Conditions for Boundedness into Hardy spaces

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We obtain the boundedness from a product of Lebesgue or Hardy spaces into Hardy spaces under suitable cancellation conditions for a large class of multilinear operators that includes the Coifman–Meyer class, sums of products of linear Calderón–Zygmund operators and combinations of these two types.

1 Introduction

In this work, we obtain the boundedness of multilinear singular operators of various types from products of Lebesgue or Hardy spaces into Hardy spaces, under suitable cancellation conditions. This particular line of investigation was initiated in the work of Coifman, Lions, Meyer and Semmes [1] who showed that certain bilinear operators with vanishing integral map $L^q \times L^{q'}$ into the Hardy space H^1 for $1 < q < \infty$ with $q' = q/(q-1)$. This result was extended by Dobyinksi [5] for Coifman–Meyer multiplier operators and by Coifman and Grafakos [4] for finite sums of products of Calderón–Zygmund operators. In [4] boundedness was extended to $H^{p_1} \times H^{p_2} \to H^p$ for the entire range $0 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$, under the necessary cancellation conditions.

Additional approaches to these results were provided by Grafakos and Li [9], Hu and Meng [13], and Huang and Liu [14]. In this work we investigate this topic via a new method based on (p,∞)-atomic decompositions. Our approach is powerful enough to encompass many types of multilinear operators that include all the previously studied (Coifman–Meyer type and finite sums of products of Calderón–Zygmund operators), as well as mixed types. An alternative approach to Hardy space estimates for bilinear operators has appeared in the recent work of Hart and Lu [12].

Recall that the Hardy space H^p with $0 < p < \infty$ is given as the space of all tempered distributions f for which

$$\|f\|_{H^p} = \Big\| \sup_{t>0} |e^{t\Delta} f| \Big\|_{L^p}$$

is finite, where $e^{t\Delta}$ denotes the heat semigroup for $0 < p \le \infty$. Note that H^p and L^p are isomorphic with norm equivalence when $1 < p \le \infty$. See [18].

In this work we study the boundedness into H^p of the following three types of operators:

- multilinear singular integral operators of Coifman–Meyer type;

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- sums of m-fold products of linear Calderón–Zygmund singular integrals;
- multilinear singular integrals of mixed type (i.e., combinations of the previous two types).

Let m, n be positive integers, and let $f_1, \dots, f_m \in \mathcal{S}$. For a bounded function σ on $(\mathbf{R}^n)^m$, we consider the multilinear operator

$$\mathcal{T}_\sigma(f_1, \dots, f_m)(x) = \int_{(\mathbf{R}^n)^m} \sigma(\xi_1, \dots, \xi_m)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\, d\xi_1\cdots d\xi_m$$

for $x \in \mathbf{R}^n$. Here $\mathcal{S}$ is the space of Schwartz functions and $\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ is the Fourier transform of a given Schwartz function f on $\mathbf{R}^n$. The space of tempered distributions is denoted by $\mathcal{S}'$.

Certain conditions on σ imply that $\mathcal{T}_\sigma$ extends to a bounded linear operator from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p as long as $1 < p_1, \dots, p_m \le \infty$ and $0 < p < \infty$ satisfies

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}. \tag{1}$$

One of these, modeled after the classical Mikhlin linear multiplier condition, is the following Coifman–Meyer condition, which says

$$|\partial^\alpha \sigma(\xi_1, \dots, \xi_m)| \le C(|\xi_1| + \cdots + |\xi_m|)^{-|\alpha|}, \quad (\xi_1, \dots, \xi_m) \in (\mathbf{R}^n)^m \setminus \{0\} \tag{2}$$

for $\alpha \in ({\mathbf{N}_0}^n)^m$ satisfying $|\alpha| \le M$ for some large M. The associated operators are called m-linear Calderón–Zygmund operators and there is a rich theory for them analogous to the linear one.

An m-linear Calderón–Zygmund operator associated with a Calderón–Zygmund kernel K on $\mathbf{R}^{mn}$ is defined by

$$\mathcal{T}_\sigma(f_1, \dots, f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x - y_1, \dots, x - y_m)f_1(y_1)\cdots f_m(y_m)\, dy_1\cdots dy_m, \tag{3}$$

where σ is the distributional Fourier transform of K on $(\mathbf{R}^n)^m$ that satisfies (2). When $m = 1$, these operators reduce to classical Calderón–Zygmund singular integral operators.

We also study another type of m-linear operators so called of *product type*. Before giving a general definition, we introduce two examples of such operators. Let σ_1, σ_2 be the classical Mihlin multipliers defined by $\sigma(\xi_1, \xi_2) = \sigma_1(\xi_1)\sigma_2(\xi_2)$. Then the first prominent example is

$$\mathcal{T}_\sigma(f_1, f_2)(x) = \int_{\mathbf{R}^n\times\mathbf{R}^n} \sigma_1(\xi_1)\sigma_2(\xi_2)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)e^{2\pi i x\cdot(\xi_1+\xi_2)}d\xi_1 d\xi_2 = \mathcal{T}_{\sigma_1}(f_1)(x)\mathcal{T}_{\sigma_2}(f_2)(x).$$

Secondly, let $n = 2, m = 2$ and for $\xi = (\eta, \rho) \in \mathbf{R}^2$, set $\sigma_1(\xi) = \frac{\eta}{|(\eta,\rho)|}$, $\sigma_2(\xi) = \frac{\rho}{|(\eta,\rho)|}$. Namely, these are multipliers associated with Riesz transform on $\mathbf{R}^2$. Then we define for $(\xi_1, \xi_2) \in \mathbf{R}^2 \times \mathbf{R}^2$, $\sigma(\xi_1, \xi_2) = \sigma_1(\xi_1)\sigma_2(\xi_2) - \sigma_2(\xi_1)\sigma_1(\xi_2)$ which implies

$$\mathcal{T}_\sigma(f_1, f_2)(x) = \mathcal{T}_{\sigma_1}(f_1)(x)\mathcal{T}_{\sigma_2}(f_2)(x) - \mathcal{T}_{\sigma_2}(f_1)(x)\mathcal{T}_{\sigma_1}(f_2)(x). \tag{4}$$

Clearly, example (4) does not satisfy the Coifman–Meyer type condition (2). The second example was motivated by the determinant of the Jacobian of a map (f, g): $J(f, g) = \partial_{x_1} f \partial_{x_2} g - \partial_{x_2} f \partial_{x_1} g$, and its mapping property into the Hardy space H^1, which was studied in [4]. It would be crucial to notice that for the second example,

$$\int_{\mathbf{R}^2} \mathcal{T}_\sigma(f_1, f_2)(x)dx = \int_{\mathbf{R}^2} (\mathcal{T}_{\sigma_1}(f_1)(x)\mathcal{T}_{\sigma_2}(f_2)(x) - \mathcal{T}_{\sigma_2}(f_1)(x)\mathcal{T}_{\sigma_1}(f_2)(x))dx = 0.$$

We now introduce operators of a more general form. An m-linear operator of *product type* on $\mathbf{R}^{mn}$ is defined by

$$\sum_{\rho=1}^{T} T_{\sigma_1^\rho}(f_1)(x)\cdots T_{\sigma_m^\rho}(f_m)(x) \quad (x \in \mathbf{R}^n), \tag{5}$$

where the $T_{\sigma_j^\rho}$'s are linear Calderón–Zygmund operators associated with the multipliers σ_j^ρ. In terms of kernels these operators can be expressed as

$$\mathcal{T}_\sigma(f_1,\dots,f_m)(x)=\sum_{\rho=1}^{T}\prod_{j=1}^{m}\int_{\mathbf{R}^n}K_{\sigma_j}^\rho(x-y_j)f_j(y_j)dy_j,$$

where $K_1^\rho,\dots,K_m^\rho$ are the Calderón–Zygmund kernels of the operator $T_{\sigma_1}^\rho,\dots,T_{\sigma_m}^\rho$, respectively for $\rho=1,\dots,T$.

With these two types of multiplier operators, Coifman–Meyer type and *product type* in mind, it seems natural to consider their mixed versions. For instance, to motivate the general case we consider the following example of a 6-linear operator: Let $I_1=\{1,2,3\},I_2=\{4,5\},I_3=\{6\}$. Also let $\mathcal{T}_{\sigma_{I_1}}$, $\mathcal{T}_{\sigma_{I_2}}$ and $\mathcal{T}_{\sigma_{I_3}}$ be respectively 3-linear, 2-linear and 1-linear Coifman–Meyer type operators-namely, all the multipliers $\sigma_{I_1},\sigma_{I_2},\sigma_{I_3}$ satisfy (2). Then we define $\sigma(\xi_1,\dots,\xi_6)=\sigma_{I_1}(\xi_1,\xi_2,\xi_3)\sigma_{I_2}(\xi_4,\xi_5)\sigma_{I_3}(\xi_6)$ and hence,

$$\mathcal{T}_\sigma(f_1,\dots,f_6)(x)=\mathcal{T}_{\sigma_{I_1}}(f_1,f_2,f_3)(x)\mathcal{T}_{\sigma_{I_2}}(f_4,f_5)(x)\mathcal{T}_{\sigma_{I_3}}(f_6)(x).$$

More generally, we consider operators of *mixed type*, i.e., of the form

$$\mathcal{T}_\sigma(f_1,\dots,f_m)(x)=\sum_{\rho=1}^{T}\sum_{I_1^\rho,\dots,I_{G(\rho)}^\rho}\prod_{g=1}^{G(\rho)}T_{\sigma_{I_g^\rho}}(\{f_l\}_{l\in I_g^\rho})(x),\tag{6}$$

where for each $\rho=1,\dots,T$, $I_1^\rho,\dots,I_{G(\rho)}^\rho$ is a partition of $\{1,\dots,m\}$ and each $T_{\sigma_{I_g^\rho}}$ is an $|I_g^\rho|$-linear Coifman–Meyer multiplier operator. We write $I_1^\rho+\cdots+I_{G(\rho)}^\rho=\{1,\dots,m\}$ to denote such partitions.

In this work, we study operators of the form (3), (5), and (6). We will be working with indices in the range

$$0<p_1,\dots,p_m\le\infty,\quad 0<p<\infty$$

that satisfy (1). Throughout this paper we reserve the letter s to denote the following index:

$$s=[n(1/p-1)]_+\tag{7}$$

and we fix $N\gg s$ a sufficiently large integer, say $N=m(n+1+2s)$.

We recall that a (p,∞)-atom is an L^∞-function a that satisfies $|a|\le\chi_Q$, where Q is a cube on $\mathbf{R}^n$ with sides parallel to the axes and

$$\int_{\mathbf{R}^n}x^\alpha a(x)\,dx=0$$

for all α with $|\alpha|\le N$. By convention, when $p=\infty$, a is called a (∞,∞)-atom if $Q=\mathbf{R}^n$ and $\|a\|_{L^\infty}\le 1$. No cancellation is required for (∞,∞)-atoms.

Before stating our main theorems, we compare the three aforementioned types of operators. One notices that the singularity of the Coifman–Meyer type multiplier is just at one point-namely, at the origin. On the other hand, the singularity of the *product type* multipliers sits on the axes. This difference creates a new difficulty in handling the *product type* operator and hence, we need to establish a new technique (in general, more complicated singularities require more delicate handling.) The special example of (bilinear) *product type* operator (4) was first studied in [4]. However, it is not easy to extend the approach used in [4] to the m-linear setting. In fact, the mapping properties of the *product type* operators into Hardy spaces in general m-linear setting were studied by Miyachi [15], but the results in [15] imposed an additional assumption which we remove. Our method to handle the m-linear *product type* operator is quite far from the one in [4,15]. Furthermore, since the structure of the *mixed type* multiplier is a mixture of the above two multipliers, the complexity of the problem increases. Nevertheless, we establish mapping properties into Hardy spaces for these three types of operators and more importantly, we provide an approach based on a unified strategy for these results.

Our main results are as follows:

Theorem 1.1 *Let $\mathcal{T}_\sigma$ be the operator defined in* (3) *and assume that it satisfies* (2). *Let* $0 < p_1, \dots, p_m \le \infty$ *and* $0 < p < \infty$ *satisfy* (1). *Assume that*

$$\int_{\mathbf{R}^n} x^\alpha \mathcal{T}_\sigma(a_1, \dots, a_m)(x)\, dx = 0, \tag{8}$$

for all $|\alpha| \le s$ *and all* (p_l, ∞)*-atoms* a_l*, or equivalently*

$$\partial^\alpha_{\xi_m} \sigma(\xi) = 0, \quad \xi \in \{(\xi_1, \dots, \xi_n) \in \mathbf{R}^n : \xi_1 + \cdots + \xi_n = 0\} \setminus \{0\}. \tag{9}$$

Then $\mathcal{T}_\sigma$ *can be extended to a bounded map from* $H^{p_1} \times \cdots \times H^{p_m}$ *to* H^p.

Theorem 1.2 *Let $\mathcal{T}_\sigma$ be the operator defined in* (5), $0 < p_1, \dots, p_m < \infty$, *and* $0 < p < \infty$ *satisfies* (1), *where each* σ^ρ_j *satisfies* (2) *with* $m = 1$. *Assume that* (8) *holds for all* $|\alpha| \le s$, *or equivalently* (9). *Then* $\mathcal{T}_\sigma$ *can be extended to a bounded map from* $H^{p_1} \times \cdots \times H^{p_m}$ *to* H^p.

Concerning Theorem 1.2, for a *product type* multiplier σ, it was proved that (8) is equivalent to the mapping property of $\mathcal{T}_\sigma$ into Hardy spaces under the additional condition in [15]. So, our main progress is twofold: first to generalize the result in [4] to the m-linear setting and remove the additional condition supposed in [15], secondly to prove the mapping property of $\mathcal{T}_\sigma$ into Hardy spaces under the easier-to-verify condition (9), compared to (8).

Theorem 1.3 *Let $\mathcal{T}_\sigma$ be the operator defined in* (6), $0 < p_1, \dots, p_m \le \infty$, *and* $0 < p < \infty$ *satisfies* (1). *Suppose that each* $\sigma_{I^\rho_g}$ *satisfies* (2) *with* $m = |I^\rho_g|$. *Assume that* (8) *holds for all* $|\alpha| \le s$, *or equivalently* (9) *and assume also that*

$$\sup_{\rho=1,\dots,T} \ \sup_{I^\rho_1 + \cdots + I^\rho_{G(\rho)} = \{1,\dots,m\}} \ \inf_{l \in I^t_g} p_l < \infty. \tag{10}$$

Then $\mathcal{T}_\sigma$ *can be extended to a bounded operator from* $H^{p_1} \times \cdots \times H^{p_m}$ *to* H^p.

Remark 1.4 (1) In Theorem 1.2, we exclude the case $p_l = \infty$ for all $l = 1, \dots, m$. In fact, one cannot expect the mapping property of $\mathcal{T}_\sigma$ with (5) if $p_l = \infty$ for some $l = 1, \dots, m$. Likewise, in Theorem 1.3, we need to make the stronger assumption (10) rather than assuming $p_l = \infty$ for some $l = 1, \dots, m$.

(2) The convergence of the integral in (8) is a consequence of Lemma 3.1 for all x outside the union of a fixed multiple of the supports of a_i, while the function $T(a_1, \dots, a_m)$ is integrable over any compact set.

(3) The equivalence between (8) and (9) is not trivial; it needs to be verified in each case.

We make a few comments about the notation. For brevity we write $d\vec{y} = dy_1 \cdots dy_m$ and we use the symbol C to denote a nonessential constant, whose value may vary at different occurrences. For $(k_1, \dots, k_m) \in \mathbf{Z}^m$, we write $\vec{k} = (k_1, \dots, k_m)$. We use the notation $A \le CB$ to indicate that $A \le C\, B$ for some constant C. We denote the Hardy-Littlewood maximal operator by M:

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, dy. \tag{11}$$

We say that $A \approx B$ if both $A \le CB$ and $B \le CA$ hold for some constant $C > 0$ independent of main parameters. The cardinality of a finite set J is denoted by either $|J|$ or $\sharp J$.

A cube Q in $\mathbf{R}^n$ has sides parallel to the axes. We denote by Q^* a centered-dilated cube of any cube Q with the length scale factor $3\sqrt{n}$; then

$$Q^* = 3\sqrt{n} Q^*, \quad Q^{**} = 9nQ. \tag{12}$$

2 Preliminary and related results

2.1 Equivalent definitions of Hardy spaces

We begin this section by recalling the definition of Hardy spaces. Let $\phi \in C^\infty_c$ satisfy

$$\operatorname{supp}(\phi) \subset \{x \in \mathbf{R}^n \ : \ |x| \le 1\} \tag{13}$$

and

$$\int_{\mathbf{R}^n} \phi(y)\, dy = 1. \tag{14}$$

For $t > 0$, we set $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. The maximal function M_ϕ associated with the smooth bump ϕ is given by:

$$M_\phi(f)(x) = \sup_{t>0} \big|(\phi_t * f)(x)\big| = \sup_{t>0} \left|t^{-n} \int_{\mathbf{R}^n} \phi\big(t^{-1}y\big) f(x-y)\, dy\right| \tag{15}$$

for $f \in \mathcal{S}'$. For $0 < p < \infty$, the Hardy space H^p is characterized as the space of all tempered distributions f for which $M_\phi(f) \in L^p$; also the H^p quasinorm of a tempered distribution f in H^p satisfies

$$\|f\|_{H^p} \approx \|M_\phi(f)\|_{L^p}.$$

We denote by $\mathcal{C}_c^\infty$ the space of all smooth functions on $\mathbf{R}^n$ with compact support. The following density property of Hardy spaces will be useful in the proof of the main theorems.

Proposition 2.1 ([18, Chapter III, 5.2(b)]) *Let $N \gg s$ be fixed. Then the following space is dense in H^p:*

$$\mathcal{O}_N = \bigcap_{\alpha \in \mathbf{N}_0^n, |\alpha| \le N} \left\{ f \in \mathcal{C}_c^\infty \,:\, \int_{\mathbf{R}^n} x^\alpha f(x)\, dx = 0 \right\},$$

where $\mathcal{C}_c^\infty$ is the space of all smooth functions with compact supports in $\mathbf{R}^n$.

Hardy spaces possess interesting properties; among them is the following *atomic decomposition*:

Theorem 2.2 ([16]) *Let $0 < p < \infty$.*

If $f \in H^p$, then there exist a collection of (p,∞)-atoms $\{a_k\}_{k=1}^\infty$ and a nonnegative sequence $\{\lambda_k\}_{k=1}^\infty$ such that

$$f = \sum_{k=1}^\infty \lambda_k a_k$$

in $\mathcal{S}'$ and that we have

$$\Big\| \sum_{k=1}^\infty \lambda_k \chi_{Q_k} \Big\|_{L^p} \le C \|f\|_{H^p}.$$

Moreover, if $f \in \mathcal{C}_c^\infty$ and $\int_{\mathbf{R}^n} x^\alpha f(x)\, dx = 0$ for all α with $|\alpha| \le [n(1/p-1)]_+$, then we can arrange that $\lambda_k = 0$ for all but finitely many k.

The following lemma, whose proof is just an application of the Fefferman-Stein vector-valued inequality for the maximal function, will be used frequently in the sequel.

Lemma 2.3 *If $\gamma > \max(1, \frac{1}{p})$, $0 < p < \infty$, $\lambda_k \ge 0$ and $\{Q_k\}_k$ are sequence of cubes, then*

$$\Big\| \sum_k \lambda_k (M\chi_{Q_k})^\gamma \Big\|_{L^p} \le C \Big\| \sum_k \lambda_k \chi_{Q_k} \Big\|_{L^p}.$$

In particular,

$$\Big\| \sum_k \lambda_k \chi_{Q_k^{**}} \Big\|_{L^p} \le C \Big\| \sum_k \lambda_k \chi_{Q_k} \Big\|_{L^p}.$$

We will also make use of the following result:

Lemma 2.4 *Let $p \in (0,\infty)$. Assume that $q \in (p,\infty] \cap [1,\infty]$. Suppose that we are given a sequence of cubes $\{Q_j\}_{j=1}^\infty$ and a sequence of non-negative L^q-functions $\{F_j\}_{j=1}^\infty$. Then*

$$\Big\| \sum_{j=1}^\infty \chi_{Q_j} F_j \Big\|_{L^p} \le C \Big\| \sum_{j=1}^\infty \left(\frac{1}{|Q_j|} \int_{Q_j} F_j(y)^q\, dy \right)^{1/q} \chi_{Q_j} \Big\|_{L^p}.$$

P r o o f. See [13] for the case of $0 < p \le 1$ and [16], [17] for the case of $1 < p < \infty$. □

2.2 Reductions in the proof of main results

To start the proof of the main results, let $p_1, \ldots, p_m$ and p be given as in Theorems 1.1, 1.2 or 1.3 and note that $H^{p_l} \cap \mathcal{O}_N$ is dense in H^{p_l} for $1 \le l \le m$ and $0 < p_l < \infty$. Recall the integer $N \gg s$ and fix $f_l \in H^{p_l} \cap \mathcal{O}_N$ for which $0 < p_l < \infty$. By Theorem 2.2, we can decompose $f_l = \sum_{k_l=1}^{\infty} \lambda_{l,k_l} a_{l,k_l}$, where $\{\lambda_{l,k_l}\}_{k_l=1}^{\infty}$ is a non-negative finite sequence and $a_{l,k_l} \in L^\infty$ is supported in a cube Q_{l,k_l} satisfying

$$|a_{l,k_l}| \le \chi_{Q_{l,k_l}}, \qquad \int_{\mathbf{R}^n} x^\alpha a_{l,k_l}(x)\, dx = 0, \qquad |\alpha| \le N$$

and that

$$\Big\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l} \chi_{Q_{l,k_l}} \Big\|_{L^{p_l}} \le C \|f_l\|_{H^{p_l}}. \tag{16}$$

If $p_l = \infty$ and $f_l \in L^\infty$, then we can conventionally rewrite $f_l = \lambda_{l,k_l} a_{l,k_l}$ where $\lambda_{l,k_l} = \|f_l\|_{L^\infty}$ and $a_{l,k_l} = \|f\|_{L^\infty}^{-1} f$ is an (∞,∞)-atom supported in $Q_{l,k_l} = \mathbf{R}^n$. In this case the summation in (16) is ignored since there is only one summand.

By the multi-sublinearity of $M_\phi \circ \mathcal{T}_\sigma$, we can estimate

$$M_\phi \circ \mathcal{T}_\sigma(f_1, \ldots, f_m) \le \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l} \Big) M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m}).$$

To prove Theorems 1.1, 1.2, and 1.3, it now suffices to establish the following result:

Proposition 2.5 *Let $\mathcal{T}_\sigma$ be the operator defined in* (3), (5) *or* (6). *Let $p_1, \ldots, p_m$ and p be given as in corresponding Theorems 1.1, 1.2 or 1.3. Then we have*

$$\Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l} \Big) M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m}) \Big\|_{L^p} \le C \prod_{l=1}^{m} \Big\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l} \chi_{Q_{l,k_l}} \Big\|_{L^{p_l}}. \tag{17}$$

Combining (16) and Proposition 2.5, yield the required estimate

$$\|\mathcal{T}_\sigma(f_1, \ldots, f_m)\|_{H^p} = \|M_\phi \circ \mathcal{T}_\sigma(f_1, \ldots, f_m)\|_{L^p} \le C \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}.$$

We may therefore focus on the proof of Proposition 2.5. In the sequel we focus on (17) whose proof depends on whether $\mathcal{T}_\sigma$ is of type (3), (5) or (6). The detailed proof for each type is discussed in subsequent sections.

3 The Coifman–Meyer type

Throughout this section, $\mathcal{T}_\sigma$ denotes for the operator defined in (3). The main purpose of this section is to establish (17) for $\mathcal{T}_\sigma$. Note that the equivalence between (8) and (9) is proved in our paper [10]. So, we assume (8) below.

3.1 Fundamental estimates for the Coifman–Meyer type

We treat the case of Coifman–Meyer multiplier operators whose symbols satisfy (2). The study of such operators was initiated by Coifman and Meyer [2], [3] and was later pursued by Grafakos and Torres [11]; see also [7] for an account. Denoting by K the inverse Fourier transform of σ, in view of (2), we have

$$|\partial_y^\beta K(y_1, \ldots, y_m)| \le C \Big(\sum_{i=1}^{m} |y_i| \Big)^{-mn-|\beta|}, \quad (y_1, \ldots, y_m) \ne (0, \ldots, 0)$$

for all $\beta = (\beta_1, \ldots, \beta_m) \in \mathbf{N}_0{}^{mn} = (\mathbf{N}_0{}^n)^m$ and $|\beta| \le N$.

Examining carefully the smoothness of the kernel, we obtain the following estimates:

Lemma 3.1 *Let a_k be (p_k,∞)-atoms supported in Q_k for all $1 \le k \le m$. Let Λ be a non-empty subset of $\{1,\ldots,m\}$. Then we have*

$$|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)| \le \frac{\min\{\ell(Q_k) : k \in \Lambda\}^{n+N+1}}{\big(\sum_{k\in\Lambda}|y-c_k|\big)^{n+N+1}}$$

for all $y \notin \cup_{k\in\Lambda} Q_k^*$.

P r o o f. We may suppose that $\Lambda = \{1,\ldots,r\}$ for some $1 \le r \le m$ and that

$$\ell(Q_1) = \min\{\ell(Q_k) : k \in \Lambda\}.$$

Let c_k be the center of Q_k and fix $y \notin \cup_{k\in\Lambda} Q_k^*$. Using the cancellation of a_1 we rewrite

$$\begin{aligned}
&\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)\\
&=\int_{\mathbf{R}^{mn}} K(y-y_1,\ldots,y-y_m)a_1(y_1)\cdots a_m(y_m)d\vec{y}\\
&=\int_{\mathbf{R}^{mn}} \big[K(y-y_1,\ldots,y-y_m)-P_N(y,y_1,y_2,\ldots,y_m)\big]a_1(y_1)\cdots a_m(y_m)d\vec{y}\\
&=\int_{\mathbf{R}^{mn}} K^1(y,y_1,y_2,\ldots,y_m)a_1(y_1)\cdots a_m(y_m)d\vec{y},
\end{aligned} \tag{18}$$

where

$$P_N(y,y_1,y_2,\ldots,y_m) = \sum_{|\alpha|\le N}\frac{1}{\alpha!}\partial_1^\alpha K(y-c_1,y-y_2,\ldots,y-y_m)(c_1-y_1)^\alpha$$

is the Taylor polynomial of degree N of $K(y-\cdot,y-y_2,\ldots,y-y_m)$ at c_1 and

$$K^1(y,y_1,\ldots,y_m) = K(y-y_1,\ldots,y-y_m) - P_N(y,y_1,y_2,\ldots,y_m). \tag{19}$$

By the smoothness condition of the kernel and the fact that

$$|y-y_k| \approx |y-c_k|$$

for all $k \in \Lambda$ and $y_k \in Q_k$, we can estimate

$$\begin{aligned}
&\big|K(y,y_1,\ldots,y_m) - P_N(y,c_1,y_2,\ldots,y_m)\big|\\
&\le C|y_1-c_1|^{N+1}\Big(\sum_{k\in\Lambda}|y-c_k| + \sum_{j=2}^m |y-y_j|\Big)^{-mn-N-1}.
\end{aligned} \tag{20}$$

Thus,

$$\begin{aligned}
|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)| &\le C\int_{\mathbf{R}^{mn}} \frac{|y_1-c_1|^{N+1}|a_1(y_1)|\cdots|a_m(y_m)|}{\Big(\sum_{k\in\Lambda}|y-c_k| + \sum_{j=2}^m|y-y_j|\Big)^{mn+N+1}}d\vec{y}\\
&\le C\int_{\mathbf{R}^{(m-1)n}} \frac{\ell(Q_1)^{n+N+1}}{\Big(\sum_{k\in\Lambda}|y-c_k| + \sum_{j=2}^m|y_j|\Big)^{mn+N+1}}dy_2\cdots dy_m\\
&\le C\frac{\ell(Q_1)^{n+N+1}}{\Big(\sum_{k\in\Lambda}|y-c_k|\Big)^{n+N+1}}.
\end{aligned}$$

□

Lemma 3.2 *Let a_k be (p_k,∞)-atoms supported in Q_k for all $1\le k\le m$. Suppose Q_1 is the cube such that $\ell(Q_1)=\min\{\ell(Q_k):1\le k\le m\}$. Then for fixed $1\le r<\infty$ and $j\in\mathbf{N}$, we have*

$$\|\mathcal{T}_\sigma(a_1,\dots,a_m)\chi_{Q_1^{**}}\|_{L^r}\le C|Q_1|^{\frac{1}{r}}\prod_{l=1}^m\inf_{z\in Q_1^*}M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}}, \tag{21}$$

$$\|M\circ\mathcal{T}_\sigma(a_1,\dots,a_m)\chi_{Q_1^{**}}\|_{L^r}\le C|Q_1|^{\frac{1}{r}}\prod_{l=1}^m\inf_{z\in Q_1^*}M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}}, \tag{22}$$

Furthermore, if Q_0 is a cube such that $\ell(Q_0)\le\ell(Q_1)$ and $2^jQ_0^{**}\cap 2^jQ_l^{**}=\emptyset$ for some l, then

$$\|\mathcal{T}_\sigma(a_1,\dots,a_m)\chi_{2^jQ_0^{**}}\|_{L^\infty}\le C\prod_{l=1}^m\inf_{z\in 2^jQ_0^*}M\chi_{2^jQ_l^{**}}(z)^{\frac{n+N+1}{mn}}. \tag{23}$$

In particular, under the above assumption on Q_0,

$$\Big(\frac{1}{|2^jQ_0^{**}|}\int_{2^jQ_0^{**}}|\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|^rdy\Big)^{\frac{1}{r}}\le C\prod_{l=1}^m\inf_{z\in 2^jQ_0^*}M\chi_{2^jQ_l^{**}}(z)^{\frac{n+N+1}{mn}}. \tag{24}$$

Proof. To check (21), we only consider $1<r<\infty$ and two following cases. First, if $Q_1^{**}\cap Q_k^{**}\ne\emptyset$ for all $2\le k\le m$, then, by the assumption $\ell(Q_1)=\min\{\ell(Q_k):1\le k\le m\}$, $Q_1^{**}\subset 3Q_k^{**}$ for all $1\le k\le m$. This implies

$$\inf_{z\in Q_1^*}M\chi_{3Q_k^{**}}(z)\ge 1,$$

for all $1\le k\le m$. Now the boundedness of $\mathcal{T}_\sigma$ from $L^r\times L^\infty\times\cdots\times L^\infty$ to L^r yields

$$\begin{aligned}\|\mathcal{T}_\sigma(a_1,\dots,a_m)\chi_{Q_1^{**}}\|_{L^r}&\le\|\mathcal{T}_\sigma(a_1,\dots,a_m)\|_{L^r}\\&\le C\|a_1\|_{L^r}\|a_2\|_{L^\infty}\cdots\|a_m\|_{L^\infty}\\&\le C|Q_1|^{\frac{1}{r}}\prod_{k=1}^m\inf_{z\in Q_1^*}M\chi_{3Q_k^{**}}(z)^{\frac{n+N+1}{mn}}.\end{aligned} \tag{25}$$

Secondly, if $Q_1^{**}\cap Q_k^{**}=\emptyset$ for some k, then the set

$$\Lambda=\{2\le k\le m:Q_1^{**}\cap Q_k^{**}=\emptyset\}$$

is a non-empty subset of $\{1,\dots,m\}$. Fix an arbitrary $y\in\mathbf{R}^n$. By the cancellation of a_1, rewrite

$$\mathcal{T}_\sigma(a_1,\dots,a_m)(y)=\int_{\mathbf{R}^{mn}}K^1(y,y_1,y_2,\dots,y_m)a_1(y_1)\cdots a_m(y_m)d\vec{y},$$

where $K^1(y,y_1,\dots,y_m)$ is defined in (19). For $y_1\in Q_1$ we estimate

$$\big|K^1(y,y_1,\dots,y_m)\big|\le C\ell(Q_1)^{N+1}\Big(|y-\xi_1|+\sum_{j=2}^m|y-y_j|\Big)^{-mn-N-1},$$

for some $\xi_1\in Q_1$ and for all $y_l\in Q_l$.

Since $Q_1^{**}\cap Q_k^{**}=\emptyset$ for all $k\in\Lambda$, $|y-\xi_1|+|y-y_k|\ge|\xi_1-y_k|\ge C|c_1-c_k|$ for all $y_k\in Q_k$ and $k\in\Lambda$. Therefore

$$\big|K^1(y,y_1,\dots,y_m)\big|\le C\ell(Q_1)^{N+1}\Big(\sum_{k\in\Lambda}|c_1-c_k|+\sum_{j=2}^m|y-y_j|\Big)^{-mn-N-1},$$

for all $y_1 \in Q_1^*$ and $y_k \in Q_k$ for $k \in \Lambda$. Insert the above inequality into (18) to obtain

$$|\mathcal{T}_\sigma(a_1,\dots,a_m)(y)| \le \frac{C\ell(Q_1)^{n+N+1}}{\left(\sum_{k\in\Lambda}|c_1-c_k|\right)^{n+N+1}} \le \frac{C\ell(Q_1)^{n+N+1}}{\sum_{k\in\Lambda}\left[\ell(Q_1)+|c_1-c_k|+\ell(Q_k)\right]^{n+N+1}}.$$

Noting that $Q_1^{**} \subset 3Q_l^{**}$ for $l \notin \Lambda$, the last inequality gives

$$\|\mathcal{T}_\sigma(a_1,\dots,a_m)\|_{L^\infty} \le C\prod_{k=1}^m \inf_{z\in Q_1^*} M\chi_{3Q_k^{**}}(z)^{\frac{n+N+1}{mn}}, \tag{26}$$

which yields

$$\|\mathcal{T}_\sigma(a_1,\dots,a_m)\chi_{Q_1^{**}}\|_{L^r} \le C|Q_1|^{\frac{1}{r}}\prod_{k=1}^m \inf_{z\in Q_1^*} M\chi_{3Q_k^{**}}(z)^{\frac{n+N+1}{mn}}. \tag{27}$$

Combining (25) and (27) and noting that $M\chi_{3Q} \le CM\chi_Q$, we obtain (21).

Similarly, we can prove (22)–(23). For example, to show (22), we again consider the case where $Q_1^{**}\cap Q_l^{**} \ne \emptyset$ holds for all l and the case where this fails. In the first case, using the boundedness of M on L^r, we arrive at the same situation as above. In the second case, we use the boundedness of M on L^∞ to see

$$\|M\circ\mathcal{T}_\sigma(a_1,\dots,a_m)\chi_{Q_1^{**}}\|_{L^r} \le C|Q_1|^{\frac{1}{r}}\|\mathcal{T}_\sigma(a_1,\dots,a_m)\|_{L^\infty}.$$

Notice that the right-hand side is already treated in (26). □

Lemma 3.2 will be used to study the behavior of the operator $M_\phi\circ\mathcal{T}_\sigma$ inside Q_1^{**}. For the region outside of Q_1^{**}, we need the following estimates.

Lemma 3.3 *Let a_k be (p_k,∞)-atoms supported in Q_k for all $1\le k\le m$. If $p_k=\infty$ then $Q_k=\mathbf{R}^n$. Suppose that Q_1 is the cube for which $\ell(Q_1)=\min\{\ell(Q_k):1\le k\le m\}$. Fix $0<t<\infty$.*

1. *If $x\notin Q_1^{**}$ and $c_1\notin B(x,100n^2t)$, then*

$$\frac{1}{t^n}\int_{B(x,t)}|\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \le C\prod_{l=1}^m M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}. \tag{28}$$

2. *If $x\notin Q_1^{**}$ and $c_1\in B(x,100n^2t)$, then*

$$\frac{\ell(Q_1)^{s+1}}{t^{n+s+1}}\int_{Q_1^*}|\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \le CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}}\prod_{l=1}^m\inf_{z\in Q_1^*}M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}}, \tag{29}$$

and

$$\frac{1}{t^{n+s+1}}\int_{(Q_1^*)^c}|y-c_1|^{s+1}|\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \le CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}}\prod_{l=1}^m\inf_{z\in Q_1^*}M\chi_{Q_l}(z)^{\frac{N-s}{mn}}. \tag{30}$$

3. *For all $x\notin Q_1^{**}$, we have*

$$\begin{aligned}&M_\phi\circ\mathcal{T}_\sigma(a_1,\dots,a_m)(x)\\ &\le C\prod_{l=1}^m M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} + CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}}\prod_{l=1}^m\inf_{z\in Q_1^*}\left(M\chi_{Q_l}(z)^{\frac{N-s}{mn}}\right).\end{aligned} \tag{31}$$

Proof. Fix $x \notin Q_1^{**}$ and denote $\Lambda = \{1 \le k \le m : x \notin Q_k^{**}\}$.

(1) Suppose $c_1 \notin B(x, 100n^2 t)$. For $y \in B(x,t)$, from (18) we rewrite

$$\mathcal{T}_\sigma(a_1, \ldots, a_m)(y) = \int_{\mathbf{R}^{mn}} K^1(y, y_1, \ldots, y_m) a_1(y_1) \cdots a_m(y_m) d\vec{y},$$

where K^1 is defined in (19). Note that for $y \in B(x,t)$, $y_1 \in Q_1$ and $c_1 \notin B(x, 100n^2 t)$, we have

$$t \le C|x - c_1| \le C|y - y_1|.$$

Since $x \notin Q_k^{**}$ for all $k \in \Lambda$,

$$|x - c_k| \le C|x - y_k| \le C(t + |y - y_k|) \le C(|y - y_1| + |y - y_k|)$$

for all $k \in \Lambda$ and $y_k \in Q_k$. Consequently,

$$\left| K^1(y, y_1, \ldots, y_m) \prod_{l=1}^m a_l(y_l) \right| \le C \frac{\ell(Q_1)^{N+1} \chi_{Q_1}(y_1)}{\left(\sum_{l=2}^m |y - y_l| + \sum_{k \in \Lambda} |x - c_k| \right)^{mn+N+1}}. \tag{32}$$

Integrating (32) over $(\mathbf{R}^n)^m$, and using that $\ell(Q_1) \le \ell(Q_l)$ for all $2 \le l \le m$, we obtain that

$$\begin{aligned} |\mathcal{T}_\sigma(a_1, \ldots, a_m)(y)| &\le C \frac{\ell(Q_1)^{n+N+1}}{\left(\sum_{l \in \Lambda} |x - c_l| \right)^{n+N+1}} \\ &\le C \prod_{l \in \Lambda} \frac{\ell(Q_l)^{\frac{n+N+1}{|\Lambda|}}}{|x - c_l|^{\frac{n+N+1}{|\Lambda|}}} \chi_{(Q_l^{**})^c}(x) \cdot \prod_{k \notin \Lambda} \chi_{Q_k^{**}}(x) \le C \prod_{l=1}^m M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}. \end{aligned}$$

This pointwise estimate proves (28).

(2) Assume $c_1 \in B(x, 100n^2 t)$. Fix $1 < r < \infty$ and estimate the left-hand side of (29) by

$$\frac{\ell(Q_1)^{s+1}}{t^{n+s+1}} |Q_1|^{1-\frac{1}{r}} \|\mathcal{T}_\sigma(a_1, \ldots, a_m) \chi_{Q_1^{**}}\|_{L^r} \le C \frac{\ell(Q_1)^{n+s+1}}{t^{n+s+1}} \prod_{l=1}^m \inf_{z \in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}},$$

where we used (21) in the above inequality. Since $x \notin Q_1^{**}$ and $c_1 \in B(x, 100n^2 t)$, $Q_1^* \subset B(x, 1000n^2 t)$ and hence, $\ell(Q_1)/t \le CM\chi_{Q_1}(x)$. This combined with the last inequality implies (29).

To verify (30), we recall the expression of $\mathcal{T}_\sigma(a_1, \ldots, a_m)(y)$ in (18) and the pointwise estimate (20) for

$$K^1(y, y_1, \ldots, y_m),$$

defined in (19). Denote $J = \{2 \le k \le m : Q_1^{**} \cap Q_k^{**} = \emptyset\}$. Using the facts that $|y - y_1| \sim |y - c_1| \ge \ell(Q_1)$ for $y \notin Q_1^*$, $y_1 \in Q_1$ and $C(|y - y_1| + |y - y_l| \ge |y_1 - y_l|) \ge |z - c_l|$ for all $z \in Q_1^*$ and $l \in J$, we now estimate

$$|\mathcal{T}_\sigma(a_1, \ldots, a_m)(y)| \le C \int_{(\mathbf{R}^n)^m} \frac{\ell(Q_1)^{N+1} \chi_{Q_1}(y_1)\, d\vec{y}}{\left(\ell(Q_1) + |y - c_1| + \sum_{l \in J} |z - c_l| + \sum_{l=2}^m |y - y_l| \right)^{mn+N+1}}$$

for all $y \in (Q_1^*)^c$ and $z \in Q_1^*$. Thus,

$$\begin{aligned} &\frac{1}{t^{n+s+1}} \int_{(Q_1^*)^c} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \ldots, a_m)(y)|\, dy \\ &\le \frac{C}{t^{n+s+1}} \int_{\mathbf{R}^n \times (\mathbf{R}^n)^m} \frac{|y - c_1|^{s+1} \ell(Q_1)^{N+1} \chi_{Q_1}(y_1)\, d\vec{y} dy}{\left(\ell(Q_1) + |y - c_1| + \sum_{l \in J} |c_1 - c_l| + \sum_{l=2}^m |y - y_l| \right)^{mn+N+1}} \\ &\le C \left(\frac{\ell(Q_1)}{t} \right)^{n+s+1} \prod_{l \in J} \left(\frac{\ell(Q_l)}{|z - c_l|} \right)^{\frac{N-s}{m}}. \end{aligned}$$

Note that $1 \leq C \inf_{z\in Q_1^*} M\chi_{2Q_l^{**}}(z)$ if $Q_1^{**} \cap Q_l^{**} \neq \emptyset$; otherwise, $\ell(Q_l)/|z - c_l| \leq CM\chi_{2Q_l^{**}}(z)^{\frac{1}{n}}$ for all $z \in Q_1^*$. Consequently,

$$\frac{1}{t^{n+s+1}} \int_{(Q_1^*)^c} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)|\, dy \leq CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^{m} \inf_{z\in Q_1^*} M\chi_{Q_l}(z)^{\frac{N-s}{mn}},$$

which yields (30).

(3) It remains to prove (31). Fix $x \notin Q_1^{**}$. To calculate $M_\phi \circ \mathcal{T}_\sigma(a_1, \dots, a_m)(x)$, we need to estimate

$$\Big| \int_{\mathbf{R}^n} \phi_t(x - y)\mathcal{T}_\sigma(a_1, \dots, a_m)(y)\, dy \Big|$$

for each $t \in (0, \infty)$. We consider two cases: $c_1 \notin B(x, 100n^2 t)$ and $c_1 \in B(x, 100n^2 t)$.

In the first case, since ϕ is supported in the unit ball, we obtain

$$\Big| \int_{\mathbf{R}^n} \phi_t(x - y)\mathcal{T}_\sigma(a_1, \dots, a_m)(y)\, dy \Big| \leq \frac{C}{t^n} \int_{B(x,t)} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)|\, dy.$$

Since $c_1 \notin B(x, 100n^2 t)$, (28) implies that

$$\Big| \int_{\mathbf{R}^n} \phi_t(x - y)\mathcal{T}_\sigma(a_1, \dots, a_m)(y)\, dy \Big| \leq C \prod_{l=1}^{m} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}. \tag{33}$$

In the second case, we exploit the moment condition of $\mathcal{T}_\sigma(a_1, \dots, a_m)$. Denote

$$\delta_1^s(t; x, y) = \phi_t(x - y) - \sum_{|\alpha|\leq s} \frac{\partial^\alpha[\phi_t](x - c_1)}{\alpha!}(c_1 - y)^\alpha. \tag{34}$$

Since $|\delta_1^s(t; x, y)| \leq Ct^{-n-s-1}$ for all x, y and (8),

$$\begin{aligned}
\Big| \int_{\mathbf{R}^n} \phi_t(x - y)\mathcal{T}_\sigma(a_1, \dots, a_m)(y)\, dy \Big| &= \Big| \int_{\mathbf{R}^n} \delta_1^s(t; x, y)\mathcal{T}_\sigma(a_1, \dots, a_m)(y)\, dy \Big| \\
&\leq \frac{C}{t^{n+s+1}} \int_{\mathbf{R}^n} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)|\, dy \\
&= \frac{C}{t^{n+s+1}} \int_{Q_1^*} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)|\, dy \\
&\quad + \frac{C}{t^{n+s+1}} \int_{(Q_1^*)^c} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)|\, dy \\
&\leq C\frac{\ell(Q_1)^{s+1}}{t^{n+s+1}} \int_{Q_1^*} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)|\, dy \\
&\quad + \frac{C}{t^{n+s+1}} \int_{(Q_1^*)^c} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)|\, dy.
\end{aligned} \tag{35}$$

Invoking (29) and (30), we obtain

$$\begin{aligned}
&\Big| \int_{\mathbf{R}^n} \phi_t(x - y)\mathcal{T}_\sigma(a_1, \dots, a_m)(y)\, dy \Big| \\
&\leq CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^{m} \inf_{z\in Q_1^*} \Big[M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} + M\chi_{Q_l}(z)^{\frac{N-s}{mn}} \Big] \\
&\leq CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^{m} \inf_{z\in Q_1^*} \Big(M\chi_{Q_l}(z)^{\frac{N-s}{mn}} \Big).
\end{aligned} \tag{36}$$

Combining (33) and (36) yields the required estimate (31). The proof of Lemma 3.3 is now complete. $\square$

3.2 The proof of Proposition 2.5 for Coifman–Meyer type operators

We now turn into the proof of (17), i.e., the estimate for

$$A = \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l}\Big) M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m}) \Big\|_{L^p}. \tag{37}$$

For each $\vec{k} = (k_1, \ldots, k_m)$, we denote by $R_{\vec{k}}$ the cube with smallest length among $Q_{1,k_1}, \ldots, Q_{m,k_m}$. Then we have $A \le C(B+G)$, where

$$B = \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l}\Big) M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m}) \chi_{R_{\vec{k}}^{**}} \Big\|_{L^p} \tag{38}$$

and

$$G = \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l}\Big) M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m}) \chi_{(R_{\vec{k}}^{**})^c} \Big\|_{L^p}. \tag{39}$$

To estimate B, for some $\max(1,p) < r < \infty$ Lemma 2.4 and (22) imply

$$\begin{aligned}
B &\le C \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l}\Big) \frac{\chi_{R_{\vec{k}}^{**}}}{|\chi_{R_{\vec{k}}^{**}}|^{\frac{1}{r}}} \|M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m}) \chi_{R_{\vec{k}}^{**}}\|_{L^r} \Big\|_{L^p} \\
&\le C \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l}\Big) \Big(\prod_{l=1}^{m} \inf_{z \in R_{\vec{k}}^{*}} M\chi_{Q_{l,k_l}}(z)^{\frac{n+N+1}{mn}}\Big) \chi_{R_{\vec{k}}^{**}} \Big\|_{L^p} \\
&= C \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \inf_{z \in R_{\vec{k}}^{*}} \lambda_{l,k_l} M\chi_{Q_{l,k_l}}(z)^{\frac{n+N+1}{mn}}\Big) \chi_{R_{\vec{k}}^{**}} \Big\|_{L^p} \\
&\le C \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \inf_{z \in R_{\vec{k}}^{*}} \lambda_{l,k_l} M\chi_{Q_{l,k_l}}(z)^{\frac{n+N+1}{mn}}\Big) \chi_{R_{\vec{k}}^{*}} \Big\|_{L^p},
\end{aligned}$$

where we used Lemma 2.3 in the last inequality. Now we can remove the infimum and apply Hölder's inequality to obtain

$$\begin{aligned}
B &\le C \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \prod_{l=1}^{m} \lambda_{l,k_l} \Big(M\chi_{Q_{l,k_l}}\Big)^{\frac{n+N+1}{mn}} \Big\|_{L^p} \\
&= C \Big\| \prod_{l=1}^{m} \sum_{k_l=1}^{\infty} \lambda_{l,k_l} \Big(M\chi_{Q_{l,k_l}}\Big)^{\frac{n+N+1}{mn}} \Big\|_{L^p} && (40) \\
&\le C \prod_{l=1}^{m} \Big\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l} \Big(M\chi_{Q_{l,k_l}}\Big)^{\frac{n+N+1}{mn}} \Big\|_{L^{p_l}} \\
&\le C \prod_{l=1}^{m} \Big\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l} \chi_{Q_{l,k_l}^{**}} \Big\|_{L^{p_l}} && (41) \\
&\le C \prod_{l=1}^{m} \Big\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l} \chi_{Q_{l,k_l}} \Big\|_{L^{p_l}}. && (42)
\end{aligned}$$

Once again, Lemma 2.3 was used in the last two inequalities.

To deal with G, we use (31) and estimate $G \le C(G_1 + G_2)$, where

$$G_1 = \Big\| \sum_{k_1,\ldots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l}\Big) \prod_{l=1}^{m} \Big(M\chi_{Q_{l,k_l}}\Big)^{\frac{n+N+1}{mn}} \Big\|_{L^p}$$

and

$$G_2 = \Big\| \sum_{k_1,\dots,k_m=1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l}\Big)\Big(\prod_{l=1}^{m} \inf_{z\in R_{\vec{k}}^*} M\chi_{Q_{l,k_l}}(z)^{\frac{N-s}{mn}}\Big)(M\chi_{R_{\vec{k}}^*})^{\frac{n+s+1}{n}} \Big\|_{L^p}.$$

Repeating the argument in estimating for B, noting that $\frac{(n+s+1)p}{n} > 1$ and $N \gg s$, we obtain

$$G \le C(G_1 + G_2) \le C \prod_{l=1}^{m} \Big\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l}\chi_{Q_{l,k_l}} \Big\|_{L^{p_l}}. \tag{43}$$

Combining (42) and (43) yields (17). This completes the proof of Proposition 2.5 for the operator $\mathcal{T}_\sigma$ of type (3).

Remark 3.4 The techniques in this paper also work for Calderón–Zygmund operators of non-convolution type; this recovers the results in [13].

4 The product type

In this entire section, we denote by $\mathcal{T}_\sigma$ the operator defined in (5) and prove Proposition 2.5 for this operator. We need to establish some results analogous to Lemmas 3.2 and 3.3. Again note that for the *product type* multiplier, the equivalence between (8) and (9) is proved in our paper [10]. So, we assume (8) below.

4.1 Fundamental estimates for product type operators

Let a_k be (p_k, ∞)-atoms supported in Q_k for all $1 \le k \le m$. Here and below $M^{(r)}$ denotes the power-maximal operator: $M^{(r)}f(x) = M(|f|^r)(x)^{\frac{1}{r}}$. Suppose Q_1 is the cube such that $\ell(Q_1) = \min\{\ell(Q_k) : 1 \le k \le m\}$, then we have the following lemmas.

Lemma 4.1 *For all $x \in Q_1^{**}$, we have*

$$M_\phi \circ \mathcal{T}_\sigma(a_1,\dots,a_m)(x)\chi_{Q_1^{**}}(x) \le C \prod_{l=1}^{m} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} \Big(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(x)\Big). \tag{44}$$

P r o o f. Fix $x \in Q_1^{**}$. We need to estimate

$$\Big| \int_{\mathbf{R}^n} \phi_t(x-y)\mathcal{T}_\sigma(a_1,\dots,a_m)(y)\,dy \le \frac{C}{t^n}\int_{B(x,t)} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy$$

for each $t \in (0,\infty)$. The proof of (44) is mainly based on the boundedness of $\mathcal{T}_\sigma$ and the smoothness condition of each Calderón–Zygmund kernel in (5). Instead of considering the whole sum in (5), for notational simplicity, it is convenient to consider one term, i.e.,

$$\mathcal{T}_\sigma(f_1,\dots,f_m) = T_{\sigma_1}(f_1)\cdots T_{\sigma_m}(f_m) \tag{45}$$

keeping in mind that this term represents the entire sum when cancellation is needed. We consider two cases: $t \le \ell(Q_1)$ and $t > \ell(Q_1)$.

Case 1: $t \le \ell(Q_1)$. By Hölder's inequality and (24), we have

$$\frac{1}{t^n}\int_{B(x,t)} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \le C \prod_{l=1}^{m} \Big(\frac{1}{t^n}\int_{B(x,t)} |T_{\sigma_l}a_l(y)|^m dy\Big)^{\frac{1}{m}}.$$

We decompose the above product depending on two sub-cases; $B(t,x) \cap Q_l^{**} = \emptyset$ or not. Then

$$\begin{aligned}
&\prod_{l=1}^{m} \Big(\frac{1}{t^n}\int_{B(x,t)} |T_{\sigma_l}a_l(y)|^m dy\Big)^{\frac{1}{m}} \\
&= \prod_{l:B(t,x)\cap Q_l^{**}=\emptyset} \Big(\frac{1}{t^n}\int_{B(x,t)} |T_{\sigma_l}a_l(y)|^m dy\Big)^{\frac{1}{m}} \prod_{l:B(t,x)\cap Q_l^{**}\ne\emptyset} \Big(\frac{1}{t^n}\int_{B(x,t)} |T_{\sigma_l}a_l(y)|^m dy\Big)^{\frac{1}{m}}.
\end{aligned}$$

In the first sub-case, we employ (24). In the second sub-case, we observe that the assumption $t \le \ell(Q_1) \le \ell(Q_l)$ implies $B(x,t) \subset 3Q_l^{**}$. As a result,

$$\prod_{l=1}^m \Big(\frac{1}{t^n}\int_{B(x,t)} |T_{\sigma_l} a_l(y)|^m dy\Big)^{\frac{1}{m}} \\ \le C \prod_{l:B(x,t)\cap Q_l^{**}\neq\emptyset} \chi_{3Q_l^{**}}(x) M^{(m)} \circ T_{\sigma_l}(a_l)(x) \prod_{l:B(x,t)\cap Q_l^{**}=\emptyset} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}.$$

Thus

$$\Big|\int_{\mathbf{R}^n} \phi_t(x-y)\mathcal{T}_\sigma(a_1,\dots,a_m)(y)\,dy\Big| \le C\prod_{l=1}^m M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\Big(1+M^{(m)}\circ T_{\sigma_l}(a_l)(x)\Big). \quad (46)$$

Case 2: $t > \ell(Q_1)$. Now we can estimate

$$\frac{1}{t^n}\int_{B(x,t)} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \le \frac{C}{|Q_1^*|}\int_{\mathbf{R}^n} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \\ = \frac{C}{|Q_1^*|}\int_{Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \\ + \frac{C}{|Q_1^*|}\int_{\mathbf{R}^n\setminus Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy.$$

By Hölder's inequality and (24), a similar technique to (46) yields

$$\frac{1}{|Q_1^*|}\int_{Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \\ \le C\prod_{l=1}^m \Big(\frac{1}{|Q_1^*|}\int_{Q_1^*} |T_{\sigma_l} a_l(y)|^m\,dy\Big)^{\frac{1}{m}} \\ \le C\prod_{l=1}^m \Big(\inf_{z\in Q_1^*} M\chi_{Q_l^{**}}(z)^{\frac{n+N+1}{mn}} + \inf_{z\in Q_1^*} M^{(m)}\circ T_{\sigma_l}(a_l)(z)\chi_{3Q_l^{**}}(z)\Big) \\ \le C\prod_{l=1}^m M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\Big(1+M^{(m)}\circ T_{\sigma_l}(a_l)(x)\Big), \quad (47)$$

since $x \in Q_1^*$. In the second term, using the decay of $T_{\sigma_1}a_1(y)$ when $y \notin Q_1^*$ as in Lemma 3.1, we obtain

$$\frac{1}{|Q_1^*|}\int_{\mathbf{R}^n\setminus Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \le \frac{C}{|Q_1^*|}\int_{\mathbf{R}^n\setminus Q_1^*} \frac{\ell(Q_1)^{n+N+1}}{|y-c_1|^{n+N+1}}\prod_{l=2}^m |T_{\sigma_l}a_l(y)|\,dy.$$

We decompose $\mathbf{R}^n \setminus Q_1^*$ into dyadic annuli and estimate

$$\frac{1}{|Q_1^*|}\int_{\mathbf{R}^n\setminus Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \\ \le C\sum_{j=1}^\infty 2^{j(-N-1)}\frac{1}{|2^jQ_1^*|}\int_{2^jQ_1^*} \chi_{2^jQ_1^*}(y)\prod_{l=2}^m |T_{\sigma_l}a_l(y)|\,dy \\ \le C\sum_{j=1}^\infty 2^{j(-N-1)}\prod_{l=2}^m \Big(\frac{1}{|2^jQ_1^*|}\int_{2^jQ_1^*} |T_{\sigma_l}a_l(y)|^m\,dy\Big)^{\frac{1}{m}} \\ \le C\sum_{j=1}^\infty 2^{j(-N-1)}\prod_{l=2}^m \Big(\inf_{z\in 2^jQ_1^*} (M\chi_{2^jQ_l^{**}})(z)^{\frac{n+N+1}{mn}} + \inf_{z\in 2^jQ_1^*} M^{(m)}\circ T_{\sigma_l}(a_l)(z)\chi_{2^{j+1}Q_l^{**}}(z)\Big),$$

where we used (24) in the last inequality.

Since $M\chi_{2^jQ} \le C2^{jn}M\chi_Q$,

$$\chi_{2^{j+1}Q_l^{**}}(x) \le (M\chi_{2^jQ_l^{**}})^{\frac{n+N+1}{mn}} \le C2^{\frac{j(n+N+1)}{m}} M\chi_{Q_l}^{\frac{n+N+1}{mn}}.$$

Inserting this inequality into the previous estimate, we deduce

$$\begin{aligned}
&\frac{1}{|Q_1^*|}\int_{\mathbf{R}^n\setminus Q_1^*} |\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy \\
&\le C\sum_{j=1}^{\infty} 2^{-j(\frac{n+N+1}{m}-n)} \prod_{l=1}^{m}\Big(M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} + M^{(m)}\circ T_{\sigma_l}(a_l)(x) M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\Big) \\
&\le C\prod_{l=1}^{m} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\Big(1+M^{(m)}\circ T_{\sigma_l}(a_l)(x)\Big), \qquad (48)
\end{aligned}$$

since $N \gg n$. Combining (46)–(48) completes the proof of (44). □

Lemma 4.2 *Assume $x \notin Q_1^{**}$ and $c_1 \notin B(x, 100n^2t)$. Then we have*

$$\frac{1}{t^n}\int_{B(x,t)} |\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy \le C\prod_{l=1}^{m} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\Big(1+M^{(m)}\circ T_{\sigma_l}(a_l)(x)\Big).$$

P r o o f. Fix any $x \notin Q_1^{**}$ and $t > 0$ such that $c_1 \notin B(x, 100n^2t)$. We denote

$$J = \{2 \le l \le m : x \notin Q_l^{**}\},\ J_0 = \{l \in J : B(x,2t)\cap Q_l^* = \emptyset\},\ J_1 = J\setminus J_0. \qquad (49)$$

As in the previous lemma, we only consider the reduced form (45) of $\mathcal{T}_\sigma$. From Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{t^n}\int_{B(x,t)} |\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|dy \\
&\le C\|T_{\sigma_1}a_1\chi_{B(x,t)}\|_{L^\infty}\prod_{l\in J_0}\|T_{\sigma_l}a_l\chi_{B(x,t)}\|_{L^\infty} \\
&\quad\times \prod_{l\in J_1}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}|T_{\sigma_l}a_l(y)|^m dy\right)^{\frac{1}{m}} \prod_{l\notin J}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}|T_{\sigma_l}a_l(y)|^m dy\right)^{\frac{1}{m}} \\
&=: C\,(\mathrm{I}\times\mathrm{II}\times\mathrm{III}\times\mathrm{IV}).
\end{aligned}$$

For I, we notice $Q_1^*\cap B(x,2t) = \emptyset$ since we have $x \notin Q_1^{**}$ and $c_1 \notin B(x,100n^2t)$. So, we have only to use the decay estimate for $T_{\sigma_1}a_1$ to get

$$\mathrm{I} = \|T_{\sigma_1}a_1\chi_{B(x,t)}\|_{L^\infty} \le C\left(\frac{\ell(Q_1)}{|x-c_1|+\ell(Q_1)}\right)^{n+N+1}.$$

For all $l \in J_1$, since $B(x,2t)\cap Q_l^* \ne \emptyset$, $Ct \ge \ell(Q_l)$; and hence, $Q_l^* \subset B(x,100n^2t)$. Therefore,

$$\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}|T_{\sigma_l}a_l(y)|^m dy\right)^{\frac{1}{m}} \le C\left(\frac{|Q_l|}{|B(x,t)|}\right)^{\frac{1}{m}} \le C \qquad (50)$$

for all $l \in J_1$. Now combining the above inequality with the estimates for I yields

$$\mathrm{I}\times\mathrm{III} \le C\left(\frac{\ell(Q_1)}{|x-c_1|+\ell(Q_1)}\right)^{\frac{n+N+1}{m}}\prod_{l\in J_1}\left(\frac{\ell(Q_1)}{|x-c_1|+\ell(Q_1)}\right)^{\frac{n+N+1}{m}}. \qquad (51)$$

As showed above $Q_l^* \subset B(x, 100n^2t)$ for all $l \in J_1$. This implies $|x-c_l| \le Ct$. Furthermore, $c_1 \notin B(x, 100n^2t)$ means $t \le C|x-c_1|$ which yields $|x-c_l| \le Ct \le C|x-c_1|$.

Recalling $\ell(Q_1) \le \ell(Q_l)$, we see that

$$\frac{\ell(Q_1)}{|x-c_1|+\ell(Q_1)} \le C\frac{\ell(Q_l)}{|x-c_l|+\ell(Q_l)}.$$

From (51), we obtain

$$\mathrm{I} \times \mathrm{III} \le CM\chi_{Q_1^{**}}(x)^{\frac{n+N+1}{mn}} \prod_{l\in J_1} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}. \tag{52}$$

Now, we turn to the estimates for II and IV. For II, we have only to employ the moment condition of a_l to get

$$\mathrm{II} = \prod_{l\in J_0} \|T_{\sigma_l}a_l \cdot \chi_{B(x,t)}\|_{L^\infty} \le C \prod_{l\in J_0} M\chi_{Q_l}(x)^{\frac{n+N+1}{n}}. \tag{53}$$

For IV, since $x \in Q_l^{**}$, we can estimate

$$\mathrm{IV} \le C \prod_{l\notin J} M^{(m)} \circ T_{\sigma_l}(a_l)(x)\chi_{Q_l^{**}}(x) \tag{54}$$

Putting (52)–(54) together, we conclude the proof of Lemma 4.2. □

Lemma 4.3 *Assume $x \notin Q_1^{**}$ and $c_1 \in B(x, 100n^2t)$. Then we have*

$$\frac{\ell(Q_1)^{s+1}}{t^{n+s+1}} \int_{Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy$$
$$\le CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^m \inf_{z\in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(z)\right). \tag{55}$$

P r o o f. It is enough to restrict $\mathcal{T}_\sigma$ to the form (45). By Hölder's inequality we have

$$\frac{\ell(Q_1)^{s+1}}{t^{n+s+1}} \int_{Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy$$
$$\le \frac{\ell(Q_1)^{n+s+1}}{t^{n+s+1}} \prod_{l=1}^m \Big(\frac{1}{|Q_1^*|}\int_{Q_1^*} |T_{\sigma_l}a_l(y)|^m dy\Big)^{\frac{1}{m}}$$
$$\le C\frac{\ell(Q_1)^{n+s+1}}{t^{n+s+1}} \prod_{l=1}^m \Big(\inf_{z\in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{n}} + \inf_{z\in Q_1^*} M^{(m)} \circ T_{\sigma_l}(a_l)(z)\chi_{2Q_l^{**}}(z)\Big),$$

where the last inequality is deduced from (24).

Since $x \notin Q_1^{**}$ and $c_1 \in B(x, 100n^2t)$, $Q_1 \subset B(x, 10000n^3t)$ which implies $\ell(Q_1)/t \le CM\chi_{Q_1}(x)^{\frac{1}{n}}$. As a result,

$$\frac{\ell(Q_1)^{s+1}}{t^{n+s+1}} \int_{Q_1^*} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy$$
$$\le CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^m \inf_{z\in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(z)\right).$$

This proves (55). □

Lemma 4.4 *Assume $x \notin Q_1^{**}$ and $c_1 \in B(x, 100n^2 t)$. Then we have*

$$\frac{1}{t^{n+s+1}} \int_{\mathbf{R}^n \setminus Q_1^*} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)| \, dy$$
$$\leq CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^{m} \inf_{z \in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(z)\right).$$

P r o o f. Using the decay of $T_{\sigma_1} a_1(y)$ when $y \notin Q_1^*$, we obtain

$$\frac{1}{t^{n+s+1}} \int_{\mathbf{R}^n \setminus Q_1^*} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)| \, dy$$
$$\leq \frac{C}{t^{n+s+1}} \int_{\mathbf{R}^n \setminus Q_1^*} |y - c_1|^{s+1} \frac{\ell(Q_1)^{n+N+1}}{|y - c_1|^{n+N+1}} \prod_{l=2}^{m} |T_{\sigma_l} a_l(y)| \, dy.$$

Using the dyadic decomposition of $\mathbf{R}^n \setminus Q_1^*$ as in the proof of Lemma 4.1, we can estimate

$$\frac{1}{t^{n+s+1}} \int_{\mathbf{R}^n \setminus Q_1^*} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)| \, dy$$
$$\leq C \frac{\ell(Q_1)^{s+1}}{t^{n+s+1}} \sum_{j=1}^{\infty} 2^{j(s-N-n)} \int_{2^j Q_1^*} \chi_{2^j Q_1^*}(y) \prod_{l=2}^{m} |T_{\sigma_l} a_l(y)| \, dy$$
$$\leq C \frac{\ell(Q_1)^{n+s+1}}{t^{n+s+1}} \sum_{j=1}^{\infty} 2^{j(s-N)} \prod_{l=2}^{m} \Big(\frac{1}{|2^j Q_1^*|} \int_{2^j Q_1^*} |T_{\sigma_l} a_l(y)|^m \, dy \Big)^{\frac{1}{m}}$$
$$\leq C \frac{\ell(Q_1)^{n+s+1}}{t^{n+s+1}} \sum_{j=1}^{\infty} 2^{j(s-N)}$$
$$\times \prod_{l=2}^{m} \Big(\inf_{z \in 2^j Q_1^*} M\chi_{2^j Q_l^{**}}(z)^{\frac{n+N+1}{mn}} + \inf_{z \in 2^j Q_1^*} M^{(m)} \circ T_{\sigma_l}(a_l)(z) \chi_{2^{j+1} Q_l^{**}}(z) \Big),$$

where we used (24) in the last inequality.

We now repeat the argument used in establishing (48) to obtain

$$\frac{1}{t^{n+s+1}} \int_{\mathbf{R}^n \setminus Q_1^*} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)| \, dy$$
$$\leq C \frac{\ell(Q_1)^{n+s+1}}{t^{n+s+1}} \prod_{l=1}^{m} \inf_{z \in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(z)\right).$$

Moreover, the assumption $x \notin Q_1^{**}$ and $c_1 \in B(x, 100n^2 t)$ implies $\frac{\ell(Q_1)}{t} \leq CM\chi_{Q_1}(x)^{\frac{1}{n}}$. Therefore,

$$\frac{1}{t^{n+s+1}} \int_{\mathbf{R}^n \setminus Q_1^*} |y - c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \dots, a_m)(y)| \, dy$$
$$\leq CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^{m} \inf_{z \in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(z)\right).$$

This proves Lemma 4.4. □

Lemma 4.5 *For all $x \in \mathbf{R}^n$, we have*

$$\begin{aligned}
&M_\phi \circ \mathcal{T}_\sigma(a_1, \dots, a_m)(x) \\
&\le C \prod_{l=1}^m M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(x)\right) \\
&\quad + CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{l=1}^m \inf_{z \in Q_1^*} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_l)(z)\right).
\end{aligned}$$

Proof. If $x \in Q_1^{**}$, the desired estimate is a consequence of Lemma 4.1. Fix $x \notin Q_1^{**}$. To estimate $M_\phi \circ \mathcal{T}_\sigma(a_1, \dots, a_m)(x)$, we need to examine

$$\left| \int_{\mathbf{R}^n} \phi_t(x-y) \mathcal{T}_\sigma(a_1, \dots, a_m)(y)\, dy \right|$$

for each $t \in (0, \infty)$. If $c_1 \notin B(x, 100n^2t)$, then we make use of Lemma 4.2; otherwise, when $c_1 \in B(x, 100n^2t)$ we recall (35) and then apply Lemma 4.3 and 4.4 to obtain the required estimate in Lemma 4.5. This completes the proof of the lemma. □

4.2 The proof of Proposition 2.5 for product type operators

For the purposes of the proof of (17), we set

$$A = \left\| \sum_{k_1, \dots, k_m = 1}^{\infty} \left(\prod_{l=1}^m \lambda_{l,k_l} \right) M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \dots, a_{m,k_m}) \right\|_{L^p}.$$

For each $\vec{k} = (k_1, \dots, k_m)$, we recall $R_{\vec{k}}$, the smallest-length cube among $Q_{1,k_1}, \dots, Q_{m,k_m}$.

In view of Lemma 4.5, we have

$$A \le CB := C \left\| \sum_{k_1, \dots, k_m = 1}^{\infty} \prod_{l=1}^m \lambda_{l,k_l} \left(M\chi_{Q_{l,k_l}} \right)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_{l,k_l})\right) \right\|_{L^p}. \tag{56}$$

In fact, our assumption imposing on s means $(n+s+1)p/n > 1$ and hence we may employ the boundedness of M to obtain

$$\begin{aligned}
A &\le CB \\
&\quad + C \left\| \sum_{k_1, \dots, k_m = 1}^{\infty} \left(M\chi_{R^*_{\vec{k}}} \right)^{\frac{n+s+1}{n}} \prod_{l=1}^m \lambda_{l,k_l} \inf_{z \in R^*_{\vec{k}}} \left(M\chi_{Q_{l,k_l}} \right)^{\frac{n+N+1}{mn}} \left(1 + M^{(m)} \circ T_{\sigma_l}(a_{l,k_l})\right) \right\|_{L^p} \\
&\le CB.
\end{aligned}$$

So, our task is to estimate B. For this, we make use of the following lemma.

Lemma 4.6 *Let $p \in (0, \infty)$ and $\alpha > \max(1, p^{-1})$. Assume that $q \in (p, \infty] \cap [1, \infty]$. Suppose that we are given a sequence of cubes $\{Q_k\}_{k=1}^\infty$ and a sequence of non-negative L^q-functions $\{F_k\}_{k=1}^\infty$. Then*

$$\left\| \sum_{k=1}^\infty (M\chi_{Q_k})^\alpha F_k \right\|_{L^p} \le C \left\| \sum_{k=1}^\infty \chi_{Q_k} M^{(q)} F_k \right\|_{L^p}.$$

Proof. By Lemma 2.4 and the fact that $M\chi_Q \le C\chi_Q + C\sum_{j=1}^\infty 2^{-jn} \chi_{2^jQ \setminus 2^{j-1}Q}$, we have

$$\begin{aligned}
\left\| \sum_{k=1}^\infty (M\chi_{Q_k})^\alpha F_k \right\|_{L^p} &\le C \left\| \sum_{j=0}^\infty \sum_{k=1}^\infty 2^{-\alpha jn} \chi_{2^jQ_k} F_k \right\|_{L^p} \\
&\le C \left\| \sum_{j=0}^\infty \sum_{k=1}^\infty 2^{-\alpha jn} \chi_{2^jQ_k} \left(\frac{1}{|2^jQ_k|} \int_{2^jQ_k} F_k(y)^q dy \right)^{\frac{1}{q}} \right\|_{L^p}.
\end{aligned}$$

Choose $\alpha > \beta > \max(1, \frac{1}{p})$ and observe the trivial estimate

$$\chi_{2^j Q_k} \leq C \left(2^{jn} M\chi_{Q_k}\right)^{\beta}.$$

Now, Lemma 2.3 gives

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}(M\chi_{Q_k})^{\alpha}F_k\right\|_{L^p} &\leq C\left\|\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\chi_{Q_k}\left(\frac{2^{(\beta-\alpha)jqn}}{|2^jQ_k|}\int_{2^jQ_k}F_k(y)^q dy\right)^{\frac{1}{q}}\right\|_{L^p} \\ &\leq C\Big\|\sum_{k=1}^{\infty}\chi_{Q_k}M^{(q)}F_k\Big\|_{L^p},\end{aligned}$$

which yields the desired estimate. □

Lemma 4.6 can be regarded as a substitute for Lemma 2.4.

Before applying Lemma 4.6 to B, we observe

$$B \leq \prod_{l=1}^{m}\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\left(M\chi_{Q_{l,k_l}}\right)^{\frac{n+N+1}{mn}}\left(1+M^{(m)}\circ T_{\sigma_l}(a_{l,k_l})\right)\right\|_{L^{p_l}}.$$

Then applying Fefferman-Stein's vector-valued inequality and Lemma 4.6, we obtain

$$\begin{aligned}&\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\left(M\chi_{Q_{l,k_l}}\right)^{\frac{n+N+1}{mn}}\left(1+M^{(m)}\circ T_{\sigma_l}(a_{l,k_l})\right)\right\|_{L^{p_l}} \\ &\leq C\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}M\big(\chi_{Q^{**}_{l,k_l}}\big)^{\frac{n+N+1}{mn}}\right\|_{L^{p_l}} + C\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\left(M\chi_{Q_{l,k_l}}\right)^{\frac{n+N+1}{mn}}M^{(m)}\circ T_{\sigma_l}(a_{l,k_l})\right\|_{L^{p_l}} \\ &\leq C\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\chi_{Q_{l,k_l}}\right\|_{L^{p_l}} + C\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\chi_{Q^{**}_{l,k_l}}M\circ M^{(m)}\circ T_{\sigma_l}(a_{l,k_l})\right\|_{L^{p_l}}.\end{aligned}$$

In the second term, we choose $q \in (m, \infty)$ and employ Lemma 2.4, and the boundedness of M and T_{σ_l} to obtain

$$\begin{aligned}&\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\chi_{Q^{**}_{l,k_l}}M\circ M^{(m)}\circ T_{\sigma_l}(a_{l,k_l})\right\|_{L^{p_l}} \\ &\leq C\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\frac{\chi_{Q^{**}_{l,k_l}}}{|Q_{l,k_l}|^{\frac{1}{q}}}\left\|M\circ M^{(m)}\circ T_{\sigma_l}(a_{l,k_l})\right\|^q_{L^q}\right\|_{L^{p_l}} \leq C\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\chi_{Q_{l,k_l}}\right\|_{L^{p_l}}.\end{aligned}$$

As a result,

$$A \leq CB \leq C\prod_{l=1}^{m}\left\|\sum_{k_l=1}^{\infty}\lambda_{l,k_l}\chi_{Q_{l,k_l}}\right\|_{L^{p_l}},$$

which completes the proof of Proposition 2.5.

5 The mixed type

In this section, we prove Proposition 2.5 for operators of type (6). The main technique to deal with an operator $\mathcal{T}_\sigma$ of mixed type is to combine the two previous types. We now establish some necessary estimates for $\mathcal{T}_\sigma$. Again note that, we can apply the result in our paper [10, Theorem 1.4] to see the equivalence between (8) and (9) for the *mixed type* multiplier operator. So, we assume (8) below. For the mixed type, we need the following lemma which can be shown in a way similar to that of Lemma 3.1.

Lemma 5.1 *Let σ be a Coifman–Meyer multiplier, a_l be (p_l,∞)-atoms supported on Q_l for $1 \le l \le m$. Assume $\ell(Q_1) = \min\{\ell(Q_l) : l = 1,\dots,m\}$ and write $\Lambda_j = \{l = 1,\dots,m : 2^jQ_1^{**} \cap 2^jQ_l^{**} = \emptyset\}$. Then for any $y \in 2^{j+1}Q_1^* \setminus 2^jQ_1^*$ we have*

$$|T_\sigma(a_1,\dots,a_m)(y)| \le C\left(\frac{\ell(Q_1)}{|y-c_1| + \sum_{l\in\Lambda_j}|y-c_l|}\right)^{n+N+1}.$$

P r o o f. The detailed proof is as follows. Fix any $y \in 2^{j+1}Q_1^*\setminus 2^jQ_1^*$. Let us use the notation $K^1(y,y_1,\dots,y_m)$ as in the proof of Lemma 3.1. Then for any $y_l \in Q_l$, $l = 1,\dots,m$, we have

$$\begin{aligned}|K^1(y,y_1,\dots,y_m)| &\le C\left(\frac{\ell(Q_1)}{|y-y_1| + \sum_{l\in\Lambda_j}|y-y_l| + \sum_{l\ge 2}|y-y_l|}\right)^{n+N+1}\\ &\le C\left(\frac{\ell(Q_1)}{|y-c_1| + \sum_{l\in\Lambda_j}|y-c_l| + \sum_{l\ge 2}|y-y_l|}\right)^{n+N+1}.\end{aligned}$$

In fact, if $l \in \Lambda_j$, $2^jQ_1^{**} \cap 2^jQ_l^{**} = \emptyset$ and hence, $y \in 2^{j+1}Q_1^*$ means $|y-y_l| \sim |y-c_l|$ for all $y_l \in Q_l$ for such l. Of course, $|y-y_1| \sim |y-c_1|$ is clear since $y \notin 2^jQ_1^*$. Using this kernel estimate, we may prove the desired estimate. □

5.1 Fundamental estimates for the mixed type operator

Let a_k be (p_k,∞)-atoms supported in Q_k for all $1 \le k \le m$. Suppose Q_1 is the cube such that $\ell(Q_1) = \min\{\ell(Q_k) : 1 \le k \le m\}$. For each $1 \le g \le G$, let $Q_{l(g)}$ be the smallest cube among $\{Q_l\}_{l\in I_g}$ and let $m_g = |I_g|$ be the cardinality of I_g. Then we have the following analogues to Lemmas 4.1–4.5. We write $m_g = \sharp I_g$ for each g.

Lemma 5.2 *For all $x \in Q_1^{**}$, we have*

$$\begin{aligned}&M_\phi \circ \mathcal{T}_\sigma(a_1,\dots,a_m)(x)\chi_{Q_1^{**}}(x)\\ &\le C\prod_{g=1}^{G}\left(M\chi_{Q_{l(g)}}(x)^{\frac{(n+N+1)m_g}{nm}} M^{(G)}\circ T_{\sigma_{I_g}}(\{a_{l,k_l}\}_{l\in I_g})(x) + \prod_{l\in I_g} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\right).\end{aligned} \tag{57}$$

P r o o f. Fix $x \in Q_1^{**}$. We need to estimate

$$\left|\int_{\mathbf{R}^n}\phi_t(x-y)\mathcal{T}_\sigma(a_1,\dots,a_m)(y)\,dy\right| \le \frac{C}{t^n}\int_{B(x,t)}|\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy$$

for each $t \in (0,\infty)$. As in the previous section, for simplicity, we only consider the following form:

$$\mathcal{T}_\sigma(f_1,\dots,f_m) = \prod_{g=1}^{G} T_{\sigma_{I_g}}(\{f_l\}_{l\in I_g}), \tag{58}$$

where $\{I_g\}_{g=1}^G$ is a partition of $\{1,\dots,m\}$ with $1 \in I_1$. By Hölder's inequality, we have

$$\frac{1}{t^n}\int_{B(x,t)}|\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\,dy \le C\prod_{g=1}^{G}\left(\frac{1}{t^n}\int_{B(x,t)}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\right)^{\frac{1}{G}}. \tag{59}$$

For each $1 \le g \le G$, we need to examine

$$\left(\frac{1}{t^n}\int_{B(x,t)}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\right)^{\frac{1}{G}}.$$

We consider two cases as in the proof of Lemma 4.1.

Case 1: $t \le \ell(Q_1)$. We observe that

$$\begin{aligned}\frac{1}{t^n}\int_{B(x,t)}|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy \le &\prod_{g:B(x,t)\cap Q^{**}_{l(g)}\neq\emptyset}\Big(\frac{1}{t^n}\int_{B(x,t)}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\Big)^{\frac{1}{G}}\\ &\times\prod_{g:B(x,t)\cap Q^{**}_{l(g)}=\emptyset}\Big(\frac{1}{t^n}\int_{B(x,t)}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\Big)^{\frac{1}{G}}.\end{aligned}$$

When $B(x,t)\cap Q^{**}_{l(g)}\neq\emptyset$, we see that $x\in 3Q^{**}_{l(g)}$. This shows

$$\begin{aligned}&\prod_{g:B(x,t)\cap Q^{**}_{l(g)}\neq\emptyset}\Big(\frac{1}{t^n}\int_{B(x,t)}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\Big)^{\frac{1}{G}} \qquad (60)\\ &\le C\prod_{g:B(x,t)\cap Q^{**}_{l(g)}\neq\emptyset}\chi_{3Q^{**}_{l(g)}}(x)M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x).\end{aligned}$$

When $B(x,t)\cap Q^{**}_{l(g)}=\emptyset$, we may use (24) to have

$$\begin{aligned}&\prod_{g:B(x,t)\cap Q^{**}_{l(g)}=\emptyset}\Big(\frac{1}{t^n}\int_{B(x,t)}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\Big)^{\frac{1}{G}} \qquad (61)\\ &\le C\prod_{g:B(x,t)\cap Q^{**}_{l(g)}=\emptyset}\prod_{l\in I_g}M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}. \qquad (62)\end{aligned}$$

These two estimates (60) and (61) yield the desired estimate in Case 1.

Case 2: $t > \ell(Q_1)$. We use the splitting

$$\begin{aligned}&\frac{1}{t^n}\int_{B(x,t)}|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy\\ &\le\frac{C}{|Q_1^*|}\int_{Q_1^*}|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy+\frac{1}{|Q_1^*|}\int_{(Q_1^*)^c}|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy.\end{aligned}$$

For the first term, (61) yields

$$\begin{aligned}&\frac{1}{|Q_1^*|}\int_{Q_1^*}|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy\\ &\le C\prod_{g=1}^{G}\left(\prod_{l\in I_g}M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}+\chi_{Q^{**}_{l(g)}}(x)M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x)\right).\end{aligned}$$

For the second term, we write

$$I_j:=\frac{1}{|Q_1^*|}\int_{2^{j+1}Q_1^*\setminus 2^jQ_1^*}|T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})(y)|\prod_{g\ge 2}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|dy.$$

Then by a dyadic decomposition of $(Q_1^*)^c$, we obtain

$$\begin{aligned}&\frac{1}{|Q_1^*|}\int_{(Q_1^*)^c}|\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy\\ &=\sum_{j=0}^{\infty}\frac{1}{|Q_1^*|}\int_{2^{j+1}Q_1^*\setminus 2^jQ_1^*}|T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})(y)|\prod_{g\ge 2}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|dy=\sum_{j=0}^{\infty}I_j.\end{aligned}$$

Now, we fix any j and evaluate each I_j. Letting $\Lambda_j = \{l = 1, \ldots, m : 2^j Q_1^{**} \cap 2^j Q_l^{**} = \emptyset\}$ and using Lemma 5.1, for $y \in 2^{j+1}Q_1^* \setminus 2^j Q_1^*$ we obtain

$$\begin{aligned}|T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})(y)| &\le C\left(\frac{\ell(Q_1)}{|y-c_1| + \sum_{l\in I_1\cap\Lambda_j}|y-c_l|}\right)^{n+N+1}\\ &\le C2^{-j(n+N+1)}\left(\frac{2^j\ell(Q_1)}{2^j\ell(Q_1) + \sum_{l\in I_1\cap\Lambda_j}|c_1-c_l|}\right)^{n+N+1}.\end{aligned}$$

We further estimate this term. If $l \in I_1 \cap \Lambda_j$, $|c_1 - c_l| \sim |x - c_l|$ since $x \in Q_1^{**}$. On the other hand, if $l \in I_1 \setminus \Lambda_j$, $\chi_{2^j Q_l^{**}}(x) = \chi_{Q_1^{**}}(x) = 1$ since $x \in Q_1^{**}$. So, we have

$$\begin{aligned}|T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})(y)| &\le C2^{-j(n+N+1)}\prod_{l\in I_1\cap\Lambda_j}\left(\frac{2^j\ell(Q_l)}{|x-c_l|}\right)^{\frac{n+N+1}{m}}\prod_{l\in I_1\setminus\Lambda_j}\chi_{2^jQ_l^{**}}(x)\\ &\le C2^{-j(n+N+1)}\prod_{l\in I_1\setminus\{1\}} M\chi_{2^jQ_l^{**}}(x)^{\frac{n+N+1}{mn}}\\ &\le C2^{-j(n+N+1)}2^{j\frac{n+N+1}{m}(m_1-1)}\prod_{l\in I_1} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}.\end{aligned}$$

This and Hölder's inequality imply that

$$\begin{aligned}&I_j \\ &\le C2^{j\frac{n+N+1}{m}(m_1-1)-j(N+1)}\prod_{l\in I_1} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\prod_{g\ge2}\left(\frac{1}{|2^jQ_1^*|}\int_{2^jQ_1^*}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\right)^{\frac{1}{G}}.\end{aligned}\tag{63}$$

In the usual way, we claim that

$$\begin{aligned}&\prod_{g\ge2}\left(\frac{1}{|2^jQ_1^*|}\int_{2^jQ_1^*}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\right)^{\frac{1}{G}}\\ &\le C2^{j\frac{n+N+1}{m}(m-m_1)}\\ &\quad\times\prod_{g\ge2}\left(M\chi_{Q_{l(g)}}(x)^{\frac{(n+N+1)m_g}{nm}}M^{(G)}\circ T_{\sigma_{I_g}}(\{a_{l,k_l}\}_{l\in I_g})(x) + \prod_{l\in I_g} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}\right).\end{aligned}\tag{64}$$

To see this, we again consider two possibilities of g for each j; $2^jQ_1^{**} \cap 2^jQ_{l(g)}^{**} \ne \emptyset$ or not. In the first case, we notice $x \in Q_1^* \subset 2^jQ_{l(g)}^{**}$ and recalling $m_g = \sharp I_g$, we obtain

$$\begin{aligned}&\left(\frac{1}{|2^jQ_1^*|}\int_{2^jQ_1^*}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\right)^{\frac{1}{G}}\\ &\le C\chi_{2^jQ_{l(g)}^{**}}(x)M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x)\\ &\le C2^{j\frac{m_g(n+N+1)}{m}}M\chi_{Q_{l(g)}}(x)^{\frac{m_g(n+N+1)}{mn}}M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x).\end{aligned}$$

In the second case; $2^jQ_1^{**} \cap 2^jQ_{l(g)}^{**} = \emptyset$, we use (24) to see

$$\begin{aligned}&\left(\frac{1}{|2^jQ_1^*|}\int_{2^jQ_1^*}|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy\right)^{\frac{1}{G}}\\ &\le C\prod_{l\in I_g} M\chi_{2^jQ_l^{**}}(x)^{\frac{n+N+1}{mn}} \le C2^{j\frac{m_g(n+N+1)}{m}}\prod_{l\in I_g} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}.\end{aligned}$$

These two estimates yield (64). Inserting (64) to (63), we arrive at

$$
\begin{aligned}
I_j &\le C 2^{-j(\frac{n+N+1}{m}-n)} \\
&\quad \times \prod_{g=1}^{G} \left(M\chi_{Q_{l(g)}}(x)^{\frac{m_g(n+N+1)}{mn}} M^{(G)} \circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x) + \prod_{l\in I_g} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} \right).
\end{aligned}
$$

Taking N sufficiently large, we can sum over $j \in \mathbf{N}$ to deduce the desired estimate. This completes the proof of Lemma 5.2 □

Lemma 5.3 *Assume $x \notin Q_1^{**}$ and $c_1 \notin B(x, 100n^2 t)$. Then we have*

$$
\begin{aligned}
&\frac{1}{t^n}\int_{B(x,t)} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\, dy \\
&\le C \prod_{g=1}^{G} \left(M\chi_{Q_{l(g)}}(x)^{\frac{m_g(n+N+1)}{mn}} M^{(G)} \circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x) + \prod_{l\in I_g} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} \right).
\end{aligned}
$$

Proof. Fix $x \notin Q_1^{**}$ and $t > 0$ such that $c_1 \notin B(x, 100n^2 t)$. Let $\mathcal{T}_\sigma$ be the operator of type (6). We may consider the reduced form (58) of $\mathcal{T}_\sigma$ and start from (59). We define

$$
J = \{g = 2,\dots,m : x \notin Q_{l(g)}^{**}\}, \quad J_0 = \{g \in J : B(x,2t) \cap Q_{l(g)}^{*} = \emptyset\}, \quad J_1 = J \setminus J_0
$$

and split the product as follows:

$$
\begin{aligned}
&\frac{1}{t^n}\int_{B(x,t)} |\mathcal{T}_\sigma(a_1,\dots,a_m)(y)|\, dy \\
&\le C \left\| T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})\chi_{B(x,t)} \right\|_{L^\infty} \prod_{l\in J_0} \left\| T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})\chi_{B(x,t)} \right\|_{L^\infty} \\
&\quad \times \prod_{g\in J_1} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} |T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy \right)^{\frac{1}{G}} \\
&\quad \times \prod_{g\in\{2,\dots,G\}\setminus J} \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} |T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(y)|^G dy \right)^{\frac{1}{G}} \\
&= C(\mathrm{I} \times \mathrm{II} \times \mathrm{III} \times \mathrm{IV}).
\end{aligned}
$$

To estimate I, we further refine the partition of I_1:

$$
\begin{aligned}
&I_1^0 = \{l \in I_1 : x \notin Q_l^{**}, B(x,2t) \cap Q_l^* = \emptyset\}, \quad I_1^1 = \{l \in I_1 : x \notin Q_l^{**}, B(x,2t) \cap Q_l^* \ne \emptyset\}, \\
&I_1^2 = I_1 \setminus (I_1^0 \cup I_1^1).
\end{aligned}
$$

Since $x \notin Q_1^{**}$ and $c_1 \notin B(x, 100n^2 t)$, we can see that $1 \in I_1^0$. From Lemma 3.1, we deduce

$$
\begin{aligned}
&|T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})(y)| \\
&\le C \frac{\ell(Q_1)^{n+N+1}}{(\sum_{l\in I_1^0} |y - c_l|)^{n+N+1}} \\
&\le C \frac{\ell(Q_1)^{n+N+1}}{(\sum_{l\in I_1^0} |x - c_l|)^{n+N+1}} \\
&\le C \Big(\frac{\ell(Q_1)}{|x - c_1| + \ell(Q_1)} \Big)^{(m-m_1)\frac{n+N+1}{m}} \\
&\quad \times \prod_{l\in I_1^0} \Big(\frac{\ell(Q_l)}{|x - c_l| + \ell(Q_l)} \Big)^{\frac{n+N+1}{m}} \prod_{l\in I_1^1} \Big(\frac{\ell(Q_1)}{|x - c_1| + \ell(Q_1)} \Big)^{\frac{n+N+1}{m}}
\end{aligned}
$$

for all $y \in B(x,t)$, where $m_1 = |I_1|$ is the cardinality of the set I_1. As in the proof of Lemma 4.2 for the product type, if $x \notin Q_l^{**}$ and $B(x,2t) \cap Q_l^* \neq \emptyset$ then $|x-c_l| \le Ct \le C|x-c_1|$. This observation implies

$$\frac{\ell(Q_1)}{|x-c_1|+\ell(Q_1)} \le C\frac{\ell(Q_l)}{|x-c_l|+\ell(Q_l)}$$

for all $l \in I_1^1$. Therefore, we can estimate

$$|T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})(y)| \le C\Big(\frac{\ell(Q_1)}{|x-c_1|+\ell(Q_1)}\Big)^{(m-m_1)\frac{n+N+1}{m}} \prod_{l\in I_1^0\cup I_1^1} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}$$

for all $y \in B(x,t)$. Obviously, $1 \le CM\chi_{Q_l}(x)$ for all $l \in I_1^2$, and hence we have

$$|T_{\sigma_{I_1}}(\{a_l\}_{l\in I_1})(y)| \le C\Big(\frac{\ell(Q_1)}{|x-c_1|+\ell(Q_1)}\Big)^{(m-m_1)\frac{n+N+1}{m}} \prod_{l\in I_1} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} \tag{65}$$

for all $y \in B(x,t)$ which gives the estimate for I. In the third term III, we simply have

$$\mathrm{III} \le \prod_{g\in J_1} M^{(G)} \circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x).$$

So, we obtain

$$\begin{aligned}
\mathrm{I}\times\mathrm{III} &\le C\prod_{l\in I_1} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} \prod_{g\in J_1}\left(\frac{\ell(Q_1)}{\ell(Q_1)+|x-c_1|}\right)^{\frac{m_g(n+N+1)}{m}} M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x)\\
&\le C\prod_{l\in I_1} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}} \prod_{g\in J_1} M\chi_{Q_{l(g)}}(x)^{\frac{m_g(n+N+1)}{mn}} M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x),
\end{aligned}$$

since $g \in J_1$ implies $|x-c_{l(g)}| \le C|x-c_1|$. For the second term II, we use estimate (63) and an argument as for estimate for I to get

$$\mathrm{II} = \prod_{l\in J_0} \left\|T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})\chi_{B(x,t)}\right\|_{L^\infty} \le C\prod_{g\in J_0}\prod_{l\in I_g} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}.$$

In the last term IV, we recall $g \notin J$ means $x \in Q_{l(g)}^{**}$ and hence,

$$\mathrm{IV} \le C\prod_{g\notin J} M\chi_{Q_{l(g)}}(x)^{\frac{m_g(n+N+1)}{mn}} M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x).$$

Combining the estimates for I, II, III and IV, we complete the proof of Lemma 5.3. □

Lemma 5.4 *Assume $x \notin Q_1^{**}$ and $c_1 \in B(x,100n^2t)$. Write*

$$A_l(x) = M\chi_{Q_{l(g)}}(x)^{\frac{m_g(n+N+1)}{mn}} M^{(G)}\circ T_{\sigma_{I_g}}(\{a_l\}_{l\in I_g})(x) + \prod_{l\in I_g} M\chi_{Q_l}(x)^{\frac{n+N+1}{mn}}$$

for $x \in \mathbf{R}^n$. Then we have

$$\begin{aligned}
&\frac{\ell(Q_1)^{s+1}}{t^{n+s+1}}\int_{Q_1^*} |\mathcal{T}_\sigma(a_1,\ldots,a_m)(y)|\,dy\\
&\le CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}}\prod_{g=1}^{G}\inf_{z\in Q_1^*} A_l(z).
\end{aligned}$$

Lemma 5.5 *Assume $x \notin Q_1^{**}$ and $c_1 \in B(x, 100n^2 t)$. Then we have*

$$\frac{1}{t^{n+s+1}} \int_{(Q_1^*)^c} |y-c_1|^{s+1} |\mathcal{T}_\sigma(a_1, \ldots, a_m)(y)|\, dy$$
$$\le CM\chi_{Q_1}(x)^{\frac{n+s+1}{n}} \prod_{g=1}^{G} \inf_{z \in Q_1^*} A_l(z),$$

where A_l is as in Lemma 5.4.

The proof of Lemmas 5.4 and 5.5 are very similar to those of Lemma 5.2, so we omit the details here.

5.2 The proof of Proposition 2.5 for the mixed type operators

Employing the above lemmas, we complete the proof of (17). For each $\vec{k} = (k_1, \ldots, k_m)$, recall the smallest-length cube $R_{\vec{k}}$ among $Q_{1,k_1}, \ldots, Q_{m,k_m}$ and write $Q_{l(g),\vec{k}(g)}$ for the cube of smallest-length among $\{Q_{l,k_l}\}_{l \in I_g}$. Combining Lemmas 5.2-5.5, we obtain the following pointwise estimate

$$M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m})(x) \le C \prod_{g=1}^{G} b_{g,\vec{k}(g)}(x) + CM\chi_{R_{\vec{k}}^*}(x)^{\frac{n+s+1}{n}} \prod_{g=1}^{G} \inf_{z \in R_{\vec{k}}^*} b_{g,\vec{k}(g)}(z),$$
$$b_{g,\vec{k}(g)}(x) = M\chi_{Q^*_{l(g),\vec{k}(g)}}(x)^{\frac{m_g(n+N+1)}{mn}} M^{(G)} \circ T_{\sigma_{I_g}}(\{a_{l,k_l}\}_{l \in I_g})(x) + \prod_{l \in I_g} M\chi_{Q_{l,k_l}}(x)^{\frac{n+N+1}{mn}}$$

for all $x \in \mathbf{R}^n$. As in the proof for the product type operator, we let

$$A = \left\| \sum_{k_1, \ldots, k_m = 1}^{\infty} \Big(\prod_{l=1}^{m} \lambda_{l,k_l} \Big) M_\phi \circ \mathcal{T}_\sigma(a_{1,k_1}, \ldots, a_{m,k_m}) \right\|_{L^p}.$$

In view of $(n+s+1)p/n > 1$, using Lemma 2.3 and Hölder's inequality, we see

$$A \le C \prod_{g=1}^{G} \left\| \sum_{k_l \ge 1 : l \in I_g} \Big(\prod_{l \in I_g} \lambda_{l,k_l} \Big) \left(\Big(M\chi_{Q^*_{l(g),\vec{k}(g)}} \Big)^{\frac{m_g(n+N+1)}{mn}} M^{(G)} \circ T_{\sigma_{I_g}}(\{a_{l,k_l}\}_{l \in I_g}) + \prod_{l \in I_g} (M\chi_{Q_{l,k_l}})^{\frac{n+N+1}{mn}} \right) \right\|_{L^{q_g}}$$
$$\le C \prod_{g=1}^{G} \Big(A_{g,1} + A_{g,2} \Big),$$

where $q_g \in (0, \infty)$ is defined by $1/q_g = \sum_{l \in I_g} 1/p_l$ and

$$A_{g,1} = \left\| \sum_{k_l \ge 1 : l \in I_g} \left(\prod_{l \in I_g} \lambda_{l,k_l} \right) \Big(M\chi_{Q^*_{l(g),\vec{k}(g)}} \Big)^{\frac{m_g(n+N+1)}{mn}} M^{(G)} \circ T_{\sigma_{I_g}}(\{a_{l,k_l}\}_{l \in I_g}) \right\|_{L^{q_g}},$$
$$A_{g,2} = \left\| \sum_{k_l \ge 1 : l \in I_g} \prod_{l \in I_g} \lambda_{l,k_l} (M\chi_{Q_{l,k_l}})^{\frac{n+N+1}{mn}} \right\|_{L^{q_g}}.$$

For $A_{g,2}$, we have only to employ Lemma 2.3 to get the desired estimate. For $A_{g,1}$, take large r and employ Lemma 4.6 to obtain

$$A_{g,1} \le C \left\| \sum_{k_l \ge 1 : l \in I_g} \left(\prod_{l \in I_g} \lambda_{l,k_l} \right) \chi_{Q^*_{l(g),\vec{k}(g)}} M^{(r)} \circ M^{(G)} [T_{\sigma_{I_g}}(\{a_{l,k_l}\}_{l \in I_g})] \right\|_{L^{q_g}}.$$

Then it follows from Lemma 2.4 and (22) that

$$A_{g,1} \le C \left\| \sum_{k_l \ge 1: l \in I_g} \left(\prod_{l \in I_g} \lambda_{l,k_l} \right) \frac{\left\| \chi_{Q^*_{l(g),\vec{k}(g)}} M^{(r)} \circ M^{(G)}[T_{\sigma_{I_g}}(\{a_{l,k_l}\}_{l \in I_g})] \right\|_{L^q} \chi_{Q^*_{l(g),\vec{k}(g)}}}{|Q_{l(g),\vec{k}(g)}|^{1/q}} \right\|_{L^{q_g}}$$

$$\le C \left\| \sum_{k_l \ge 1: l \in I_g} \left(\prod_{l \in I_g} \lambda_{l,k_l} \right) \chi_{Q^*_{l(g),\vec{k}(g)}} \inf_{z \in Q^*_{l(g),\vec{k}(g)}} \prod_{l \in I_g} M\chi_{Q_l}(z)^{\frac{n+N+1}{mn}} \right\|_{L^{q_g}}$$

$$\le C \prod_{l \in I_g} \left\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l} (M\chi_{Q_l})^{\frac{n+N+1}{mn}} \right\|_{L^{p_l}} \le C \prod_{l \in I_g} \left\| \sum_{k_l=1}^{\infty} \lambda_{l,k_l} \chi_{Q_l} \right\|_{L^{p_l}},$$

which completes the proof of Lemma 2.5 for operators of mixed type.

6 Examples

We provide examples of operators of the kinds discussed in this paper: All of the following are symbols of trilinear operators acting on functions on the real line, thus they are functions on $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

The symbol

$$\sigma_1(\xi_1, \xi_2, \xi_3) = \frac{(\xi_1 + \xi_2 + \xi_3)^2}{\xi_1^2 + \xi_2^2 + \xi_3^2}$$

is associated with an operator of type (3).

The symbol

$$\begin{aligned}\sigma_2(\xi_1, \xi_2, \xi_3) &= \frac{\xi_1^3}{(1+\xi_1^2)^{\frac{3}{2}}} \frac{1}{(1+\xi_2^2+\xi_3^2)^{\frac{3}{2}}} + \frac{1}{(1+\xi_1^2)^{\frac{3}{2}}} \frac{\xi_2^3}{(1+\xi_2^2+\xi_3^2)^{\frac{3}{2}}} \\ &\quad + \frac{1}{(1+\xi_1^2)^{\frac{3}{2}}} \frac{\xi_3^3}{(1+\xi_2^2+\xi_3^2)^{\frac{3}{2}}} - \frac{3\xi_1}{(1+\xi_1^2)^{\frac{3}{2}}} \frac{\xi_2\xi_3}{(1+\xi_2^2+\xi_3^2)^{\frac{3}{2}}} \\ &= \frac{(\xi_1+\xi_2+\xi_3)(\xi_1^2+\xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_2\xi_3-\xi_3\xi_1)}{(1+\xi_1^2)^{\frac{3}{2}}(1+\xi_2^2+\xi_3^2)^{\frac{3}{2}}}\end{aligned}$$

provides an example of an operator of type (6). Note that each term is given as a product of a multiplier of ξ_1 times a multiplier of (ξ_2, ξ_3).

The symbol

$$\begin{aligned}\sigma_3(\xi_1, \xi_2, \xi_3) &= \frac{\xi_1^4}{(1+\xi_1^2)^2} \frac{\xi_2^2}{(1+\xi_2^2)^2} \frac{\xi_3}{(1+\xi_3^2)^2} - \frac{\xi_1^4}{(1+\xi_1^2)^2} \frac{\xi_2}{(1+\xi_2^2)^2} \frac{\xi_3^2}{(1+\xi_3^2)^2} \\ &\quad - \frac{\xi_1^2}{(1+\xi_1^2)^2} \frac{\xi_2^4}{(1+\xi_2^2)^2} \frac{\xi_3}{(1+\xi_3^2)^2} + \frac{\xi_1}{(1+\xi_1^2)^2} \frac{\xi_2^4}{(1+\xi_2^2)^2} \frac{\xi_3^2}{(1+\xi_3^2)^2} \\ &\quad + \frac{\xi_1^2}{(1+\xi_1^2)^2} \frac{\xi_2}{(1+\xi_2^2)^2} \frac{\xi_3^4}{(1+\xi_3^2)^2} - \frac{\xi_1}{(1+\xi_1^2)^2} \frac{\xi_2^2}{(1+\xi_1^2)^2} \frac{\xi_3^4}{(1+\xi_3^2)^2} \\ &= -\frac{\xi_1\xi_2\xi_3(\xi_1-\xi_2)(\xi_2-\xi_3)(\xi_3-\xi_1)(\xi_1+\xi_2+\xi_3)}{(1+\xi_1^2)^2(1+\xi_2^2)^2(1+\xi_3^2)^2}\end{aligned}$$

yields an example of an operator of type (5). The next example:

$$\sigma_4(\xi_1, \xi_2, \xi_3) = \frac{\xi_1\xi_2}{\xi_1^2+\xi_2^2+(\xi_1+\xi_2)^2} \cdot 1 - \frac{\xi_1\xi_2}{\xi_1^2+\xi_2^2+\xi_3^2}$$

shows that the integer $G(\rho)$ varies according to ρ. Notice that all four examples satisfy

$$\sigma_1(\xi_1, \xi_2, \xi_3) = \sigma_2(\xi_1, \xi_2, \xi_3) = \sigma_3(\xi_1, \xi_2, \xi_3) = \sigma_4(\xi_1, \xi_2, \xi_3) = 0$$

when $\xi_1 + \xi_2 + \xi_3 = 0$. This yields condition (8) when $s = 0$; see [10]. For higher order cancellation $s \in \mathbf{Z}^+$, we consider ${\sigma_1}^{s+1}, {\sigma_2}^{s+1}, {\sigma_3}^{s+1}$, for example.

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