropic Hardy-Littlewood maximal function associated with the norm $\|\cdot\|$ by

$$(\mathbb{M}f)(z) = \sup_{I} \frac{1}{|I|} \int_{I} |f(y)|\, dy, \tag{0.6}$$

where the supremum is taken over all sets $I = \{w : \|w - z\| \le N\}$ with $N > 0$. Let $d = |\alpha| = \sum_{j=1}^{n} \alpha_j$ be the homogeneous dimension of the norm $\|\cdot\|$. Below L^q will always be $L^q(\mathbb{R}^n)$ for some fixed $n \ge 2$. We have the following

Lemma. *Let $0 < \beta \le 1$ and $\frac{d}{\beta} < r < \infty$. Then for all p with $\frac{d}{\beta} < p < r$, there exists a constant $C_{p,\beta} > 0$ such that for all A on $\mathbb{R}^n$ with $\mathbb{D}^{\beta} A = a \in L^r$, we have*

$$|A(u) - A(v)| \le C_{p,\beta} \|u - v\|^{\beta} \big\{ [\mathbb{M}(|a|^p)(u)]^{\frac{1}{p}} + \mathbb{M}(|a|)(u) + (R_* a)(u) \big\}, \tag{0.7}$$

where $\mathcal{R}_*$ denotes a "nice" nonisotropic maximal singular integral which is bounded on L^p for all $p > 1$.

1. The case $\beta = 1$. By definition $A = \mathcal{J}_{\beta} a$, where $\mathcal{J}_{\beta}$ is the nonisotropic Riesz potential defined by

$$\widehat{\mathcal{J}_{\beta} f}(\zeta) = \|\zeta\|^{-\beta} \hat{f}(\zeta). \tag{1.1}$$

We set $\mathcal{J}_1 = \mathcal{J}$. If J is the kernel of $\mathcal{J}$, then by [FR], J is in $C^{\infty}(\mathbb{R}^n - \{0\})$, and

$$J(\lambda^{\alpha} z) = \lambda^{(1-d)} J(z). \tag{1.2}$$

We introduce a smooth cutoff function η with $\eta = 1$ on $[-10, 10]$ and $\eta = 0$ on the complement of $[-20, 20]$. Set $\delta = \|u - v\|$. We have

$$\begin{aligned} A(u) - A(v) &= \int [J(u-w) - J(v-w)]a(w)\, dw \\ &= \int J^1(u-w)a(w)\, dw - \int J^1(v-w)a(w)\, dw + \int [J^2(u-w) - J^2(v-w)]a(w)\, dw \\ &= I + II + III, \end{aligned}$$

where $J^1(z) = J(z)\eta(\frac{\|z\|}{\delta})$ and $J^2(z) = J(z)[1 - \eta(\frac{\|z\|}{\delta})]$. To estimate term I, we use (1.2) and a routine modification of the argument in [St] pp 62-64. We obtain that

$$|I| \leq C\delta\, \mathbb{M}(|a|)(u) = C\|u - v\|\, \mathbb{M}(|a|)(u). \tag{1.3}$$

By Hölder's inequality and (1.2), we have

$$|II| \leq \Big(\int_{\|u-w\| \leq 20\delta} |a(w)|^p\, dw \Big)^{\frac{1}{p}} \Big(\int_{\|u-w\| \leq 20\delta} \|v - w\|^{(1-d)p'}\, dw \Big)^{\frac{1}{p'}} \leq C\delta\, [\mathbb{M}(|a|^p)(u)]^{\frac{1}{p}}, \tag{1.4}$$

since $\|u - w\| \leq 20\delta$ implies $\|v - w\| \leq 21\delta$ and the required inequality $(1 - d)p' > -d$ follows from $p > d$. This completes the estimate for term II. We now write term III as follows:

$$\begin{aligned} &\int \big\{J^2(u-w) - J^2(v-w) - (u' - v') \cdot (\nabla_{u'} J^2)(u-w)\big\}a(w)\, dw \\ &\qquad + (u' - v') \int (\nabla_{u'} J^2)(u-w)a(w)\, dw \\ &= III_1 + III_2. \end{aligned} \tag{1.5}$$

Note that $J^2(u - w)$ is supported where $\|u - w\| \geq 10\delta$. By Taylor's Theorem and (1.2) we have that the expression inside curly brackets in III_1 is bounded in absolute value by

$$\begin{aligned} &|(u'' - v'') \cdot (\nabla_{u''} J^2)(u-w)| + O\Big(\frac{\delta^2}{(\delta + \|u - w\|)^{d+1}}\Big) \\ &\leq C \sum_{j=k_0+1}^{n} \frac{\delta^{\alpha_j}}{(\delta + \|u - w\|)^{d+\alpha_j-1}} + O\Big(\frac{\delta^2}{(\delta + \|u - w\|)^{d+1}}\Big) \\ &\leq C\delta\, \frac{\delta^{\varepsilon}}{(\delta + \|u - w\|)^{d+\varepsilon}} \qquad \text{for some } \varepsilon > 0. \end{aligned} \tag{1.6}$$

Clearly the integral of (1.6) is bounded by $C\delta\,\mathbb{M}(|a|)(u)$.

Finally, let $\vec{R}(z) = (\nabla_{z'}J)(z)$. $\vec{R}$ is a nonisotropic Calderón-Zygmund kernel, that is homogeneous of degree of degree $-d$ and C^∞ away from the origin. We have $(\nabla_{z'}J^2)(z) = J_1^2 + J_2^2$, where $J_1^2 = [1-\eta(\frac{\|z\|}{\delta})](\nabla_{z'}J)(z)$ and $J_2^2 = -J(z)\nabla_{z'}(\eta(\frac{\|z\|}{\delta}))$. The operator with kernel J_2^2 can be shown as before to be pointwise bounded by $C\,\mathbb{M}(|a|)(u)$. The operator with kernel J_1^2 is dominated by

$$(1.7)\qquad \mathcal{R}_* a(u) = \sup_{\delta>0}\left|\int \vec{R}(u-w)[1-\eta(\tfrac{\|u-w\|}{\delta})]a(w)\,dw\right|,$$

and $\mathcal{R}_*$ is a "nice" maximal singular integral which maps L^p to L^p for all $p>1$. Since $\|u'-v'\|\le\delta$, it follows that term III_2 satisfies the estimate (0.7).

2. The case of fractional differentiation. We now take up the case $0<\beta<1$. As before, we have

$$\begin{aligned} &A(u)-A(v) = \int [J_\beta(u-w)-J_\beta(v-w)]a(w)\,dw \\ &= \int J_\beta^1(u-w)a(w)\,dw - \int J_\beta^1(v-w)a(w)\,dw + \int [J_\beta^2(u-w)a(w) - J_\beta^2(v-w)]a(w)\,dw \\ &= I + II + III, \end{aligned}$$

where $J_\beta^1(z) = J_\beta(z)\eta(\frac{\|z\|}{\delta})$, $J_\beta^2(z) = J_\beta(z) - J_\beta^1(z)$, $\delta = \|u-v\|$, and η is the bump introduced in the previous section. It is easy to see that $|J_\beta(z)|\le C\|z\|^{\beta-d}$. We certainly have that $|I|\le C\delta^\beta\mathbb{M}(|a|)(u)$. Also, Hölder's inequality gives

$$|II| \le \Big(\int_{\|u-w\|\le 20\delta} |a(w)|^p\,dw\Big)^{\frac{1}{p}}\Big(\int_{\|u-w\|\le 20\delta} \|v-w\|^{(\beta-d)p'}\,dw\Big)^{\frac{1}{p'}} \le C\delta\,[\mathbb{M}(|a|^p)(u)]^{\frac{1}{p}},$$

if $(\beta-d)p' > -d$, i.e. $p>\frac{d}{\beta}$. Finally in term III, $\|v-w\|\sim\|u-w\| >> \delta$, so

$$|J_\beta(u-w)-J_\beta(v-w)| \le C\frac{\|u-v\|}{\|u-w\|^{d+1-\beta}}\chi_{\|u-w\|>\delta} \le C\frac{\delta^{1-\beta}}{(\delta+\|u-w\|)^{d+1-\beta}}.$$

Since $0 < 1-\beta < 1$, integrating with respect to w, we obtain that

$$|III| \leq C\delta^\beta \mathbb{M}(|a|)(u)$$

This concludes the proof of the Lemma.

3. Remarks and applications. The analogous formulation of (0.7) for $1 < \beta < n$ is

$$|A(y) - \sum_{|\gamma| \leq [\beta]} \frac{1}{\gamma!}\frac{\partial^\gamma A}{\partial x^\gamma}(x)\,(y-x)^\gamma| \leq C_{p,\beta}\|x-y\|^\beta \big\{[\mathbb{M}(|a|^p)(x)]^{\frac{1}{p}} + \mathbb{M}(|a|)(x) + (\mathcal{R}_* a)(x)\big\}, \tag{3.1}$$

where $\frac{n}{\beta} < p < r$ and $a = \mathbb{D}^\beta A$ is in L^r. Here $[\beta]$ is the greatest integer $\leq \beta$.

Let us sketch the proof of (3.1) in the special case where $\alpha_j = 1$ for all j. Fix x and y and let $t = y - x$. The left hand side of (3.1) is the sum of the following three expressions:

$$\int_{|z-x|\geq 10|t|} \big(|y-z|^{-n+\beta} - \sum_{|\gamma|\leq[\beta]} \frac{t^\gamma}{\gamma!}\frac{\partial^\gamma}{\partial x^\gamma}(|x-z|^{-n+\beta})\big)a(z)\,dz \tag{3.2}$$

$$-\int_{|z-x|\leq 10|t|} \sum_{|\gamma|\leq[\beta]} \frac{t^\gamma}{\gamma!}\frac{\partial^\gamma}{\partial x^\gamma}(|x-z|^{-n+\beta})a(z)\,dz \tag{3.3}$$

$$\int_{|z-x|\leq 10|t|} |y-z|^{-n+\beta} a(z)\,dz. \tag{3.4}$$

By Taylor's theorem there exists a ξ_z on the line segment joining $x-z$ to $y-z$ such that

$$|y-z|^{-n+\beta} - \sum_{|\gamma|\leq[\beta]} \frac{t^\gamma}{\gamma!}\frac{\partial^\gamma}{\partial x^\gamma}(|x-z|^{-n+\beta}) = \sum_{|\gamma|=[\beta]+1} \frac{t^\gamma}{\gamma!}\frac{\partial^\gamma}{\partial x^\gamma}(|x-z|^{-n+\beta})\big|_{x=\xi_z}.$$

The γ^{th} derivative of $|x-z|^{-n+\beta}$ decays like $|x-z|^{-n+\beta-|\gamma|}$ near infinity and since $|\xi_z - z|$ is comparable to $|z-x|$, we estimate (3.2) by

$$\sum_{|\gamma|=[\beta]+1} C_\gamma\ |t|^{|\gamma|} \int_{|z-x|\geq 10|t|} |z-x|^{-n-(|\gamma|-\beta)}|a(z)|\,dz \leq C|t|^\beta \mathbb{M}(|a|)(x), \tag{3.5}$$

where we used that $1 + [\beta] - \beta > 0$. To estimate (3.3) note that for any fixed $|\gamma| \leq [\beta]$, we have

$$\begin{aligned} &\Big|\int_{|z-x|\leq 10|t|} t^\gamma \frac{\partial^\gamma}{\partial x^\gamma}(|z-x|^{-n+\beta})a(z)\,dz\Big| \\ \leq\, &C_\gamma |t|^{|\gamma|} \int_{|z-x|\leq 10|t|} |z-x|^{-n+\beta-|\gamma|}|a(z)|\,dz \leq C_\gamma |t|^\beta \mathbb{M}(|a|)(x), \end{aligned} \tag{3.6}$$

provided that $\beta - |\gamma| > 0$, which certainly holds unless β is an integer and $|\gamma| = \beta$. In this exceptional case we argue differently. Suppose that β is an integer and fix $\gamma = (\gamma_1, \cdots, \gamma_n)$ with $|\gamma| = \beta$. Let $\mathcal{R}_\gamma = R_1^{\gamma_1} R_2^{\gamma_2} \dots R_n^{\gamma_n}$, where R_j is the usual j^{th} Riesz transform. Also let $(\mathcal{R}_\gamma)_*$ be the maximal truncated singular integral of $\mathcal{R}_\gamma$. If K_γ is the kernel of the operator $\mathcal{R}_\gamma$, an easy calculation gives that $\frac{\partial^\gamma}{\partial x^\gamma}(|x|^{-n+\beta}) = c_\gamma K_\gamma(x)$. Therefore, when β is an integer, we estimate the part of the sum in (3.3) with $|\gamma| = \beta$ by

$$\left| t^\beta \left((f * K_\gamma)(x) - \int_{|z-x| \geq 10|t|} K_\gamma(x-z) a(z)\, dz \right) \right| \leq C|t|^\beta [(\mathcal{R}_\gamma)_*(a)(x) + (\mathcal{R}_\gamma a)(x)].$$

Finally, note the the domain of integration of the integral in (3.4) is contained in the set $\{z : |z - y| \leq 11|t|\}$. We apply Hölder's iequality to the functions $|z-y|^{-n+\beta} \chi_{|z-y| \leq 11|t|}$ and $a(z)\chi_{|z-x| \leq 10|t|}$ with exponents p' and p respectively. Since $\frac{n}{\beta} < p < r$, the function $|z-y|^{-n+\beta}\chi_{|z-y|\leq 11|t|}$ is in $L^{p'}$. We deduce that (3.4) is bounded by $C|t|^\beta [\mathbb{M}(|a|^p)]^{\frac{1}{p}}(x)$.

As a consequence, we obtain the following

Corollary. *Let $0 < \beta < n$ and $\frac{n}{\beta} < r < \infty$. Suppose that $a = \mathbb{D}^\beta A$ is in L^r. Then*

$$A_*(u) = \sup_{v \in \mathbb{R}^n - \{u\}} \frac{|A(v) - \sum_{|\gamma| \leq [\beta]} \frac{1}{\gamma!} \frac{\partial^\gamma A}{\partial u^\gamma}(u)\,(v-u)^\gamma|}{|u-v|^\beta} \tag{3.7}$$

is also in L^r (with norm $\leq C_{r,\beta} \|a\|_{L^r}$.)

Let A be a function as in the corollary. When $\beta \leq 1$, (3.7) implies that for all $\varepsilon > 0$, there exists a set S_A, whose complement has measure less than ε, on which A is Hölder continuous of order β in the following sense: there exists a constant C, which depends on A, n, r and ε, such that

$$\text{for all } u \in S_A \text{ and all } v \in \mathbb{R}^n, \text{ we have } |A(u) - A(v)| \leq C|u-v|^\beta. \tag{3.8}$$

When $r = \infty$, S_A can be taken to be the whole space. For $r < \infty$, (3.8) gives a weaker version of Hölder continuity.

One might guess that A_* could be in weak L^r if $r = \frac{n}{\beta}$. This turns out to be false as the example $A(x) = (\log\log \frac{1}{|x|})\chi_{|x|\leq 1}$ shows in $\mathbb{R}^2$ when $r = 2$ and $\beta = 1$.

References

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