L^p BOUNDS FOR A MAXIMAL DYADIC SUM OPERATOR

LOUKAS GRAFAKOS, TERENCE TAO, AND ERIN TERWILLEGER

Abstract. The authors prove L^p bounds in the range $1 < p < \infty$ for a maximal dyadic sum operator on $\mathbf{R}^n$. This maximal operator provides a discrete multidimensional model of Carleson's operator. Its boundedness is obtained by a simple twist of the proof of Carleson's theorem given by Lacey and Thiele [6] adapted in higher dimensions [8]. In dimension one, the L^p boundedness of this maximal dyadic sum implies in particular an alternative proof of Hunt's extension [3] of Carleson's theorem on almost everywhere convergence of Fourier integrals.

1. The Carleson-Hunt theorem

A celebrated theorem of Carleson [1] states that the Fourier series of a square-integrable function on the circle converges almost everywhere to the function. Hunt [3] extended this theorem to L^p functions for $1 < p < \infty$. Alternative proofs of Carleson's theorem were provided by C. Fefferman [2] and by Lacey and Thiele [6]. The last authors proved the theorem on the line, i.e. they showed that for f in $L^2(\mathbf{R})$ the sequence of functions

$$S_N(f)(x) = \int_{|\xi| \le N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$$

converges to $f(x)$ for almost all $x \in \mathbf{R}$ as $N \to \infty$. This result was obtained as a consequence of the boundedness of the maximal operator

$$\mathcal{C}(f) = \sup_{N>0} |S_N(f)|$$

from $L^2(\mathbf{R})$ into $L^{2,\infty}(\mathbf{R})$. In view of the transference theorem of Kenig and Tomas [4] the above result is equivalent to the analogous theorem for Fourier series on the circle. Lacey and Thiele [5] have also obtained a proof of Hunt's theorem by adapting the techniques in [6] to the L^p case but this proof is rather complicated compared with the relatively short and elegant proof they gave for $p = 2$.

Investigating higher dimensional analogues, Pramanik and Terwilleger [8] recently adapted the proof of Carleson's theorem by Lacey and Thiele [6] to prove weak type $(2,2)$ bounds for a discrete maximal operator on $\mathbf{R}^n$ similar to the one which arises in the aforementioned proof. After a certain averaging procedure, this result provides an alternative proof of Sjölin's [10] theorem on the weak L^2 boundedness of

Date: December 11, 2002.

1991 *Mathematics Subject Classification.* Primary 42A20. Secondary 42A24.

Key words and phrases. Fourier series, almost every convergence.

Grafakos is supported by the NSF. Tao is a Clay Prize Fellow and is supported by a grant from the Packard Foundation.

maximally modulated Calderón-Zygmund operators on $\mathbf{R}^n$. The purpose of this note is to extend the result of Pramanik and Terwilleger [8] to the range $1 < p < \infty$ via a variation of the $L^2 \to L^{2,\infty}$ case. Particularly in dimension 1, the theorem below yields a new proof of Hunt's theorem (i.e. the L^p boundedness of $\mathcal{C}$ for $1 < p < \infty$) using a variation of the proof of Lacey and Thiele [6].

2. Reduction to two estimates

We use the notation introduced in [6] and expanded in [8]. A tile in $\mathbf{R}^n \times \mathbf{R}^n$ is a product of dyadic cubes of the form

$$\prod_{j=1}^{n} I^j = \prod_{j=1}^{n} [m_j 2^k, (m_j+1)2^k),$$

where k and m_j are integers for all $j = 1, 2, \cdots, n$. We denote a *tile* by $s = I_s \times \omega_s$, where $|I_s||\omega_s| = 1$. The cube I_s will be called the time projection of s and ω_s the frequency projection of s. For a tile s with $\omega_s = \omega_s^1 \times \omega_s^2 \times \ldots \times \omega_s^n$, we can divide each dyadic interval ω_s^j into two intervals of the form

$$\omega_s^j = (\omega_s^j \cap (-\infty, c(\omega_s^j)) \cup (\omega_s^j \cap [c(\omega_s^j), \infty))$$

for $j = 1, 2, \ldots, n$. Then ω_s can be decomposed into 2^n subcubes formed from all combinations of cross products of these half intervals. We number these subcubes using the lexicographical order on the centers and denote the subcubes by $\omega_{s(i)}$ for $i = 1, 2, \ldots, 2^n$. A tile s is then the union of 2^n *semi-tiles* given by $s(i) = I_s \times \omega_{s(i)}$ for $i = 1, 2, \ldots, 2^n$.

We let ϕ be a Schwartz function such that $\widehat{\phi}$ is real, nonnegative, and supported in the cube $[-1/10, 1/10]^n$. Define

$$\phi_s(x) = |I_s|^{-\frac{1}{2}} \phi\left(\frac{x - c(I_s)}{|I_s|^{\frac{1}{n}}}\right) e^{2\pi i c(\omega_{s(1)})\cdot x},$$

where $c(J)$ is the center of a cube J. As in [6] and [8], we will consider the dyadic sum operator

$$\mathcal{D}_r(f) = \sum_{s \in D} \langle f, \phi_s \rangle (\chi_{\omega_{s(r)}} \circ N) \phi_s,$$

where $2 \le r \le 2^n$ is a fixed integer, $N : \mathbf{R}^n \to \mathbf{R}^n$ is a fixed measurable function, D is a set of tiles, and $\langle f, g \rangle$ is the complex inner product $\int_{\mathbf{R}} f(x)\overline{g(x)}\, dx$.

The following theorem is the main result of this article.

Theorem 1. *Let $1 < p < \infty$. Then there is a constant $C_{n,p}$ independent of the measurable function N, of the set D, and of r such that for all $f \in L^p(\mathbf{R}^n)$ we have*

$$\|\mathcal{D}_r(f)\|_{L^p(\mathbf{R}^n)} \le C_{n,p} \|f\|_{L^p(\mathbf{R}^n)}. \tag{1}$$

In one-dimension, using the averaging procedure introduced in [6], it follows that the norm estimate (1) implies

$$\|\mathcal{C}(f)\|_{L^p} \le C_p \|f\|_{L^p},$$

which is the Carleson-Hunt theorem. Using the Marcinkiewicz interpolation theorem [11] and the restricted weak type reduction of Stein and Weiss [12], estimate (1) will be a consequence of the restricted weak type estimate

$$\|\mathcal{D}_r(\chi_F)\|_{L^{p,\infty}(\mathbf{R}^n)} \le C_{n,p}|F|^{\frac{1}{p}}, \qquad 1<p<\infty \tag{2}$$

which is supposed to hold for all n-dimensional sets F of finite measure. But to show that a function g lies in $L^{p,\infty}$, it suffices to show that for every measurable set E of finite measure, there is a subset E' of E which satisfies $|E'|\ge \frac{1}{2}|E|$ and also

$$\left|\int_{E'} g(x)\,dx\right| \le A\,|E|^{\frac{p-1}{p}};$$

this implies that $\|g\|_{L^{p,\infty}(\mathbf{R}^n)} \le c_p\,A$, where c_p is a constant that depends only on p.

Let $C(n,q)$ be the weak type (q,q) operator norm for the Hardy-Littlewood maximal operator. Given a set E of finite measure we set

$$\Omega = \Big\{M(\chi_F) > \Big(2\frac{|F|}{|E|}\Big)^{\frac{1}{q}} C(n,q)\Big\},$$

where we choose q so that $p<q\le\infty$ if $|F|>|E|$ and $1\le q<p$ if $|F|\le|E|$. Note that in the first case the set Ω is empty. Using the L^q to $L^{q,\infty}$ boundedness of the Hardy-Littlewood maximal operator, we have $|\Omega|\le\frac{1}{2}|E|$ and hence $|E'|\ge\frac{1}{2}|E|$. Thus estimate (2) will follow from

$$\left|\int_{E'} \mathcal{D}_r(\chi_F)(x)\,dx\right| \le C_{n,p}|E|^{\frac{p-1}{p}}|F|^{\frac{1}{p}}, \tag{3}$$

where $C_{n,p}$ depends only on p and dimension n. The required estimate (3) will then be a consequence of the following two estimates:

$$\left|\int_{E'} \sum_{\substack{s\in D\\ I_s\subseteq\Omega}} \langle\chi_F,\phi_s\rangle(\chi_{\omega_{s(r)}}\circ N)\phi_s(x)\,dx\right| \le C_{n,p,q}|E|^{\frac{p-1}{p}}|F|^{\frac{1}{p}}, \tag{4}$$

and

$$\sum_{\substack{s\in D\\ I_s\not\subseteq\Omega}} |\langle\chi_F,\phi_s\rangle|\,|\langle\chi_{E'\cap N^{-1}[\omega_{s(r)}]},\phi_s\rangle| \le C_{n,p,q}|E|^{\frac{p-1}{p}}|F|^{\frac{1}{p}}. \tag{5}$$

3. The proof of estimate (4)

Following [7], we denote by $I(D)$ the dyadic grid which consists of all the time projections of tiles in D. For each dyadic cube J in $I(D)$ we define

$$D_J := \{s\in D : I_s = J\}$$

and a function

$$\psi_J(x) := |J|^{-\frac{1}{2}}\left(1+\frac{|x-c(J)|}{|J|^{\frac{1}{n}}}\right)^{-\gamma},$$

where γ is a large integer to be chosen shortly. For each $k=0,1,2,\dots$ we introduce families

$$\mathcal{F}_k = \big\{J\in I(D):\ 2^kJ\subseteq\Omega,\ 2^{k+1}J\not\subseteq\Omega\big\}.$$

We may assume $|F| \leq |E|$, otherwise the set Ω is empty and (4) is trivial.

We begin by controlling the left hand side of (4) by

$$\begin{aligned} &\sum_{\substack{J\in I(D)\\ J\subseteq\Omega}} \Big| \sum_{s\in D(J)} \int_{E'} \big\langle \chi_F \mid \phi_s \big\rangle\, \chi_{\omega_{s(2)}}(N(x))\phi_s(x)\,dx \Big| \\ &\leq \sum_{k=0}^{\infty} \sum_{\substack{J\in I(D)\\ J\in\mathcal{F}_k}} \Big| \int_{E'} \sum_{s\in D(J)} \big\langle \chi_F \mid \phi_s \big\rangle\, \chi_{\omega_{s(r)}}(N(x))\phi_s(x)\,dx \Big| \end{aligned} \tag{6}$$

Using the fact that the function $M(\chi_F)^{\frac{1}{2}}$ is an A_1 weight with A_1-constant bounded above by a quantity independent of F, it is easy to find a constant $C_0 < \infty$ such that for each $k = 0, 1, \ldots$ and $J \in \mathcal{F}_k$ we have

$$\big\langle \chi_F, \psi_J \big\rangle \leq |J|^{\frac{1}{2}} \inf_J M(\chi_F) \leq |J|^{\frac{1}{2}}\, C_0^k \inf_{2^{k+1}J} M(\chi_F) \leq C(n,q) 2^{\frac{1}{q}}\, C_0^k |J|^{\frac{1}{2}} \Big(\frac{|F|}{|E|}\Big)^{\frac{1}{q}} \tag{7}$$

since $2^{k+1}J$ meets the complement of Ω. For $J \in \mathcal{F}_k$ one also has that $E' \cap 2^k J = \emptyset$ and hence

$$\int_{E'} \psi_J(y)\,dy \leq \int_{(2^k J)^c} \psi_J(y)\,dy \leq |J|^{\frac{1}{2}} C_\gamma 2^{-k\gamma}\,. \tag{8}$$

Next we note that for each $J \in I(D)$ and $x \in \mathbf{R}^n$ there is at most one $s = s_x \in D_J$ such that $N(x) \in \omega_{s_x(r)}$. Using this observation along with (7) and (8) we can therefore estimate the expression on the right in (6) as follows

$$\begin{aligned} &\leq \sum_{k=0}^{\infty} \sum_{\substack{J\in I(D)\\ J\in\mathcal{F}_k}} \Big| \int_{E'} \big\langle \chi_F \mid \phi_{s_x} \big\rangle\, \chi_{\omega_{s_x(r)}}(N(x))\phi_{s_x}(x)\,dx \Big| \\ &\leq C \sum_{k=0}^{\infty} \sum_{\substack{J\in I(D)\\ J\in\mathcal{F}_k}} \int_{E'} \big\langle \chi_F, \psi_J \big\rangle \psi_J(xt)\,dx \\ &\leq C \Big(\frac{|F|}{|E|}\Big)^{\frac{1}{q}} \sum_{k=0}^{\infty} C_0^k \sum_{J\in\mathcal{F}_k} |J|^{\frac{1}{2}} \int_{E'} \psi_J(x)\,dx \\ &\leq C \Big(\frac{|F|}{|E|}\Big)^{\frac{1}{q}} \sum_{k=0}^{\infty} (C_0 2^{-\gamma})^k \sum_{J\in\mathcal{F}_k} |J| \end{aligned} \tag{9}$$

and at this point we pick γ so that $C_0 2^{-\gamma} < 1$. It remains to control $\sum_{J\in\mathcal{F}_k} |J|$ for each nonnegative integer k. In doing this we let $\mathcal{F}_k^*$ be all elements of $\mathcal{F}_k$ which are maximal under inclusion. Then we observe that if $J \in \mathcal{F}_k^*$ and $J' \in \mathcal{F}_k$ satisfy $J' \subseteq J$ then dist $(J', J^c) = 0$ (otherwise $2J'$ would be contained in J and thus $2^{k+1}J' \subseteq 2^k J \subseteq \Omega$.) But for any fixed J in $\mathcal{F}_k^*$ and any scale m, all the cubes J' in $J' \in \mathcal{F}_k$ of sidelength 2^m that touch J are concentrated near the boundary of J and have total measure at most $2^m \cdot 2^n (|J|^{\frac{1}{n}})^{n-1}$. Summing over all integers m with

$2^m \leq |J|^{\frac{1}{n}}$, we obtain a bound which is at most a multiple of $|J|$. We conclude that

$$\sum_{J\in\mathcal{F}_k} |J| = \sum_{J\in\mathcal{F}_k^*} \sum_{\substack{J'\in\mathcal{F}_k\\ J'\subseteq J}} |J'| \leq \sum_{J\in\mathcal{F}_k^*} c_n\,|J| \leq c_n\,|\Omega|$$

since elements of $\mathcal{F}_k^*$ are disjoint and contained in Ω. Inserting this estimate in (9) and using that the Hardy-Littlewood maximal operator is of weak type $(1,1)$, we obtain the required bound

$$C\Big(\frac{|F|}{|E|}\Big)^{\frac{1}{q}}\,|\Omega| \leq C'\,|F| \leq C'\,|E|^{\frac{p-1}{p}}\,|F|^{\frac{1}{p}}$$

for the expression on the right in (6) and hence for the expression on the left in (4).

4. The proof of estimate (5)

In proving estimate (5) we may assume that $\frac{1}{2} \leq |E| \leq 1$ by a simple scaling argument. (The scaling changes the sets D, Ω, and the measurable function N but note that the final constants are independent of these quantities.) In addition all constants in the sequel are allowed to depend on n and p as described above. We may also assume that the set D is finite. Note that under the normalization of the set E, our choice of q is as follows: $1 \leq q < p$ if $|F| \leq c_0$ and $p < q \leq \infty$ when $|F| > c_0$ where c_0 is a fixed number in the interval $(\frac{1}{2},1)$, (in fact $c_0 = |E|$)

We recall that a finite set of tiles T is called a tree if there exists a tile $t\in T$ such that all $s\in T$ satisfy $s<t$ (which means $I_s\subset I_t$ and $\omega_t\subset\omega_s$.) In this case we call t the top of T and we denote it by $t=t(T)$. A tree T is called an r-tree if

$$\omega_{t(T)(r)} \subset \omega_{s(r)}$$

for all $s\in T$. For a finite set of tiles Q we define the energy of a nonzero function f with respect to Q by

$$\mathcal{E}(f;Q) = \frac{1}{\|f\|_{L^2(\mathbf{R}^n)}} \sup_T \left(\frac{1}{|I_{t(T)}|}\sum_{s\in T} |\langle f,\phi_s\rangle|^2\right)^{\frac{1}{2}},$$

where the supremum is taken over all r-trees T contained in Q. We also define the mass of a set of tiles Q by

$$\mathcal{M}(Q) = \sup_{s\in Q}\sup_{\substack{u\in Q\\ s<u}} \int_{E'\cap N^{-1}[\omega_{u(r)}]} \frac{|I_u|^{-1}}{\left(1+\frac{|x-c(I_u)|}{|I_u|^{1/n}}\right)^{\gamma}}\,dx.$$

We now fix a set of tiles D and sets E and F with finite measure (recall $\frac{1}{2}\leq|E|\leq 1$). We define P to be the set of all tiles in D with the property $I_s \nsubseteq \Omega$. Given a finite set of tiles P, find a very large integer m_0 one can construct a sequence of pairwise disjoint sets P_{m_0}, P_{m_0-1}, P_{m_0-2}, P_{m_0-3}, ... such that

$$P = \bigcup_{j=-\infty}^{m_0} P_j$$

and such that the following properties are satisfied

(a) $\mathcal{E}(\chi_F; P_j) \leq 2^{(j+1)n}$ for all $j \leq m_0$.
(b) $\mathcal{M}(P_j) \leq 2^{(2j+2)n}$ for all $j \leq m_0$.
(c) $\mathcal{E}\big(\chi_F; P \setminus (P_{m_0} \cup \cdots \cup P_j)\big) \leq 2^{jn}$ for all $j \leq m_0$.
(d) $\mathcal{M}\big(P \setminus (P_{m_0} \cup \cdots \cup P_j)\big) \leq 2^{2jn}$ for all $j \leq m_0$.
(e) P_j is a union of trees T_{jk} such that $\sum_k |I_{t(T_{jk})}| \leq C_0 2^{-2jn}$ for all $j \leq m_0$.

This can be done by induction, see [2], [6], and is based on an energy and a mass lemma shown in [8].

The following lemma is the main ingredient of the proof and will be proved in the next section.

Lemma 1. *There is a constant C such that for all measurable sets F and all finite set of tiles P which satisfy $I_s \nsubseteq \Omega$ for all $s \in P$, we have*

$$\mathcal{E}(\chi_F; P) \leq C|F|^{\frac{1}{q}-\frac{1}{2}}$$

Note that this gives us decay no matter if $|F|$ is large or small due to the choice of q (the reader is reminded that if $|F| \leq c_0$ then $q \in [1,p)$ while if $|F| \geq c_0$ then $q \in (p,\infty]$.) We also recall the estimate below from [8].

Lemma 2. *There is a finite constant C_1 such that for all trees T, all $f \in L^2(\mathbf{R}^n)$, and all measurable sets E' with $|E'| \leq 1$ we have*

$$\sum_{s\in T} \big|\langle f, \phi_s\rangle \langle \chi_{E'\cap N^{-1}[\omega_{s(r)}]}, \phi_s\rangle\big| \leq C_1\, |I_{t(T)}|\, \mathcal{E}(f;T)\, \mathcal{M}(T) \|f\|_{L^2(\mathbf{R}^n)}. \tag{10}$$

Given the sequence of sets P_j as above, we use (a), (b), (e), the observation that the mass is always bounded by 1, and Lemmata 1 and 2 to obtain

$$\begin{aligned}
&\sum_{s\in P} \big|\langle \chi_F, \phi_s\rangle \langle \chi_{E'\cap N^{-1}[\omega_{s(r)}]}, \phi_s\rangle\big| \\
=\ & \sum_j \sum_{s\in P_j} \big|\langle \chi_F, \phi_s\rangle \langle \chi_{E'\cap N^{-1}[\omega_{s(r)}]}, \phi_s\rangle\big| \\
\leq\ & \sum_j \sum_k \sum_{s\in T_{jk}} \big|\langle \chi_F, \phi_s\rangle \langle \chi_{E'\cap N^{-1}[\omega_{s(r)}]}, \phi_s\rangle\big| \\
\leq\ & C_1 \sum_j \sum_k |I_{t(T_{jk})}|\, \mathcal{E}(T_{jk})\, \mathcal{M}(T_{jk}) |F|^{\frac{1}{2}} \\
\leq\ & C_1\, |F|^{\frac{1}{2}} \sum_j \sum_k |I_{t(T_{jk})}|\, \min(2^{(j+1)n}, C|F|^{\frac{1}{q}-\frac{1}{2}})\, \min(1, 2^{(2j+2)n}) \\
\leq\ & C'|F|^{\frac{1}{2}} \sum_j 2^{-2jn}\, \min(2^{jn}, |F|^{\frac{1}{q}-\frac{1}{2}}) \min(1, 2^{2jn}) \\
\leq\ & C''|F|^{\frac{1}{q}} \big(1 + \big|\log |F|^{\frac{1}{2}-\frac{1}{q}}\big|\big) \\
\leq\ & C''' \min(1, |F|) \big(1 + \big|\log |F|\big|\big) \\
\leq\ & C_p |F|^{\frac{1}{p}}
\end{aligned}$$

for all $1 < p < \infty$. We observe that the choice of q was made to deal with the logarithmic presence in the estimate above. Had we taken $q = p$ throughout, we would have obtained the sought estimates with the extra factor of $1 + \big|\log|F|\big|$.

Looking at the penultimate inequality above, we note that we have actually obtained a stronger estimate than the one claimed in (3). Rescaling the set E and taking q to be either 1 or ∞, we have actually proved that for every measurable set E of finite measure, there is a subset E' of E such that for all measurable sets F of finite measure we have

$$\Big|\int_{E'} \mathcal{D}_r(\chi_F)\,dx\Big| \le C\,|E|\,\min\Big(1, \frac{|F|}{|E|}\Big)\Big(1 + \Big|\log\frac{|F|}{|E|}\Big|\Big)\,.$$

This will be of use to us in section 6.

5. The proof of Lemma 1

It remains to prove Lemma 1. Because of our normalization of the set E we may assume that $\Omega = \{M(\chi_F) > c\,|F|^{\frac{1}{q}}\}$ for some $c > 0$. Fix an r-tree T contained in P and let $I_t = I_{t(T)}$ be the time projection of its top.

We write the function χ_F as $\chi_{F\cap 3I_t} + \chi_{F\cap(3I_t)^c}$. We begin by observing that for s in P one has

$$|\langle \chi_{F\cap(3I_t)^c}, \phi_s\rangle| \le \frac{C_\gamma |I_s|^{\frac{1}{2}} \inf\limits_{I_s} M(\chi_F)}{\Big(1 + \dfrac{\mathrm{dist}((3I_t)^c, c(I_s))}{|I_s|^{\frac{1}{n}}}\Big)^{\gamma}} \le C_\gamma |I_s|^{\frac{1}{2}} |F|^{\frac{1}{q}} \left(\frac{|I_s|}{|I_t|}\right)^{\frac{\gamma}{n}}$$

since I_s meets the complement of Ω for every $s \in P$. Square this inequality and sum over all s in T to obtain

$$\sum_{s\in T} |\langle \chi_{F\cap(3I_t)^c}, \phi_s\rangle|^2 \le C\,|I_t|\,|F|^{\frac{2}{q}}\,,$$

where the last estimate follows by placing the I_s 's into groups $\mathcal{G}_m$ of cardinality at most 2^{mn} so that each element of $\mathcal{G}_m$ has size $2^{-mn}|I_t|$.

We now turn to the corresponding estimate for the function $\chi_{F\cap 3I_t}$. At this point it will be convenient to distinguish the case $|F| > c_0$ from the case $|F| \le c_0$. In the case $|F| > c_0$ the set Ω is empty and therefore

$$\sum_{s\in T} |\langle \chi_{F\cap 3I_t}, \phi_s\rangle|^2 \le C\,\|\chi_{F\cap 3I_t}\|_{L^2}^2 \le C\,|I_t| \;\le C\,|I_t|\,|F|^{\frac{2}{q}}\,,$$

where the first estimate follows follows from the Bessel inequality (13) which holds on any r-tree T; the reader may consult [8] or prove it directly.

We therefore concentrate on the case $|F| \le c_0$. In proving Lemma 1 we may assume that there exists a point $x_0 \in I_t$ such that $M(\chi_F)(x_0) \le c\,|F|^{\frac{1}{q}}$, otherwise there is nothing to prove. We may also assume that the center of $\omega_{t(T)}$ is zero, i.e. $c(\omega_{t(T)}) = 0$, otherwise we may work with a suitable modulation of the function $\chi_{F\cap 3I_t}$ in the Calderón-Zygmund decomposition below.

We write the set $\Omega = \{M(\chi_F) > c\,|F|^{\frac{1}{q}}\}$ as a disjoint union of dyadic cubes J'_ℓ such that the dyadic parent $\widetilde{J}'_\ell$ of J'_ℓ is not contained in Ω and therefore

$$|F \cap J'_\ell| \le |F \cap \widetilde{J}'_\ell| \le 2\,c\,|F|^{\frac{1}{q}}\,|J'_\ell|\,.$$

Now some of these dyadic cubes may have size larger than or equal to $|I_t|$. Let J'_ℓ be such a cube. Then we split J'_ℓ in $\frac{|J'_\ell|}{|I_t|}$ cubes $J'_{\ell,m}$ each of size exactly $|I_t|$. Since there is an $x_0 \in I_t$ with $M(\chi_F)(x_0) \le c\,|F|^{\frac{1}{q}}$, it follows that

$$|F \cap J'_{\ell,m}| \le 2\,c\,|F|^{\frac{1}{q}}\,|I_t| \left(1 + \frac{\mathrm{dist}(I_t, J'_{\ell,m})}{|I_t|^{\frac{1}{n}}}\right)^n . \tag{11}$$

We now have a new collection of dyadic cubes $\{J_k\}_k$ contained in Ω consisting of all the previous J'_ℓ when $|J'_\ell| < |I_t|$ and the $J'_{\ell,m}$'s when $|J'_{\ell,m}| \ge |I_t|$. In view of the construction we have

$$|F \cap J_k| \le \begin{cases} 2\,c\,|F|^{\frac{1}{q}}\,|J_k| & \text{when } |J_k| < |I_t| \\ 2\,c\,|F|^{\frac{1}{q}}\,|J_k| \left(1 + \dfrac{\mathrm{dist}(I_t, J_k)}{|I_t|}\right)^n & \text{when } |J_k| = |I_t| \end{cases} \tag{12}$$

for all k. We now define the "bad functions"

$$b_k = \chi_{J_k \cap 3I_t \cap F} - \frac{|J_k \cap 3I_t \cap F|}{|J_k|}\chi_{J_k}$$

which are supported in J_k, have mean value zero, and they satisfy

$$\|b_k\|_{L^1(\mathbf{R}^n)} \le 2\,c\,|F|^{\frac{1}{q}}\,|J_k| \left(1 + \frac{\mathrm{dist}(I_t, J_k)}{|I_t|}\right)^n .$$

We also set

$$g = \chi_{F \cap 3I_t} - \sum_k b_k$$

the "good function" of the above Calderón-Zygmund decomposition. We check that that $\|g\|_{L^\infty(\mathbf{R}^n)} \le C|F|^{\frac{1}{q}}$. Indeed, for x in J_k we have

$$g(x) = \frac{|F \cap 3I_t \cap J_k|}{|J_k|} \le \begin{cases} \dfrac{|F \cap J_k|}{|J_k|} & \text{when } |J_k| < |I_t| \\[2ex] \dfrac{|F \cap 3I_t|}{|I_t|} & \text{when } |J_k| = |I_t| \end{cases}$$

and both of the above are at most a multiple of $|F|^{\frac{1}{q}}$; the latter is because there is an $x_0 \in I_t$ with $M(\chi_F)(x_0) \le c\,|F|^{\frac{1}{q}}$. Also for $x \in (\cup_k J_k)^c = \Omega^c$, $g(x) = \chi_{F \cap 3I_t}(x)$ which is at most $M(\chi_F)(x) \le c\,|F|^{\frac{1}{q}}$. We conclude that $\|g\|_{L^\infty(\mathbf{R}^n)} \le C\,|F|^{\frac{1}{q}}$. Moreover

$$\|g\|_{L^1(\mathbf{R}^n)} \le \sum_k \int_{J_k} \frac{|F \cap 3I_t \cap J_k|}{|J_k|}\,dx + \|\chi_{F \cap 3I_t}\|_{L^1(\mathbf{R}^n)} \le C\,|F \cap 3I_t| \le C\,|F|^{\frac{1}{q}}\,|I_t|$$

since the J_k are disjoint. It follows that

$$\|g\|_{L^2(\mathbf{R}^n)} \le C\,|F|^{\frac{1}{2q}}|F|^{\frac{1}{2q}}\,|I_t|^{\frac{1}{2}} = C|F|^{\frac{1}{q}}\,|I_t|^{\frac{1}{2}}\,.$$

Using the simple Bessel inequality

$$\sum_{s\in T} |\langle g, \phi_s\rangle|^2 \le C\,\|g\|^2_{L^2(\mathbf{R}^n)} \tag{13}$$

we obtain the required conclusion for the function g.

For a fixed $s \in P$ and J_k we will denote by

$$d(k,s) = \text{dist }(J_k, I_s)\,.$$

Then we have the following estimate for all s and k:

$$|\langle b_k, \phi_s\rangle| \le C_\gamma\,|F|^{\frac{1}{q}}\,|J_k|\Big(1+\frac{d(k,t)}{|I_t|^{\frac{1}{n}}}\Big)^n \frac{|J_k|\,|I_s|^{-\frac{3}{2}}}{(1+\frac{d(k,s)}{|I_s|^{\frac{1}{n}}})^{\gamma+n}} \le \frac{C_\gamma\,|F|^{\frac{1}{q}}\,|J_k|^2\,|I_s|^{-\frac{3}{2}}}{(1+\frac{d(k,s)}{|I_s|^{\frac{1}{n}}})^{\gamma}} \tag{14}$$

since $1+\frac{d(k,t)}{|I_t|^{\frac{1}{n}}} \le 1+\frac{d(k,s)}{|I_s|^{\frac{1}{n}}}$.

We also have the estimate

$$|\langle b_k, \phi_s\rangle| \le \frac{C_\gamma\;|F|^{\frac{1}{q}}\,|I_s|^{\frac{1}{2}}}{(1+\frac{d(k,s)}{|I_s|^{\frac{1}{n}}})^{\gamma}}\,. \tag{15}$$

To prove (14) we use the fact that the center of $\omega_{t(T)} = 0$ (which implies that ϕ'_s obeys size estimates similar to $|I_s|^{-1}|\phi_s|$) and the mean value property of b_k to obtain

$$\big|\langle b_k, \phi_s\rangle\big| = \Big|\int_{J_k} b_k(y)\big(\phi_s(y)-\phi_s(c(J_k))\big)\,dy\Big| \le \|b_k\|_{L^1}|J_k|\,\sup_{\xi\in J_k} \frac{C_\gamma |I_s|^{-\frac{3}{2}}}{(1+\frac{|\xi-c(I_s)|}{|I_s|^{\frac{1}{n}}})^{\gamma}}\,.$$

To prove estimate (15) we note that

$$|\langle b_k, \phi_s\rangle| \le C_\gamma\,|I_s|^{\frac{1}{2}}\,\big(\inf_{I_s} M(b_k)\big)\frac{1}{(1+\frac{d(k,s)}{|I_s|^{\frac{1}{n}}})^{\gamma}}$$

and that

$$M(b_k) \le M(\chi_F) + \frac{|F\cap 3I_t\cap J_k|}{|J_k|} M(\chi_{J_k})$$

and since $I_s \nsubseteq \Omega$ we have $\inf_{I_s} M(\chi_F) \le c\,|F|^{\frac{1}{q}}$ while the second term in the sum above was observed earlier to be at most $C\,|F|^{\frac{1}{q}}$.

Finally we have the estimate

$$|\langle b_k, \phi_s\rangle| \le \frac{C_\gamma\,|F|^{\frac{1}{q}}\,|J_k|\,|I_s|^{-\frac{1}{2}}}{(1+\frac{d(k,s)}{|I_s|^{\frac{1}{n}}})^{\gamma}} \tag{16}$$

which follows by taking the geometric mean of (14) and (15).

Now for a fixed $s \in P$ we may have either $J_k \subseteq I_s$ or $J_k \cap I_s = \emptyset$ (since I_s is not contained in Ω.) Therefore for fixed $s \in P$ there are only three possibilities for J_k:

(a) $J_k \subseteq 3I_s$

(b) $J_k \cap 3I_s = \emptyset$
(c) $J_k \cap I_s = \emptyset$, $J_k \cap 3I_s \neq \emptyset$, and $J_k \nsubseteq 3I_s$.
Observe that case (c) is equivalent to the following statement:
(c) $J_k \cap I_s = \emptyset$, $d(k,s) = 0$, and $|J_k| \geq 2^n |I_s|$.

Let us start with case (c). Note that for each I_s there exists at most $2^n - 1$ choices of J_k with the above properties. Thus for each s in the sum below we can pick one $J_{k(s)}$ at a cost of $2^n - 1$, which is harmless. Also note that since $d(k,s) = 0$ and $|J_k| \geq 2^n |I_s|$, we must have that $I_s \subset 2J_k$. But $I_s \subset I_t$ and $|J_k| \leq |I_t|$ implies that $J_k \subset 3I_t$. Now for a given J_k and a fixed scale $m \geq 1$, there are at most $2^m \times (\#\text{ of sides}) + 2^n$ possibilities of I_s such that $2^{-mn}|J_k| = |I_s|$ and $d(k,s) = 0$. Using (15) we obtain

$$\begin{aligned}\sum_{s\in T}\Big|\sum_{k:\ J_k \text{ as in (c)}} \langle b_k, \phi_s\rangle\Big|^2 &\leq (2^n-1)^2 \sum_{s\in T}\Big|\langle b_{k(s)}, \phi_s\rangle\Big|^2 \\ &\leq C_n\,|F|^{\frac{2}{q}} \sum_{\substack{s\in T \text{ for which} \\ \exists\ J_k \text{ as in (c)}}} |I_s| \\ &\leq C_n\,|F|^{\frac{2}{q}} \sum_{m\geq 1} \sum_{\substack{s\in T \\ 2^{-mn}|J_{k(s)}|=|I_s|}} 2^{-mn}|J_{k(s)}| \\ &\leq C_n\,|F|^{\frac{2}{q}} \sum_{m\geq 1}(2^m \times (\#\text{ of sides}) + 2^n)2^{-mn} \sum_k |J_k| \\ &\leq C_n\,|F|^{\frac{2}{q}}|I_t|,\end{aligned}$$

where we have used the disjointness of the J_k's. This finishes case (c).

We now consider case (a). Using (14) we can write

$$\bigg(\sum_{s\in T}\Big|\sum_{k:\ J_k \text{ as in (a)}} \langle b_k, \phi_s\rangle\Big|^2\bigg)^{\frac12} \leq C_\gamma\,|F|^{\frac1q}\bigg(\sum_{s\in T}\Big|\sum_{k:\ J_k\subseteq 3I_s} |J_k|^{\frac32}\frac{|J_k|^{\frac12}}{|I_s|^{\frac32}}\Big|^2\bigg)^{\frac12}$$

and we control the expression inside the parenthesis above by

$$\sum_{s\in T}\bigg(\sum_{k:\ J_k\subseteq 3I_s} |J_k|^3\bigg)\bigg(\sum_{k:\ J_k\subseteq 3I_s}\frac{|J_k|}{|I_s|^3}\bigg) \leq \sum_{k:\ J_k\subseteq 3I_t} |J_k|^3 \sum_{\substack{s\in T \\ J_k\subseteq 3I_s}} \frac{1}{|I_s|^2}$$

in view of the Cauchy-Schwarz inequality and of the fact that the dyadic cubes J_k are disjoint and contained in $3I_s$. Finally note that the last sum above adds up to at most $C_n\,|J_k|^{-2}$ since for every dyadic cube J_k there exist at most $2^n+1+(\#\text{ of sides})$ dyadic cubes of a given size whose triples contain it. The required estimate $C_{n,\gamma}\,|F|^{\frac1q}\,|I_t|^{\frac12}$ now follows.

Finally we deal with case (b) which is the most difficult case. We split the set of k into two subsets, those for which $J_k \subseteq 3I_t$ and those for which $J_k \nsubseteq 3I_t$, (recall $|J_k| \leq |I_t|$.) Whenever $J_k \nsubseteq 3I_t$ we have $d(k,s) \approx d(k,t)$. In this case we use

Minkowski's inequality below and estimate (16) with $\gamma > n$ to obtain the estimate

$$\begin{aligned}
\Big(\sum_{s\in T}\Big|\sum_{k:\ J_k\nsubseteq 3I_t}\langle b_k,\phi_s\rangle\Big|^2\Big)^{\frac12} &\le \sum_{k:\ J_k\nsubseteq 3I_t}\Big(\sum_{s\in T}|\langle b_k,\phi_s\rangle|^2\Big)^{\frac12}\\
&\le C_\gamma|F|^{\frac1q}\sum_{k:\ J_k\nsubseteq 3I_t}|J_k|\Big(\sum_{s\in T}\frac{|I_s|^{\frac{2\gamma}{n}-1}}{d(k,s)^{2\gamma}}\Big)^{\frac12}\\
&\le C_\gamma|F|^{\frac1q}\sum_{k:\ J_k\nsubseteq 3I_t}\frac{|J_k|}{d(k,t)^\gamma}\Big(\sum_{s\in T}|I_s|^{\frac{2\gamma}{n}-1}\Big)^{\frac12}\\
&\le C_\gamma|F|^{\frac1q}\,|I_t|^{\frac{\gamma}{n}-\frac12}\sum_{k:\ J_k\nsubseteq 3I_t}\frac{|J_k|}{d(k,t)^\gamma}\\
&\le C_\gamma|F|^{\frac1q}\,|I_t|^{\frac{\gamma}{n}-\frac12}\sum_{l=1}^{\infty}\sum_{k:d(k,t)\approx 2^l|I_t|^{\frac1n}}\frac{|J_k|}{(2^l|I_t|^{\frac1n})^\gamma}\,.
\end{aligned}$$

But note that all the J_k with $d(k,t)\approx 2^l|I_t|^{\frac1n}$ are contained in $2^{l+2}I_t$ and since they are disjoint we can estimate the last sum above by $C2^{lm}|I_t|(2^l|I_t|^{\frac1n})^{-\gamma}$. The required estimate $C_\gamma|F|^{\frac1q}\,|I_t|^{\frac12}$ now follows.

Next we consider the sum below in which we use estimate (14)

$$\begin{aligned}
&\Big(\sum_{s\in T}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\langle b_k,\phi_s\rangle\Big|^2\Big)^{\frac12}\\
&\le C_\gamma\,|F|^{\frac1q}\Big(\sum_{s\in T}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}|J_k|^2|I_s|^{-\frac32}\Big(\frac{|I_s|^{\frac1n}}{d(k,s)}\Big)^\gamma\Big|^2\Big)^{\frac12}
\end{aligned}$$

$$\le C_\gamma\,|F|^{\frac1q}\Big\{\sum_{s\in T}\Big[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|^3}{|I_s|^2}\Big(\frac{|I_s|^{\frac1n}}{d(k,s)}\Big)^\gamma\Big]\Big[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|}{|I_s|}\Big(\frac{d(k,s)}{|I_s|^{\frac1n}}\Big)^{-\gamma}\Big]\Big\}^{\frac12}. \tag{17}$$

The second sum above can be estimated by

$$\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\int_{J_k}\Big(\frac{|x-c(I_s)|}{|I_s|^{\frac1n}}\Big)^{-\gamma}\frac{dx}{|I_s|}\le\int_{(3I_s)^c}\Big(\frac{|x-c(I_s)|}{|I_s|^{\frac1n}}\Big)^{-\gamma}\frac{dx}{|I_s|}\le C_\gamma.$$

Putting this estimate into (17), we have

$$
\begin{aligned}
&\leq C_\gamma\,|F|^{\frac{1}{q}}\Big\{\sum_{s\in T}\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\leq|I_s|}}|J_k|^3|I_s|^{-2}\bigg(\frac{|I_s|^{\frac{1}{n}}}{d(k,s)}\bigg)^{\gamma}\Big\}^{\frac12}\\
&\leq C_\gamma\,|F|^{\frac{1}{q}}\Big\{\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\leq|I_s|}}|J_k|^3\sum_{m\geq\frac{\log|J_k|}{n}}2^{-2mn}\sum_{\substack{s\in T\\ |I_s|=2^{mn}}}\bigg(\frac{d(k,s)}{2^m}\bigg)^{-\gamma}\Big\}^{\frac12}\\
&\leq C_\gamma\,|F|^{\frac{1}{q}}\Big\{\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\leq|I_s|}}|J_k|^3\sum_{m\geq\frac{\log|J_k|}{n}}2^{-2mn}\Big\}^{\frac12}\\
&\leq C_\gamma\,|F|^{\frac{1}{q}}\Big\{\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\leq|I_s|}}|J_k|^3|J_k|^{-2}\Big\}^{\frac12}\\
&\leq C_\gamma\,|F|^{\frac{1}{q}}\,|I_t|^{\frac12}\,.
\end{aligned}
$$

There is also the subcase of case (b) in which $|J_k|\geq|I_s|$. Here we have the two special subcases: $I_s\cap 3J_k=\emptyset$ and $I_s\subseteq 3J_k=\emptyset$. We begin with the first of these special subcases in which we use estimate (15). We have

$$
\bigg(\sum_{s\in T}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\langle b_k,\phi_s\rangle\Big|^2\bigg)^{\frac12} \tag{18}
$$

$$
\leq C_\gamma\,|F|^{\frac{1}{q}}\bigg(\sum_{s\in T}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}|I_s|^{\frac12}\frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^{\gamma}}\Big|^2\bigg)^{\frac12}
$$

$$
\leq C_\gamma\,|F|^{\frac{1}{q}}\bigg(\sum_{s\in T}\bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|I_s|^2}{|J_k|}\frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^{\gamma}}\bigg]\bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|J_k|}{|I_s|}\frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^{\gamma}}\bigg]\bigg)^{\frac12}. \tag{19}
$$

Since $I_s \cap 3J_k = \emptyset$ we have that $d(k,s) \approx |x - c(I_s)|$ for every $x \in J_k$. Therefore the second term inside square brackets above satisfies

$$\sum_{\substack{k:\ J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \cap 3J_k = \emptyset}} \frac{|J_k|}{|I_s|} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^\gamma} \leq \sum_k \int_{J_k} \Big(\frac{|x - c(I_s)|}{|I_s|^{\frac{1}{n}}} \Big)^{-\gamma} \frac{dx}{|I_s|} \leq C_\gamma .$$

Putting this estimate into (19), we obtain

$$C_\gamma |F|^{\frac{1}{q}} \Bigg(\sum_{s\in T} \sum_{\substack{k:\ J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \cap 3J_k = \emptyset}} \frac{|I_s|^2}{|J_k|} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^\gamma} \Bigg)^{\frac{1}{2}}$$

$$\leq C_\gamma |F|^{\frac{1}{q}} \Bigg(\sum_{s\in T} |I_s| \sum_{\substack{k:\ J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \subseteq 3J_k}} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^\gamma} \Bigg)^{\frac{1}{2}}$$

$$\leq C_\gamma |F|^{\frac{1}{q}} \Bigg(\sum_{k: J_k \subseteq 3I_t} |J_k| \sum_{m=0}^{\infty} 2^{-mn} \sum_{\substack{s:\ I_s \subseteq 3J_k \\ J_k \cap 3I_s = \emptyset \\ |I_s| = 2^{-mn}|J_k|}} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^\gamma} \Bigg)^{\frac{1}{2}} .$$

Since the last sum above is at most a constant (18) satisfies the estimate $C_\gamma \, |F|^{\frac{1}{q}} \, |I_t|^{\frac{1}{2}}$.

Finally there is the subcase of case (b) in which $|J_k| \geq |I_s|$ and $I_s \subseteq 3J_k = \emptyset$. Here again we use estimate (15). We have

$$\Bigg(\sum_{s\in T} \Big| \sum_{\substack{k:\ J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \subseteq 3J_k}} \langle b_k, \phi_s \rangle \Big|^2 \Bigg)^{\frac{1}{2}} \leq C_\gamma \, |F|^{\frac{1}{q}} \Bigg(\sum_{s\in T} |I_s| \Big| \sum_{\substack{k:\ J_k \subseteq 3I_t \\ J_k \cap 3I_s = \emptyset \\ |J_k| > |I_s| \\ I_s \subseteq 3J_k}} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^\gamma} \Big|^2 \Bigg)^{\frac{1}{2}} . \tag{20}$$

Let us make some observations. For a fixed s there exists at most finitely many J_k's contained in $3I_t$ with size at least $|I_s|$. Consider the following sets for $\alpha \in \{0, 1, 2, \dots\}$,

$$\mathcal{J}^\alpha := \{ J_k \text{ as in the sum above} : 2^\alpha |I_s|^{\frac{1}{n}} \leq d(k,s) < 2^{\alpha+1} |I_s|^{\frac{1}{n}} \}.$$

We would like to know that for all α the cardinality of $\mathcal{J}^\alpha$ is bounded by a fixed constant depending only on dimension. This would allow us to work with a single cube $J^\alpha(s)$ from each set at the cost of a constant in the sum below. Fix $\alpha \in \{0, 1, 2, \dots\}$ and note that $I_s \subseteq 3J_k$ and $d(k,s) > 2^\alpha |I_s|^{\frac{1}{n}}$ implies that $|J_k| > 2^{\alpha n} |I_s|$. It is clear that the cardinality of $\mathcal{J}^\alpha$ would be largest if we had $|J_k| = 2^{\alpha+1}|I_s|$ for all $J_k \in \mathcal{J}^\alpha$. Then the cube of size $7^n 2^{\alpha n} |I_s|$ centered at I_s would contain all elements of $\mathcal{J}_k$. This bounds the number of such elements by $\left(\frac{7}{2}\right)^n$.

Then using the Cauchy-Schwarz inequality we obtain

$$\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\subseteq 3J_k}} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^\gamma}\Big|^2 \le \Big(\frac{7}{2}\Big)^{2n}\Big|\sum_{\alpha=1}^{\infty}\frac{|I_s|^{\frac{\gamma}{2n}}}{\operatorname{dist}(J^\alpha(s),I_s)^{\frac{\gamma}{2}}}\frac{1}{2^{\frac{\alpha\gamma}{2}}}\Big|^2$$

$$\le C_n\sum_{\alpha=1}^{\infty}\frac{|I_s|^{\frac{\gamma}{n}}}{\operatorname{dist}(J^\alpha(s),I_s)^{\gamma}}$$

$$\le C_n\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\subseteq 3J_k}} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k,s)^\gamma}$$

Putting this estimate into the right hand side of (20), the estimate $C_{n,\gamma}\,|F|^{\frac{1}{q}}\,|I_t|^{\frac{1}{2}}$ now follows as in the previous case. This concludes the proof of Lemma 1.

6. APPLICATIONS

We conclude by discussing some applications. We show how one can strengthen the results of the previous sections to obtain distributional estimates for the function $\mathcal{D}_r(\chi_F)$ similar to those in the paper of Sjölin [10].

We showed in section 4 that for any measurable set E there is a set E' of at least half the measure of E such that

$$\Big|\int_{E'}\mathcal{D}_r(\chi_F)\,dx\Big| \le C\,\min(|E|,|F|)\Big(1+\Big|\log\frac{|F|}{|E|}\Big|\Big) \tag{21}$$

for some constant C depending only on the dimension. For $\lambda>0$ we define

$$E_\lambda = \big\{|\mathcal{D}_r(\chi_F)|>\lambda\big\}$$

and also

$$E_\lambda^1 = \big\{\mathrm{Re}\ \mathcal{D}_r(\chi_F)>\lambda\big\} \qquad E_\lambda^2 = \big\{\mathrm{Re}\ \mathcal{D}_r(\chi_F)<-\lambda\big\}$$
$$E_\lambda^3 = \big\{\mathrm{Im}\ \mathcal{D}_r(\chi_F)>\lambda\big\} \qquad E_\lambda^4 = \big\{\mathrm{Im}\ \mathcal{D}_r(\chi_F)<-\lambda\big\}.$$

We apply (21) to each set E_λ^j to obtain

$$\lambda\,|E_\lambda^j| \le \min(|E_\lambda^j|,|F|)\Big(1+\Big|\log\frac{|F|}{|E_\lambda^j|}\Big|\Big).$$

Using this fact in combination with the easy observation that for $a>1$

$$\frac{a}{\log a}\le\frac{1}{\lambda} \implies a\le\frac{10}{\lambda}\log(\frac{1}{\lambda}),$$

to obtain that

$$|E_\lambda^j|\le C'|F|\begin{cases}\frac{1}{\lambda}\log(\frac{1}{\lambda}) & \text{when } \lambda<\frac12\\ e^{-c\lambda} & \text{when } \lambda\ge\frac12.\end{cases}$$

Since $|E_{2\sqrt{2}\,\lambda}|\le\sum_{j=1}^4|E_\lambda^j|$ we conclude a similar estimate for E_λ.

Next we obtain similar distributional estimates for maximally modulated singular integrals $\mathcal{M}$ such as the maximally modulated Hilbert transform (i.e. Carleson's operator) or the maximally modulated Riesz transforms

$$\mathcal{M}(f)(x) = \sup_{\xi\in\mathbf{R}^n} \left| \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} e^{2\pi i \xi\cdot y} f(y)\, dy \right|.$$

To achieve this in the one dimensional setting, one applies an averaging argument similar to that in [6] to both terms of estimate (21) to recover a similar estimate with the Carleson operator. For more general homogeneous singular integrals with sufficiently smooth kernels, one applies the averaging argument to suitable modifications of the operators $\mathcal{D}_r$ as in [8]. Then one obtains a version of estimate (21) in which $\mathcal{D}_r(\chi_F)$ is replaced by $\mathcal{M}(\chi_F)$. The same procedure as above then yields the distributional estimate

$$\left|\left\{|\mathcal{M}(\chi_F)| > \lambda\right\}\right| \le C_n' |F| \begin{cases} \frac{1}{\lambda}\log(\frac{1}{\lambda}) & \text{when } \lambda < \frac{1}{2} \\ e^{-c\lambda} & \text{when } \lambda \ge \frac{1}{2}. \end{cases}$$

which recovers Lemma 1.2 in [10]. It should be noted that the corresponding estimate

$$\left|\left\{|\mathcal{D}_r(\chi_F)| > \lambda\right\}\right| \le C_n |F| \begin{cases} \frac{1}{\lambda}\log(\frac{1}{\lambda}) & \text{when } \lambda < \frac{1}{2} \\ e^{-c\lambda} & \text{when } \lambda \ge \frac{1}{2}. \end{cases} \tag{22}$$

obtained here for $\mathcal{D}_r$ is stronger as it concerns an "unaveraged version" of all the aforementioned maximally modulated singular integrals $\mathcal{M}$.

Using the idea employed in Sjölin [9] we can obtain the following result as a consequence of (22). Let B be a ball in $\mathbf{R}^n$.

Proposition 1. *(i) If $\int_B |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)|\, dx < \infty$, then $\mathcal{D}_r(f)$ is finite a.e. on B.*
(ii) If $\int_B |f(x)| (\log^+ |f(x)|)^2\, dx < \infty$, then $\mathcal{D}_r(f)$ is integrable over B.
(iii) For all $\lambda > 0$ we have

$$\left|\left\{x \in \mathbf{R}^n :\ |\mathcal{D}_r(f)(x)| > \lambda\right\}\right| \le C\, e^{-c\lambda/\|f\|_{L^\infty}}$$

where C, c only depend on the dimension (in particular they are independent of the measurable function $N : \mathbf{R}^n \to \mathbf{R}^n$.)

References

[1] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157.

[2] C. Fefferman, *Pointwise convergence of Fourier series*, Ann. of Math. **98** (1973), 551–571.

[3] R. Hunt, *On the convergence of Fourier series*, Orthogonal Expansions and their continuous analogues, (Edwardsville, IL 1967), D. T. Haimo (ed), Southern Illinois Univ. Press, Carbondale IL, 1968, 235–255.

[4] C. Kenig and P. Tomas, *Maximal operators defined by Fourier multipliers*, Studia Math. **68** (1980), 79–83.

[5] M. Lacey and C. Thiele, *Convergence of Fourier series*, preprint.

[6] M. Lacey and C. Thiele, *A proof of boundedness of the Carleson operator*, Math. Res. Lett. **7** (2000), 361–370.

[7] M. Lacey and C. Thiele, *On Calderón's conjecture*, Ann. of Math. **149** (1999), 475–496.

[8] M. Pramanik and E. Terwilleger, *A weak L^2 estimate for a maximal dyadic sum operator on* $\mathbf{R}^n$, Ill. J. Math., to appear.
[9] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh-Fourier series*, Ark. Matematik **7** (1968), 551–570.
[10] P. Sjölin, *Convergence almost everywhere of certain singular integrals and multiple Fourier series*, Ark. Matematik **9** (1971), 65–90.
[11] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton NJ, 1970.
[12] E. M. Stein and G. Weiss, *An extension of theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. **8** (1959), 263–284.

Loukas Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: loukas@math.missouri.edu

Terence Tao, Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90024, USA
E-mail address: tao@math.ucla.edu

Erin Terwilleger, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 USA
E-mail address: erin@math.gatech.edu