IMPROVED WEIGHTED ESTIMATES FOR HÖRMANDER MULTIPLIERS VIA INTERPOLATION

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ABSTRACT. We prove an interpolation theorem for a class of Fourier multipliers on weighted spaces. This allows us to improve the results of Beltran–Roos–Seeger [2] and Bernicot–Frey–Petermichl [3] concerning weighted estimates for Hörmander multiplier operators.

1. INTRODUCTION

For a Schwartz function f on $\mathbb{R}^n$ we denote by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ its Fourier transform. The inverse Fourier transform of f is denoted by $f^\vee$ and equals $\widehat{f}(-\xi)$. A Fourier multiplier is an operator of the form $f \mapsto (\sigma\widehat{f})^\vee$, where σ is a bounded function on $\mathbb{R}^n$. A Fourier multiplier associated with a function σ is denoted by T_σ and is called an L^p Fourier multiplier if it admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself, for some $p \in (1,\infty)$.

A condition on bounded functions σ which imply that T_σ is an L^p Fourier multiplier for some $p \in (1,\infty)$ is the following one due to Mikhlin [21]:

$$|\partial^\alpha\sigma(\xi)| \le C_\alpha|\xi|^{-|\alpha|}, \qquad |\alpha| \le \Big[\frac{n}{2}\Big] + 1. \tag{1}$$

Inspired by the work of Hörmander [14], condition (1) has been relaxed in many ways, for instance, if $rs > n$, the condition

$$\sup_{R>0}\left(R^{r|\alpha|-n}\int_{R<|x|<2R}|\partial^\alpha\sigma(x)|^r\,dx\right)^{1/r} < +\infty \quad \text{for all } |\alpha| \le s, \tag{2}$$

implies that the associated operator T_σ is L^p bounded. A natural generalization of (2) is

$$\sup_{j\in\mathbb{Z}}\|(I-\Delta)^{\frac{s}{2}}(\widehat{\Psi}\,\sigma(2^j\cdot))\|_{L^r} < \infty, \qquad \text{where } rs > n. \tag{3}$$

Here $(I-\Delta)^{\frac{s}{2}}$ is a fractional power of the Laplacian Δ, and Ψ is a Schwartz function whose Fourier transform is supported in annulus $\{\xi \in \mathbb{R}^n : 1/2 \le |\xi| < 2\}$ and which satisfies $\sum_{j\in\mathbb{Z}}\hat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$.

It is known that the condition (3) implies that T_σ is bounded on L^p for the range

$$\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{1}{r} < \frac{s}{n}, \qquad 1 < p < \infty, \tag{4}$$

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as shown by Calderón–Torchinsky [4]. The appearance of r in (4) is not necessary. In fact, if $1 < p < \infty$, $0 < s < n$, and (3) holds, then T_σ is bounded on L^p whenever

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}. \tag{5}$$

Some estimates near L^1 involving Besov spaces [25] were obtained by Seeger [22], [23] who showed that

$$\|T_\sigma f\|_{L^{1,r}} \le C \sup_{j\in\mathbb{Z}} \|\widehat{\Psi}\, \sigma(2^j\cdot)\|_{B^{n/2}_{2,1}} \|f\|_{H^1}. \qquad r \ge 2. \tag{6}$$

On the critical line, Grafakos–He–Honzík–Nguyen [11] proved that if $1 < p < 2$, $0 < s < n/2$, and $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, then T_σ is bounded from L^p to $L^{p,2}$ provided that

$$\|\widehat{\Psi}\, \sigma(2^j\cdot)\|_{B^s_{n/s,1}} < \infty.$$

Grafakos and Slavíková [10] later improved the space in (3) from L^r to $L^{n/s,1}$, proving that under the Lorentz–Sobolev condition

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{s/2}(\widehat{\Psi}\, \sigma(2^j\cdot))\|_{L^{n/s,1}(\mathbb{R}^n)} < \infty, \tag{7}$$

the operator T_σ is bounded on L^p for all

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}. \tag{8}$$

Moreover, a counterexample of Slavíková [24] indicates that the boundedness of T_σ fails on the critical line $|1/p - 1/2| = s/n$.

In contrast with the unweighted case, our understanding of the weighted theory for Fourier multipliers remains rather limited. Kurtz and Wheeden [18] proved that if $1 < p < \infty$, $1 < r < 2$, $rs > n$ and $ps > n$, and σ satisfies (2), then the operator T_σ is bounded on $L^p(w)$ for every weight $w \in A_{ps/n}$. Additionally, Kurtz and Wheeden identified the sharp range of power weights $|x|^\beta$ ensuring the boundedness of T_σ under (2). Later, Muckenhoupt–Wheeden–Young [20] obtained a complete characterization of Fourier multipliers on power-weighted spaces $L^2(|x|^{2a})$. In the radial setting, Lee and Seeger [19] further showed that for quasi-radial multipliers, weighted L^2 boundedness with power weights is equivalent to a one-dimensional Sobolev condition on the multiplier.

Weighted extrapolation results for operators [6] [7] show that weighted boundedness of an operator T at a single exponent extrapolates to weighted boundedness for all admissible exponents. In particular, for the multiplier operator T_σ with $s < n/2$, the boundedness under conditions (3) and (7) necessarily involves RH classes; otherwise the boundedness for power weights would be incompatible with their extrapolation theory.

More recently, the work of Beltrán-Roos-Seeger [2] shows that T_σ satisfies a sparse bound whenever (3) holds. Combining their result with Proposition 6.4 of Bernicot-Frey-Petermichl [3], one obtains the weighted estimate

$$T_\sigma : L^p(w) \to L^p(w) \quad \text{for all } w \in A_{p(\frac{1}{r}+\frac{1}{2})} \cap RH_q,$$

where $q > 1$ satisfies

$$\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{p} + \frac{1}{pq}. \tag{9}$$

In this paper, we investigate the relationship among the aforementioned parameters and improve the range of the weights for which T_σ is bounded from $L^p(w)$ to $L^p(w)$ under condition (3). We also provide examples that indicate the best possible results one might expect for Fourier multipliers on weighted spaces. See Theorem 2.3.

Definition 1.1 (Muckenhoupt A_p weights)**.** Let $1 < p < \infty$. A weight w (a locally integrable, positive a.e. function) belongs to the class A_p if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $p = 1$, we say $w \in A_1$ if

$$[w]_{A_1} := \sup_Q \frac{1}{|Q|} \int_Q w(x)\,dx \cdot (\operatorname*{ess\,inf}_Q w)^{-1} < \infty.$$

Definition 1.2 (Reverse Hölder RH_q weights)**.** Let $1 < q < \infty$. We say $w \in RH_q$ if

$$[w]_{RH_q} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^q\,dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right)^{-1} < \infty.$$

For $q = \infty$, we say $w \in RH_\infty$ if

$$[w]_{RH_\infty} := \sup_Q \frac{\operatorname{ess\,sup}_Q w}{\frac{1}{|Q|}\int_Q w(x)\,dx} < \infty.$$

In this work, we prove an interpolation result (Theorem 5.1) that allows us to vary the smoothness of a multiplier. This theorem is the main result of Section 5. As an application of Theorem 5.1 we obtain an improvement of the best (up to now) known results concerning weighted estimates for Fourier multipliers satisfying Hörmander's condition. These applications are given in Section 6. In particular, one of these theorems (Theorem 6.1) concerns the case $p > 2$ and $0 < s < \frac{n}{2}$ in which we prove that if $\frac{1}{2} - \frac{1}{p} < \frac{s}{n}$, $rs > n$ and

$$\bar{q} = 1 + \frac{1 - \frac{2s}{n}}{\frac{1}{2} - \frac{1}{p}}, \tag{10}$$

then we have

$$\|T_\sigma f\|_{L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}} [\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^r} \|f\|_{L^p(w)} \qquad \text{for} \qquad w \in A_{\frac{p}{2}} \cap RH_{\bar{q}}.$$

In view of the work of Beltrán–Roos–Seeger [2] and Bernicot–Frey–Petermichl [3], until now, the best available value of $\bar{q}$ in (10) grew rapidly and tended to infinity as $r \to n/s$. In contrast, our index $\bar{q}$ in (10) remains finite and is independent of r. We also obtain an analogous result (Theorem 6.2) in the case $p = 2$.

Additionally, we provide counterexamples (Theorem 2.3) indicating that $A_{\frac{ps}{n}+1} \cap RH_{\frac{n}{ps}}$ may be the optimal space of weights for which positive results hold.

Notation: We use the symbol $A \lesssim B$ to denote that the quantity A is bounded by a constant multiple of B, where the constant only depends on inessential parameters. We use $A \sim B$ to indicate both $A \lesssim B$ and $B \lesssim A$.

2. Counterexample

To motive our study, we exhibit counterexamples that indicate restrictions on the weights w if a Fourier multiplier T_σ maps $L^p(w)$ to itself. These restrictions can be expressed in terms of the weight classes A_p and RH_q.

Lemma 2.1. *Let $s > 0$. Consider the multiplier*

$$\sigma(\xi) = e^{-2\pi i \xi_1}(1 + 4\pi^2|\xi|^2)^{-\frac{s}{2}} \tag{11}$$

defined on $\mathbb{R}^n$, where $\xi = (\xi_1, \ldots, \xi_n)$. Then

$$\sup_{j\in\mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}}[\sigma\widehat{\Psi}(2^j\cdot)]\|_{L^{\frac{n}{s},1}} < \infty. \tag{12}$$

Here $\widehat{\Psi}$ is a smooth function supported in an annulus centered at zero.

Proof. Case I: If s is an even integer, then we have

$$(I - \Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot)) = \sum_{|\alpha|\le s} \partial^\alpha(\hat{\Psi}\sigma(2^j\xi)) = \sum_{|\alpha|\le s}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}\hat{\Psi}(\xi))2^{j|\beta|}(\partial^\beta\sigma)(2^j\xi).$$

Moreover,

$$|\partial^\beta\sigma(\xi)| = \left|\sum_{\gamma\le\beta}\partial^{\beta-\gamma}(e^{-2\pi i\xi_1})\partial^\gamma(\frac{1}{(1+4\pi^2|\xi|^2)^{s/2}})\right| \le \sum_{\gamma\le\beta}\binom{\beta}{\gamma}C_\gamma(1+|\xi|)^{-s-\gamma}$$

where $\eta = (1, 0, ...0)$. By two equations above, we get

$$\begin{aligned} |(I - \Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))| &\le \sum_{|\alpha|\le s}\sum_{\beta\le\alpha}\sum_{\gamma\le\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}C_\gamma(\partial^{\alpha-\beta}\hat{\Psi}(\xi))2^{j(|\beta|-s-\gamma)}(\frac{1}{2^j}+|\xi|)^{-s-\gamma} \\ &\le \sum_{|\alpha|\le s}\sum_{\beta\le\alpha}\sum_{\gamma\le\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}C_\gamma\,\partial^{\alpha-\beta}\hat{\Psi}(\xi)\,|\xi|^{-s-\gamma}. \end{aligned} \tag{13}$$

The term $2^{j(|\beta|-s-\gamma)}$ is removed since $|\beta| \le |\alpha| \le s$ implies that $2^{j(|\beta|-s-\gamma)} \le 1$. Now we have $|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))|$ could be bounded by some compactly supported Schwartz function which is in $L^{\frac{n}{s},1}$ and its norm is independent on j. Thus we have

$$\sup_{j\in\mathbb{Z}}\|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{\frac{n}{s},1}} < \infty.$$

Case II. If s is not an even integer, we find an even integer N such that $N > s$. Define

$$F(z) = \int_{\mathbb{R}^n}(I-\Delta)^{\frac{zN}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi$$

where $\psi \in L^{(\frac{n}{s})',\infty}$ with $\|\psi\|_{L^{(\frac{n}{s})',\infty}} = 1$.

When $z = it$,

$$\begin{aligned} |F(it)| &\le \left|\int_{\mathbb{R}^n}(I-\Delta)^{\frac{itN}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi\right| \\ &\le \|\psi\|_{L^{(\frac{n}{s})',\infty}}\|(I-\Delta)^{\frac{itN}{2}}(\hat{\Psi}\sigma)\|_{L^{\frac{n}{s},1}} \\ &\le C_N(1+|t|)^{[\frac{n}{s}]+2}\|\hat{\Psi}\sigma(2^j\cdot)\|_{L^{\frac{n}{s},1}} \\ &\le \|\sigma\|_{L^\infty}C_{N,s}(1+|t|)^{[\frac{n}{s}]+2}. \end{aligned} \tag{14}$$

The first inequality follows from Hölder's inequality. By [9, Corollary 5.3.3], we have $\|(I-\Delta)^{it}\|_{L^p\to L^p} \lesssim_{n,p} (1+|t|)^{[\frac{n}{2}]+2}$ $\forall 1<p<\infty$. For any $p\in(0,\infty)$ (in this problem, we can let p be $\frac{n}{s}$) we choose p_0, p_1 such that $1<p_0<p<p_1<\infty$ and apply the above lemma to get $\|(I-\Delta)^{it}\|_{L^{p,1}\to L^{p,1}} \lesssim_{n,p,p_0,p_1} (1+|t|)^{[\frac{n}{2}]+2}$, so the second inequality follows.

Similarly, when $z=1+it$,

$$
\begin{aligned}
|F(1+it)| &= \left|\int_{\mathbb{R}^n} (I-\Delta)^{\frac{itN}{2}+\frac{N}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi\right| \\
&\le \|\psi\|_{L^{(\frac{n}{s})',\infty}} \|(I-\Delta)^{\frac{itN}{2}+\frac{N}{2}}[\hat{\Psi}\sigma]\|_{L^{\frac{n}{s},1}} \\
&\le C_N(1+|t|)^{[\frac{n}{s}]+2}\|(I-\Delta)^{\frac{N}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\|_{L^{\frac{n}{s},1}} \\
&\le C'_{N,s}(1+|t|)^{[\frac{n}{s}]+2}
\end{aligned}
\tag{15}
$$

Here we applied case I to get the last inequality.

Claim: $F(z)$ is analytic on the unit strip S and continuous on $\bar{S}$.
The claim allows us to apply [9, C.0.4] to obtain

$$|F(\theta)| \le C_{n,\theta} C_{N,s}^{1-\theta} C_{N,s}'^{\theta} \quad \forall\theta\in(0,1).$$

Thus

$$
\begin{aligned}
\sup_{\psi\in L^{\frac{n}{s},\infty}} F(\theta_0) &= \sup_{\psi\in L^{\frac{n}{s},\infty}} \int_{\mathbb{R}^n} (I-\Delta)^{\frac{\theta_0 N}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi \\
&= \|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))(\xi)\|_{L^{\frac{n}{s},1}} \\
&\le C_{n,\theta_0} C_{N,s}^{1-\theta_0} C_{N,s}'^{\theta_0}
\end{aligned}
$$

Proof of claim:

$$
\begin{aligned}
F(z) &= \int_{\mathbb{R}^n} (I-\Delta)^{\frac{zN}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\psi(\xi)d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+4\pi^2|\eta|^2)^{\frac{zN}{2}} (\widehat{\hat{\Psi}\sigma(2^j\cdot)})(\eta) e^{2\pi i\xi\cdot\eta} d\eta\psi(\xi)d\xi
\end{aligned}
\tag{16}
$$

The double integral converges absolutely and the continuity of $F(z)$ on $\bar{S}$ is straightforward. Now we show $F(z)$ is analytic on S. For any point z_0 in S, pick $\delta>0$ such that

$$2\delta < \frac{N}{2} - \Re(\frac{z_0 N}{2}).$$

By [9, lemma 2.7.5] for $0<\frac{N}{2}|z|<\delta$, we obtain

$$\left|\frac{(1+4\pi^2|\eta|^2)^{\frac{zN}{2}}-1}{z}\right| \le \frac{2N}{\delta}\max\{(1+4\pi^2|\eta|^2)^{2\delta}, (1+4\pi^2|\eta|^2)^{-2\delta}\}$$

which gives

$$\left|\frac{(1+4\pi^2|\eta|^2)^{\frac{zN+z_0N}{2}} - (1+4\pi^2|\eta|^2)^{\frac{z_0N}{2}}}{z}\right| \le \frac{2N}{\delta}(1+4\pi^2|\eta|^2)^{\frac{N}{2}}.$$

In addition, $\hat{\Psi}$ and ψ are Schwartz functions, so we can apply the Lebesgue dominated convergence theorem when the z derivative hits the term $(1+4\pi^2|\eta|^2)^{\frac{zN}{2}}$. This yields the analyticity of $F(z)$. □

Proposition 2.2. *Let $1 < p < \infty$ and let σ be as in (11). Suppose that T_σ is bounded from $L^p(|x|^\beta)$ to itself for some $\beta \in \mathbb{R}$. Then we must have*

$$-sp \le \beta \le sp.$$

Proof. Let $e_1 = (1, 0, \dots, 0)$. Consider the function

$$f(x) = |x+e_1|^{-n/p} \left(\log \frac{1}{|x+e_1|}\right)^{-\delta} \chi_{\{|x+e_1| \le 1/5\}}(x)$$

for some $1/p < \delta < 1$. Then

$$\int_{\mathbb{R}^n} |f(x)|^p |x|^\beta \, dx \approx \int_{|x+e_1| \le 1/5} |x+e_1|^{-n} \left(\log \frac{1}{|x+e_1|}\right)^{-p\delta} dx < \infty.$$

Notice that

$$\sigma^\vee(x) = G_s(x-e_1),$$

where G_s is the Bessel potential. Then $T_\sigma(f)(x)$ is nonnegative (since G_s is positive) and equals

$$\begin{aligned} T_\sigma(f)(x) &= \int_{|y+e_1| \le 1/5} |y+e_1|^{-n/p} \left(\log \frac{1}{|y+e_1|}\right)^{-\delta} G_s(x-e_1-y) \, dy \\ &= \int_{|y| \le 1/5} |y|^{-n/p} \left(\log \frac{1}{|y|}\right)^{-\delta} G_s(x-y) \, dy \\ &\ge c \int_{\substack{|y| \le 1/5 \\ |x-y| \le 2}} |y|^{-n/p} \left(\log \frac{1}{|y|}\right)^{-\delta} |x-y|^{-n+s} \, dy. \end{aligned}$$

Let us now consider $|x| \le \frac{2}{5}$. Then when $|y| \le \frac{1}{5}$ we have $|x-y| \le 2$ and $|x-y| \approx |x|$, so

$$\begin{aligned} T_\sigma(f)(x) &\ge c \int_{|y| \le 1/5} |y|^{-n/p} \left(\log \frac{1}{|y|}\right)^{-\delta} |x-y|^{-n+s} \, dy \\ &\ge c \int_{\substack{|y| \le 1/5 \\ |x| \ge 2|y|}} |y|^{-n/p} \left(\log \frac{1}{|y|}\right)^{-\delta} |x-y|^{-n+s} \, dy \\ &\approx c|x|^{-n+s} \int_{|y| \le |x|/2} |y|^{-n/p} \left(\log \frac{1}{|y|}\right)^{-\delta} dy. \end{aligned}$$

Since $|x|/2 \le 1/5$, the last integral is equivalent to

$$|x|^{-n/p+n} \left(\log \frac{1}{|x|}\right)^{-\delta},$$

so we obtain

$$T_\sigma(f)(x) \ge c|x|^{-n/p+s} \left(\log \frac{1}{|x|}\right)^{-\delta}, \quad \text{when } |x| \le \frac{2}{5}.$$

Now, recall that we are assuming T_σ maps $L^p(|x|^\beta)$ to itself. This implies that

$$\int_{|x| \le 2/5} \left[|x|^{-n/p+s} \left(\log \tfrac{1}{|x|}\right)^{-\delta}\right]^p |x|^\beta dx < \infty,$$

and this implies that $\beta \ge -ps$.

We now turn to the other assertion of the proposition. Consider the function

$$g(x) = |x|^{-(n+\beta)/p} \left(\log \frac{1}{|x|}\right)^{-\delta} \chi_{\{|x|\le 1/5\}}(x)$$

for some $1/p < \delta < 1$. Since $\delta p > 1$, we have $g \in L^p(|x|^\beta)$.

Arguing as in the preceding example,

$$\begin{aligned} T_\sigma(g)(x) \ge& c \int g(y) G_s(x - e_1 - y)\, dy \\ \ge& c \int_{\substack{|y|\le 1/5 \\ |x-e_1-y|\le 2}} |y|^{-(n+\beta)/p} \left(\log \frac{1}{|y|}\right)^{-\delta} |x - e_1 - y|^{-n+s} dy. \end{aligned}$$

Let us consider x in the region $|x - e_1| \le \frac{2}{5}$. Then if $|x - e_1| \ge 2|y|$, it follows that $|x - e_1 - y| \approx |x - e_1|$. Moreover, if $|y| \le \frac{1}{5}$ then $|x - e_1 - y| \le 2$. Then restricting the integral to $|x - e_1| \ge 2|y|$, we obtain

$$T_\sigma(g)(x) \ge c|x - e_1|^{-n+s} \int_{|y|\le |x-e_1|/2} |y|^{-(n+\beta)/p} \left(\log \frac{1}{|y|}\right)^{-\delta} dy.$$

Hence,

$$T_\sigma(g)(x) \ge c|x - e_1|^{s-(n+\beta)/p} \left(\log \frac{1}{|x - e_1|}\right)^{-\delta}, \quad \text{when } |x - e_1| \le \frac{2}{5}.$$

Now, notice that the assumption

$$\int_{|x-e_1|\le 2/5} T_\sigma(g)(x)^p |x|^\beta dx < \infty$$

implies that $ps - \beta \ge 0$. □

Using the information from Proposition 2.2 we obtain the following result.

Theorem 2.3. *Let $0 < s < n$ and $1 < p < \frac{n}{s}$. Suppose that $q > \frac{ps}{n} + 1$ [or $1 \le r < \frac{n}{ps}$]. Then there is a weight w in A_q [or w in RH_r] and a bounded function σ on $\mathbb{R}^n$ satisfying*

$$\sup_{j\in\mathbb{Z}} \|(I - \Delta)^{s/2}[\hat{\Psi}\sigma(2^j\cdot)]\|_{L^{\frac{n}{s},1}} < \infty \tag{17}$$

such that T_σ is unbounded from $L^p(w)$ to itself.

Proof. If $q > \frac{ps}{n} + 1$, then we choose β satisfying $ps < \beta < n(q-1)$ and we note that $|x|^\beta$ lies in A_q but by Proposition 2.2, T_σ is unbounded from $L^p(w)$ to itself. Likewise, if $1 \le r < \frac{n}{ps}$, then we choose β such that $\frac{n}{r} > \beta > sp$. Then the weight $|x|^{-\beta}$ lies in RH_r but $-\beta < -sp$ and by by Proposition 2.2, T_σ is unbounded from $L^p(w)$ to itself. □

In view of Theorem 2.3 one may surmise that the largest class of weights w for which boundedness of T_σ on $L^p(w)$ holds is $A_{\frac{ps}{n}+1} \cap RH_{\frac{n}{ps}}$. This is unknown as of this writing.

We also remark that condition (17) in Theorem 2.3 can be replaced by the more restrictive condition

$$\sup_{j\in\mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}}(\widehat{\Psi}\sigma(2^j\cdot))\|_{L^r} < \infty,$$

for some $r > n/s$. This follows by a small adjustment in the proof of Lemma 2.1.

The boundedness of T_σ on $L^p(w)$ whenever w lies is $A_{\frac{ps}{n}+1} \cap RH_{\frac{n}{ps}}$ remains an open question as of this writing. In this paper we make some advances in this direction, improving known results, in particular, those of Beltran–Roos–Seeger [2] and Bernicot–Frey–Petermichl [3].

3. Preliminaries

In this section we present some useful facts about the A_p and RH classes.

Lemma 3.1 ([8, Corollary 7.2.6])**.** *Let $1 < p < \infty$, then we have*

$$A_p = \bigcup_{q \in (1,p)} A_q.$$

Lemma 3.2 ([15, Lemma 3.1])**.** *$w \in RH_s$ iff $w^s \in A_\infty := \cup_{p>1} A_p$ with*

$$\frac{[w^s]_{A_\infty}^{\frac{1}{s}}}{[w]_{A_\infty}} \le [w]_{RH_s} \le [w^s]_{A_\infty}^{\frac{1}{s}}.$$

Lemma 3.3 ([16, Section I])**.** *Define*

$$\mathcal{A}_p^q := \{\, w : [w^q]_{A_p} < \infty \,\} \text{ for } 1 < p < \infty \text{ and } 1 < q < \infty.$$

Let $0 < \sigma < \infty$, $1 < p < \infty$ and $v \in \mathcal{A}_p^\sigma$, then $v \in A_{1+\frac{p-1}{\sigma}}$ iff $v \in A_\infty$. In fact, we have

$$[v]_{A_\infty} \le [v]_{A_{1+\frac{p-1}{\sigma}}} \le [v^\sigma]_{A_p}^{\frac{1}{\sigma}} \text{ if } \sigma \ge 1,$$

and

$$[v]_{A_\infty} \le [v]_{A_{1+\frac{p-1}{\sigma}}} \le [v^\sigma]_{A_p}^{\frac{1}{\sigma}} [v^\sigma]_{RH_{\frac{1}{\sigma}}}^{\frac{1}{\sigma}} \text{ if } 0 < \sigma < 1.$$

Lemma 3.4 ([5, Theorem2.2] [16, Section I])**.** *Let $1 \le s < \infty$, then we have $\mathcal{A}_p^s = A_{1+\frac{p-1}{s}} \cap RH_s$ with*

$$([w]_{RH_s}[w]_{A_{1+\frac{p-1}{s}}})^{s/2} \le [w^s]_{A_p} \le ([w]_{RH_s}[w]_{A_{1+\frac{p-1}{s}}})^s.$$

Proof. This follows by combining Lemmas 3.2 and 3.3. □

Lemma 3.5 ([9, Theorem 8.6.9])**.** *For any $w \in A_p, 1 < p < \infty$, and $K > 1$, and any r satisfying*

$$1 \le r \le 1 + \frac{K-1}{K} \frac{1}{2^{3/2} e^{1/e} 6^n [w]_{A_p}}$$

we have $[w]_{RH_r} \le K$, i.e., for any cube Q in $\mathbf{R}^n$,

$$\left(\frac{1}{|Q|} \int_Q w^r dx \right)^{\frac{1}{r}} \le \frac{K}{|Q|} \int_Q w dx$$

Lemma 3.6. *If $1 < q < \infty$, then we have*

$$\bigcup_{\bar{q} > q} RH_{\bar{q}} = RH_q.$$

Proof. Let $w \in RH_q$ be arbitrary. By Lemma 3.2, we know that

$$w^q \in A_l \quad \text{for some } l > 1.$$

It follows from Lemma 3.5 that there exists $r > 1$ such that

$$(w^q)^r = w^{qr} \in A_l.$$

Applying Lemma 3.2 again, we conclude that

$$w \in RH_{qr}.$$

Since $r > 1$, we have $qr > q$, and hence for every $w \in RH_q$ there exists $\bar{q} > q$ such that $w \in RH_{\bar{q}}$. On the other hand, it is well known that the Reverse Hölder classes are monotone, namely

$$RH_{\bar{q}} \subset RH_q \quad \text{for all } \bar{q} > q.$$

Therefore,

$$\bigcup_{\bar{q} > q} RH_{\bar{q}} = RH_q,$$

which completes the proof. □

Lemma 3.7. *Let $1 < p, p_i < \infty$, $0 < s^*, s_i^* < n$ and $1 < q, q_i < \infty$ for $i \in \{0, 1\}$. Suppose $w_0 \in A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0}$ and $w_1 \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1}$ with $\frac{p_i s_i^*}{n} > 1$, and these parameters satisfy*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s^* = (1-\theta)s_0^* + \theta s_1^*, \quad q\left(\frac{s^*}{n} - \frac{1}{p}\right) = (1-\theta)q_0\left(\frac{s_0^*}{n} - \frac{1}{p_0}\right) + \theta q_1\left(\frac{s_1^*}{n} - \frac{1}{p_1}\right)$$

where $\theta \in (0, 1)$. Then

$$w := w_0^{\frac{q_0}{q}\frac{1-\theta}{p_0}p} w_1^{\frac{q_1}{q}\frac{\theta}{p_1}p} \in A_{\frac{ps^*}{n}} \cap RH_q.$$

Proof. We finally use Hölder's inequality and Lemma 3.4 to conclude the proof. □

Lemma 3.8 (The converse to Lemma 3.7)**.** *Let $1 < p, q < \infty$ and $0 < s^* < n$. For any fixed parameters*

$$q_0, q_1, p_0, p_1 \in (1, \infty), \qquad s_0^*, s_1^* \in (0, n), \qquad \theta \in (0, 1),$$

with $\frac{p_i s_i^*}{n} > 1$ for $i = 0, 1$, satisfying

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad s^* = (1-\theta)s_0^* + \theta s_1^*,$$

$$q\left(\frac{s^*}{n} - \frac{1}{p}\right) = (1-\theta)\, q_0\left(\frac{s_0^*}{n} - \frac{1}{p_0}\right) + \theta\, q_1\left(\frac{s_1^*}{n} - \frac{1}{p_1}\right),$$

and any $w \in A_{\frac{ps^*}{n}} \cap RH_q$, there exist weights

$$w_0 \in A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0} \quad \text{and} \quad w_1 \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1}$$

such that

$$w = w_0^{\alpha_0} w_1^{\alpha_1}, \qquad \alpha_0 := \frac{q_0}{q}\frac{1-\theta}{p_0}p, \qquad \alpha_1 := \frac{q_1}{q}\frac{\theta}{p_1}p.$$

Proof. For a given $w \in A_{\frac{ps^*}{n}} \cap RH_q = \mathcal{A}^q_{q(\frac{ps^*}{n}-1)+1}$ by Lemma 3.4, then there exist appropriate choices of $v_1, v_2 \in A_1$ [17] such that

$$\begin{aligned} w^q &= v_1\, v_2^{-q(\frac{ps^*}{n}-1)} \\ &= \big(v_1\, v_2^{-q_0(\frac{p_0 s_0^*}{n}-1)}\big)^{\frac{1-\theta}{p_0}p} \big(v_1\, v_2^{-q_1(\frac{p_1 s_1^*}{n}-1)}\big)^{\frac{\theta}{p_1}p} \\ &= \big(w_0^{q_0}\big)^{\frac{1-\theta}{p_0}p} \big(w_1^{q_1}\big)^{\frac{\theta}{p_1}p}, \end{aligned}$$

where we defined

$$w_0 := \big(v_1\, v_2^{-q_0(\frac{p_0 s_0^*}{n}-1)}\big)^{\frac{1}{q_0}}, \qquad w_1 := \big(v_1\, v_2^{-q_1(\frac{p_1 s_1^*}{n}-1)}\big)^{\frac{1}{q_1}}.$$

Consequently,

$$w_0 \in \mathcal{A}^{q_0}_{q_0(\frac{p_0 s_0^*}{n}-1)+1} = A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0}, \qquad w_1 \in \mathcal{A}^{q_1}_{q_1(\frac{p_1 s_1^*}{n}-1)+1} = A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1}.$$

This concludes the proof. □

Lemma 3.9. *Let $n > s > 0$, $\frac{n}{s} < r_s < r_l < \infty$, and suppose that the multiplier σ satisfies*

$$\sup_{j\in\mathbb{Z}} \Big\| (I-\Delta)^{s/2} \Big[\hat{\Psi}\, \sigma(2^j\cdot)\Big] \Big\|_{L^{r_l}} < \infty.$$

Then we have

$$\sup_{j\in\mathbb{Z}} \Big\| (I-\Delta)^{s/2} \Big[\hat{\Psi}\, \sigma(2^j\cdot)\Big] \Big\|_{L^{r_s}} < \infty.$$

Proof. Let $s > 0$ and $1 < p_i, q_i < \infty$ with $i \in \{1, 2\}$. we have

$$\|(I-\Delta)^{s/2}(fg)\|_{L^{p,1}} \lesssim \|(I-\Delta)^{s/2} f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} + \|f\|_{L^{p_1,q_1}} \|(I-\Delta)^{s/2} g\|_{L^{p_2,q_2}}.$$

for

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad 1 = \frac{1}{q_1} + \frac{1}{q_2}$$

by [12, Page 21]. Let $p = r_s$, $p_1 = r_l$, $q_1 = r_l$, $q_2 = r_l'$, $f = \hat{\Psi}\sigma(2^j\cdot)$, $g = \hat{\theta}$ which is compactly supported and $\hat{\theta} \equiv 1$ on supp $(\hat{\Psi})$. Note that $\|\cdot\|_{L^r} = \|\cdot\|_{L^{r,r}}$ for $1 < r < \infty$. Choosing $p_2 > 1$ satisfies $\frac{1}{r_s} = \frac{1}{r_l} + \frac{1}{p_2}$ so that

$$\begin{aligned} &\|(I-\Delta)^{s/2}(\hat{\theta}\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_s}} \\ &\qquad \le \|(I-\Delta)^{s/2}(\hat{\theta}\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_s,1}} \\ &\qquad \lesssim \|(I-\Delta)^{s/2}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_l}} \|\hat{\theta}\|_{L^{p_2,r_l'}} + \|\hat{\Psi}\sigma(2^j\cdot)\|_{L^{r_l}} \|(I-\Delta)^{s/2}\hat{\theta}\|_{L^{p_2,r_l'}} \\ &\qquad \lesssim \|(I-\Delta)^{s/2}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_l}} < \infty. \end{aligned}$$

This provides the required conclusion. □

Theorem 3.10 ([1, Weighted Littlewood–Paley theory])**.** *Let $1 < p < \infty$, and let w be a weight. If $w \in A_p$ and $f \in L^p(w)$, then*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \sim \|f\|_{L^p(w)}.$$

Lemma 3.11 ([10, Theorem 8.6.9]). *Assume that $0 < s < n$ and $q > n/s$. Then we have*

$$\sup_{j\in\mathbb{Z}} \left\| \frac{f(x+2^{-j}\cdot)}{(1+|\cdot|)^s} \right\|_{L^{\frac{n}{s},\infty}} \leq CM_q f(x), \qquad x \in \mathbb{R}^n,$$

where $M_q f = (M[|f|^q])^{1/q}$.

Lemma 3.12 ([10, Lemma 3.3]). *Let $1 < p, p_0, p_1 < \infty$ be related as in*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

for some $\theta \in (0,1)$. Given $f \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist smooth functions h_j^ε, $j = 1, \ldots, N_\varepsilon$, supported in cubes on $\mathbb{R}^n$ with pairwise disjoint interiors, and nonzero complex constants c_j^ε such that

$$f_z^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z} h_j^\varepsilon, \tag{18}$$

$$\|f_\theta^\varepsilon - f\|_{L^2} < \varepsilon, \quad \|f_{k+it}^\varepsilon\|_{L^{p_k}} \leq (\|f\|_{L^p} + \varepsilon)^{\frac{p}{p_k}}, \quad k = 0, 1.$$

Having completed this introductory material, we now state some recent results concerning estimates of multipliers on weighted spaces. Our goal is to improve some of these results via interpolation. This will be achieved in Section 6.

Proposition 3.13 ([2, Proposition 7.5]). *Let $1 \leq p_0 \leq 2$, $n > s > 0$, $1/r = 1/p_0 - 1/2$, and σ satisfies*

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{s/2}[\hat{\Psi}\sigma(2^j\cdot)]\|_{L^r} < A$$

for some $A < \infty$. Suppose one of these followings holds:
(i) $1 \leq p_0 \leq q_0 \leq 2$, and $s > n(1/p_0 - 1/2)$ i.e. $s/n > 1/r$.
(ii) $2 \leq q_0 < \infty$ and $s > n(1/p_0 - 1/q_0)$ i.e. $s/n > 1/r + 1/2 - 1/q_0$.
Then we must have

$$\|T_\sigma\|_{\mathrm{Sp}_\gamma(p_0,q_0')} \lesssim_{p_0,r,q_0,\alpha,\gamma} A.$$

Remark : (i) and (ii) both imply that $s/n > 1/r$.

Proposition 3.14 ([3, Proposition 6.4]). *Let $1 < p_0 < p < q_0 < \infty$. Suppose a linear operator T on L^p is bounded and*

$$\|T\|_{\mathrm{Sp}_\gamma(p_0,q_0')} < \infty,$$

then

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim ([w]_{A_{\frac{p}{p_0}}}[w]_{RH_{(\frac{q_0}{p})'}})^\alpha$$

where $\alpha = \max(\frac{1}{p-p_0}, \frac{q_0-1}{q_0-p})$.

Combing Proposition 3.13 and Theorem 3.14 yields the following theorem.

Theorem 3.15 ([2] [3]). *Let $n/2 > s > 0$, $s/n > 1/r$, and suppose that the multiplier σ satisfies*

$$\sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{s/2} \left[\hat{\Psi}\,\sigma(2^j\cdot)\right] \right\|_{L^r} < \infty.$$

Then the operator T_σ is bounded on $L^p(w)$ in the following three cases:
Case 1: $p=2$. *If $q_0^*>2$ satisfies*

$$\frac{s}{n}-\frac{1}{r}>\frac{1}{2}-\frac{1}{q_0^*},$$

then T_σ is bounded on $L^2(w)$ for any weight

$$w\in A_{\frac{2}{r}+1}\cap RH_{(\frac{q_0^*}{2})'}.$$

Case 2: $p>2$ *and* $\frac{s}{n}-\frac{1}{r}>\frac{1}{2}-\frac{1}{p}$. *If $q_0^*>p$ satisfies*

$$\frac{s}{n}-\frac{1}{r}>\frac{1}{2}-\frac{1}{q_0^*},$$

then T_σ is bounded on $L^p(w)$ for any weight

$$w\in A_{\frac{p}{r}+\frac{p}{2}}\cap RH_{(\frac{q_0^*}{p})'}.$$

Case 3: $p<2$. *The operator T_σ is bounded on $L^p(w)$ for any weight*

$$w\in A_{\frac{p}{r}+\frac{p}{2}}\cap RH_{(\frac{q_0^*}{p})'},$$

provided that one of the following holds:

(i) $1\le \frac{1}{\frac{1}{r}+\frac{1}{2}}<p<q_0^*\le 2$;

(ii) $2\le q_0^*$ *and*

$$\frac{s}{n}-\frac{1}{r}>\frac{1}{2}-\frac{1}{q_0^*}.$$

Remark: $\frac{s}{n}-\frac{1}{r}>\frac{1}{2}-\frac{1}{q_0^*}$ and $q_0^*>p$ imply $\frac{s}{n}>\frac{s}{n}-\frac{1}{r}>\frac{1}{2}-\frac{1}{q_0^*}>\frac{1}{2}-\frac{1}{p}$.

4. Weighted boundedness for higher order of smoothness

In this section we consider the weighted boundedness of multipliers for large s, in fact $s>n/2$.

Theorem 4.1. *Let $1<p<\infty$ and $0<s<n$. If*

$$s>\frac{n}{2},\quad w\in A_{\frac{ps}{n}},\quad \sup_{j\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^{\frac{n}{s},1}}<\infty,$$

then, for any bounded compactly supported function f on $\mathbb{R}^n$,

$$\|T_\sigma f\|_{L^p(w)}\lesssim_{p,s,w}\sup_{j\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^{\frac{n}{s},1}}\|f\|_{L^p(w)}.$$

Proof. Let

$$K=\sup_{j\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^{\frac{n}{s},1}}.$$

Introduce the function Θ defined by

$$\hat{\Theta}=\hat{\Psi}\left(\frac{\xi}{2}\right)+\hat{\Psi}(\xi)+\hat{\Psi}(2\xi),$$

and observe that

$$\hat{\Theta}(\xi)=1\quad\text{on }\operatorname{supp}\hat{\Psi}.$$

Let Δ_j and Δ_j^Θ denote the Littlewood–Paley operators associated with Ψ and Θ, respectively.

$$\Delta_j f(x) \equiv \int_{\mathbb{R}^n} \hat{\Psi}(2^{-j}\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi, \quad \Delta_j^\Theta f(x) \equiv \int_{\mathbb{R}^n} \hat{\Theta}(2^{-j}\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi.$$

Then,

$$\begin{aligned}
\Delta_j T_\sigma(f)(x) &= \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{\Psi}\left(2^{-j}\xi\right)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi \\
&= \int_{\mathbb{R}^n} \left(\Delta_j^\Theta f\right)^\wedge(\xi)\widehat{\Psi}\left(2^{-j}\xi\right)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi \\
&= 2^{jn}\int_{\mathbb{R}^n} \left(\Delta_j^\Theta f\right)^\wedge\left(2^j\xi'\right)\widehat{\Psi}\left(\xi'\right)\sigma\left(2^j\xi'\right)e^{2\pi i x\cdot 2^j\xi'}d\xi' \\
&= 2^{jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\Delta_j^\Theta f\right)(y)e^{-2\pi i y\cdot 2^j\xi'}dy\widehat{\Psi}\left(\xi'\right)\sigma\left(2^j\xi'\right)e^{2\pi i x\cdot 2^j\xi'}d\xi' \\
&= 2^{jn}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-2\pi i y\cdot 2^j\xi'}\widehat{\Psi}\left(\xi'\right)\sigma\left(2^j\xi'\right)e^{2\pi i x\cdot 2^j\xi'}d\xi'\right)\left(\Delta_j^\Theta f\right)(y)dy \\
&= \int_{\mathbb{R}^n} \left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]^\wedge(y)dy \\
&= \int_{\mathbb{R}^n} \frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}(1+|y|)^s\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]^\wedge(y)dy.
\end{aligned}$$

By the Hölder and the Hausdorff–Young inequalities in Lorentz spaces,

$$\begin{aligned}
|\Delta_j T_\sigma(f)(x)| &\le \left\|\frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n)}\left\|(1+|y|)^s\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]^\wedge(y)\right\|_{L^{(\frac{n}{s})',1}(\mathbb{R}^n)} \\
&\le C'\left\|\frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n)}\left\|(I-\Delta)^{\frac{s}{2}}\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \\
&\le C\left\|\frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n)}
\end{aligned}$$

Note that by Lemma 3.1, there exists $q \in (n/s, \min(2,p))$ which is close to $\frac{n}{s}$ such that $w \in A_{p/q}$. Then for this q, by applying Lemma 3.11 we get,

$$\left\|\frac{\Delta_j^\Theta f(x+2^{-j}\cdot)}{(1+|\cdot|)^s}\right\|_{L^{\frac{n}{s},\infty}} \lesssim M_q[\Delta_j^\Theta f](x).$$

By Theorem 3.10,

$$\begin{aligned}
\|T_\sigma f\|_{L^p(w)} &\le C\left\|\left(\sum_{j\in\mathbb{Z}}|\Delta_j T_\sigma f|^2\right)^{\frac{1}{2}}\right\|_{L^p(w)} \le CK\left\|\left(\sum_{j\in\mathbb{Z}}(M_q[\Delta_j^\Theta f])^2\right)^{\frac{1}{2}}\right\|_{L^p(w)} \\
&= CK\left\|\left(\sum_{j\in\mathbb{Z}}(M|\Delta_j^\Theta f|^q)^{\frac{2}{q}}\right)^{\frac{q}{2}}\right\|_{L^{\frac{p}{q}}(w)}^{\frac{1}{q}}
\end{aligned}$$

$$\leq C(p,s,n)K \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\Theta} f|^{q\cdot\frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(w)}^{\frac{1}{q}}$$

$$= C(p,s,n)K \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\Theta} f|^{2} \right)^{\frac{1}{2}} \right\|_{L^{p}(w)} \leq C(p,s,n)K\|f\|_{L^p(w)},$$

where in the third inequality, since $q \in (n/s, \min(2,p))$ which is close to $\frac{n}{s}$ such that $w \in A_{p/q}$, which implies $\frac{p}{q} > 1$ and $\frac{2}{q} > 1$, we can apply the weighted vector-valued maximal inequality. [1] □

5. Interpolation theorem for Fourier multipliers

The main result of this article is the following interpolation theorem for families of Fourier multipliers on weighted spaces which depend analytically on the smoothness parameter s. This is based on an extension of the 3-lines lemma due to Hirschmann [13].

For the purposes of the following theorem, we define

$$\mathcal{B} = \{f \in L^\infty : \mathbb{R}^n \to \mathbb{C} |\ \operatorname{supp} f \subset \overline{B(0,R)} \setminus B(0,\epsilon), \quad \text{for some } 0 < \epsilon < R < \infty\}$$

the space of all bounded functions whose support is compact and misses the origin. Note that for any fixed $p \in (1,\infty)$, $\mathcal{B}$ is a dense subset of L^p. Moreover, we have

$$\int_{\mathbb{R}^n} |f|^p w\, dx \leq \|f\|_{L^\infty}^p \int_{suppf} w\, dx < \infty,$$

and $\mathcal{B}$ is dense in $L^p(w)$ for any weight w by [9, Proposition 8.2.2], so $\mathcal{B}$ is a dense subset of $L^p(w)$ for any weight w.

Theorem 5.1. *Let* $0 < \theta < 1, 1 < p, p_i < \infty$, $1 < q, q_i < \infty$, $0 < s, s_i, s^*, s_i^* < n$, $\frac{s_i^*}{n} \geq \frac{1}{p_i}$ *and* $r_i > \frac{n}{s_i}$. *Suppose that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1, \quad s^* = (1-\theta)s_0^* + \theta s_1^*,$$

$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad q(\tfrac{s^*}{n} - \tfrac{1}{p}) = (1-\theta)q_0(\tfrac{s_0^*}{n} - \tfrac{1}{p_0}) + \theta q_1(\tfrac{s_1^*}{n} - \tfrac{1}{p_1}), \qquad i \in \{0,1\}.$$

Assume that for each $i \in \{0,1\}$ *we have*

$$\|T_\sigma f\|_{L^{p_i}(w_i)} \lesssim \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s_i}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_i}} \|f\|_{L^{p_i}(w_i)} \tag{19}$$

for all σ *satisfying* $\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s_i}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_i}} < \infty$, *all* $w_i \in A_{\frac{p_i s_i^*}{n}} \cap RH_{q_i}$ *and all* $f \in \mathcal{B}$. *Then we have*

$$\|T_\sigma f\|_{L^{p}(w)} \lesssim \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r}} \|f\|_{L^{p}(w)} \tag{20}$$

for all σ *satisfying* $\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r}} < \infty$, *all* $w \in A_{\frac{ps^*}{n}} \cap RH_q$ *and all* $f \in \mathcal{B}$.

Proof. For any function σ satisfying

$$\sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)\right\|_{L^r} < \infty \quad \text{with} \quad r > \frac{n}{s}, \tag{21}$$

we denote

$$\varphi_j = (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)], \quad j\in\mathbb{Z}.$$

For a complex number z with $0\le \Re(z)\le 1$, we define

$$\sigma_z(\xi) = \sum_{k\in\mathbb{Z}} (I-\Delta)^{-\frac{(1-z)s_0+zs_1}{2}} \left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k} \right] (2^{-k}\xi)\hat{\Phi}(2^{-k}\xi), \tag{22}$$

where $\hat{\Phi}$ is a Schwartz function supported in the set $\{\xi\in\mathbb{R}^n : 1/4\le|\xi|\le 4\}$ and $\hat{\Phi}\equiv 1$ in the support of $\hat{\Psi}$.

Fix $f,g\in\mathcal{B}$, Given $\varepsilon>0$, by the density we can choose $f_*, g_*\in C_0^\infty(\mathbb{R}^n)$ such that

$$\|f_*-f\|_{L^p}<\varepsilon, \quad \|g_*-g\|_{L^{p'}}<\varepsilon, \quad \text{and} \, \|f_*-f\|_{L^2}<\varepsilon, \quad \|g_*-g\|_{L^2}<\varepsilon \tag{23}$$

for any fixed $p>1$. Let f_z^ε and g_z^ε be functions having the form (18), satisfying

$$\|f_\theta^\varepsilon - f_*\|_{L^2}<\varepsilon, \quad \|g_\theta^\varepsilon - g_*\|_{L^2}<\varepsilon, \tag{24}$$

and

$$\begin{aligned} \|f_{it}^\varepsilon\|_{L^{p_0}} &\le (\|f_*\|_{L^p}+\varepsilon)^{\frac{p}{p_0}}, \quad \|f_{1+it}^\varepsilon\|_{L^{p_1}} \le (\|f_*\|_{L^p}+\varepsilon)^{\frac{p}{p_1}}, \\ \|g_{it}^\varepsilon\|_{L^{p_0}} &\le (\|g_*\|_{L^{p'}}+\varepsilon)^{\frac{p'}{p_0'}}, \quad \|g_{1+it}^\varepsilon\|_{L^{p_1'}} \le (\|g_*\|_{L^{p'}}+\varepsilon)^{\frac{p'}{p_1'}}. \end{aligned} \tag{25}$$

Combining (23), (24) and (25), we have

$$\|f_\theta^\varepsilon - f\|_{L^2}<2\varepsilon, \quad \|g_\theta^\varepsilon - g\|_{L^2}<2\varepsilon, \tag{26}$$

and

$$\begin{aligned} \|f_{it}^\varepsilon\|_{L^{p_0}} &\le (\|f\|_{L^p}+2\varepsilon)^{\frac{p}{p_0}}, \quad \|f_{1+it}^\varepsilon\|_{L^{p_1}} \le (\|f\|_{L^p}+2\varepsilon)^{\frac{p}{p_1}}, \\ \|g_{it}^\varepsilon\|_{L^{p_0}} &\le (\|g\|_{L^{p'}}+2\varepsilon)^{\frac{p'}{p_0'}}, \quad \|g_{1+it}^\varepsilon\|_{L^{p_1'}} \le (\|g\|_{L^{p'}}+2\varepsilon)^{\frac{p'}{p_1'}}. \end{aligned} \tag{27}$$

Set

$$w_{\delta,M} := \delta\chi_{\{w\le\delta\}} + w\chi_{\{\delta<w<M\}} + M\chi_{\{w\ge M\}}$$

for $0<\delta<M<\infty$. Then notice that

$$\lim_{\delta\to0, M\to\infty} w_{\delta,M} = w \quad \text{pointwise}.$$

Define

$$F(z) = \int_{\mathbb{R}^n} [\widetilde{T}_z f_z^\varepsilon(x)] g_z^\varepsilon(x)\mathrm{d}x,$$

where

$$\widetilde{T}_z f := T_{\sigma_z}\left[f\cdot w_{0,\delta,M}^{-\frac{q_0}{q}\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\frac{z}{p_1}} \right] \cdot w_{0,\delta,M}^{\frac{q_0}{q}\frac{1-z}{p_0}} w_{1,\delta,M}^{\frac{q_1}{q}\frac{z}{p_1}}$$

and z is a complex number with $0\le\Re(z)\le1$. It is straightforward to verify that F is analytic on the strip $S=\{z\in\mathbb{C} : 0<\Re(z)<1\}$ and continuous on its closure. Let

$z \in \tau + it$ satisfy $0 \le \tau < \theta$ and $t \in \mathbb{R}$. set $s_\tau = (1-\tau)s_0 + \tau s_1$ and $\frac{1}{r_\tau} = \frac{1-\tau}{r_0} + \frac{\tau}{r_1}$ where $\tau \in (0,1)$ By applying [9, Theorem 5.4.9(c) and Corollary 5.3.3] we obtain

$$\begin{aligned}
\|\sigma_z\|_{L^\infty} &\le \sup_{k\in\mathbb{Z}} \left\| (I-\Delta)^{-\frac{(1-z)s_0+zs_1}{2}} \left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k} \right] \right\|_{L^\infty} \\
&\le C(n,r_0,r_1,s_0,s_1) \sup_{k\in\mathbb{Z}} \left\| (I-\Delta)^{-\frac{(1-z)s_0+zs_1}{2}} \left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k} \right] \right\|_{L^{r_\tau}_{s_\tau}} \\
&\le C(n,r_0,r_1,s_0,s_1) \sup_{k\in\mathbb{Z}} \left\| (I-\Delta)^{it\frac{s_0-s_1}{2}} \left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k} \right] \right\|_{L^{r_\tau}} \\
&\le C'(n,r_0,r_1,s_0,s_1)(1+|t|)^{[\frac{n}{2}]+1} \sup_{k\in\mathbb{Z}} \left\| |\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k} \right\|_{L^{r_\tau}} \\
&= C'(n,r_0,r_1,s_0,s_1)(1+|t|)^{[\frac{n}{2}]+1} \sup_{k\in\mathbb{Z}} \|\varphi_k\|_{L^r}^{\frac{r}{r_\tau}} .
\end{aligned}$$

Here the constant $C(n,r_0,r_1,s_0,s_1)$ is obtained from [9, Theorem 5.4.9(c) and Proposition 5.2.3]. Additionally, we have the estimate

$$\|f_z^\varepsilon\|_{L^2}^2 = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{(1-\Re(z))\frac{2p}{p_0}+\Re(z)\frac{2p}{p_1}} \|h_j^\varepsilon\|_{L^2} \le C(p_0,p_1,\varepsilon,f) < \infty.$$

Similarly,

$$\|g_z^\varepsilon\|_{L^2}^2 \le C(p_0,p_1,\varepsilon,g) < \infty.$$

Then

$$\begin{aligned}
|F(z)| &= \left| \int_{\mathbb{R}^n} \sigma_z \left[f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{z}{p_1}} \right]^\wedge (\xi) \left[g_z^\varepsilon \cdot w_{0,\delta,M}^{\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{\frac{q_1}{q}\cdot\frac{z}{p_1}} \right]^\vee (\xi)\, d\xi \right| \\
&\le \|\sigma_z\|_{L^\infty} \int_{\mathbb{R}^n} \left| \left[f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{z}{p_1}} \right]^\wedge (\xi) \left[g_z^\varepsilon \cdot w_{0,\delta,M}^{\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{\frac{q_1}{q}\cdot\frac{z}{p_1}} \right]^\vee (\xi) \right| d\xi \\
&\le \|\sigma_z\|_{L^\infty} \left\| f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-\Re(z)}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{\Re(z)}{p_1}} \right\|_{L^2} \left\| g_z^\varepsilon \cdot w_{0,\delta,M}^{\frac{q_0}{q}\cdot\frac{1-\Re(z)}{p_0}} w_{1,\delta,M}^{\frac{q_1}{q}\cdot\frac{\Re(z)}{p_1}} \right\|_{L^2} \\
&\le C(n,r_0,r_1,s_0,s_1,\varepsilon,f,g,\delta,M)(1+|t|)^{[\frac{n}{2}]+1} \sup_{j\in\mathbb{Z}} \|\varphi_j\|_{L^r}^{\frac{r}{r_\tau}} .
\end{aligned} \tag{28}$$

Here the Fourier and inverse Fourier transform above are well defined. In fact,

$$f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{z}{p_1}}$$

is a bounded and compactly supported function, so

$$f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{z}{p_1}}$$

lies in $L^2(\mathbb{R}^n)$. This implies

$$\left[f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{z}{p_1}} \right]^\wedge \in L^2(\mathbb{R}^n).$$

Moreover, $\sigma_z \in L^\infty$, then we have

$$\sigma_z \left[f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{z}{p_1}} \right]^\wedge \in L^2(\mathbb{R}^n).$$

Hence we can get the first equality of (28) by [9, Prop 2.3.7].

For any point z_0 in S, pick $\delta > 0$ such that

$$2\delta < \Re(\frac{q_1}{q} \cdot \frac{z_0}{p_1}) < 1.$$

By [9, lemma 2.7.5] for $0 < \frac{q_1}{q} \cdot \frac{z}{p_1} < \delta$, we obtain

$$\left| \frac{w_{1,\delta,M}^{\frac{q_1}{q} \cdot \frac{z}{p_1}} - 1}{z} \right| \leq \frac{q}{q_1} \cdot \frac{p_1}{\delta} \max\{(w_{1,\delta,M}^{2\delta}, w_{1,\delta,M}^{-2\delta}\},$$

which gives

$$\left| \frac{w_{1,\delta,M}^{\frac{q_1}{q} \cdot \frac{z+z_0}{p_1}} - w_{1,\delta,M}^{\frac{q_1}{q} \cdot \frac{z_0}{p_1}}}{z} \right| \leq \frac{q}{q_1} \cdot \frac{p_1}{\delta} \max\{w_{1,\delta,M}^{2}, 1\} \leq \frac{p_1}{\delta} \max\{M^2, 1\}.$$

In addition, from the argument above of (28), the Fourier transform and the inverse Fourier transform at the first line is in L^∞, so we can apply the Lebesgue dominated convergence theorem when the z derivative hits the term $w_{1,\delta,M}^{\frac{q_1}{q} \cdot \frac{z}{p_1}}$. Also we use the similar way to deal with other terms in $F(z)$ involving with z. This yields the analyticity of $F(z)$.

Then, we consider $F(1+it)$ and $F(it)$. For the function $\widehat{\Psi}\, \sigma_{1+it}(2^j \cdot)$, only finitely many indices k near j contribute nonzero terms in (22).

Therefore, for each $j \in \mathbb{Z}$,

$$\begin{aligned}
&\left\| (I-\Delta)^{\frac{s_1}{2}} [\hat{\Psi} \sigma_{1+it}(2^j \cdot)] \right\|_{L^{r_1}} \\
&\leq \left\| (I-\Delta)^{\frac{s_1}{2}} \left[\widehat{\Psi} (I-\Delta)^{-\frac{s_1+it(s_1-s_0)}{2}} \left[|\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)} e^{iArg\varphi_j} \right] \right] \right\|_{L^{r_1}} \\
&\leq C \left\| (I-\Delta)^{\frac{s_1}{2}} \left[(I-\Delta)^{-\frac{s_1+it(s_1-s_0)}{2}} \left[|\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)} e^{iArg\varphi_j} \right] \right] \right\|_{L^{r_1}} \\
&\leq C \left\| (I-\Delta)^{-\frac{it(s_1-s_0)}{2}} \left[|\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)} e^{iArg\varphi_j} \right] \right\|_{L^{r_1}} \\
&\leq C(s_0,s_1,n)(1+|t|)^{\frac{n}{2}+1} \left\| |\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)} \right\|_{L^{r_1}} \\
&= C(s_0,s_1,n)(1+|t|)^{\frac{n}{2}+1} \|\varphi_j\|_{L^r}^{\frac{r}{r_1}}
\end{aligned} \tag{29}$$

where we have used [9, Corollary 5.3.3] for the fourth inequality. By Hölder's inequality and (19) we obtain

$$\begin{aligned}
|F(1+it)| &\leq \|\widetilde{T}_{1+it} f_{1+it}^{\varepsilon}\|_{L^{p_1}} \|g_{1+it}^{\varepsilon}\|_{L^{p_1'}} \\
&= \|T_{\sigma_{1+it}}(f_{1+it}^{\varepsilon} w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{it}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{1+it}{p_1}})\|_{L^{p_1}(w_{1,\delta,M}^{\frac{q_1}{q}})} \|g_{1+it}^{\varepsilon}\|_{L^{p_1'}} \\
&\lesssim \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s_1}{2}} [\hat{\Psi}\sigma_{1+it}(2^j \cdot)] \right\|_{L^{r_1}} \|f_{1+it}^{\varepsilon}\|_{L^{p_1}} \|g_{1+it}^{\varepsilon}\|_{L^{p_1'}}.
\end{aligned} \tag{30}$$

Note that $f_{1+it}^{\varepsilon} \in C_0^\infty(\mathbb{R}^n)$ implies $f_{1+it}^{\varepsilon} w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{it}{p_0}} w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{1+it}{p_1}} \in \mathcal{B}$, and the truncation gives $w_{1,\delta,M}^{\frac{q_1}{q}} \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1}$. Combining (27), (29) and (30) we deduce

$$|F(1+it)| \lesssim C(s_0,s_1,n)(1+|t|)^{\frac{n}{2}+1} \sup_{j\in\mathbb{Z}} \|\varphi_j\|_{L^r}^{\frac{r}{r_1}} (\|f\|_{L^p}+\varepsilon)^{\frac{p}{p_1}} (\|g\|_{L^{p'}}+\varepsilon)^{\frac{p'}{p_1'}}. \tag{31}$$

Similarly, for $\hat{\Psi}\sigma_{it}(2^j\cdot)$, only finitely many indices k near j contribute in (22). Therefore, for each $j \in \mathbb{Z}$,

$$\begin{aligned}
&\left\|(I-\Delta)^{\frac{s_0}{2}}[\hat{\Psi}\sigma_{it}(2^j\cdot)]\right\|_{L^{r_0}} \\
&\leq \left\|(I-\Delta)^{\frac{s_0}{2}}\left[\widehat{\Psi}(I-\Delta)^{-\frac{s_0+it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it}e^{iArg\varphi_j}\right]\right]\right\|_{L^{r_0}} \\
&\leq C\left\|(I-\Delta)^{\frac{s_0}{2}}\left[(I-\Delta)^{-\frac{s_0+it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it}e^{iArg\varphi_j}\right]\right]\right\|_{L^{r_0}} \\
&= C\left\|(I-\Delta)^{-\frac{it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it}e^{iArg\varphi_j}\right]\right\|_{L^{r_0}} \\
&\leq C(s_0,s_1,n)(1+|t|)^{\frac{n}{2}+1}\left\||\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it}e^{iArg\varphi_j}\right\|_{L^{r_0}} \\
&= C(s_0,s_1,n)(1+|t|)^{\frac{n}{2}+1}\|\varphi_j\|_{L^r}^{\frac{r}{r_0}}.
\end{aligned} \tag{32}$$

Now we can apply the Hölder's inequality and (19) to obtain

$$\begin{aligned}
|F(it)| &\leq \|\widetilde{T}_{it}f_{it}^{\varepsilon}\|_{L^{p_0}}\|g_{it}^{\varepsilon}\|_{L^{p_0'}} \\
&\lesssim \sup_{j\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{s_0}{2}}[\hat{\Psi}\sigma_{it}(2^j\cdot)]\right\|_{L^{r_0}}\|f_{it}^{\varepsilon}\|_{L^{p_0}}\|g_{it}^{\varepsilon}\|_{L^{p_0'}}.
\end{aligned}$$

Here $f_{it}^{\varepsilon} \in C_0^{\infty}(\mathbb{R}^2)$ implies $f_{it}^{\varepsilon}w_{0,\delta,M}^{-\frac{q_0}{q}\cdot\frac{1-it}{p_0}}w_{1,\delta,M}^{-\frac{q_1}{q}\cdot\frac{it}{p_1}} \in \mathcal{B}$ and the truncation gives that $w_{0,\delta,M}^{\frac{q_0}{q}}$ lies in $A_{\frac{p_0s_0^*}{n}} \cap RH_{q_0}$. Therefore,

$$|F(it)| \lesssim C(s_0,s_1,n)(1+|t|)^{\frac{n}{2}+1}\sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}^{\frac{r}{r_0}}(\|f\|_{L^p}+\varepsilon)^{\frac{p}{p_0}}(\|g\|_{L^{p'}}+\varepsilon)^{\frac{p'}{p_0'}}. \tag{33}$$

A combination of (31), (33) and [9, Corollary C.0.3] yields

$$|F(\theta)| \lesssim \sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}(\|f\|_{L^p}+\varepsilon)(\|g\|_{L^{p'}}+\varepsilon) \quad \text{for all } \ f,g\in\mathcal{B}. \tag{34}$$

Consider the following inequality

$$\left|\int_{\mathbb{R}^n}(\widetilde{T}_\theta f)g\,\mathrm{d}x\right| \leq |F(\theta)| + \left|\int_{\mathbb{R}^n}(\widetilde{T}_\theta f)g\,\mathrm{d}x - \int_{\mathbb{R}^n}(\widetilde{T}_\theta f_\theta^{\varepsilon})g_\theta^{\varepsilon}\,\mathrm{d}x\right|. \tag{35}$$

where $f,g\in\mathcal{B}$ and $f_\theta^{\varepsilon}, g_\theta^{\varepsilon} \in C_0^{\infty}(\mathbb{R}^n)$. Set

$$w_{\theta,\delta,M} = w_{0,\delta,M}^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}}w_{1,\delta,M}^{\frac{q_1}{q}\cdot\frac{\theta}{p_1}} \in (\delta^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}+\frac{q_1}{q}\cdot\frac{\theta}{p_1}}, M^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}+\frac{q_1}{q}\cdot\frac{\theta}{p_1}})$$

and compute the second term above as follows

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}(\widetilde{T}_\theta f)g\,\mathrm{d}x - \int_{\mathbb{R}^n}(\widetilde{T}_\theta f_\theta^{\varepsilon})g_\theta^{\varepsilon}\,\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^n}(\sigma_\theta\widehat{fw_{\theta,\delta,M}^{-1}})^{\vee}\cdot gw_{\theta,\delta,M} - (\sigma_\theta\widehat{f_\theta^{\varepsilon}w_{\theta,\delta,M}^{-1}})^{\vee}\cdot g_\theta^{\varepsilon}w_{\theta,\delta,M}\,\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^n}\sigma_\theta\widehat{fw_{\theta,\delta,M}^{-1}}\cdot(gw_{\theta,\delta,M})^{\vee} - \sigma_\theta\widehat{f_\theta^{\varepsilon}w_{\theta,\delta,M}^{-1}}\cdot(g_\theta^{\varepsilon}w_{\theta,\delta,M})^{\vee}\,\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^n}\sigma_\theta\left[\widehat{f_\theta^{\varepsilon}w_{\theta,\delta,M}^{-1}}(g_\theta^{\varepsilon}w_{\theta,\delta,M}-gw_{\theta,\delta,M})^{\vee} + (gw_{\theta,\delta,M})^{\vee}(\widehat{f_\theta^{\varepsilon}w_{\theta,\delta,M}^{-1}}-\widehat{fw_{\theta,\delta,M}^{-1}})\right]\,\mathrm{d}x\right|
\end{aligned}$$

$$\leq \|\sigma_\theta\|_\infty(\|f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}\|_{L^2}\|g_\theta^\varepsilon w_{\theta,\delta,M} - gw_{\theta,\delta,M}\|_{L^2} + \|gw_{\theta,\delta,M}\|_{L^2}\|f_\theta^\varepsilon w_{\theta,\delta,M}^{-1} - fw_{\theta,\delta,M}^{-1}\|_{L^2})$$
$$\leq C(\delta, M, p_0, p_1, q_0, q_1, q)\|\sigma_\theta\|_\infty(\|f_\theta^\varepsilon\|_{L^2}\|g_\theta^\varepsilon - g\|_{L^2} + \|g\|_{L^2}\|f_\theta^\varepsilon - f\|_{L^2}).$$

In the calculation above, $\widehat{fw_{\theta,\delta,M}^{-1}}$, $\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}}$, $\sigma_\theta\widehat{fw_\theta^{-1}}$, $\sigma_\theta\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}}$ $gw_{\theta,\delta,M}$ and $g_\theta^\varepsilon w_{\theta,\delta,M}$ are all in L^2, so the corresponding Fourier transform and the inverse Fourier transform are well defined. Letting $\varepsilon \to 0$ the second term of (35) goes to 0, by using the duality argument and the fact that $\mathcal{B}$ is dense in $L^p(\mathbb{R}^n)$ we have

$$\|\widetilde{T}_\theta f\|_{L^p} \lesssim \sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}\|f\|_{L^p} \text{ for all } f \in \mathcal{B},$$

equivalently, replacing $f \mapsto f \cdot w_{0,\delta,M}^{\frac{q_0}{q}\frac{1-\theta}{p_0}} w_{1,\delta,M}^{\frac{q_1}{q}\frac{\theta}{p_1}} \in \mathcal{B}$, we have

$$\|T_\sigma f\|_{L^p(w_{0,\delta,M}^{\frac{q_0}{q}\frac{1-\theta}{p_0}p} w_{1,\delta,M}^{\frac{q_1}{q}\frac{\theta}{p_1}p})} = \|T_{\sigma_\theta} f\|_{L^p(w_{0,\delta,M}^{\frac{q_0}{q}\frac{1-\theta}{p_0}p} w_{1,\delta,M}^{\frac{q_1}{q}\frac{\theta}{p_1}p})} \lesssim \sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}\|f\|_{L^p(w_{0,\delta,M}^{\frac{q_0}{q}\frac{1-\theta}{p_0}p} w_{1,\delta,M}^{\frac{q_1}{q}\frac{\theta}{p_1}p})},$$

where $f \in \mathcal{B} \subset L^p(w_{0,\delta,M}^{\frac{q_0}{q}\frac{1-\theta}{p_0}p} w_{1,\delta,M}^{\frac{q_1}{q}\frac{\theta}{p_1}p})$. Now we have prove that

$$\lim_{M\to\infty}\lim_{\delta\to 0}\|f\|_{L^p(w_{\theta,\delta,M}^p)} = \|f\|_{L^p(w_\theta^p)} \quad \text{for all } f \in \mathcal{B}$$

where

$$w_{\theta,\delta,M} := w_{0,\delta,M}^{\frac{q_0}{q}\frac{1-\theta}{p_0}} w_{1,\delta,M}^{\frac{q_1}{q}\frac{\theta}{p_1}} \qquad w_\theta := w_0^{\frac{q_0}{q}\frac{1-\theta}{p_0}} w_1^{\frac{q_1}{q}\frac{\theta}{p_1}}.$$

First, fix M, we have that $|fw_{\theta,\delta,M}|^p$ is bounded by an integrable function, $|fM^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}+\frac{q_1}{q}\cdot\frac{\theta}{p_1}}|^p$, so we can apply the Lebesgue dominated convergence theorem to get

$$\lim_{\delta\to 0}\|f\|_{L^p(w_{\theta,\delta,M}^p)} = \|f\|_{L^p(w_{\theta,M}^p)} \quad \text{with} \quad w_{\theta,M} := w_{0,M}^{\frac{q_0}{q}\frac{1-\theta}{p_0}} w_{1,M}^{\frac{q_1}{q}\frac{\theta}{p_1}}.$$

Second, since $|fw_{\theta,M}|^p$ converges to $|fw_\theta|^p$ increasingly as $M \to \infty$, applying the Lebesgue monotone convergence theorem we have

$$\lim_{M\to\infty}\|f\|_{L^p(w_{\theta,M}^p)} = \|f\|_{L^p(w_\theta^p)}.$$

By Fatou's lemma,

$$\|T_\sigma f\|_{L^p(w_0^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}p} w_1^{\frac{q_1}{q}\cdot\frac{\theta}{p_1}p})} \leq \liminf_{\delta\to 0, M\to\infty}\|T_\sigma f\|_{L^p(w_{0,\delta,M}^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}p} w_{1,\delta,M}^{\frac{q_1}{q}\cdot\frac{\theta}{p_1}p})}.$$

Consequently, we obtain

$$\|T_\sigma f\|_{L^p(w_0^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}p} w_1^{\frac{q_1}{q}\cdot\frac{\theta}{p_1}p})} \lesssim \sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^{\frac{n}{s},1}}\|f\|_{L^p(w_0^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}p} w_1^{\frac{q_1}{q}\cdot\frac{\theta}{p_1}p})} \quad \text{for all } f \in \mathcal{B}.$$

Here $w_0^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}p} w_1^{\frac{q_1}{q}\cdot\frac{\theta}{p_1}p} \in A_{\frac{ps^*}{n}} \cap RH_q$ by Lemma 3.7.

Finally, any given $w \in A_{\frac{ps^*}{n}} \cap RH_q$ can be factored by Lemma 3.8 as $w = w_0^{\frac{q_0}{q}\cdot\frac{1-\theta}{p_0}p} w_1^{\frac{q_1}{q}\cdot\frac{\theta}{p_1}p}$ with $w_0 \in A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0}$ and $w_1 \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1}$. Then the conclusion (20) holds for w in view of this factorization. ☐

6. Applications

In this section we use our interpolation Theorem 5.1 to improve results concerning weighted bounds of Fourier multipliers.

Theorem 6.1. *Let* $2 < p < \infty$, $0 < s < \frac{n}{2}$, $\frac{1}{2} - \frac{1}{p} < \frac{s}{n}$, *and* $rs > n$. *Define*

$$\bar{q} = 1 + \frac{1 - \frac{2s}{n}}{\frac{1}{2} - \frac{1}{p}}.$$

Let $w \in A_{\frac{p}{2}} \cap RH_{\bar{q}}$ *and* σ *satisfy*

$$\sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi} \sigma(2^j \cdot)] \right\|_{L^r} < \infty. \tag{36}$$

Then for any bounded compactly supported function f on $\mathbb{R}^n$ we have

$$\|T_\sigma f\|_{L^p(w)} \lesssim \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi} \sigma(2^j \cdot)] \right\|_{L^r} \|f\|_{L^p(w)}.$$

Proof. We fix $2 < p < \infty$ and $0 < s < \frac{n}{2}$. For small $\varepsilon_0, \varepsilon_1 > 0$ we define the following parameters

$$s_0 = \varepsilon_0 n, \quad \frac{1}{r_0} = \frac{\varepsilon_0}{2}, \quad p_0 = 2, \quad \frac{s_0^*}{n} = \frac{1}{r_0} + \frac{1}{2} = \frac{\varepsilon_0}{2} + \frac{1}{2}, \quad q_0^* = \frac{4}{2 - \varepsilon_0}, \quad q_0 = \left(\frac{q_0^*}{2}\right)' = \frac{2}{\varepsilon_0},$$
$$s_1 = s_1^* = \left(\frac{1}{2} + \varepsilon_1\right) n, \quad \frac{1}{r_1} = \left(\frac{1}{2} + \frac{\varepsilon_1}{2}\right), \quad q_1 = 1, \quad \theta = \frac{\frac{s}{n} - \varepsilon_0}{\frac{1}{2} + \varepsilon_1 - \varepsilon_0}.$$

This choice of parameters gives

$$s = (1 - \theta) s_0 + \theta s_1.$$

Next we choose $p_1 > 2$, $s^*(\varepsilon_0, \varepsilon_1)$, $r(\varepsilon_0, \varepsilon_1)$, and $q(\varepsilon_0, \varepsilon_1)$ such that

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \qquad s^*(\varepsilon_0, \varepsilon_1) = (1 - \theta) s_0^* + \theta s_1^*,$$
$$\frac{1}{r(\varepsilon_0, \varepsilon_1)} = (1 - \theta) \frac{1}{r_0} + \theta \frac{1}{r_1}, \quad q(\varepsilon_0, \varepsilon_1) \left(\frac{s^*}{n} - \frac{1}{p}\right) = (1 - \theta) q_0 \left(\frac{s_0^*}{n} - \frac{1}{p_0}\right) + \theta q_1 \left(\frac{s_1^*}{n} - \frac{1}{p_1}\right).$$

We observe that as $\varepsilon_0, \varepsilon_1 \to 0$ we have

$$\theta = \frac{\frac{s}{n} - \varepsilon_0}{\frac{1}{2} + \varepsilon_1 - \varepsilon_0} \longrightarrow \frac{2s}{n},$$
$$\frac{s^*(\varepsilon_0, \varepsilon_1)}{n} = \frac{1}{2} + \frac{\varepsilon_0}{2}(1 - \theta) + \varepsilon_1 \theta \longrightarrow \frac{1}{2},$$
$$\frac{1}{r(\varepsilon_0, \varepsilon_1)} = (1 - \theta) \frac{\varepsilon_0}{2} + \theta \left(\frac{1}{2} + \frac{\varepsilon_1}{2}\right) \longrightarrow \frac{s}{n},$$
$$q(\varepsilon_0, \varepsilon_1) = \frac{\frac{3}{2} + \theta + \varepsilon_1 \theta - \frac{1}{p}}{\frac{1}{2} - \frac{1}{p} + \varepsilon_1 \theta + \frac{\varepsilon_0}{2}(1 - \theta)} \longrightarrow \frac{\frac{3}{2} - \frac{2s}{n} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{p}} = 1 + \frac{1 - \frac{2s}{n}}{\frac{1}{2} - \frac{1}{p}} = \bar{q}.$$

For any given $w \in A_{\frac{p}{2}} \cap RH_{\bar{q}}$ and given $r \in (\frac{n}{s}, \infty)$, we choose sufficiently small $\varepsilon_0, \varepsilon_1$ by Lemma 3.6, such that for $q = q(\varepsilon_0, \varepsilon_1)$, $s^* = s^*(\varepsilon_0, \varepsilon_1)$ we have

$$w \in A_{\frac{ps^*}{n}} \cap RH_q$$
$$r > r(\varepsilon_0, \varepsilon_1) > \frac{n}{s}.$$

We explain why we have

$$\|T_\sigma f\|_{L^{p_i}(w_i)} \leq \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s_i}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_i}} \|f\|_{L^{p_i}(w_i)} \tag{37}$$

for all σ satisfying $\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s_i}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_i}} < \infty$, all $w_i \in A_{\frac{p_i s_i^*}{n}} \cap RH_{q_i}$ and $i \in \{0,1\}$. In particular, when $i = 0$, one can verify that

$$\frac{1}{r_0} < \frac{s_0}{n}, \quad \frac{s_0}{n} - \frac{1}{r_0} > \frac{1}{2} - \frac{1}{q_0^*},$$

so (37) is derived from Theorem 3.15 (case 1), and it is derived from Theorem 4.1 when $i = 1$. Finally, via Theorem 5.1 we obtain

$$\|T_\sigma f\|_{L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r(\varepsilon_0,\varepsilon_1)}} \|f\|_{L^p(w)}$$

for all σ satisfying

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r(\varepsilon_0,\varepsilon_1)}} < \infty. \tag{38}$$

Finally, we notice that all σ that satisfy (36) must also satisfy (38) by Lemma 3.9. □

We have the following result in the case $p = 2$.

Theorem 6.2. *$0 < s < \frac{n}{2}$, $rs > n$ and $\bar{q} = \frac{2n}{s} - 1$. If*

$$\sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^r} < \infty, \tag{39}$$

then, for any bounded compactly supported function f on $\mathbb{R}^n$,

$$\|T_\sigma f\|_{L^2(w)} \lesssim \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^r} \|f\|_{L^2(w)}$$

for any weight $w \in A_{\frac{s}{n}+1} \cap RH_{\bar{q}}$.

Proof. For any fixed $0 < s < \frac{n}{2}$ and $w \in A_{\frac{s}{n}+1} \cap RH_{\bar{q}}$, one may choose the following parameters:

$$p_0 = 2, \quad s_0 = \varepsilon_0 n, \quad \frac{1}{r_0} = \frac{\varepsilon_0}{2}, \quad \frac{s_0^*}{n} = \frac{1}{r_0} + \frac{1}{2} = \frac{\varepsilon_0}{2} + \frac{1}{2}, \quad q_0^* = \frac{4}{2-\varepsilon_0}, \quad q_0 = \left(\frac{q_0^*}{2}\right)' = \frac{2}{\varepsilon_0},$$

$$p_1 = 2, \quad s_1 = s_1^* = (1-\varepsilon_1)n, \quad \frac{1}{r_1} = 1 - 2\varepsilon_1, \quad q_1 = 1, \quad \theta = \frac{\frac{s}{n} - \varepsilon_0}{1 - \varepsilon_1 - \varepsilon_0},$$

such that $s = (1-\theta)s_0 + \theta s_1$, and then we choose p_1, $s^*(\varepsilon_0, \varepsilon_1)$, $r(\varepsilon_0, \varepsilon_1)$, and $q(\varepsilon_0, \varepsilon_1)$ such that

$$\frac{1}{2} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s^*(\varepsilon_0, \varepsilon_1) = (1-\theta)s_0^* + \theta s_1^*,$$

$$\frac{1}{r(\varepsilon_0, \varepsilon_1)} = (1-\theta)\frac{1}{r_0} + \theta\frac{1}{r_1}, \quad q(\varepsilon_0, \varepsilon_1)\Big(\frac{s^*}{n} - \frac{1}{2}\Big) = (1-\theta)q_0\Big(\frac{s_0^*}{n} - \frac{1}{2}\Big) + \theta q_1\Big(\frac{s_1^*}{n} - \frac{1}{2}\Big).$$

Letting $\varepsilon_0, \varepsilon_1 \to 0$ we obtain

$$\theta = \frac{\frac{s}{n} - \varepsilon_0}{1 - \varepsilon_1 - \varepsilon_0} \ \longrightarrow\ \frac{s}{n},$$

$$\frac{s^*(\varepsilon_0, \varepsilon_1)}{n} = \frac{1}{2}(1+\theta) + \frac{\varepsilon_0}{2}(1-\theta) - \varepsilon_1\theta \ \longrightarrow\ \frac{1}{2} + \frac{s}{2n},$$

$$\frac{1}{r(\varepsilon_0, \varepsilon_1)} = (1-\theta)\frac{\varepsilon_0}{2} + \theta\,(1 - 2\varepsilon_1) \ \longrightarrow\ \frac{s}{n},$$

$$q(\varepsilon_0, \varepsilon_1) = \frac{1 - \frac{\theta}{2} - \varepsilon_1\theta}{\frac{\theta}{2} - \varepsilon_1\theta + \frac{\varepsilon_0}{2}(1-\theta)} \ \longrightarrow\ \frac{1 - \frac{s}{2n}}{\frac{s}{2n}} = \frac{2n}{s} - 1$$

when $0 < s < \frac{n}{2}$. For a given weight $w \in A_{\frac{s}{n}+1} \cap RH_{\bar{q}}$, we choose $\varepsilon_0, \varepsilon_1 > 0$ sufficiently small according to Lemma 3.6 such that

$$w \in A_{\frac{2s^*(\varepsilon_0,\varepsilon_1)}{n}} \cap RH_{q(\varepsilon_0,\varepsilon_1)} \quad \text{and} \quad \frac{n}{s} < r(\varepsilon_0, \varepsilon_1) < r.$$

This allows us to apply Lemma 3.8, yielding weights

$$w_0 \in A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0}, \qquad w_1 \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1},$$

such that

$$w = w_0^{\frac{q_0}{q}(1-\theta)} w_1^{\frac{q_1}{q}\theta}.$$

We now explain why for $i \in \{0,1\}$ and any σ satisfying

$$\sup_{j\in\mathbb{Z}} \big\| (I-\Delta)^{\frac{s_i}{2}} (\widehat{\Psi}\,\sigma(2^j \cdot)) \big\|_{L^{r_i}} < \infty,$$

we have the estimate

$$\|T_\sigma f\|_{L^2(w_i)} \lesssim \sup_{j\in\mathbb{Z}} \big\| (I-\Delta)^{\frac{s_i}{2}} (\widehat{\Psi}\,\sigma(2^j \cdot)) \big\|_{L^{r_i}} \, \|f\|_{L^2(w_i)}, \tag{40}$$

for all $w_i \in A_{\frac{p_i s_i^*}{n}} \cap RH_{q_i}$. In particular, when $i = 0$, one can verify that

$$\frac{1}{r_0} < \frac{s_0}{n}, \quad \frac{s_0}{n} - \frac{1}{r_0} > \frac{1}{2} - \frac{1}{q_0^*},$$

so (40) is derived from Theorem 3.15 (case 1), and it is derived from Theorem 4.1 when $i = 1$. Finally, via Theorem 5.1 we obtain

$$\|T_\sigma f\|_{L^2(w)} \lesssim \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}} (\hat{\Psi}\sigma(2^j \cdot))\|_{L^{r(\varepsilon_0,\varepsilon_1)}} \|f\|_{L^2(w)}$$

for all σ satisfying $\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}} (\hat{\Psi}\sigma(2^j \cdot))\|_{L^{r(\varepsilon_0,\varepsilon_1)}} < \infty$. Finally, we conclude

$$\|T_\sigma f\|_{L^2(w)} \lesssim \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}} (\hat{\Psi}\sigma(2^j \cdot))\|_{L^r} \|f\|_{L^2(w)}$$

by Lemma 3.9 since $r(\varepsilon_0, \varepsilon_1) < r$. □

Remark 6.3. (1) In both cases $p > 2$ and $p = 2$, the limiting exponent $q(\varepsilon_0, \varepsilon_1)$ satisfies

$$q(\varepsilon_0, \varepsilon_1) \ \longrightarrow\ \bar{q} \ \text{ as } \ \varepsilon_0, \varepsilon_1 \ \longrightarrow\ 0$$

where

$$\bar{q} = \begin{cases} 1 + \dfrac{1-\frac{2s}{n}}{\frac{1}{2}-\frac{1}{p}}, & p > 2, \\ \dfrac{2n}{s} - 1, & p = 2, \end{cases}$$

and in either case we have $\bar{q} > \frac{n}{ps}$. Consequently, the conclusions of Theorems 6.1 and Theorem 6.2 are consistent with Proposition 2.2.

(2) We now compare Theorems 6.1 and 6.2 with Theorem 3.15. In Theorem 3.15, there is the following constraint

$$\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{q_0^*}, \quad q_0^* > p.$$

When $\frac{1}{r}$ approaches $\frac{s}{n}$, the corresponding reverse Hölder exponent

$$q = \left(\frac{q_0^*}{p}\right)' \longrightarrow \infty \quad \Longleftrightarrow \quad RH_q \longrightarrow RH_\infty.$$

As a result, the boundedness of T_σ can only be obtained for a very small class of weights.

In contrast, Theorems 6.1and 6.2 show that even if $\frac{1}{r}$ is close to $\frac{s}{n}$, boundedness of the operator T_σ still holds for weights w in the class

$$w \in \begin{cases} A_{\frac{p}{2}} \cap RH_{\bar{q}}, & p > 2, \\ A_{1+\frac{s}{n}} \cap RH_{\bar{q}}, & p = 2, \end{cases}$$

where the exponent $\bar{q}$ remains a finite constant once s and p are fixed. This yields an improvement of Theorem 3.15.

Theorem 6.4 (The case $p<2$)**.** *Let $1 < p < 2$, $0 < s < \frac{n}{2}$, $\frac{1}{p} - \frac{1}{2} < \frac{s}{n}$, and $rs > n$. Define*

$$\bar{q}_1 = 1 + \frac{1-\frac{2s}{n}}{\frac{1}{2}-\frac{1}{p'}}, \quad \tilde{p} = \frac{1}{(p'-1)\bar{q}_1} + 1, \quad \tilde{q} = \frac{p'-1}{\frac{p'}{2}-1}. \tag{41}$$

Let $w \in A_{\tilde{p}} \cap RH_{\tilde{q}}$ and σ satisfy

$$\sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^r} < \infty.$$

Then for any bounded compactly supported function f on $\mathbb{R}^n$ we have

$$\|T_\sigma f\|_{L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^r} \|f\|_{L^p(w)}.$$

Proof. Fix $1 < p < 2$. Let

$$\bar{q}_1 = 1 + \frac{1-\frac{2s}{n}}{\frac{1}{2}-\frac{1}{p'}}, \qquad \tilde{w} = w^{-\frac{1}{p-1}} \in A_{\frac{p'}{2}} \cap RH_{\bar{q}_1}.$$

We use a duality argument to prove that if

$$T_\sigma : L^{p'}(\mathbb{R}^n, \tilde{w}) \to L^{p'}(\mathbb{R}^n, \tilde{w})$$

then

$$T_\sigma : L^p(\mathbb{R}^n, w) \to L^p(\mathbb{R}^n, w).$$

First, we recall that the dual space of $L^p(\mathbb{R}^n, w)$ is $L^{p'}(\mathbb{R}^n, \tilde{w})$ under the pairing

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

In fact, one direction of this assertion follow by Hölder's inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx = \int_{\mathbb{R}^n} |f(x)|w(x)^{1/p}\,|g(x)|w(x)^{-1/p}\,dx \le \|f\|_{L^p(w)}\,\|g\|_{L^{p'}(\tilde{w})}. \tag{42}$$

The adjoint (or transpose) T^* of a a linear operator T defined on $\mathcal{B}$ satisfies

$$\int_{\mathbb{R}^n} (Tf)(x)\,g(x)\,dx = \int_{\mathbb{R}^n} f(x)\,(T^*g)(x)\,dx \tag{43}$$

for all $f, g \in \mathcal{B}$. Then by density, one extends (43) to all $f \in L^p(w)$ and $g \in L^{p'}(\tilde{w})$.

For $f \in L^p(w)$,we have

$$\|Tf\|_{L^p(w)} = \sup_{\substack{g\in L^{p'}(\tilde{w})\\ \|g\|_{L^{p'}(\tilde{w})}=1}} \left|\int_{\mathbb{R}^n} (Tf)(x)\,g(x)\,dx\right|. \tag{44}$$

By the adjoint relation (43) and (42) we deduce

$$\left|\int_{\mathbb{R}^n} (Tf)\,g\,dx\right| = \left|\int_{\mathbb{R}^n} f\,(T^*g)\,dx\right| \le \|f\|_{L^p(w)}\,\|T^*g\|_{L^{p'}(\tilde{w})}. \tag{45}$$

Combining (44) and (45) we have

$$\|Tf\|_{L^p(w)} \le \|T^*\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})}\,\|f\|_{L^p(w)}.$$

Hence

$$\|T\|_{L^p(w)\to L^p(w)} \le \|T^*\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})}.$$

Applying the same argument with p replaced by p' and w replaced by $\tilde{w}$, we obtain the reverse inequality, and therefore

$$\|T\|_{L^p(w)\to L^p(w)} = \|T^*\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})}.$$

In particular, if $T = T_\sigma$ then

$$\|T_\sigma\|_{L^p(w)\to L^p(w)} = \|T_\sigma^*\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})}.$$

(Notice that if σ is real-valued, then $T_\sigma^* = T_\sigma$.)

Moreover, we observe that

$$\begin{aligned} w^{-\frac{1}{p-1}} \in A_{\frac{p'}{2}} \cap RH_{\bar{q}_1} \quad &\iff \quad w^{-\frac{1}{p-1}\bar{q}_1} \in A_{\bar{q}_1\left(\frac{p'}{2}-1\right)+1} = A_P, \quad \text{where } P = \bar{q}_1\left(\frac{p'}{2}-1\right)+1 \\ &\iff \quad w^{-\frac{1-p'}{P-1}\bar{q}_1} \in A_{P'} \\ &\iff \quad w^{\frac{p'-1}{P-1}\bar{q}_1} \in A_{P'} \\ &\iff \quad w \in A_{\tilde{p}} \cap RH_{\tilde{q}}, \quad \text{where } \tilde{q} = \frac{p'-1}{P-1}\bar{q}_1 \text{ and } \tilde{p} = \frac{1}{\tilde{q}}(P'-1)+1. \end{aligned}$$

Eliminating the auxiliary parameter P, we obtain

$$\tilde{q} = \frac{p'-1}{\frac{p'}{2}-1} \qquad \tilde{p} = \frac{1}{(p'-1)\bar{q}_1} + 1,$$

where

$$\bar{q}_1 = 1 + \frac{1 - \frac{2s}{n}}{\frac{1}{2} - \frac{1}{p'}}.$$

These are exactly the relationships (41) of the indices in the statement of the theorem. □

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