THE MARCINKIEWICZ MULTIPLIER CONDITION FOR BILINEAR OPERATORS

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ABSTRACT. This article is concerned with the question of whether Marcinkiewicz multipliers on $\mathbb{R}^{2n}$ give rise to bilinear multipliers on $\mathbb{R}^n \times \mathbb{R}^n$. We show that this is not always the case. Moreover we find necessary and sufficient conditions for such bilinear multipliers to be bounded. These conditions in particular imply that a slight logarithmic modification of the Marcinkiewicz condition gives multipliers for which the corresponding bilinear operators are bounded on products of Lebesgue and Hardy spaces.

1. INTRODUCTION

In this article we study bilinear multipliers of Marcinkiewicz type. Recall that a function $\sigma(\xi,\eta) = \sigma(\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n)$ defined away from the coordinate axes on $\mathbb{R}^{2n}$, which satisfies the conditions

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta} |\xi_1|^{-\alpha_1} \dots |\xi_n|^{-\alpha_n} |\eta_1|^{-\beta_1} \dots |\eta_n|^{-\beta_n} \tag{1.1}$$

for sufficiently large multi-indices $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$, is called a Marcinkiewicz multiplier. It is a classical result, see for instance [18], that Marcinkiewicz multipliers give rise to bounded linear operators M_σ from $L_p(\mathbb{R}^{2n})$ into itself for $1 < p < \infty$. Here M_σ is the multiplier operator with symbol σ, that is

$$M_\sigma(F)(x) = \int_{\mathbb{R}^{2n}} \widehat{F}(\xi)\sigma(\xi)e^{2\pi i \langle x,\xi\rangle} d\xi,$$

where F is a Schwartz function on $\mathbb{R}^{2n}$ and $\widehat{F}(\xi)$ is the Fourier transform of F, defined by $\widehat{F}(\xi) = \int_{\mathbb{R}^{2n}} F(x)e^{-2\pi i\langle x,\xi\rangle}dx$. (We will use the notation $\langle x,y\rangle = \sum_{k=1}^m x_k y_k$ for $x = (x_1,\dots,x_m)$ and $y = (y_1,\dots,y_m)$ elements of $\mathbb{R}^m$.) The Marcinkiewicz condition (1.1) is less restrictive than the Hörmander-Mihlin condition

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}, \tag{1.2}$$

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which is also known to imply boundedness for the linear operator W_σ from $L_p(\mathbb{R}^{2n})$ into itself when $1 < p < \infty$. The advantage of condition (1.2) is that it is supposed to hold for multi-indices up to order $|\alpha| + |\beta| \le n+1$ versus up to order $|\alpha| + |\beta| \le 2n$ for condition (1.1).

In this paper we study bilinear multiplier operators whose symbols satisfy similar conditions. More precisely, we are interested in boundedness properties of bilinear operators

$$W_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\sigma(\xi,\eta)e^{2\pi i\langle x,\xi\rangle}e^{2\pi i\langle x,\eta\rangle}\,d\xi\,d\eta,$$

originally defined for f, g Schwartz functions on $\mathbb{R}^n$ and σ a function on $\mathbb{R}^{2n}$. A well-known theorem of Coifman and Meyer [4] says that if the function σ on $\mathbb{R}^{2n}$ satisfies (1.2) for sufficiently large multi-indices α and β, then the bilinear map $W_\sigma(f,g)$ extends to a bounded operator from $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n)$ into $L_{p_0,\infty}(\mathbb{R}^n)$ when $1 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p_0$ and $p_0 \ge 1$. ($L_{p_0,\infty}$ here denotes the space weak L_{p_0}.) This result was later extended to the range $1 > p_0 \ge 1/2$ by Grafakos and Torres [9] and Kenig and Stein [11]. The extension into L_{p_0} for $p_0 < 1$ was stimulated by the recent work of Lacey and Thiele [12] who showed that the discontinuous symbol $\sigma(\xi,\eta) = -i\mathrm{sgn}\ (\xi-\eta)$ on $\mathbb{R}^2$ gives rise to a bounded bilinear operator W_σ from $L_{p_1}(\mathbb{R}) \times L_{p_2}(\mathbb{R})$ into $L_{p_0}(\mathbb{R})$ for $2/3 < p_0 < \infty$ when $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p_0$.

In this article we address the question of whether the Marcinkiewicz condition (1.1) on $\mathbb{R}^{2n}$ gives rise to a bounded bilinear operator W_σ on $\mathbb{R}^n \times \mathbb{R}^n$. We answer this question negatively. More precisely, we show that there exist examples of bounded functions $\sigma(\xi,\eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ which satisfy the stronger condition

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}|\xi|^{-|\alpha|}|\eta|^{-|\beta|} \tag{1.3}$$

for all multi-indices α and β, for which the corresponding bilinear operators W_σ do not map $L_{p_1} \times L_{p_2}$ into $L_{p_0,\infty}$ for any triple of exponents satisfying $1/p_1 + 1/p_2 = 1/p_0$ and $1 < p_1, p_2 < \infty$.

We reduce this problem to the study of bilinear operators of the type

$$(f,g) \to \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\,\widetilde{\Delta}_j f\,\widetilde{\Delta}_k g, \tag{1.4}$$

where a_{jk} is a bounded sequence of scalars depending on σ and $\widetilde{\Delta}_j$ are the Littlewood-Paley operators given by multiplication on the Fourier transform side by a smooth bump supported near the frequency $|\xi| \sim 2^j$. In section 6, in particular Theorem 6.5, we find a necessary and sufficient condition on the infinite matrix $A = (a_{jk})_{j,k}$ so that the bilinear operator in (1.4) maps $L_{p_1} \times L_{p_2}$ into $L_{p_0,\infty}$. This condition is expressed in terms of an Orlicz space norm of the sequence $(a_{jk})_{j,k}$. It turns out that this condition is independent of the exponents p_1, p_2, p_0 and depends only on quantities intrinsic to the matrix A, (although the actual norm of the operator in (1.4) from $L_{p_1} \times L_{p_2}$ into $L_{p_0,\infty}$ does depend on the indices p_1, p_2, p_0).

The results of section 6 are transferred to multiplier theorems for bilinear operators in section 7. This transference is achieved using a Fourier expansion of the symbol σ on products of dyadic cubes. Theorem 7.2 is the main result of this section and Theorem 7.3 shows that this theorem is best possible. Theorem 7.2 allows us to derive that the estimates

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta} |\xi|^{-|\alpha|} |\eta|^{-|\beta|} \big(\log(1 + |\log \tfrac{|\xi|}{|\eta|}|)\big)^{-\theta} \tag{1.5}$$

do give rise to a bounded bilinear operator W_σ on products of L_p spaces when $\theta > 1$, while we show that this is not the case when $0 < \theta < \frac{1}{2}$. We obtain similar results when the expression $\big(\log(1 + |\log \frac{|\xi|}{|\eta|}|)\big)^{-\theta}$ in (1.5) is replaced by the expression $\big(\log(1 + |\log \frac{|\xi|}{|\eta|}|)\big)^{-1}\big(\log(1 + \log(1 + |\log \frac{|\xi|}{|\eta|}|))\big)^{-\theta}$ for $\theta > 1$.

We find more convenient to work with the martingale difference operators Δ_k associated with the $\sigma-$algebra of all dyadic cubes of size 2^k in $\mathbb{R}^n$ and later transfer our results to the Littlewood-Paley operators $\widetilde{\Delta}_k$. This point of view is introduced in the next section.

We end this article with a short discussion on paraproducts, see section 8. These are operators of the type (1.4) for specific sequences $(a_{jk})_{j,k}$ of zeros and ones.

2. A maximal operator

Let $(\Omega, \Sigma, \mathbb{P})$ be any probability space and let $(\Sigma_k)_{k\ge 0}$ be a *filtration* i.e. an increasing sequence of sub-$\sigma-$algebras of Σ. We say that (Σ_k) is a *dyadic filtration* if each Σ_k is atomic and has precisely 2^k atoms each with probability 2^{-k}. We say (Σ_k) is a 2^n-*adic filtration* if each Σ_k is atomic with precisely 2^{nk} atoms each with probability 2^{-nk}.

Associated to Σ_k we define the conditional expectation operators $\mathcal{E}_k f = \mathbb{E}(f|\Sigma_k)$ and the martingale difference operators $\Delta_k f = \mathcal{E}_k f - \mathcal{E}_{k-1} f$ for $k \ge 1$, and $f \in L_1(\Omega)$.

Let $A = (a_{jk})$ be a complex $M \times N$ matrix, and let $(\Omega, \Sigma, \mathbb{P})$ be a probability space with a dyadic filtration $(\Sigma_k)_{k\ge 0}$. For $1 \le p < \infty$ we define $h_p(A)$ to be the least constant so that for all $f \in L_p(\Omega)$ we have

$$\big\| \max_{1\le j\le M} |\sum_{k=1}^N a_{jk} \Delta_k f| \big\|_{L_p} \le h_p(A) \|f\|_{L_p}. \tag{2.1}$$

We also define the corresponding weak constants, i.e. the least constants so that for all $f \in L_p(\Omega)$ we have

$$\big\| \max_{1\le j\le M} |\sum_{k=1}^N a_{jk}\Delta_k f|\big\|_{L_{p,\infty}} \le h_p^w(A)\|f\|_{L_p}. \tag{2.2}$$

Finally for $0<q<p<\infty$ we define the mixed constants $h_{p,q}(A)$ as the least constants such that for all $f\in L_p(\Omega)$ we have

$$\big\| \max_{1\le j\le M} |\sum_{k=1}^N a_{jk}\Delta_k f|\big\|_{L_q} \le h_{p,q}(A)\|f\|_{L_p}. \tag{2.3}$$

Note that these definitions are independent of the choice of the probability space and of the dyadic filtration. Indeed if A is fixed, it suffices to take $f \in L_p(\Sigma_N)$ and hence we can consider a finite probability space with 2^N points and a finite dyadic filtration $(\Sigma_k)_{k=0}^N$. We also note that $h_p(A)$ is the operator norm of the map $T_A : L_p(\Omega) \to L_p(\Omega;\ell_\infty^M)$ defined by

$$T_A f = \big(\sum_{k=1}^N a_{jk}\Delta_k f\big)_{j=1}^M.$$

Similarly $h_p^w(A)$ is the norm of the operator $T_A : L_p \to L_{p,\infty}(\Omega;\ell_\infty^M)$.

Our first result is that all these constants are mutually equivalent, when $1<p<\infty$:

Theorem 2.1. *If $1<p,q<\infty$ then there is a constant $0<C=C(p,q)<\infty$ such that for all complex $M\times N$ matrices A we have*

$$\frac{1}{C}h_p(A) \le h_q^w(A) \le h_q(A) \le Ch_p(A).$$

Proof. It suffices to prove an estimate of the type $h_p(A) \le Ch_q^w(A)$ for any choice of $1<p,q<\infty$. We first prove a weak type $(1,1)$ estimate for T_A, i.e. that $h_1^w(A) \le Ch_q^w(A)$. Suppose $f\in L_1$ with $\|f\|_{L_1}=1$. Then if $\lambda,\gamma>0$, with $\lambda\gamma>1$, we can use an appropriate Calderón-Zygmund decomposition to find finite sets $D_1,\cdots,D_m$ so that each D_l is an atom of some Σ_l,

$$\gamma\lambda \le \mathbb{P}(D_l)^{-1}\int_{D_l}|f|\,d\mathbb{P} = \operatorname*{Ave}_{D_l} f \le 2\gamma\lambda,$$

and $|f(\omega)|\le\gamma\lambda$ if $\omega\notin\cup_{l=1}^m D_l$. Let

$$g=\sum_{l=1}^m (\operatorname*{Ave}_{D_l} f)\chi_{D_l}$$

and $E=\cup_{l=1}^m D_l$. Then $T_A(f\chi_E-g)$ is supported in E and thus

$$\mathbb{P}(\|T_A(f\chi_E-g)\|_{\ell_\infty^M} > \lambda/2) \le \mathbb{P}(E) \le (\gamma\lambda)^{-1}. \tag{2.4}$$

On the other hand $\|f - f\chi_E + g\|_{L_\infty} \le 3\gamma\lambda$ and $\|f - f\chi_E + g\|_{L_1} \le 1$. Hence $\|f - f\chi_E + g\|_{L_q} \le 3^{1/q'}(\gamma\lambda)^{1/q'}$ and so

$$\|T_A(f - f\chi_E + g)\|_{L_{q,\infty}(\ell_\infty^M)} \le h_q^w(A)3^{1/q'}(\gamma\lambda)^{1/q'}, \tag{2.5}$$

which implies that

$$\mathbb{P}(\|T_A(f - f\chi_E + g)\|_{\ell_\infty^M} > \lambda/2) \le \frac{2^q}{\lambda^q}(h_q^w(A))^q 3^{q-1}(\gamma\lambda)^{q-1}. \tag{2.6}$$

Selecting $\gamma = 1/h_q^w(A)$ and combining with (2.4) we obtain (for $\lambda > h_q^w(A)$)

$$\lambda\,\mathbb{P}(\|T_A f\|_{\ell_\infty^M} > \lambda) \le C h_q^w(A) \tag{2.7}$$

where $C = C(p,q)$. This gives the weak-type (1,1) estimate for T_A. Now by the Marcinkiewicz interpolation theorem (applied to the sublinear map $f \mapsto \|T_A f(\omega)\|_{\ell_\infty^M}$) we obtain that $h_p(A) \le C(p,q)h_q^w(A)$ as long as $1 < p < q$.

We now prove that $h_p(A) \le C(p,q)h_q^w(A)$ when $1 < q < p < \infty$. We consider the dual map $T_A^* : L_1(\Omega;\ell_1^M) \to L_1$ defined by

$$T_A^*\mathbf{f} = \sum_{j=1}^M \sum_{k=1}^N a_{jk}\Delta_k f_j$$

where $\mathbf{f}(\omega) = (f_j(\omega))_{j=1}^M$. We have that $T_A^* : L_r(\Omega;\ell_1^M) \to L_r$ has norm bounded by $C(q,r)h_q^w(A)$ as long as $1 < r' < q$ i.e. $q' < r < \infty$. Using this r as a starting point, we repeat the argument above to show that $T_A^* : L_1(\Omega;\ell_1^M) \to L_{1,\infty}$ has norm bounded by $Ch_q^w(A)$. The Marcinkiewicz interpolation theorem can again be used to show that $T_A^* : L_{p'}(\Omega,\ell_1^M) \to L_{p'}$ has norm bounded by $Ch_q^w(A)$ for all $1 < p' < r$, and thus in particular when $1 < p' < q'$. Therefore we obtain that $h_p(A) \le Ch_q^w(A)$ when $1 < q < p < \infty$. □

Remark. From now we will write $h(A) = h_2(A)$ so that each $h_p(A)$ for $1 < p < \infty$ is equivalent to $h(A)$.

It is of some interest to observe that even the corresponding mixed constants are also equivalent to $h(A)$.

Theorem 2.2. *Suppose $0 < q < p$ and $1 < p < \infty$. Then there is a constant $C = C(p,q)$ so that*

$$\frac{1}{C}h(A) \le h_{p,q}(A) \le Ch(A).$$

Proof. This will depend on the following Lemma:

Lemma 2.3. *Suppose $1 \le p < \infty$ and $0 < q < p$. Then there is a constant $C = C(p,q)$ so that if $r = \min(p,2)$ we have*

$$\|T_A\|_{L_p \to L_{r,\infty}(\ell_\infty^M)} \le Ch_{p,q}(A). \tag{2.8}$$

Proof. (Lemma 2.3) We may assume $q<r$. This is a fairly standard application of Nikishin's theorem, see [16]. Here we use a version given in [17]. It is simplest to consider the case when Ω is finite with $|\Omega|=2^N$. Consider the map $T_A: L_p\to L_q(\Omega;\ell_\infty^M)$. For each $f\in L_p$ with $\|f\|_{L_p}\le 1$, let $F_f(x)=\|T_Af(x)\|_{\ell_\infty^M}$. For $\|f_j\|_{L_p}\le 1$ with $1\le j\le J$, $\sum_{j=1}^J|b_j|^r=1$, and $(\epsilon_j)_{j=1}^J$ a sequence of independent Bernoulli random variables on some probability space, we have

$$\big\|\max_{1\le j\le J}|b_j|F_{f_j}\big\|_{L_q}\le \mathbb{E}\Big(\big\|\sum_{j=1}^J\epsilon_jb_jT_Af_j\big\|_{L_q(\ell_\infty^M)}\Big)\le Ch_{p,q}(A),$$

since L_p has type r. It follows from [17] that there is a function $w\in L_1$, with $\int w\,d\mathbb{P}=1$, and $w\ge 0$ a.e such that for any set $E\subset\Omega$

$$\Big(\int_E F_f^q d\mathbb{P}\Big)^{\frac1q}\le Ch_{p,q}(A)\Big(\int_E w\,d\mathbb{P}\Big)^{\frac1q-\frac1r}.$$

Now consider the set S of all permutations of Ω which induce permutations of the atoms of each Σ_k for $1\le k\le N$; there are 2^{2^N-1} such permutations φ. For $\varphi\in S$ we have

$$\Big(\int_E F_{f\circ\varphi}^q\,d\mathbb{P}\Big)^{\frac1q}\le Ch_{p,q}(A)\Big(\int_E w\,d\mathbb{P}\Big)^{\frac1q-\frac1r}$$

or equivalently

$$\Big(\int_E F_f^q\,d\mathbb{P}\Big)^{\frac1q}\le Ch_{p,q}(A)\Big(\int_E w\circ\varphi^{-1}\,d\mathbb{P}\Big)^{\frac1q-\frac1r}.$$

Raising to the power $(\frac1q-\frac1r)^{-1}$, averaging over S, and then raising to the power $\frac1q-\frac1r$ gives

$$\Big(\int_E F_f^q d\mathbb{P}\Big)^{\frac1q}\le Ch_{p,q}(A)\Big(\frac{1}{|S|}\sum_{\varphi\in S}\Big(\int_E w\circ\varphi^{-1}\,d\mathbb{P}\Big)\Big)^{\frac1q-\frac1r}.$$

But this implies

$$\Big(\int_E F_f^q d\mathbb{P}\Big)^{\frac1q}\le Ch_{p,q}(A)\,\mathbb{P}(E)^{\frac1q-\frac1r}$$

which gives the required weak type estimate (2.8). □

We now return to the proof of Theorem 2.2. We first observe that we always have $h_{p,q}(A)\le Ch_p^w(A)$ since $q<p$. If $1<p\le 2$, Lemma 2.3 gives that $h_p^w(A)\le Ch_{p,q}(A)$ and the required conclusion follows from Theorem 2.1. Assume therefore that $p>2$ and that T_A maps $L_p\to L_q(\ell_\infty^M)$ with norm $h_{p,q}(A)$. Fix f with $\|f\|_{L_1}=1$ and use the Calderón-Zygmund decomposition of Theorem 2.1, to obtain (2.4) as before, but instead of (2.5) the estimate

$$\|T_A(f-f\chi_E+g)\|_{L_q}\le h_{p,q}(A)3^{1/p'}(\gamma\lambda)^{1/p'}, \tag{2.9}$$

which implies

$$\mathbb{P}(\|T_A(f - f\chi_E + g)\|_{\ell_\infty^M} > \lambda/2) \le \frac{2^q}{\lambda^q}(h_{p,q}(A))^q 3^{q/p'}(\gamma\lambda)^{q/p'}. \tag{2.10}$$

Selecting $\gamma = h_{p,q}(A)^{-s}\lambda^{s-1}$ with $\frac{1}{s} = \frac{1}{p'} + \frac{1}{q}$ and combining with (2.4) we obtain

$$\lambda\,\mathbb{P}(\|T_A f\|_{\ell_\infty^M} > \lambda)^{\frac{1}{s}} \le Ch_{p,q}(A). \tag{2.11}$$

This says that T_A maps L_1 into $L_{s,\infty}(\ell_\infty^M)$ with norm at most $Ch_{p,q}(A)$, in particular that T_A maps L_1 into $L_t(\ell_\infty^M)$ as long as $0 < t < s$. Lemma 2.3 gives that T_A maps L_p into $L_{2,\infty}(\ell_\infty^M)$ and also L_1 into $L_{2,\infty}(\ell_\infty^M)$ with norms at most a multiple of $h_{p,q}(A)$. By interpolation it follows that T_A maps L_r into $L_{2,\infty}(\ell_\infty^M) \subset L_{r,\infty}(\ell_\infty^M)$ for $1 \le r \le 2$. We conclude that $h_r^w(A) \le Ch_{p,q}(A)$ for $1 < r < 2$ but since $h_r^w(A)$ is comparable to $h_p^w(A)$, we finally obtain $h_p^w(A) \le Ch_{p,q}(A)$. Since the converse inequality is always valid when $q < p$, we apply Theorem 2.1 to conclude the proof. □

We next prove the elementary observation for $1 < p < \infty$, that $h(A)$ remains unchanged when interpolating extra columns or extra rows of zeros.

Lemma 2.4. *Let A be a complex $M \times N$ matrix and $(m_r)_{r=1}^M, (n_s)_{s=1}^N$ be two increasing finite sequences of natural numbers. Suppose $M_1 \ge m_M$ and $N_1 \ge n_N$. Let $B = (b_{jk})$ be the $M_1 \times N_1$-matrix defined by $b_{jk} = a_{rs}$ when $j = m_r$ and $k = n_s$, and $b_{jk} = 0$ otherwise. Then $h(A) = h(B)$.*

Proof. Interpolating extra rows of zeros is trivial, so we can assume $m_r = m$ for all r. For the case of columns, we only need to show that $h(B) \le h(A)$. We may suppose that Ω is a finite set with 2^{N_1} points and that $(\Sigma_k)_{k=0}^{N_1}$ is a finite dyadic filtration of Ω. It is then possible to write $\Omega = \Omega_1 \times \Omega_2$ where $|\Omega_1| = 2^{N_1-N}$ and $|\Omega_2| = 2^N$, and find a dyadic filtration $(\Sigma_k^{(1)})_{k=0}^{N_1-N}$ of Ω_1 and a dyadic filtration $(\Sigma_k^{(2)})_{k=0}^N$ of Ω_2 so that $\Sigma_k^{(1)} \times \Sigma_k^{(2)} = \Sigma_{n_k}$, for $0 \le k \le N$ and $\Sigma_{k+1}^{(1)} \times \Sigma_k^{(2)} = \Sigma_{n_{k+1}-1}$ for $0 \le k \le N-1$. Then for $f \in L_2(\Omega_1 \times \Omega_2)$ let $g = \sum_{k=1}^N \Delta_k f$ and note that

$$\Delta_{n_k} f(\omega_1, \omega_2) = \Delta_k^{(2)} g_{\omega_1}(\omega_2),$$

where $g_{\omega_1}(\omega_2) = g(\omega_1, \omega_2)$. Hence

$$\int_{\Omega_2} \sup_j \Big|\sum_{k=1}^N a_{j,n_k}\Delta_{n_k} f(\omega_1,\omega_2)\Big|^2 d\omega_2 \le h_p(A)\int_{\Omega_2} |g(\omega_1,\omega_2)|^2 d\omega_2.$$

Integrating over Ω_1 gives

$$\big\|\sup_j |\sum_{k=1}^N a_{j,n_k}\Delta_{n_k} f|\big\|_{L_2} \le h_p(A)\|g\|_{L_2} \le h_p(A)\|f\|_{L_2}.$$

This completes the proof. ∎

We can now extend our definitions, replacing dyadic filtrations by 2^n-adic filtrations:

Proposition 2.5. *Suppose $n \in \mathbb{N}$ and $1 < p < \infty$. Then there is a constant $C(p,n)$ with the following property. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and suppose $(\Sigma_k)_{k=0}^{\infty}$ is a 2^n-adic filtration. Let A be any $M \times N$ matrix and let $h_p(A;n)$ be the least constant so that*

$$\big\| \sup_j |\sum_{k=1}^{N} a_{jk}\Delta_k f| \big\|_{L_p} \le h_p(A;n)\|f\|_{L_p},$$

and $h_p^w(A;n)$ be the least constant so that

$$\big\| \sup_j |\sum_{k=1}^{N} a_{jk}\Delta_k f| \big\|_{L_{p,\infty}} \le h_p^w(A;n)\|f\|_{L_p}.$$

Then $h_p^w(A) \le h_p^w(A;n)$, $h_p(A) \le h_p(A;n)$, and $h_p^w(A;n) \le h_p(A;n) \le Ch(A)$.

Proof. This is essentially trivial; we need only to prove that $h_p(A;n) \le Ch(A)$. To do this note that $h_p(A;n) = h_p(B)$ where B is obtained from A by repeating each column n times. The proposition follows then by the triangle law from Lemma 2.4. ∎

3. Estimates for $h(A)$

We next turn to the problem of estimating $h(A)$. We shall assume that $(\Omega, \mathbb{P})$ is a fixed probability space with a dyadic filtration $(\Sigma_k)_{k=0}^{\infty}$. Our first estimate is trivial.

Proposition 3.1. *There is a constant C so that for any $M \times N$ matrix $A = (a_{jk})$ we have*

$$h(A) \le C \sup_{1\le j\le M} \sum_{k=0}^{N} |a_{jk} - a_{j,k+1}|,$$

where we set $a_{j0} = a_{j,N+1} = 0$ for all $1 \le j \le M$.

Proof. Suppose $f \in L_2$. Summation by parts gives

$$\sum_{k=1}^{N} a_{jk}\Delta_k f = \sum_{k=0}^{N} (a_{jk} - a_{j,k+1})\mathcal{E}_k f,$$

thus

$$|\sum_{k=1}^{N} a_{jk}\Delta_k f| \le (\sup_{1\le j\le M} \sum_{k=0}^{N} |a_{jk} - a_{j,k+1}|) \sup_k |\mathcal{E}_k f|,$$

and the result follows because of the maximal estimate

$$\|\sup_k |\mathcal{E}_k f|\|_{L_2} \le C\|f\|_{L_2},$$

proved in [8]. □

We next turn to the problem of getting a more delicate estimate. To do this we need the theory of a certain Lorentz space. Let $w = (w_k)_{k=1}^\infty$ be a decreasing sequence of positive real numbers. We will consider the following two conditions on w :

$$\exists C > 0,\ \exists \theta > 0, \qquad w_k \le C\Big(\frac{\log(j+1)}{\log(k+1)}\Big)^\theta w_j \quad \text{when} \quad 1 \le j \le k, \tag{3.1}$$

(where throughout this paper log denotes the logarithm with base 2) and

$$\sum_{k=1}^\infty \frac{w_k}{k} < \infty. \tag{3.2}$$

Roughly speaking (3.1) means that w_k decays logarithmically while (3.2) implies that it decays reasonably fast. Note that $w_k = (\log(k+1))^{-\theta}$ satisfies (3.1) if $\theta > 0$ and (3.2) if $\theta > 1$. The sequence $w_k = (\log(k+1))^{-1}(\log\log(k+2))^{-\theta}$ satisfies both (3.1) and (3.2) when $\theta > 1$.

Now let $d = d(w,1)$ be the Lorentz sequence space of all complex sequences $\mathbf{u} = (u_k)_{k\in\mathbb{Z}}$ such that

$$\|\mathbf{u}\|_d = \sup_\pi \sum_{k\in\mathbb{Z}} w_{\pi(k)}|u_k| < \infty$$

where the supremum is taken over all one-one maps $\pi : \mathbb{Z} \to \mathbb{N}$. The dual of $d(w,1)$ can be naturally identified as the space $d^* = d^*(w,1)$ consisting of all sequences $(v_k)_{k\in\mathbb{Z}}$ so that

$$\sup_{k\in\mathbb{N}} \frac{v_1^* + \cdots + v_k^*}{w_1 + \cdots + w_k} = \|\mathbf{v}\|_{d^*} < \infty$$

where $(v_k^*)_{k=1}^\infty$ is the decreasing rearrangement of $(|v_k|)_{k\in\mathbb{Z}}$. We refer to [13] p. 175 for properties of Lorentz spaces. Note that under condition (3.1), $d(w,1)$ is also an Orlicz sequence space (see [13] p. 176).

The following Lemma is surely well-known to specialists, but we include a proof.

Lemma 3.2. *Under condition (3.1), the Lorentz space $d(w,1)$ has cotype two.*

Proof. By combining Proposition 1.f.3 (p.82) and Theorem 1.f.7 (p.84) of [14] one sees that it is only necessary to show that $d(w,1)$ has a lower q-estimate for some $q < 2$. To do this observe that if $\mathbf{v}_1, \cdots, \mathbf{v}_N$ are disjointly supported sequences, then

$$\|\sum_{j=1}^N \mathbf{v}_j\|_d \geq \inf_{k\geq 1} \frac{w_k}{w_{kN}} \sum_{j=1}^N \|\mathbf{v}_j\|_d.$$

Hence

$$\sum_{j=1}^N \|\mathbf{v}_j\|_{d(w,1)} \leq C(\log(N+1))^\theta \|\sum_{j=1}^N \mathbf{v}_j\|_d.$$

Now suppose $1 < q < 2$ and $\|\sum \mathbf{v}_j\|_d = 1$. Then for each $s \in \mathbb{N}$, let m_s be the number of j so that $2^{-s} < \|\mathbf{v}_k\|_d \leq 2^{-s+1}$. Then

$$m_s 2^{-s} \leq C(\log(m_s+1))^\theta.$$

This in turn implies that

$$m_s^{1-\rho} \leq C2^s$$

where $\rho > 0$ is chosen so that $(1-\rho)^{-1} < q$. Then we obtain an estimate

$$\sum_{j=1}^N \|\mathbf{v}_j\|_d^q \leq C \sum_{s=1}^\infty m_s 2^{-sq} \leq C'.$$

This establishes a lower q-estimate. □

The norms $\|\cdot\|_d$ and $\|\cdot\|_{d^*}$ are of course defined for finite sequences with M elements and thus can be thought as norms on $\mathbb{C}^M$. We denote these spaces $d(w,1)^{(M)}$ and $d^*(w,1)^{(M)}$.

Proposition 3.3. *If (w_n) satisfies both (3.1) and (3.2) then given $2 < p < \infty$ there is a constant C so that for any sequence $\epsilon_k = \pm 1$ and any $M, N \in \mathbb{N}$ we have the estimate*

$$\Big(\mathbb{E}\big(\|\sum_{k=1}^N \epsilon_k \Delta_k \mathbf{f}\|_{\ell_\infty}^2\big)\Big)^{\frac{1}{2}} \leq C\big(\mathbb{E}\big(\|\mathbf{f}\|_{d^*}^p\big)\big)^{\frac{1}{p}},$$

for any $\mathbf{f} \in L_p(\Omega; d^*(w,1)^{(M)})$.

Proof. We start by using an argument due to Muckenhoupt [15], see also [20]. For any fixed $\epsilon_1, \cdots, \epsilon_N$ let $S = \sum_{k=1}^N \epsilon_k \Delta_k$. Now fix $f \in L_\infty$. Then by a result of Burkholder [2], $\|S\|_{L_p \to L_p} = p-1$ if $2 \leq p < \infty$. Then for any $\alpha > 0$ we have

$$\mathbb{E}(\cosh(\alpha|Sf|)) \leq 1 + \sum_{m=1}^\infty \frac{\alpha^{2m}}{(2m)!}(2m-1)^{2m}\|f\|_{L_{2m}}^{2m}. \tag{3.3}$$

Since $\|f\|_{L_{2m}}^{2m} \leq \|f\|_{L_2}^2 \|f\|_{L_\infty}^{2m-2}$ and since for $m \geq 1$ we have

$$\frac{(2m-1)^{2m}}{(2m)!} \leq \frac{(2m)^{2m}}{(2m)!} \leq e^{2m},$$

it follows from (3.3) that

$$\mathbb{E}(\cosh(\alpha|Sf|)-1) \le (\alpha e)^2\|f\|_{L_2}^2 \sum_{k=0}^{\infty}(\alpha e)^{2k}\|f\|_{L_\infty}^{2k}.$$

In particular if $\alpha e\|f\|_\infty \le \frac{1}{2}$ we have

$$\mathbb{E}(\cosh(\alpha|Sf|)-1) \le 2e^2\alpha^2\|f\|_{L_2}^2. \tag{3.4}$$

At this point we return to the Lorentz space $d(w,1)$. Let us define $\gamma_0 = 0$, $\gamma_1 = 1$, and $\gamma_k = 2^{2^{k-2}}$ for $k \ge 2$. Let $W_k = w_{\gamma_k}$. It will be convenient to normalize condition (3.2) so that we have

$$\sum_{k=1}^{\infty} \gamma_k W_k = 1. \tag{3.5}$$

We also note that (3.1) implies the existence of a constant C so that we have

$$|w_1 + \cdots + w_k| \le Ckw_k \tag{3.6}$$

for $k \ge 1$.

Now suppose $\mathbf{f} = (f_j)_{j=1}^M \in L_\infty(\Omega;\mathbb{C}^M)$. Suppose that $\mathbf{f}$ is supported on a measurable set E and satisfies $\|\mathbf{f}(\omega)\|_{d^*} \le 1$ everywhere. Then we can define a measurable map π from Ω into the set of permutations of $\{1,2,\cdots,M\}$ so that $|f_{\pi(\omega)(1)}(\omega)| \ge |f_{\pi(\omega)(2)}(\omega)| \ge \cdots \ge |f_{\pi(\omega)(M)}(\omega)|$ for all $\omega \in \Omega$. Thus

$$|f_{\pi(\omega)(j)}(\omega)| \le Cw_j$$

for all $1 \le j \le M$. Let $E_{jk} = \{\omega \in E : \ \pi(\omega)(k) = j\}$ when $j,k \in \{1,\dots,M\}$ and $E_{jk} = \emptyset$ otherwise. Now for $1 \le j \le M$ and $l = 1,2,3,\dots,$ let

$$f_j^{(l)} = \sum_{k=\gamma_{l-1}}^{\gamma_l - 1} f_j \chi_{E_{jk}}$$

so that $f_j = \sum_{l=1}^{\infty} f_j^{(l)}$. If $0 < \alpha e \le \frac{1}{2C}$ we can estimate

$$\begin{aligned}
\mathbb{E}(\cosh(\alpha|Sf_j|)-1) &= \mathbb{E}\Big(\cosh\big(\Big|\sum_{l=1}^{\infty}\alpha S f_j^{(l)}\Big|\big)-1\Big) \\
&\le \mathbb{E}\Big(\max_{l\ge 1}\big(\cosh(\alpha\gamma_l^{-1}W_l^{-1}|Sf_j^{(l)}|)-1\big)\Big) \\
&\le e^2\alpha^2\sum_{l=1}^{\infty}\gamma_l^{-2}W_l^{-2}\|f_j^{(l)}\|_{L_2}^2,
\end{aligned}$$

in view of (3.4) since $\|f_j^{(l)}\|_{L_\infty} \le CW_l$ and $\alpha\gamma_l^{-1}W_l^{-1}\|f_j^{(l)}\|_{L_\infty} \le \frac{1}{2}$. Thus

$$\mathbb{E}(\cosh(\alpha|Sf_j|)-1) \le e^2C^2\alpha^2\sum_{l=1}^{\infty}\gamma_l^{-2}\sum_{k=\gamma_{l-1}}^{\gamma_l-1}\mathbb{P}(E_{jk}).$$

It follows that

$$\mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty})-1) \le e^2C^2\alpha^2\sum_{j=1}^{M}\sum_{l=1}^{\infty}\gamma_l^{-2}\sum_{k=\gamma_{l-1}}^{\gamma_l-1}\mathbb{P}(E_{jk}).$$

Note that for each $k \in \mathbb{N}$, $\sum_{j=1}^{M}\mathbb{P}(E_{jk}) \le \mathbb{P}(E)$. Hence we obtain that if $\mathbf{f}$ is supported on E with $\|\mathbf{f}(\omega)\|_{d^*} \le 1$ everywhere and $\alpha e < \frac{1}{2C}$, then

$$\mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty})-1) \le e^2C^2\alpha^2\sum_{l=1}^{\infty}\gamma_l^{-1}\mathbb{P}(E) = C_1\alpha^2\mathbb{P}(E) \tag{3.7}$$

for a suitable constant C_1. Let us next refine (3.7). For $n \ge 0$, let

$$G_n = \{\omega \in E:\ 4^{-n-1} < \|(\omega)\|_{d^*} \le 4^{-n}\}.$$

Then by (3.7) we have if $\alpha < (4Ce)^{-1}$

$$\mathbb{E}(\cosh(2^{n+1}\alpha\|S(\mathbf{f}\chi_{G_n})\|_{\ell_\infty})-1) \le C_1\alpha^2 4^{-n}\mathbb{P}(G_n)$$

and as

$$\mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty})-1) \le \mathbb{E}\Big(\sup_{n\ge 0}\big(\cosh(2^{n+1}\alpha\|S(\mathbf{f}\chi_{G_n})\|_{\ell_\infty})-1\big)\Big),$$

we obtain, under the assumptions $\|\mathbf{f}(\omega)\|_{d^*}\le 1$ everywhere and $\alpha<(4C)^{-1}$,

$$\mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty})-1) \le C_1\alpha^2\sum_{n=0}^{\infty}4^{-n}\mathbb{P}(G_n) \le C_2\mathbb{E}(\|\mathbf{f}\|_{d^*}). \tag{3.8}$$

If we use a fixed value of α and the estimate $x^2 \le 2(\cosh x - 1)$ we find that

$$\mathbb{E}(\|S\mathbf{f}\|_{\ell_\infty}^2) \le C_3\mathbb{E}(\|\mathbf{f}\|_{d^*})$$

if $\|\|\mathbf{f}\|_{d^*}\|_\infty \le 1$. This in turn gives us for every $\mathbf{f} \in L_\infty(\Omega; d^*(w,1)^{(M)})$

$$\mathbb{E}(\|S\mathbf{f}\|_{\ell_\infty}^2) \le C_3\|\|\mathbf{f}\|_{d^*}\|_\infty\,\mathbb{E}(\|\mathbf{f}\|_{d^*}). \tag{3.9}$$

Now let $2 < p < \infty$ and fix $\mathbf{f}$ with $\mathbb{E}(\|\mathbf{f}\|_{d^*}^p) = 1$. We set $E_0 = \{\|\mathbf{f}\|_{d^*} \le 1\}$ and $E_n = \{2^{n-1} < \|\mathbf{f}\|_{d^*} \le 2^n\}$ for $n \ge 1$. Applying (3.9) we obtain

$$\begin{aligned}(\mathbb{E}(\|S\mathbf{f}\|_{\ell_\infty}^2))^{\frac{1}{2}} &\le \Big(C_3\sum_{n=0}^{\infty}2^n\mathbb{P}(E_n)\mathbb{E}(\|\mathbf{f}\|_{d^*})\Big)^{\frac{1}{2}} \le C_3^{\frac{1}{2}}\sum_{n=0}^{\infty}2^{\frac{n}{2}}\mathbb{P}(E_n)^{\frac{1}{2}}\\ &\le C_3^{\frac{1}{2}}\Big(\sum_{n=0}^{\infty}2^{(2-p)n}\Big)^{\frac{1}{2}}\Big(\sum_{n=0}^{\infty}2^{np}\mathbb{P}(E_n)\Big)^{\frac{1}{2}} \le C_4,\end{aligned}$$

which completes the proof under the assumption $\mathbb{E}(\|\mathbf{f}\|_{d^*}^p) = 1$. The general case follows by scaling. □

We now establish our main estimate for $h(A)$.

Theorem 3.4. *Let $w = (w_n)_{n=1}^\infty$ be a sequence satisfying (3.1) and (3.2). Then there is a constant C so that for any $M \times N$ matrix $A = (a_{kj})_{j,k}$ we have*

$$h(A) \le C \max_{1 \le k \le N} \|\mathbf{a}_k\|_{d^*}$$

where $\mathbf{a}_k = (a_{kj})_{j=1}^M$. In particular we have

$$h(A) \le C \max_{j,k} \frac{|a_{jk}|}{w_{|j-k|+1}}.$$

Proof. We suppose $p > 2$ and that A is a matrix satisfying $\max_{1 \le k \le N} \|\mathbf{a}_k\|_{d^*} \le 1$. Consider the operator $T_A : L_p(\Omega) \to L_2(\Omega; \ell_\infty^M)$. The adjoint operator is $T_A^* :\ L_2(\Omega; \ell_1^M) \to L_{p'}(\Omega)$ given by

$$T_A^*(\mathbf{f}) = \sum_{k=1}^N \langle \Delta_k \mathbf{f}, \mathbf{a}_k \rangle.$$

The dual statement of the result in Proposition 3.3 gives that for any sequence of ± 1's, $\epsilon_1, \cdots, \epsilon_N$ we have the estimate

$$\big(\mathbb{E}\big(\| \sum_{k=1}^N \epsilon_k \Delta_k \mathbf{f} \|_d^{p'}\big)\big)^{\frac{1}{p'}} \le C(\mathbb{E}(\|\mathbf{f}\|_{\ell_1}^2))^{\frac{1}{2}} \tag{3.10}$$

where C depends only on (w_n). Now let $\epsilon_1, \cdots \epsilon_N$ be a sequence of independent Bernoulli random variables on some probability space $(\Omega', \mathbb{P}')$. We use $\mathbb{E}'$ to denote expectations on Ω'. Using Lemma 3.2 we obtain

$$\begin{aligned}
(\mathbb{E}(\|T_A^* \mathbf{f}\|_d^{p'}))^{\frac{1}{p'}} &\le C_0 \big(\mathbb{E}\big(\sum_{k=1}^N |\langle \Delta_k \mathbf{f}, \mathbf{a}_k \rangle|^2\big)^{\frac{p'}{2}}\big)^{\frac{1}{p'}} \\
&\le C_0 \big(\mathbb{E}\big(\sum_{k=1}^N \|\Delta_k \mathbf{f}\|_d^2\big)^{\frac{p'}{2}}\big)^{\frac{1}{p'}} \\
&\le C_1 \big(\mathbb{E}\mathbb{E}'\big(\| \sum_{k=1}^N \epsilon_k \Delta_k \mathbf{f}\|_d^{p'}\big)\big)^{\frac{1}{p'}} \\
&\le C_2 (\mathbb{E}\|\mathbf{f}\|_{\ell_1}^2)^{\frac{1}{2}}.
\end{aligned}$$

This gives $h_{p,2}(A) \le C_2$ which completes the proof by using Theorem 2.2. □

Remark. Theorem 3.4 implies that given any $\theta > 1$ there is a constant C_θ so that

$$h(A) \le C_\theta \tag{3.11}$$

whenever $A = (a_{kj})_{j,k}$ is a matrix satisfying

$$|a_{jk}| \le 2(\log(2 + |j - k|))^{-\theta}. \tag{3.12}$$

We show that this is not the case when $0 < \theta < \frac{1}{2}$. Let N be any natural number and define $A = (a_{jk})$ to be a $2^N \times N$ matrix given by $a_{jk} = b_{jk}N^{-\theta}$, where $b_{jk} = \pm 1$ and the set $(b_{jk})_{j=1}^{2^N}$ runs through all 2^N choices of signs. Choose f real so that $|\Delta_k f| = 1$ for $1 \le k \le N$. Then $\|f\|_{L_2} = \sqrt{N}$. On the other hand

$$\max_{1\le j\le 2^N} |\sum_{k=1}^N a_{jk}\Delta_k f| = N^{1-\theta}\,\chi_\Omega,$$

which implies that $h(A) \ge N^{\frac{1}{2}-\theta}$. However

$$|a_{jk}| \le N^{-\theta} \le 2(N+1)^{-\theta} \le 2(\log(2 + |j-k|))^{-\theta}$$

but $h(A) \ge N^{\frac{1}{2}-\theta} \to \infty$ as $N \to \infty$. Thus (3.11) fails when $0 < \theta < \frac{1}{2}$.

4. The harmonic version of the maximal operator

We shall now fix $n \in \mathbb{N}$ and work with $\mathbb{R}^n$. Let $\mathcal{D}_0$ be the collection of all unit cubes of the form $\prod_{j=1}^n [m_j, m_j + 1]$ where $m_j \in \mathbb{Z}$ and let $\mathcal{D}_k$ be the set of all cubes of the form $\prod_{j=1}^n [2^{-k}m_j, 2^{-k}(m_j + 1)]$ where $m_j \in \mathbb{Z}$. For $k \in \mathbb{Z}$, let Σ_k denote the $\sigma-$algebra generated by the dyadic cubes $\mathcal{D}_k$. We define the corresponding conditional expectation operators

$$\mathcal{E}_k f = \sum_{Q\in\mathcal{D}_k} (\operatorname*{Ave}_Q f)\chi_Q$$

for $f \in L_1^{loc}(\mathbb{R}^n)$ and the martingale difference operators $\Delta_k f = \mathcal{E}_k f - \mathcal{E}_{k-1} f$ for $k \in \mathbb{Z}$.

Now let $A = (a_{jk})_{j,k\in\mathbb{Z}}$ be any infinite complex matrix. We shall call A a $c_{00}-$matrix if it has only finitely many non-zero entries. For a $c_{00}-$matrix define $h_p[A; n]$ to be the least constant such that for all $f \in L_p(\mathbb{R}^n)$ we have

$$\|\max_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} a_{jk}\Delta_k f|\|_{L_p} \le h_p[A; n]\|f\|_{L_p}. \tag{4.1}$$

Also let $h_p^w[A; n]$ be the corresponding weak-type constant, i.e. the least constant such that for all $f \in L_p(\mathbb{R}^n)$ we have

$$\|\max_{j\in\mathbb{Z}} |\sum_{k=1}^N a_{jk}\Delta_k f|\|_{L_{p,\infty}} \le h_p^w[A; n]\|f\|_{L_p}. \tag{4.2}$$

The following Lemma is easily verified and we omit its proof.

Lemma 4.1. *Let $h_p^w(A;n)$ and $h_p(A;n)$ be as in Proposition 2.5. For any $1<p<\infty$ and any infinite c_{00}-matrix A we have $h_p[A;n]=h_p(B;n)$ and $h_p^w[A;n]=h_p^w(B;n)$, where B is any $M\times N$ matrix of the form $b_{jk}=a_{j+r,k+s}$ for some $r,s\in\mathbb{Z}$ such that $a_{j+r,k+s}=0$ unless $1\le j\le M$ and $1\le k\le N$.*

Now for any infinite matrix A we define

$$h(A)=\sup_N h\big((a_{j-N,k-N})_{1\le j\le 2N}^{1\le k\le 2N}\big).$$

The following is an immediate consequence of Lemma 4.1 and Proposition 2.5.

Corollary 4.2. *For any $1<p<\infty$ and any $n\in\mathbb{N}$ there is a constant $C=C(p,N)$ so that for any infinite c_{00}-matrix we have*

$$C^{-1}h(A)\le h_p^w[A;n]\le h_p[A;n]\le Ch(A).$$

We now turn to the harmonic model of the maximal operator studied in section 2. Let $\mathcal{S}(\mathbb{R}^n)$ denote the set of all Schwartz functions on $\mathbb{R}^n$ and for $f\in\mathcal{S}(\mathbb{R}^n)$ let

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle\xi,x\rangle}dx$$

denote the Fourier transform of f. We will denote by $f^\vee(\xi)=\widehat{f}(-\xi)$ the inverse Fourier transform of f. We shall fix a radial function $\psi\in\mathcal{S}(\mathbb{R}^n)$ whose Fourier transform is real-valued and satisfies $\widehat{\psi}(\xi)=1$ for $|\xi|\le 1$ and $\widehat{\psi}(\xi)=0$ for $|\xi|\ge 2$. We define a Schwartz function ϕ by setting $\widehat{\phi}(\xi)=\widehat{\psi}(\xi)-\widehat{\psi}(2\xi)$. Then $\widehat{\phi}$ is supported in the annulus $2^{-1}\le|\xi|\le 2$. We then define $\psi_j(x)=2^{nj}\psi(2^jx)$ and $\phi_j(x)=2^{nj}\phi(2^jx)$ for $j\in\mathbb{Z}$. Note that $\widehat{\phi_j}(\xi)=\widehat{\phi}(2^{-j}\xi)$ is supported in the annulus $2^{j-1}\le|\xi|\le 2^{j+1}$. We also define operators

$$\widetilde{S}_jf=\psi_j*f\quad\text{and}\quad\widetilde{\Delta}_jf=\phi_j*f$$

for $f\in L_1+L_\infty$. The $\widetilde{\Delta}_j$'s are called the Littlewood-Paley operators. Now if $A=(a_{jk})_{(j,k)\in\mathbb{Z}^2}$ is an infinite c_{00}-matrix and $1<p<\infty$, we let $\widetilde{h}_p(A)$ be the least constant so that for all $f\in L_p$ we have

$$\big\|\sup_{j\in\mathbb{Z}}\big|\sum_{k\in\mathbb{Z}}a_{jk}\widetilde{\Delta}_kf\big|\big\|_{L_p}\le\widetilde{h}_p(A)\|f\|_{L_p}. \tag{4.3}$$

We also define $\widetilde{h}_p^w(A)$ to be the least constant such that for all $f\in L_p$ we have

$$\big\|\sup_{j\in\mathbb{Z}}\big|\sum_{k\in\mathbb{Z}}a_{jk}\widetilde{\Delta}_kf\big|\big\|_{L_{p,\infty}}\le\widetilde{h}_p^w(A)\|f\|_{L_p}. \tag{4.4}$$

We now have the following.

Lemma 4.3. *Suppose $r \in \mathbb{Z}$. Then if $1 < p < \infty$ and $A = (a_{jk})$ is any infinite c_{00}-matrix, then $\widetilde{h}_p(A) = \widetilde{h}_p(B)$ and $\widetilde{h}_p^w(A) = \widetilde{h}_p^w(B)$, where $B = (b_{jk})$ and $b_{jk} = a_{j,k+r}$.*

Proof. Consider the dilation operator $D_r f(x) = f(2^{-r}x)$. Then $D_r^{-1}\widetilde{\Delta}_k D_r f = \widetilde{\Delta}_{k-r} f$ and we have

$$\begin{aligned}&\big\|\sup_j \big|\sum_k a_{j,k+r}\widetilde{\Delta}_k f\big|\big\|_{L_p} = \big\|\sup_j \big|\sum_k a_{jk}\widetilde{\Delta}_{k-r} f\big|\big\|_{L_p}\\ =&2^{-rn/p}\big\|\sup_j \big|\sum_k a_{jk}\widetilde{\Delta}_k D_r f\big|\big\|_{L_p} \le 2^{-rn/p} h_p(A)\|D_r f\|_{L_p} = h_p(A)\|f\|_{L_p},\end{aligned}$$

which implies $\widetilde{h}_p(B) \le \widetilde{h}_p(A)$. Likewise we obtain $\widetilde{h}_p(A) \le \widetilde{h}_p(B)$. The corresponding result for the weak type constants follows similarly. □

Next we prove that the Littlewood-Paley operators $\widetilde{\Delta}_j$ and the martingale difference operators Δ_k are essentially orthogonal on L_2 when $k \neq j$.

Proposition 4.4. *There exists a constant C so that for every k, j in $\mathbb{Z}$ we have the following estimate on the operator norm of $\Delta_j\widetilde{\Delta}_k : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$*

$$\|\Delta_k\widetilde{\Delta}_j\|_{L_2\to L_2} \le C2^{-\frac{9}{20}|j-k|}. \tag{4.5}$$

Proof. By a simple dilation argument it suffices to prove (4.5) when $k = 0$. In this case we have the estimate

$$\begin{aligned}&\|\Delta_0\widetilde{\Delta}_j\|_{L_2\to L_2} = \|\mathcal{E}_0\widetilde{\Delta}_j - \mathcal{E}_{-1}\widetilde{\Delta}_j\|_{L_2\to L_2}\\ \le&\|\mathcal{E}_0\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2\to L_2} + \|\mathcal{E}_{-1}\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2\to L_2}\end{aligned}$$

and also by the self-adjointness of the Δ_k's and $\widetilde{\Delta}_j$'s we have

$$\begin{aligned}&\|\Delta_0\widetilde{\Delta}_j\|_{L_2\to L_2} = \|\widetilde{\Delta}_j\Delta_0\|_{L_2\to L_2} = \|\widetilde{\Delta}_j\mathcal{E}_0 - \widetilde{\Delta}_j\mathcal{E}_{-1}\|_{L_2\to L_2}\\ \le&\|\widetilde{\Delta_j\mathcal{E}_0}\|_{L_2\to L_2} + \|\widetilde{\Delta}_j\mathcal{E}_{-1}\|_{L_2\to L_2}.\end{aligned}$$

The required estimate (4.5) (when $k = 0$) will be a consequence of the pair of inequalities

$$\|\mathcal{E}_0\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2\to L_2} + \|\mathcal{E}_{-1}\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2\to L_2} \le C2^j \qquad \text{when } j \le 0, \tag{4.6}$$

$$\|\widetilde{\Delta_j\mathcal{E}_0}\|_{L_2\to L_2} + \|\widetilde{\Delta}_j\mathcal{E}_{-1}\|_{L_2\to L_2} \le C2^{-\frac{1}{2}j} \qquad \text{when } j \ge 0. \tag{4.7}$$

We start by proving (4.6). We only consider the term $\mathcal{E}_0\widetilde{\Delta}_j - \widetilde{\Delta}_j$ since the term $\mathcal{E}_{-1}\widetilde{\Delta}_j - \widetilde{\Delta}_j$ is similar. Let $f \in L_2(\mathbb{R}^n)$. Then

$$\|\mathcal{E}_0\widetilde{\Delta}_j f - \widetilde{\Delta}_j f\|_{L_2}^2 = \sum_{Q\in\mathcal{D}_0} \|f * \phi_j - \operatorname*{Ave}_Q(f*\phi_j)\|_{L_2(Q)}^2$$

$$\le \sum_{Q\in\mathcal{D}_0} \int_Q\int_Q |(f*\phi_j)(x) - (f*\phi_j)(t)|^2\, dt\, dx$$

$$\le \sum_{Q\in\mathcal{D}_0} \int_Q\int_Q \Big(\int_{3Q} |f(y)||\phi_j(x-y)|\, dy\Big)^2\, dt\, dx$$

$$+ \sum_{Q\in\mathcal{D}_0} \int_Q\int_Q \Big(\int_{3Q} |f(y)||\phi_j(t-y)|\, dy\Big)^2\, dt\, dx$$

$$+ \sum_{Q\in\mathcal{D}_0} \int_Q\int_Q \Big(\int_{(3Q)^c} |f(y)|2^{jn+j}|\nabla\phi(2^j(\xi_{x,t}-y))|\, dy\Big)^2\, dt\, dx,$$

where $\xi_{x,t}$ lies on the line segment between x and t. It is now easy to see that the sum of the last three expressions above is bounded by

$$C2^{2jn}\sum_{Q\in\mathcal{D}_0}\int_{3Q}|f(y)|^2\,dy + C_M 2^{2j}\sum_{Q\in\mathcal{D}_0}\int_Q\Big(\int_{\mathbb{R}^n}\frac{2^{jn}|f(y)|\,dy}{(1+2^j|x-y|)^M}\Big)^2 dx$$

which is clearly controlled by $C2^{2j}\|f\|_{L_2}^2$. This estimate is useful when $j \le 0$.

We now turn to the proof of (4.7). We work only with the term $\widetilde{\Delta_j}\mathcal{E}_0$ as the other one is similar.

$$\big\|\widetilde{\Delta_j}\mathcal{E}_0(f)\big\|_{L^2}^2 = \Big\|\sum_{Q\in\mathcal{D}_0}(\operatorname*{Ave}_Q f)\,(\psi_{2^{-j}}*\chi_Q)\Big\|_{L_2}^2$$

$$\le 2\Big\|\sum_{Q\in\mathcal{D}_0}(\operatorname*{Ave}_Q f)\,(\psi_{2^{-j}}*\chi_Q)\chi_{5\sqrt{n}Q}\Big\|_{L_2}^2$$

$$+ 2\Big\|\sum_{Q\in\mathcal{D}_0}(\operatorname*{Ave}_Q f)\,(\psi_{2^{-j}}*\chi_Q)\chi_{(5\sqrt{n}Q)^c}\Big\|_{L_2}^2.$$

Since the functions appearing inside the sum in the first term above have supports with bounded overlap we obtain

$$\Big\|\sum_{Q\in\mathcal{D}_0}(\operatorname*{Ave}_Q f)(\psi_{2^{-j}}*\chi_Q)\chi_{5\sqrt{n}Q}\Big\|_{L_2}^2 \le C\sum_{Q\in\mathcal{D}_0}(\operatorname*{Ave}_Q|f|)^2\big\|\psi_{2^{-j}}*\chi_Q\big\|_{L_2}^2,$$

and the crucial observation is that for any $Q \in \mathcal{D}_0$ we have

$$\big\|\psi_{2^{-j}}*\chi_Q\big\|_{L^2}^2 \le C'\sum_{r=1}^n\int_{|\xi_r|\approx 2^j}\frac{|e^{2\pi i\xi_r}-1|^2}{|2\pi i\xi_r|^2}d\xi_r\Big[\prod_{l\ne r}\int\frac{|e^{2\pi i\xi_l}-1|^2}{|2\pi i\xi_l|^2}d\xi_l\Big] \le C\,2^{-\frac{9}{10}j},$$

which can be obtained by Plancherel's identity and of the fact that in the region where ξ_r is the largest variable of $\xi = (\xi_1, \dots, \xi_n)$ we have $|\xi_r| \approx |\xi| \approx 2^j$ on the support of $\widehat{\psi_{2^{-j}}}(\xi)$.

Putting these observations together, we deduce

$$\Big\| \sum_{Q\in\mathcal{D}_0} (\operatorname*{Ave}_Q f)\,(\psi_{2^{-j}} * \chi_Q)\chi_{3Q}\Big\|_{L_2}^2 \le C \sum_{Q\in\mathcal{D}_0} (\operatorname*{Ave}_Q |f|)^2 2^{-\frac{9}{10}j} \le C\, 2^{-\frac{9}{10}j}\big\|f\big\|_{L_2}^2,$$

and the required conclusion will be proved if we can show that

$$\Big\| \sum_{Q\in\mathcal{D}_0} (\operatorname*{Ave}_Q f)\,(\psi_j * \chi_Q)\chi_{(3Q)^c}\Big\|_{L_2}^2 \le C 2^{-2j}\|f\|_{L_2}^2. \tag{4.8}$$

We prove (4.8) by using a purely size estimate. Let c_Q be the center of the dyadic cube Q. For $x \notin 3Q$ we have the easy estimate

$$|(\psi_j * \chi_Q)(x)| \le \frac{C_M 2^{jn}}{(1+2^j|x-c_Q|)^M} \le \frac{C_M 2^{jn}}{(1+2^j)^{M/2}}\frac{1}{(1+|x-c_Q|)^{M/2}}$$

since both $2^j \ge 1, |x-c_Q| \ge 1$. We now control the left hand side of (4.8) by

$$\begin{aligned}
&2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0}\sum_{Q'\in\mathcal{D}_0} (\operatorname*{Ave}_Q |f|)(\operatorname*{Ave}_{Q'} |f|) \int_{\mathbb{R}^n} \frac{C_M\, dx}{(1+|x-c_Q|)^{\frac{M}{2}}(1+|x-c_{Q'}|)^{\frac{M}{2}}}\\
\le &2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0}\sum_{Q'\in\mathcal{D}_0} \frac{(\operatorname*{Ave}_Q |f|)(\operatorname*{Ave}_{Q'} |f|)}{(1+|c_Q-c_{Q'}|)^{\frac{M}{4}}} \int_{\mathbb{R}^n} \frac{C_M\, dx}{(1+|x-c_Q|)^{\frac{M}{4}}(1+|x-c_{Q'}|)^{\frac{M}{4}}}\\
\le &2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0}\sum_{Q'\in\mathcal{D}_0} \frac{C'_M}{(1+|c_Q-c_{Q'}|)^{\frac{M}{4}}}\Big(\int_Q |f(y)|^2\,dy + \int_{Q'} |f(y)|^2\,dy\Big)\\
\le &C''_M 2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0} \int_Q |f(y)|^2\,dy = C''_M 2^{j(2n-M)}\|f\|_{L_2}^2.
\end{aligned}$$

By taking M large enough we obtain (4.8) and thus (4.7). □

We have the following result relating $h(A)$ and $\widetilde{h}_p(A)$.

Theorem 4.5. *For every $1 < p < \infty$, there is a constant C depending only on ψ and p so that for any $c_{00}-$matrix A we have*

$$\widetilde{h}_p^w(A) \le \widetilde{h}_p(A) \le Ch(A).$$

Moreover, there are three bumps ϕ^1, ϕ^2, ϕ^3 whose Fourier transforms are supported in the annuli $1/2 \le |\xi| \le 4$ and which satisfy $\sum_k \widehat{\phi^i}(2^{-k}\xi) = 1$ when $\xi \neq 0$ for all $i = 1, 2, 3$ such that

$$h(A) \le C\big[\widetilde{h^1}_p^w(A) + \widetilde{h^2}_p^w(A) + \widetilde{h^3}_p^w(A)\big],$$

where $\widetilde{h^i}_p^w(A)$ is the constant $\widetilde{h}_p^w(A)$ associated with ϕ^i in place of ϕ.

Proof. Consider the operators $V_r,\ r \in \mathbb{Z}$ defined by

$$V_r = \sum_{j\in\mathbb{Z}} \Delta_j \widetilde{\Delta}_{j+r}.$$

Then

$$V_r V_r^* = \sum_{j,k} \Delta_j \widetilde{\Delta}_{j+r} \widetilde{\Delta}_{k+r} \Delta_k = \sum_{|j-k|\le 1} \Delta_j \widetilde{\Delta}_{j+r} \widetilde{\Delta}_{k+r} \Delta_k.$$

Hence by splitting into 3 pieces and using Proposition 4.4 we obtain the estimate

$$\|V_r\|_{L_2\to L_2} \le C2^{-|r|}.$$

Next pick q so that $1 < q < \infty$ and $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}$ where $0 < \theta < 1$. Let $(\epsilon_j)_{j\in\mathbb{Z}}$ be a sequence of independent Bernoulli random variables on some probability space $(\Omega, \mathbb{P})$. Then for $f \in L_q(\Omega)$ we have

$$V_r f = \int_\Omega \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \epsilon_j(\omega)\epsilon_{k-r}(\omega)\Delta_j \widetilde{\Delta}_k f \, d\mathbb{P}.$$

Averaging now gives

$$\|V_r f\|_{L_q} \le (\max_\omega \| \sum_{j\in\mathbb{Z}} \epsilon_j(\omega)\Delta_j\|_{L_q\to L_q})(\max_\omega \| \sum_{k\in\mathbb{Z}} \epsilon_{k-r}(\omega)\widetilde{\Delta}_k\|_{L_q\to L_q})\|f\|_{L_q}.$$

Hence $\|V_r\|_{L_q\to L_q} \le C$ where C depends only on q and ψ. Similarly $\|V_r^*\|_{L_q\to L_q} \le C$. By interpolation we obtain $\|V_r\|_{L_p\to L_p}, \|V_r^*\|_{L_p\to L_p} \le C2^{-|r|(1-\theta)}$.

Finally let us write

$$\sup_{j\in\mathbb{Z}} \big| \sum_{k\in\mathbb{Z}} a_{jk}\widetilde{\Delta}_k f\big| = \sup_{j\in\mathbb{Z}} \big| \sum_{k\in\mathbb{Z}} a_{jk} \sum_{r\in\mathbb{Z}} \Delta_{k-r}\widetilde{\Delta}_k f\big|$$
$$\le \sum_{r\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \big| \sum_{k\in\mathbb{Z}} a_{j,k+r}\Delta_k\widetilde{\Delta}_{k+r} f\big| = \sum_{r\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \big| \sum_{k\in\mathbb{Z}} a_{j,k+r}\Delta_k \big[\sum_{l} \Delta_l \widetilde{\Delta}_{l+r} f\big]\big|.$$

Thus by Proposition 2.5,

$$\| \sup_{j\in\mathbb{Z}} | \sum_{k\in\mathbb{Z}} a_{jk}\widetilde{\Delta}_k f|\|_{L_p} \le Ch(A) \sum_{r\in\mathbb{Z}} \|V_r f\|_{L_p} \le Ch_p(A)\|f\|_{L_p}.$$

This shows that $\widetilde{h}_p(A) \le Ch(A)$.

For $0 < \delta \le 1/10$, we let $\widetilde{\Delta}_k^{[a,b]}$ be a Littlewood-Paley operator associated with a smooth bump whose Fourier transform is supported in the annulus $a \le |\xi| \le b$. (Here $a > 0$.) If this Fourier transform is equal to 1 on the smaller annulus $c \le |\xi| \le d$, then we denote it by $\widetilde{\Delta}_{k,[c,d]}^{[a,b]}$.

Pick a Littlewood-Paley operator such that $\sum_r \widetilde{\Delta}_{r,[1,2-2\delta]}^{[1-\delta,2]} = I$. For the converse direction we use V_r^* and Lemma 4.3. We have

$$\sup_{j\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} a_{jk}\Delta_k f\Big| \le \sum_{r\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} a_{j,k+r} \widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]} \Delta_{k+r} f\Big|.$$

The definitions of such bumps give

$$\sum_k a_{j,k+r}\widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]}\Delta_{k+r} f = \sum_k a_{j,k+r}\widetilde{\Delta}_{k,[1-\delta,2]}^{[1-2\delta,2+\delta]}\widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]}\Delta_{k+r} f.$$

We will make use of the decomposition

$$\widetilde{\Delta}_{k,[1-\delta,2]}^{[1-2\delta,2+\delta]} = \widetilde{\Delta}_k^{[1-2\delta,1+\delta]} + \widetilde{\Delta}_k^{[1,2-2\delta]} + \widetilde{\Delta}_k^{[2-3\delta,2+\delta]}.$$

In view of the orthogonality property:

$$\widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]}\widetilde{\Delta}_l^{[1,2-2\delta]} = 0 \quad \text{when } l\neq k,$$

we have

$$\sum_k a_{j,k+r}\widetilde{\Delta}_k^{[1,2-2\delta]}\widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]}\Delta_{k+r} f = \sum_k a_{j,k+r}\widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]}\Big[\sum_l \widetilde{\Delta}_l^{[1,2-2\delta]}\Delta_{l+r} f\Big],$$

so the required conclusion for this term follows by the boundedness of V_r^*. To handle the part of the sum associated with $\widetilde{\Delta}_k^{[1-2\delta,1+\delta]}$ we set

$$\widetilde{\Delta}_k^{[1-2\delta,1+\delta]}\widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]} = \approx{\Delta}_k^{[1-2\delta,1+\delta]}$$

and we pick a Littlewood-Paley operator such that

$$\sum_k \widetilde{\Delta}_{k,[1+\delta,2-6\delta]}^{[1-3\delta,2+2\delta]} = I.$$

This satisfies the orthogonality property:

$$\widetilde{\Delta}_{k,[1+\delta,2-6\delta]}^{[1-3\delta,2+2\delta]}\approx{\Delta}_l^{[1-2\delta,1+\delta]} = \begin{cases} 0 & \text{when } l\neq k,\\ \approx{\Delta}_k^{[1-2\delta,1+\delta]} & \text{when } l=k,\end{cases}$$

and so

$$\sum_k a_{j,k+r}\approx{\Delta}_k^{[1-2\delta,1+\delta]}\Delta_{k+r} f = \sum_k a_{j,k+r}\widetilde{\Delta}_{k,[1+\delta,2-6\delta]}^{[1-3\delta,2+2\delta]}\Big[\sum_l \approx{\Delta}_l^{[1-2\delta,1+\delta]}\Delta_{l+r} f\Big],$$

so the required estimate for this term also follows by the boundedness of V_r^*.

Finally, for the part of the sum associated with $\widetilde{\Delta}_k^{[2-3\delta,2+\delta]}$ we set

$$\widetilde{\Delta}_k^{[2-3\delta,2+\delta]}\widetilde{\Delta}_{k,[1,2-2\delta]}^{[1-\delta,2]} = \approx{\Delta}_k^{[2-3\delta,2]},$$

and we pick a Littlewood-Paley operator such that

$$\sum_k \widetilde{\Delta}^{[1-\frac{3}{2}\delta,2+2\delta]}_{k,[1+\delta,2-3\delta]} = I.$$

This satisfies the orthogonality property:

$$\widetilde{\Delta}^{[1-\frac{3}{2}\delta,2+2\delta]}_{k,[1+\delta,2-3\delta]} \widetilde{\widetilde{\Delta}}^{[2-3\delta,2]}_l = \begin{cases} 0 & \text{when } l \neq k, \\ \widetilde{\widetilde{\Delta}}^{[2-3\delta,2]}_k & \text{when } l = k, \end{cases}$$

and so

$$\sum_k a_{j,k+r} \widetilde{\widetilde{\Delta}}^{[2-3\delta,2]}_k \Delta_{k+r} f = \sum_k a_{j,k+r} \widetilde{\Delta}^{[1-\frac{3}{2}\delta,2+2\delta]}_{k,[1+\delta,2-3\delta]} \big[\sum_l \widetilde{\widetilde{\Delta}}^{[1-2\delta,1+\delta]}_l \Delta_{l+r} f \big],$$

so the required assertion follows as before. Combining these estimates we obtain

$$\| \sup_{j\in\mathbb{Z}} | \sum_{k\in\mathbb{Z}} a_{jk} \Delta_k f | \|_{L_{p,\infty}} \leq C \big[\widetilde{h1}^w_p(A) + \widetilde{h2}^w_p(A) + \widetilde{h3}^w_p(A) \big] \sum_{r\in\mathbb{Z}} \|V_r^* f\|_{L_p}$$

which leads to the claim $h(A) \leq C \big[\widetilde{h1}^w_p(A) + \widetilde{h2}^w_p(A) + \widetilde{h3}^w_p(A) \big]$. □

We next extend the definition of $\widetilde{h}_p(A)$ to the case when $0 < p \leq 1$. For such p's we define $\widetilde{h}_p(A)$ to be the least constant so that for $f \in \mathcal{S}$ we have

$$\| \sup_{j\in\mathbb{Z}} | \sum_{k\in\mathbb{Z}} a_{jk} \widetilde{\Delta}_k f | \|_{L_p} \leq C \|f\|_{H_p}. \tag{4.9}$$

The space H_p that appears on the right of (4.9) when $0 < p \leq 1$ is the classical real Hardy space of Fefferman and Stein [7] and the expression $\| \ \|_{H_p}$ is its quasi-norm.

Theorem 4.6. *If $0 < p < 1$ then there is constant $C = C(p, \psi)$ so that $C^{-1} h(A) \leq \widetilde{h}_p(A) \leq C h(A)$.*

Proof. First we show the estimate $\widetilde{h}_p(A) \leq C h(A)$. Using the atomic characterization of H_p, [3], we note that it suffices to get an estimate for a function $f \in \mathcal{S}$ supported in a cube Q so that $|f(x)| \leq |Q|^{-\frac{1}{p}}$ for $x \in Q$ and $\int x^\alpha f(x) = 0$ if $|\alpha| \leq N = [n(\frac{1}{p} - 1)]$. It is then easy to see that if $x \notin 2Q$

$$| \sum_{k\in\mathbb{Z}} a_{jk} \widetilde{\Delta}_k f(x) | \leq C h(A) |x - c_Q|^{-n-N-1}$$

since $|a_{jk}| \leq C h(A)$ for each j, k. (Here $2Q$ is the cube with twice the length and the same center c_Q as usually.) This gives the estimate

$$\int_{\mathbb{R}^n \setminus 2Q} \sup_j | \sum_k a_{jk} \widetilde{\Delta}_k f(x) |^p dx \leq C^p h(A)^p.$$

On the other hand,

$$\int_{2Q} \sup_j \Big| \sum_k a_{jk} \widetilde{\Delta}_k f(x) \Big|^p dx \le C|Q|^{1-\frac{p}{2}} h(A)^p \bigg(\int_Q |f(x)|^2 dx \bigg)^{\frac{p}{2}}$$

and combining with the previous estimate we obtain $\widetilde{h}_p(A) \le Ch(A)$.

Complex interpolation gives that $\widetilde{h}_q(A) \le \widetilde{h}_2(A)^\theta \widetilde{h}_p(A)^{1-\theta}$ when $1 < q < 2$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. Since $\widetilde{h}_q(A) \ge C^{-1}h(A)$ we deduce the estimate $\widetilde{h}_p(A) \ge C^{-1}h(A)$. □

5. Bilinear operators

Let σ be a bounded measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ we define a bilinear operator $W_\sigma(f,g)$ with multiplier σ by setting

$$W_\sigma(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi,\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle x, \xi+\eta \rangle} \, d\xi d\eta. \tag{5.1}$$

If (5.1) is satisfied we say that σ is the bilinear symbol (or multiplier) of W_σ. Now suppose $1 < p_1, p_2 < \infty$ and let p_0 be defined by $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$. We say that W_σ is strongly $(p_1,p_2)-$bounded if W_σ extends to a bounded bilinear operator from $L_{p_1} \times L_{p_2} \to L_{p_0}$. In this case we denote its norm by $\|W_\sigma\|_{L_{p_1} \times L_{p_2} \to L_{p_0}}$ (we define this be expression to be ∞ if W_σ is not bounded). Similarly we say W_σ is weakly $(p_1,p_2)-$bounded if it extends to a bounded bilinear operator from $L_{p_1} \times L_{p_2} \to L_{p_0,\infty}$ and its norm is then denoted $\|W_\sigma\|_{L_{p_1} \times L_{p_2} \to L_{p_0,\infty}}$.

We extend these definitions to the case $0 < p_1, p_2 < \infty$ by replacing the L_p spaces by the corresponding Hardy spaces when $0 < p_j \le 1$. In the definition below we set $H_p = L_p$ for $1 < p < \infty$. Given $0 < p_1, < p_2 < \infty$ and p_0 defined by $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$, we say that W_σ is strongly $(p_1,p_2)-$bounded if it extends to a bounded bilinear operator from $H_{p_1} \times H_{p_2} \to L_{p_0}$, and we denote its norm by $\|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0}}$. We say that W_σ is weakly $(p_1,p_2)-$bounded if it extends to a bounded bilinear operator from $H_{p_1} \times H_{p_2} \to L_{p_0,\infty}$, and in this case we denote its norm by $\|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0,\infty}}$. Now for a bounded function σ on $\mathbb{R}^n \times \mathbb{R}^n$ and $0 < p_1, p_2 < \infty$ we define its corresponding strong and weak (p_1,p_2)-multiplier norm by

$$\|\sigma\|_{\mathcal{M}_{p_1,p_2}} = \|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0}} \quad \text{and} \quad \|\sigma\|_{\mathcal{M}^w_{p_1,p_2}} = \|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0,\infty}},$$

where $1/p_0 = 1/p_1 + 1/p_2$.

This definition of multiplier norm is analogous to that in the linear case. If $\upsilon \in L_\infty(\mathbb{R}^n)$, $\|\upsilon\|_{\mathcal{M}_p}$ denotes the norm of υ as a multiplier from H_p into L_p that is

$$\|\upsilon\|_{\mathcal{M}_p} = \|M_\upsilon\|_{H_p \to L_p}, \qquad \text{where} \quad M_\upsilon f = (\upsilon \widehat{f}\,)^\vee,$$

when $0 < p < \infty$. Next we mention a few properties of multipliers.

Proposition 5.1. *Suppose $\sigma \in L_\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 < p_1, p_2 < \infty$. Then:*
(i) If $\sigma'(\xi,\eta)=\sigma(\xi-\xi_0,\eta-\eta_0)$ for some fixed ξ_0, η_0 then $\|\sigma'\|_{\mathcal{M}_{p_1,p_2}} = \|\sigma\|_{\mathcal{M}_{p_1,p_2}}$.
(ii) If $L : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear operator and $\sigma_L(\xi,\eta) = \sigma(L\xi, L\eta)$ then $\|\sigma_L\|_{\mathcal{M}_{p_1,p_2}} = \|\sigma\|_{\mathcal{M}_{p_1,p_2}}$.
(iii) If $\mu, \upsilon \in L_\infty(\mathbb{R}^n)$ and $\sigma'(\xi,\eta) = \mu(\xi)\sigma(\xi,\eta)\upsilon(\eta)$, then

$$\|\sigma'\|_{\mathcal{M}_{p_1,p_2}} \le \|\mu\|_{\mathcal{M}_{p_1}} \|\sigma\|_{\mathcal{M}_{p_1,p_2}} \|\upsilon\|_{\mathcal{M}_{p_2}}.$$

Proof. For (i) note that $W_{\sigma'}(f,g) = e^{2\pi i\langle x,\xi_0+\eta_0\rangle} W(e^{-2\pi i\langle x,\xi_0\rangle} f, e^{-2\pi i\langle x,\eta_0\rangle} g)$. For (ii) note that $W_{\sigma_L}(f,g)\circ(L^t)^{-1} = W(f\circ(L^t)^{-1}, g\circ(L^t)^{-1})$. (iii) is trivial. □

Lemma 5.2. *Let $\sigma \in L_\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose that either $p_0 \ge 1$, or that σ is locally Riemann-integrable (i.e. continuous except on a set of measure zero). Then $\|\sigma\|_{L_\infty} \le \|\sigma\|_{\mathcal{M}_{p_1,p_2}}$ whenever $p_0 = p_1p_2/(p_1+p_2)$ and $0 < p_1, p_2 < \infty$.*

Proof. Suppose that σ is locally Riemann-integrable and let (ξ_0, η_0) be a point of continuity of σ. Then if we put $\sigma'_\lambda(\xi,\eta) = \sigma(\xi_0 + \lambda\xi, \eta_0 + \lambda\eta)$, Proposition 5.1 gives that $\|W_{\sigma'_\lambda}\|_{H_{p_1}\times H_{p_2}\to L_{p_0}} = \|W_\sigma\|_{H_{p_1}\times H_{p_2}\to L_{p_0}}$. Now if $f, g \in \mathcal{S}$ it is easy to see that as $\lambda \to 0$ we have convergence in L_2 (and even pointwise) of $W_{\sigma'_\lambda}(f,g)$ to $\sigma(\xi_0,\eta_0)f(x)g(x)$.

If $p_0 \ge 1$ let Q_k be a cube of side 2^{-k} centered at $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$. Let

$$\sigma_k(\xi,\eta) = \frac{1}{|Q_k|}\int_{Q_k} \sigma(\xi+\xi_0, \eta+\eta_0)d\xi_0\, d\eta_0.$$

Proposition 5.1 and the fact that $p_0 \ge 1$ easily imply that $\|W_{\sigma_k}\|_{L_{p_1}\times L_{p_2}\to L_{p_0}} \le \|W_\sigma\|_{L_{p_1}\times L_{p_2}\to L_{p_0}}$. Since σ_k is continuous we have $\|\sigma_k\|_{L_\infty} \le \|W_\sigma\|_{L_{p_1}\times L_{p_2}\to L_{p_0}}$. Taking limits as $k \to \infty$ yields the conclusion. □

Next we require a lemma on series in L_p.

Lemma 5.3. *Let $0 < p < \infty$. Suppose that for some $(f_{jk})_{(j,k)\in\mathbb{Z}^2}$ sequence of L_p functions and for all pairs of sequences $(\delta_j)_{j\in\mathbb{Z}}, (\delta'_k)_{k\in\mathbb{Z}}$ with $\sup_{j\in\mathbb{Z}} |\delta_j| \le 1$ and $\sup_{j\in\mathbb{Z}} |\delta'_j| \le 1$, we have*

$$\sup_{N\in\mathbb{N}} \Big\| \sum_{|j|\le N} \sum_{|k|\le N} \delta_j \delta'_k f_{jk} \Big\|_{L_p} \le M.$$

Then there is a constant $C = C(p)$ such that
(i) $\sup_{|\delta_j|\le 1} \| \sum_{j\in\mathbb{Z}} \delta_j f_{jj}\|_{L_p} \le CM$ (and the series converges unconditionally),
(ii) $\|(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |f_{jk}|^2)^{\frac{1}{2}}\|_{L_p} \le CM$.

Proof. In fact (ii) follows immediately from Khintchine's inequality by taking ϵ_j, ϵ'_k two mutually independent sequences of Bernoulli random variables. To obtain (i), take ϵ_j be a sequence of Bernoulli random variables and for any finite subset $\mathcal{F} \subset \mathbb{Z}$ write

$$\sum_{j\in\mathcal{F}} \delta_j f_{jj} = \sum_{j\in\mathcal{F}}\sum_{k\in\mathcal{F}} \delta_j\epsilon_j\epsilon_k f_{jk} - \sum_{\substack{j,k\in\mathcal{F}\\ j\neq k}} \delta_j\epsilon_j\epsilon_k f_{jk}. \tag{5.2}$$

Now for all $|\delta_j| \le 1$, (see also [10], proof of Theorem 4.6),

$$\begin{aligned}\mathbb{E}(\|\sum_{\substack{j,k\in\mathcal{F}\\ j\neq k}} \delta_j\epsilon_j\epsilon_k f_{jk}\|_{L_p}^p)^{1/p} &\le C\big\|(\sum_{\substack{j,k\in\mathcal{F}\\ j<k}} |\delta_j f_{jk} + \delta_k f_{kj}|^2)^{\frac12}\big\|_{L_p}\\ \le & C\big\|(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |f_{jk}|^2)^{\frac12}\big\|_{L_p} \le CM\end{aligned}$$

by a generalization of Khintchine's inequality due to Bonami [1] and part (ii). The same estimate is also valid for $\sum\limits_{j\in\mathcal{F}}\sum\limits_{k\in\mathcal{F}} \delta_j\epsilon_j\epsilon_k f_{jk}$ by our assumptions. These estimates together with (5.2) give (i). □

We now introduce some notation that will be useful in the sequel. For $(j,k) \in \mathbb{Z}$ let $D_{jk} = \{(\xi,\eta) : 2^{j-1} \le |\xi| \le 2^{j+1},\ 2^{k-1} \le |\eta| \le 2^{k+1}\}$. Also for $\theta > 0$ let $D_{jk}(\theta) = \{(\xi,\eta) : 2^{j-\theta} \le |\xi| \le 2^{j+\theta},\ 2^{k-\theta} \le |\eta| \le 2^{k+\theta}\}$.

Proposition 5.4. *For any $1 < p_1, p_2 < \infty$ there is a constant $C = C(p_1, p_2)$ so that whenever $(\sigma_{jk})_{j,k\in\mathbb{Z}}$ is a family of bilinear symbols with supp $\sigma_{jk} \subset D_{jk}$ which satisfy*

$$\sup_{|\delta_j|\le 1}\sup_{|\delta'_k|\le 1} \|\sum_j\sum_k \delta_j\delta'_k\sigma_{jk}\|_{\mathcal{M}_{p_1,p_2}} \le M,$$

then the following statements are valid:
(i) For any scalar sequence (δ_j) with $\sup_j |\delta_j| \le 1$ and any $r \in \mathbb{Z}$ we have

$$\|\sum_{j\in\mathbb{Z}} \delta_j\sigma_{j,j+r}\|_{\mathcal{M}_{p_1,p_2}} \le CM.$$

(ii) For all $r \ge 3$ we have,

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\le j-r} \sigma_{jk}\|_{\mathcal{M}_{p_1,p_2}} + \|\sum_{k\in\mathbb{Z}}\sum_{j\le k-r} \sigma_{jk}\|_{\mathcal{M}_{p_1,p_2}} \le C(1 + r^{\max(\frac{1}{p_0},1)})M.$$

(iii) For every $r \ge 3$, $p_0 \le 1$ and for all $f, g \in \mathcal{S}$, we have

$$\big\|\sum_{j\in\mathbb{Z}}\sum_{k\le j-r} W_{\sigma_{jk}}\big\|_{L_{p_1}\times L_{p_2}\to H_{p_0}} \le C(1 + r^{\max(\frac{1}{p_0},1)})M$$

$$\big\|\sum_{k\in\mathbb{Z}}\sum_{j\le k-r} W_{\sigma_{jk}}\big\|_{L_{p_1}\times L_{p_2}\to H_{p_0}} \le C(1 + r^{\max(\frac{1}{p_0},1)})M.$$

Proof. For simplicity we write $W_{jk} = W_{\sigma_{jk}}$ below. (i) follows directly from Lemma 5.3. To prove (ii) and (iii) it is enough to consider the case $r = 3$, since the other cases follow trivially by applying (i) and the known case $r = 3$. We therefore suppose $r \geq 3$ and establish both (ii) and (iii). An easy calculation gives that for f, g Schwartz, the function $W_{jk}(f, g)$ has Fourier transform supported in the annulus $2^{j-2} \leq |\zeta| \leq 2^{j+2}$ when $k \leq j - 3$. It follows that

$$\begin{aligned} \|\sum_{j\in\mathbb{Z}}\sum_{k\leq j-3} W_{jk}(f,g)\|_{L_{p_0}} &\leq \|\sum_{j\in\mathbb{Z}}\sum_{k\leq j-3} W_{jk}(f,g)\|_{H_{p_0}} \\ &\leq C\|(\sum_{j\in\mathbb{Z}}|\sum_{k\leq j-3} W_{jk}(f,g)|^2)^{\frac{1}{2}}\|_{L_{p_0}} \\ &\leq C\mathbb{E}(\|\sum_{j\in\mathbb{Z}}\epsilon_j\sum_{k\leq j-3} W_{jk}(f,g)\|_{L_{p_0}}^{p_0})^{1/p_0} \end{aligned} \tag{5.3}$$

where as usual $(\epsilon_j)_{j\in\mathbb{Z}}$ is a sequence of independent Bernoulli random variables. (If $p_0 > 1$ then $H_{p_0} = L_{p_0}$.) We need to control the last term in (5.3).

Our hypothesis gives the estimate

$$\mathbb{E}(\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\epsilon_j W_{jk}(f,g)\|_{L_{p_0}}^{p_0})^{1/p_0} \leq CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}, \tag{5.4}$$

while we can apply (i) to obtain

$$\mathbb{E}(\|\sum_{j\in\mathbb{Z}}\sum_{|k-j|\leq 2}\epsilon_j W_{jk}(f,g)\|_{L_{p_0}}^{p_0})^{1/p_0} \leq CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}. \tag{5.5}$$

It remains to estimate

$$\begin{aligned} \mathbb{E}\big(\|\sum_{j\in\mathbb{Z}}\sum_{k\geq j+3}\epsilon_j W_{jk}(f,g)\|_{L_{p_0}}^{p_0}\big)^{1/p_0} &\leq \mathbb{E}\big(\|\sum_{j\in\mathbb{Z}}\sum_{k\geq j+3}\epsilon_j W_{jk}(f,g)\|_{H_{p_0}}^{p_0}\big)^{1/p_0} \\ &\leq C\mathbb{E}\big(\|\sum_{k\in\mathbb{Z}}(\sum_{j\leq k-3}\epsilon_j W_{jk}(f,g)\|_{L_{p_0}}^{p_0}\big)^{1/p_0} \\ &\leq C\mathbb{E}\big(\|\sum_{k\in\mathbb{Z}}\epsilon'_k\sum_{j\leq k-3}\epsilon_j W_{jk}(f,g)\|_{L_{p_0}}^{p_0}\big)^{1/p_0} \end{aligned}$$

where ϵ'_k is a second (independent) sequence of independent Bernoulli random variables. Hence using again Khintchine's inequality we have

$$\begin{aligned} \mathbb{E}\big(\|\sum_{j\in\mathbb{Z}}\sum_{k\geq j+3}\epsilon_j W_{jk}(f,g)\|_{L_{p_0}}^{p_0}\big)^{1/p_0} &\leq C\|(\sum_{k\in\mathbb{Z}}\sum_{j\leq k-3}|W_{jk}(f,g)|^2)^{\frac{1}{2}}\|_{L_{p_0}} \\ &\leq C\|(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|W_{jk}(f,g)|^2)^{\frac{1}{2}}\|_{L_{p_0}} \\ &\leq CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}} \end{aligned} \tag{5.6}$$

in view of Lemma 5.3. Using (5.4), (5.5), and (5.6) we obtain

$$\mathbb{E}\big(\|\sum_{j\in\mathbb{Z}}\epsilon_j\sum_{k\le j-3}W_{jk}(f,g)\|_{L_{p_0}}^{p_0}\big)^{1/p_0}\le CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}$$

which combined with (5.3) gives the first of the assertions (ii) and (iii) for $r=3$. The second assertions are derived similarly by symmetry. □

We will need one further preliminary lemma.

Lemma 5.5. *For any $1<p_1,p_2<\infty$ there is a constant $C=C(p_1,p_2)$ such that for any family of symbols $(\sigma_{jk})_{j,k\in\mathbb{Z}}$ with $\operatorname{supp}\sigma_{jk}\subset D_{jk}$ and for any μ,υ C^∞ functions on the annulus $\frac14\le|\xi|\le 4$ we have*

$$\sup_{|\delta_j|}\sup_{|\delta'_k|\le1}\big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_k\tau_{jk}\big\|_{\mathcal{M}_{p_1,p_2}}\le CK_\mu K_\upsilon\sup_{|\delta_j|}\sup_{|\delta'_k|\le1}\big\|\sum_j\sum_k\delta_j\delta'_k\sigma_{jk}\big\|_{\mathcal{M}_{p_1,p_2}},$$

where $\tau_{jk}(\xi,\eta)=\mu(2^{-j}\xi)\sigma_{jk}(\xi,\eta)\upsilon(2^{-k}\eta)$,

$$K_\mu=\sup_{\substack{|\alpha|\le m\\ \frac14\le|\xi|\le4}}\left|\frac{\partial^\alpha\mu}{\partial\xi^\alpha}\right|,\qquad K_\upsilon=\sup_{\substack{|\alpha|\le m\\ \frac14\le|\xi|\le4}}\left|\frac{\partial^\alpha\upsilon}{\partial\xi^\alpha}\right|,$$

and $m=[(n+1)/2]$.

Proof. Recalling the definition of ϕ from section 4 we note that the function

$$\big(\sum_{l=j-2}^{j+2}\widehat{\phi}_l(\xi)\big)\big(\sum_{l=j-2}^{k+2}\widehat{\phi}_l(\eta)\big)$$

is compactly supported and is equal to 1 on the support of $\sigma_{jk}(\xi,\eta)$. For any sequence δ_j with $\sup|\delta_j|\le1$ we observe that

$$\big\|\big(\sum_{j\in\mathbb{Z}}\delta_j\mu(2^{-j}\xi)\big)\big(\sum_{l=j-2}^{j+2}\widehat{\phi}_l(\xi)\big)\big\|_{\mathcal{M}_{p_1}}\le CK_\mu \tag{5.7}$$

$$\big\|\big(\sum_{k\in\mathbb{Z}}\delta'_k\mu(2^{-jk}\eta)\big)\big(\sum_{l=k-2}^{k+2}\widehat{\phi}_l(\eta)\big)\big\|_{\mathcal{M}_{p_2}}\le CK_\upsilon \tag{5.8}$$

by the Hörmander multiplier theorem. Let U_{j_1,j_2,k_1,k_2} be the bilinear operator with symbol

$$\Big(\delta_{j_1}\mu(2^{-j_1}\xi)\sum_{l=j_1-2}^{j_1+2}\widehat{\phi}_l(\xi)\Big)\sigma_{j_2,k_2}(\xi,\eta)\Big(\delta'_{k_1}\upsilon(2^{-k_1}\eta)\sum_{l=k_1-2}^{k_1+2}\widehat{\phi}_l(\eta)\Big),$$

for some fixed $|\delta_j|,|\delta'_k|\le1$. Let

$$M=\sup_{|\delta_j|}\sup_{|\delta'_k|\le1}\big\|\sum_j\sum_k\delta_j\delta'_k\sigma_{jk}\big\|_{\mathcal{M}_{p_1,p_2}}$$

and let $(\epsilon_j), (\epsilon'_k)$ be two sequences of mutually independent Bernoulli random variables. Then for $f, g \in \mathcal{S}$ we have

$$\mathbb{E}\Big(\|\sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \epsilon_{j_1}\epsilon_{j_2}\epsilon'_{k_1}\epsilon'_{k_2} U_{j_1,j_2,k_1,k_2}(f,g)\|_{L_{p_0}}^{p_0}\Big)^{\frac{1}{p_0}}$$
$$\leq CMK_\mu K_v \|f\|_{L_{p_1}}\|f\|_{L_{p_2}}$$

by our hypothesis, (5.7), and (5.8). We now use Lemma 5.3 twice to deduce that

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} U_{j,j,k,k}(f,g)\|_{L_{p_0}} \leq CK_\mu K_v M\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}.$$

This proves the required assertion. □

6. Bilinear operators and infinite matrices

Recall from section 4 that $\phi_j(x) = 2^{nj}\phi(2^j x)$ are smooth bumps whose Fourier transforms are supported in the annuli $2^{j-1} \leq |\xi| \leq 2^{j+1}$. In this section we will consider symbols σ of the form

$$\sigma_A(\xi,\eta) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\widehat{\phi_j}(\xi)\widehat{\phi_k}(\eta) \tag{6.1}$$

where $A = (a_{jk})_{(j,k)\in\mathbb{Z}^2}$ is a bounded infinite matrix. We let $W_A = W_{\sigma_A}$ and $\|A\|_\infty = \sup_{j,k}|a_{jk}|$.

If A is such an infinite matrix we define A_L to be its lower-triangle and A_U to be its upper-triangle i.e. $A_L = (a_{jk}\theta_{jk})_{j,k}$ and $A_U = (a_{jk}\theta_{kj})_{j,k}$ where $\theta_{jk} = 1$ if $k < j$ and 0 otherwise. We let A_D be the diagonal $A - A_U - A_L$. Now define

$$H(A) = h(A_L) + h(A_U^t) + \|A\|_\infty \tag{6.2}$$

Notice that $H(A) \geq \|A\|_\infty$ and that H is a norm on the space of $\{A : H(A) < \infty\}$ which makes it a Banach space.

Our objective will be to show that for any choice of $0 < p_1, p_2 < \infty$ we have $\|W_A\|_{H_{p_1}\times H_{p_2}\to L_{p_0}} \approx H(A)$. This will provide us with an equivalent expression for the norm of the multiplier σ_A defined in (6.1).

We start by proving the simple upper estimate below.

Lemma 6.1. *If $0 < p_1, p_2 < \infty$ there is a constant $C = C(p_1,p_2)$ so that for any matrix A we have $\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \leq CH(A)$.*

Proof. We give the proof in the case $p_1, p_2 > 1$; the only real alteration for the other cases would be to replace the appropriate $L_{p_j}-$norm with the $H_{p_j}-$

norm and use Theorem 4.6. Suppose $f, g \in \mathcal{S}$ and consider

$$W_A(f,g) = \sum_{j\in\mathbb{Z}} \sum_{k\le j-3} a_{jk}\widetilde{\Delta}_j f \widetilde{\Delta}_k g + \sum_{k\in\mathbb{Z}} \sum_{j\le k-3} a_{jk}\widetilde{\Delta}_j f \widetilde{\Delta}_k g + \sum_{j\in\mathbb{Z}} \sum_{k=j-2}^{j+2} a_{jk}\widetilde{\Delta}_j f \widetilde{\Delta}_k g. \tag{6.3}$$

We estimate the first term by noticing that for fixed j the Fourier transform of $\widetilde{\Delta}_j f \sum_{k\le j-3} a_{jk}\widetilde{\Delta}_k g$ is contained in the set $\{\zeta :\ 2^{j-2} \le |\zeta| \le 2^{j+2}\}$. Hence if $p_0 > 1$ we have

$$\|\sum_{j\in\mathbb{Z}} \widetilde{\Delta}_j f \sum_{k\le j-3} a_{jk}\widetilde{\Delta}_k g\|_{L_{p_0}} \le C\|(\sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f|^2)^{\frac12}(|\sum_{k\le j-3} a_{jk}\widetilde{\Delta}_k|^2)^{\frac12}\|_{L_{p_0}}.$$

If $0 < p_0 \le 1$ we obtain the same estimate by noticing that

$$\|\sum_{j\in\mathbb{Z}} \widetilde{\Delta}_j f \sum_{k\le j-3} a_{jk}\widetilde{\Delta}_k g\|_{L_{p_0}} \le \|\sum_{j\in\mathbb{Z}} \widetilde{\Delta}_j f \sum_{k\le j-3} a_{jk}\widetilde{\Delta}_k g\|_{H_{p_0}}$$

and using the corresponding square-function estimates in H_{p_0}. Now we have

$$\begin{aligned}\big\|(\sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f|^2 |\sum_{k\le j-3} a_{jk}\widetilde{\Delta}_k g|^2)^{\frac12}\big\|_{L_{p_0}} \\ \le \big\|(\sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f|^2)^{\frac12} \sup_{j\in\mathbb{Z}} |\sum_{k\le j-3} a_{jk}\widetilde{\Delta}_k g|\big\|_{L_{p_0}}.\end{aligned} \tag{6.4}$$

If we let A_{LL} be the matrix with entries a_{jk} if $k \le j-3$ and 0 otherwise, then $h(A_{LL}) \le h(A_L) + h(B)$ where B is the matrix with entries a_{jk} if $j-2 \le k \le j-1$ and 0 otherwise. It is trivial to see that one has the estimate $h(B) \le 2\|A\|_\infty$ so that $h(A_{LL}) \le Ch(A_L)$. Hence (6.4) and Theorem 4.5 give

$$\begin{aligned}\|\sum_{j\in\mathbb{Z}} \sum_{k\le j-3} a_{jk}\widetilde{\Delta}_j f \widetilde{\Delta}_k g\|_{L_{p_0}} &\le C\|(\sum_{j\in\mathbb{Z}} \widetilde{\Delta}_j f)^{\frac12}\|_{L_{p_1}} \|\sup_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} a_{jk}\widetilde{\Delta}_k g|\|_{L_{p_2}} \\ &\le Ch(A_L)\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}.\end{aligned}$$

The same argument shows that the third term in (6.3) is controlled by $Ch(A_U^t)\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}$. The middle term in (6.3) is easy. For $-2 \le r \le 2$ we have

$$\begin{aligned}&\big\|\sum_{j\in\mathbb{Z}} a_{j,j+r}\widetilde{\Delta}_j f \widetilde{\Delta}_{j+r} g\big\|_{L_{p_0}} \\ \le&\big\|(\sum_{j\in\mathbb{Z}} |a_{j,j+r}||\widetilde{\Delta}_j f|^2)^{\frac12}\big\|_{L_{p_1}} \big\|(\sum_{k\in\mathbb{Z}} |a_{j,j+r}||\widetilde{\Delta}_{j+r} g|^2)^{\frac12}\big\|_{L_{p_2}} \\ \le& C\max_j |a_{j,j+r}|\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}.\end{aligned}$$

Combining we obtain the required upper estimate: $\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \le CH(A)$. $\square$

To obtain the converse is somewhat more complicated. First we prove a general result which we will use in other situations as well.

Proposition 6.2. *For any $1 < p_1, p_2 < \infty$ with $p_0 = (1/p_1 + 1/p_2)^{-1} \geq 1$, there is a constant $C = C(p_1, p_2)$ with the following property. Whenever $(\sigma_{jk})_{(j,k)\in\mathbb{Z}^2}$ is a family of symbols with supp $\sigma_{jk} \subset D_{jk}$ which satisfy*

$$\sup_{|\delta_j|\leq 1} \sup_{|\delta'_k|\leq 1} \|\sum_j \sum_k \delta_j \delta'_k W_{\sigma_{jk}}\|_{L_{p_1}\times L_{p_2}\to L_{p_0}} \leq M,$$

then

$$\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \leq CM,$$

where $A = (a_{jk})_{j,k}$ and

$$a_{jk} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma_{jk}(2^j\xi, 2^k\eta) d\xi\, d\eta.$$

Proof. As before we write $W_{jk} = W_{\sigma_{jk}}$. Let us consider first the case when $\sigma_{jk} = 0$ unless $k \leq j - 5$. Let υ be a $C^\infty-$function on $\mathbb{R}^n$ supported on $2^{-4} \leq |\xi| \leq 2^4$ and such that $\upsilon(\xi) = 1$ on $2^{-3} \leq |\xi| \leq 2^3$. Fix $\xi_0 \in \mathbb{R}^n$ and consider the symbol

$$\tau_{jk}(\xi_0; \xi, \eta) = \upsilon(2^{-j}\xi)\sigma_{jk}(\xi + 2^j\xi_0, \eta).$$

Note that τ_{jk} is supported in $D_{jk}(4)$. Let T_{jk} be bilinear operator with symbol τ_{jk}. For any sequences $(\delta_j)_{j\in\mathbb{Z}}, (\delta'_k)_{k\in\mathbb{Z}}$ with $\sup|\delta_j|, \sup|\delta'_k| \leq 1$ and $f, g \in \mathcal{S}$ we have

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \delta_j\delta'_k T_{jk}(f,g)\|_{L_{p_0}} \leq C\|(\sum_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} \delta'_k T_{jk}(f,g)|^2)^{\frac{1}{2}}\|_{L_{p_0}}$$

by considering the supports of the Fourier transforms. But then for fixed j,

$$\sum_{k\in\mathbb{Z}} \delta'_k T_{jk}(f,g)(x) = e^{-2\pi i\langle x, 2^j\xi_0\rangle} \sum_{k\in\mathbb{Z}} \delta_k W_{jk}(f,g)(x),$$

hence

$$\begin{aligned}\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \delta_j\delta'_k T_{jk}(f,g)\|_{L_{p_0}} &\leq C\|(\sum_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} \delta'_k W_{jk}(f,g)|^2)^{\frac{1}{2}}\|_{L_{p_0}} \\ &\leq C\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \delta'_k W_{jk}(f,g)\|_{H_{p_0}} \\ &\leq CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}\end{aligned}$$

using Proposition 5.4.

Now note that if $|\xi_0| > 18$ then all T_{jk} vanish. Since $p_0 \geq 1$, we integrate over $|\xi_0| \leq 18$ to obtain symbols

$$\tau'_{jk}(\xi, \eta) = \int_{|\xi_0|\leq 18} \tau_{jk}(\xi, \eta)\, d\xi_0 = \upsilon(2^{-j}\xi)\int_{\mathbb{R}^n} \sigma_{jk}(\xi + 2^j\xi_0, \eta) d\xi_0$$

with corresponding bilinear operators T'_{jk} satisfying

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_kT'_{jk}\|_{L_{p_1}\times L_{p_2}\to L_{p_0}}\le CM$$

whenever $|\delta_j|,|\delta'_k|\le 1$.

Note that τ'_{jk} is supported on $D_{jk}(3)$. Also if $2^{j-3}\le|\xi|\le 2^{j+3}$ we have that $\tau'_{jk}(\xi,\eta)$ is constant in ξ.

Next let O_n be the orthogonal group of $\mathbb{R}^n$ and let dL denote the Haar measure on this group. Define

$$\tau^{\#}_{jk}(\xi,\eta)=\int_{\frac14}^4\lambda^{n-1}\int_{O_n}\tau'_{jk}(\lambda L\xi,\lambda L\eta)dL\,d\lambda,$$

and let $T^{\#}_{jk}$ be the corresponding bilinear operator. If $(\xi,\eta)\in D_{jk}$ we can compute that

$$\tau^{\#}_{jk}(\xi,\eta)=c2^{nk}|\eta|^{-n}a_{jk}$$

where c is a constant depending only on dimension. On the other hand, since $p_0\ge 1$, Proposition 5.1 (ii) gives that

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_kT^{\#}_{jk}\|_{L_{p_1}\times L_{p_2}\to L_{p_0}}\le CM$$

whenever $|\delta_j|,|\delta'_k|\le 1$.

Note that supp $\tau^{\#}_{jk}\subset D_{jk}(6)$. Let us take $\mathbb{M}_1$ and $\mathbb{M}_2$ to be residue classes modulo 10. Then if we replace δ_j by $\delta_j\chi_{\mathbb{M}_1}(j)$ and δ'_k by $\delta'_k\chi_{\mathbb{M}_2}(k)$ we obtain a bilinear operator whose symbol coincides with $a_{jk}2^{nk}|\eta|^{-n}\delta_j\delta'_k$ on D_{jk} for $(j,k)\in\mathbb{M}_1\times\mathbb{M}_2$. Using Proposition 5.1 (iii) and the multipliers $\sum_{j\in\mathbb{M}_1}\widehat{\phi_j}$ and $\sum_{k\in\mathbb{M}_2}\widehat{\phi_k}$ we obtain that the bilinear operator V with symbol

$$\sum_{j\in\mathbb{M}_1}\sum_{k\in\mathbb{M}_2}\delta_j\delta'_k2^{nk}|\eta|^{-n}a_{jk}\widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta),$$

satisfies $\|V\|_{L_{p_1}\times L_{p_2}\to L_{p_0}}\le CM$. Summing over 100 different pairs of residue classes gives a similar estimate for the symbol

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_k2^{nk}|\eta|^{-n}a_{jk}\widehat{\phi_j}(\xi)\widehat{\phi_k}(\eta).$$

The last step is to remove the factor $2^{nk}|\eta|^{-n}$. But this can be done by using Lemma 5.5 since $|\eta|^{-n}$ is C^∞ on $\frac14\le|\eta|\le 4$. □

We will use this result to make an important estimate on the effect of translation in the computation of $\|W_A\|_{L_{p_1}\times L_{p_2}\to L_{p_0}}$. Let us define $A^{[r,s]}$ to be the matrix $(a_{j+r,k+s})_{j,k}$.

Lemma 6.3. *(i) There is a constant C so that for all matrices A we have*

$$\|\sigma_{A^{[r,s]}}\|_{\mathcal{M}_{2,2}} \leq C^{|r-s|}\|\sigma_A\|_{\mathcal{M}_{2,2}}$$

(ii) For all $1 < p_1, p_2 < \infty$ with $p_0 = p_1p_2/(p_1+p_2) \geq 1$, there is a constant $C = C(p_1,p_2)$ so that if $|\delta_j|, |\delta'_k| \leq 1$ then

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_k a_{jk}\widehat{\phi}(2^{-j}\xi)\widehat{\phi}(2^{-k}\eta)\|_{\mathcal{M}_{p_1,p_2}} \leq C\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}},$$

i.e. $\|\sigma_D\|_{M_{p_1,p_2}} \leq C\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}}$, where $D = (d_{jk})_{j,k} = (\delta_j\delta'_k a_{jk})_{j,k}$.

Proof. It is clear from Proposition 5.1 that for any $r \in \mathbb{Z}$ we have

$$\|W_{A^{[r,r]}}\|_{L_2\times L_2\to L_1} = \|W_A\|_{L_2\times L_2\to L_1}.$$

Thus it suffices to consider the case $r = 0$ and $s = \pm 1$ and establish a bound in this case. To do this we consider the symbols

$$\sigma_{jk}(\xi,\eta) = \sigma_A(\xi,\eta)\mu(2^{-j}\xi)\upsilon(2^{-k}\eta)\widehat{\phi_j}(\xi)\widehat{\phi_k}(\eta),$$

where μ, υ are $C^\infty-$functions satisfying $|\mu(\xi)|, |\upsilon(\eta)| \leq 1$ for all ξ,η. Since $\|\sum_{j\in\mathbb{Z}}\delta_j\mu(2^{-j}\xi)\widehat{\phi_j}(\xi)\|_{\mathcal{M}_2}$ is bounded by 3 whenever $\sup_j|\delta_j| \leq 1$, and there is a similar bound for $\sum_{k\in\mathbb{Z}}\delta'_k\upsilon(2^{-k}\eta)\widehat{\phi_k}(\eta)$ we have an immediate estimate;

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_k W_{\sigma_{jk}}\|_{L_2\times L_2\to L_1} \leq 9\|W_A\|_{L_2\times L_2\to L_1}.$$

Now let

$$b_{jk} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma_{jk}(2^j\xi, 2^k\eta)d\xi\, d\eta.$$

Then we can compute

$$b_{jk} = \sum_{r=-1}^{1}\sum_{s=-1}^{1} c_{rs}a_{j+r,k+s}$$

where

$$c_{rs} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mu(\xi)\upsilon(\eta)\widehat{\phi_{-r}}(\xi)\widehat{\phi_{-s}}(\eta)\widehat{\phi}_0(\xi)\widehat{\phi}_0(\eta)d\xi\, d\eta.$$

Since the functions $\widehat{\phi}_r$ for $-1 \leq r \leq 1$ are linearly independent on the support of $\widehat{\phi}_0$ we can use the above estimate for a linear combination of a finite number of choices of υ and ξ so that $c_{rs} = 0$ except when $r = 0$ and $s = 1$, so that $B = cA^{[0,1]}$ for some fixed constant $c \neq 0$. By Proposition 6.2 we have $\|W_B\|_{L_2\times L_2\to L_1} \leq C\|W_A\|_{L_2\times L_2\to L_1}$. This and the similar argument for the case $s = -1$ gives the result (i).

For (ii) we observe that the above argument actually also yields a bound on $\|W_D\|_{L_2\times L_2\to L_1}$ when $D = (d_{jk}) = (\delta_j\delta'_k b_{jk})$ (since $\delta_j\delta'_k\sigma_{jk}$ also verifies the hypotheses of Proposition 6.2. By choosing a similar linear combination we can then ensure that $b_{jk} = ca_{jk}$ and obtain the desired result. □

The next step is to consider a discrete model of the bilinear operator W_{σ_A}. We restrict ourselves to $p_1 = p_2 = 2$ for this, although our calculations can be done in more generality. If A is a $c_{00}-$matrix we define $V_A:\ L_2 \times L_2 \to L_1$ by

$$V_A(f,g) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\Delta_j f\Delta_k g,$$

where Δ_j are the martingale difference operators as defined in section 4. We then have

Lemma 6.4. *There is a constant C so that if A is a (strictly) lower-triangular matrix we have $h(A) \le C\|V_A\|_{L_2\times L_2\to L_1}$.*

Proof. This is a stopping time argument. Suppose $f \in L_2$ with $\|f\|_{L_2} = 1$. Note that for each j the function $f_j = \sum_{k\in\mathbb{Z}} a_{jk}\Delta_k f$ is Σ_{j-1}-measurable where Σ_{j-1} is the $\sigma-$algebra generated by the dyadic cubes in $\mathcal{D}_{j-1}$. Fix $\lambda > 0$. For each j let $\mathcal{Q}_j$ be the collection of cubes $Q \in \mathcal{D}_{j-1}$ so that $|f_j| > \lambda$ on Q and for each $j_1 < j$ we have $|f_{j_1}| \le \lambda$ on Q. It is not difficult to see that

$$\{x:\ \max_{j\in\mathbb{Z}} |f_j(x)| > \lambda\} = \bigcup_{j\in\mathbb{Z}}\bigcup_{Q\in\mathcal{Q}_j} Q$$

and this is a disjoint union. Also note the left-hand side has finite measure.

For each j be u_j be a Σ_j-measurable function such that $|u_j| = 1$ everywhere and $\mathcal{E}_{j-1}u_j = 0$. Let

$$g = \sum_{j\in\mathbb{Z}} u_j \sum_{Q\in\mathcal{Q}_j} \chi_Q.$$

Then

$$\|g\|_{L_2}^2 = |\{x : \max_{j\in\mathbb{Z}} |f_j(x)|\}|$$

and

$$V_A(f,g) = \sum_{j\in\mathbb{Z}} f_j\Delta_j g = \sum_{j\in\mathbb{Z}} f_j u_j \sum_{Q\in\mathcal{Q}_j} \chi_Q.$$

Hence

$$|V_A(f,g)| \ge \lambda\chi_{(\max_j |f_j|>\lambda)}$$

so that we have

$$\lambda|\{\max_j |f_j| > \lambda\} \le \|V_A\|_{L_2\times L_2\to L_1}.$$

This implies that $h_2^w(A) \le \|V_A\|_{L_2\times L_2\to L_1}$ and the result follows from Theorem 2.1. □

We are now ready for the main result:

Theorem 6.5. *Suppose* $0 < p_1, p_2 < \infty$. *Then there is a constant* $C = C(p_1, p_2)$ *so that for any infinite matrix* A *we have*

$$\frac{1}{C}H(A) \le \|\sigma_A\|_{\mathcal{M}^w_{p_1,p_2}} \le \|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \le CH(A).$$

Proof. The upper bound is proved in Lemma 6.1 so we only need to prove the lower bound. It suffices to prove the results for the case when A is a $c_{00}-$matrix. We start by considering the case $p_1 = p_2 = 2$, when A is strictly lower-triangular.

In this case let us estimate the norm of the discrete model V_A. In fact

$$\begin{aligned} V_A(f,g) &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\Delta_j f\Delta_k g \\ &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\widetilde{\Delta}_{j-r}\Delta_j f\widetilde{\Delta}_{k-s}\Delta_k g \\ &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_{j+r,k+s}\widetilde{\Delta}_j\Delta_{j+r} f\widetilde{\Delta}_k\Delta_{k+s} g \\ &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} W_{A^{[r,s]}}(\sum_{j\in\mathbb{Z}}\widetilde{\Delta}_j\Delta_{j+r} f, \sum_{k\in\mathbb{Z}}\widetilde{\Delta}_k\Delta_{k+s} g) \\ &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} W_{A^{[r,s]}}(V^*_{-r}f, V^*_{-s}g), \end{aligned}$$

where V_r is defined in the proof of Theorem 4.5. Using Proposition 4.4 we obtain

$$\|V_A\|_{L_2\times L_2\to L_1} \le C\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} 2^{-|r|-|s|}\|W_{A^{[r,s]}}\|_{L_2\times L_2\to L_1}.$$

(All these quantities are finite since A has only finitely many non-zero entries, and so there is a uniform bound on $W_{A^{[r,s]}}$.)

It follows that we have an estimate (for a suitable C_0,)

$$h(A) \le C_0\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} 2^{-|r|-|s|}\|W_{A^{[r,s]}}\|_{L_2\times L_2\to L_1}. \tag{6.5}$$

Next we estimate $H(A^{[r,s]})$. If $s \ge r$ it is clear that A remains lower-triangular and the invariance properties of $h(A)$ imply that $H(A^{[r,s]}) \le H(A)$. If $s < r$ then it is easy to estimate

$$h(A_L^{[r,s]}) \le h(A_L) + (r-s)\|A\|_\infty$$

and

$$h((A_U^{[r,s]})^t) \le (r-s)\|A\|_\infty.$$

We deduce that

$$H(A^{[r,s]}) \le h(A) + |r-s|\|A\|_\infty$$

for all r, s. Thus we have for a suitable constant C_0

$$\|W_{A^{[r,s]}}\|_{L_2\times L_2\to L_1} \le C_1(1+|r-s|)h(A). \tag{6.6}$$

Now we may pick an integer N large enough so that

$$C_1C_0\sum_{|r|>N}\sum_{|s|>N}(1+|r-s|)2^{-|r|-|s|} \le \frac{1}{2}.$$

Then we can combine (6.5) and (6.6) to obtain

$$h(A) \le C_2\sum_{|r|\le N}\sum_{|s|\le N}\|W_{A^{[r,s]}}\|_{L_2\times L_2\to L_1}. \tag{6.7}$$

At this point Lemma 6.3 gives the conclusion that

$$h(A) \le C\|W_A\|_{L_2\times L_2\to L_1}.$$

Now suppose A is arbitrary. If we let W_{jk} be the bilinear operator with symbol $a_{jk}\widehat{\phi_j}(\xi)\widehat{\phi_k}(\eta)$, Lemma 6.3 (ii) implies that we can use Proposition 5.4 (ii) to deduce that $\|W_{A_L}\|_{L_2\times L_2\to L_1} \le C\|W_A\|_{L_2\times L_2\to L_1}$ for some absolute constant C. Thus the above argument yields $h(A_L) \le C\|W_A\|_{L_2\times L_2\to L_1}$. Similarly $h(A_U^t) \le C\|W_A\|_{L_2\times L_2\to L_1}$ and Lemma 5.2 is enough to show that $\|A\|_\infty \le C\|W_A\|_{L_2\times L_2\to L_1}$. Combining these we have the estimate

$$H(A) \le C\|W_A\|_{L_2\times L_2\to L_1}.$$

The proof is completed by a simple interpolation technique. We will argue first that an estimate of the type

$$H(A) \le C(p_1,p_2)\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \tag{6.8}$$

for some fixed $1 < p_1, p_2 < \infty$ implies the estimate

$$H(A) \le C(q,p_2)\|\sigma_A\|_{\mathcal{M}^w_{p_1,q}} \tag{6.9}$$

for every $1 < q < \infty$. We only need to consider the first case and $q \ne p_2$ (when $q = p_2$ one repeats the step). Then we may find $1 < r < \infty$ and $0 < \theta < 1$ so that

$$\frac{1}{p_2} = \frac{1-\theta}{q} + \frac{\theta}{r}.$$

The Marcinkiewicz interpolation theorem yields

$$\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \le C(p_1,p_2,\theta)(\|\sigma_A\|_{\mathcal{M}^w_{p_1,q}})^{1-\theta}(\|\sigma_A\|_{\mathcal{M}_{p_2,r}})^{\theta}. \tag{6.10}$$

Since $\|\sigma_A\|_{\mathcal{M}_{p_2,r}} \le C(p_2,r)H(A)$, using (6.10), and (6.8) we obtain estimate (6.9) as required (recall that we assume A is a c_{00}-matrix so that all these quantities are finite).

Repeated use of this argument starting from $p_1 = p_2 = 2$ gives the theorem in the cases $1 < p_1, p_2 < \infty$.

Finally in the case where either $p_1 \le 1$ or $p_2 \le 1$ (or both) one can use complex interpolation to deduce

$$\|\sigma_A\|_{\mathcal{M}^w_{q_1,q_2}} \le C(\|\sigma_A\|_{\mathcal{M}^w_{p_1,p_2}})^{1-\theta}(\|\sigma_A\|_{\mathcal{M}_{2,2}})^\theta$$

where $q_1, q_2 > 1$ and

$$\frac{1}{q_1} = \frac{1-\theta}{p_1} + \frac{\theta}{2}, \qquad \frac{1}{q_2} = \frac{1-\theta}{p_2} + \frac{\theta}{2}.$$

This clearly extends the lower estimate to the cases $p_1, p_2 \le 1$. □

7. Applications to bilinear multipliers

We will now consider the boundedness of the bilinear operator W_σ under conditions of Marcinkiewicz type on the symbol σ. We will say that a symbol σ is C^N if it is C^N on the set $\{(\xi,\eta) : |\xi|, |\eta| > 0\}$. We first give an example to show that conditions (1.3) for a function σ on $\mathbb{R}^{2n}$ do not imply boundedness for the corresponding bilinear map on $\mathbb{R}^n \times \mathbb{R}^n$.

Example. There is a $C^\infty-$symbol σ so that for every pair of multi-indices (α, β) there is a constant $C_{\alpha,\beta}$ so that

$$|\xi|^{|\alpha|}|\eta|^{|\beta|}|\partial^\alpha_\xi \partial^\beta_\eta \sigma(\xi,\eta)| \le C_{\alpha,\beta} \tag{7.1}$$

but W_σ is not of weak type (p_1, p_2) for any $0 < p_1, p_2 < \infty$.

Indeed if we let A be a bounded infinite matrix and $\sigma(\xi,\eta) = \sigma_A(\xi,\eta)$, then σ satisfies the condition (7.1). However W_A is of weak type (p_1,p_2) if and only if $H(A) < \infty$ by theorem 6.5. At the end of Section 3 we showed that there are examples (with A lower-triangular) where $H(A) = \infty$.

In fact more is true. It is shown that the condition $0 < \theta < \frac{1}{2}$ in (3.12) is insufficient to give a bound on $h(A)$ or $H(A)$ when A is lower-triangular. This means that if $0 < \theta < \frac{1}{2}$ we can construct a symbol σ which is C^∞, with W_σ not of weak type (p_1, p_2) for any $0 < p_1, p_2 < \infty$ and such that for each pair of multi-indices (α, β) there is a constant $C_{\alpha,\beta}$ with

$$|\xi|^{|\alpha|}|\eta|^{|\beta|}|\partial^\alpha_\xi \partial^\beta_\eta \sigma(\xi,\eta)| \le C_{\alpha,\beta}\big(\log(1 + |\log \tfrac{|\xi|}{|\eta|}|)\big)^{-\theta} \tag{7.2}$$

but W_σ is not of weak type (p_1,p_2) for any $p_1, p_2 > 0$.

These examples indicate that the Marcinkiewicz-type conditions (7.1) need to be modified if they are to imply boundedness for bilinear operators on $\mathbb{R}^n \times \mathbb{R}^n$.

In order to formulate some general results, let us introduce the following notation. For $\sigma \in L_\infty$ we define

$$\|\sigma\|_H = \sup_{1\le|\xi|\le 2} \sup_{1\le|\eta|\le 2} H((\sigma(2^j\xi, 2^k\eta)_{j,k}). \tag{7.3}$$

If σ is of class C^N we define

$$\|\sigma\|_H^{(N)} = \sum_{|\alpha|\le N} \||\xi|^{|\alpha|}\partial_\xi^\alpha \sigma\|_H + \sum_{|\beta|\le N} \||\eta|^{|\beta|}\partial_\eta^\beta \sigma\|_H. \tag{7.4}$$

It will also be useful to define in this case

$$\|\sigma\|_{\mathcal{M}_{p_1,p_2}}^{(N)} = \sum_{|\alpha|\le N} \||\xi|^{|\alpha|}\partial_\xi^\alpha \sigma\|_{\mathcal{M}_{p_1,p_2}} + \sum_{|\beta|\le N} \||\eta|^{|\beta|}\partial_\eta^\beta \sigma\|_{\mathcal{M}_{p_1,p_2}}. \tag{7.5}$$

Now consider an arbitrary L^∞ symbol σ of class C^{n+1}. Let

$$\sigma_{jk}(\xi,\eta) = \sigma(\xi,\eta)\widehat{\phi}(2^{-j}\xi)\widehat{\phi}(2^{-k}\eta). \tag{7.6}$$

Set $\widehat{\zeta}(\xi) = \widehat{\phi_{-2}}(\xi) + \widehat{\phi_{-3}}(\xi) + \widehat{\phi_{-4}}(\xi)$. Then $\widehat{\zeta}$ is equal to 1 on the annulus $1/16 \le |\xi| \le 1/4$ and vanishes off the annulus $1/32 \le |\xi| \le 1/2$. Thus the function $\widehat{\zeta}(\xi)\widehat{\zeta}(\eta)$ is supported in the unit cube $[0,1]^{2n}$ and is equal to one on the support of

$$(\xi,\eta) \to \sigma_{jk}(2^{j+3}\xi, 2^{k+3}\eta)$$

which is also contained in $[0,1]^{2n}$. Inspired by [5], we expand the function above in Fourier series on $[0,1]^{2n}$. We have

$$\sigma_{jk}(2^{j+3}\xi, 2^{k+3}\eta) = \sum_{\nu\in\mathbb{Z}^n}\sum_{\rho\in\mathbb{Z}^n} a_{jk}(\nu,\rho)e^{2\pi i(\langle\xi,\nu\rangle+\langle\eta,\rho\rangle)}\widehat{\zeta}(\xi)\widehat{\zeta}(\eta),$$

where for $(\nu,\rho)\in\mathbb{Z}^n\times\mathbb{Z}^n$ we set

$$a_{jk}(\nu,\rho) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(2^{j+3}t, 2^{k+3}s)\widehat{\phi}(8t)\widehat{\phi}(8s)e^{-2\pi i(\langle t,\nu\rangle+\langle s,\rho\rangle)}dt\,ds. \tag{7.7}$$

We will denote by $A(\nu,\rho)$ the matrix with entries $a_{jk}(\nu,\rho)$. Now setting

$$\tau^{\nu,\rho}(\xi,\eta) = \Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}(\nu,\rho)e^{\frac{\pi i}{4}(2^{-j}\langle\xi,\nu\rangle+2^{-k}\langle\eta,\rho\rangle)}\Big)\widehat{\zeta}(2^{-j-3}\xi)\widehat{\zeta}(2^{-k-3}\eta), \tag{7.8}$$

we can write a symbol σ of class C^{n+1} as

$$\sigma(\xi,\eta) = \sum_{\nu\in\mathbb{Z}^n}\sum_{\rho\in\mathbb{Z}^n}\tau^{\nu,\rho}(\xi,\eta). \tag{7.9}$$

In the next lemma we obtain some elementary estimates based on this expansion.

Lemma 7.1. *Suppose $0 < p_1, p_2 < \infty$ and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$. Then:*
(i) There is a constant $C = C(p_1,p_2)$ so that for any (ν,ρ)

$$\|\tau^{\nu,\rho}\|_{\mathcal{M}_{p_1,p_2}} \le C(1+|\nu|+|\rho|)^{2m}H(A(\nu,\rho))$$

where $m = [(n+1)/2]$.
(ii) There is a constant $C = C(N, p_1, p_2)$ *such that if* σ *is of class* C^N*, and* $|\nu| + |\rho| > 0$*, then*

$$H(A(\nu,\rho)) \le C(1 + |\nu| + |\rho|)^{-N} \|\sigma\|_H^{(N)},$$

while

$$H(A(0,0)) \le C\|\sigma\|_H.$$

(iii) If $p_0 \ge 1$ *and* σ *is of class* C^N *then there is a constant* $C = C(N, p_1, p_2)$ *such that*

$$H(A(\nu,\rho)) \le C(1 + |\nu| + |\rho)^{2m-N} \|\sigma\|_{\mathcal{M}_{p_1,p_2}}^{(N)}.$$

Proof. Observe that $\widehat{\zeta}(2^{-j-3}\xi) = \widehat{\phi}(2^{-j-1}\xi) + \widehat{\phi}(2^{-j}\xi) + \widehat{\phi}(2^{-j+1}\xi)$ and therefore $\tau^{\nu,\rho}(\xi,\eta)$ is the sum of nine terms of the form

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} a_{j,k}(\nu,\rho) \big(e^{\frac{\pi i}{4}\langle 2^{-j}\xi,\nu\rangle} \widehat{\phi}(2^{-j-r}\xi)\big) \big(e^{\frac{\pi i}{4}\langle 2^{-k}\eta,\rho\rangle} \widehat{\phi}(2^{-k-s}\eta)\big)$$

where $r, s \in \{-1, 0, +1\}$. We now use Lemma 5.5, Lemma 6.3 (ii), and Lemma 6.1 in that order to obtain

$$\|\tau^{\nu,\rho}\|_{\mathcal{M}_{p_1,p_2}} \le C(1 + |\nu|)^m (1 + |\rho|)^m H(A(\nu,\rho))$$

where $m = [(n+1)/2]$. This proves (i).

For (ii) note that if $|\alpha|, |\beta| \le N$ integration by parts gives

$$a_{jk}(\nu,\rho) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\xi^\alpha \big(\sigma(2^{j+3}\xi, 2^{k+3}\eta) \widehat{\phi}(8\xi) \widehat{\phi}(8\eta)\big) \frac{e^{-2\pi i(\langle\xi,\nu\rangle + \langle\eta,\rho\rangle)}}{(-2\pi i\nu)^\alpha} d\xi d\eta, \tag{7.10}$$

$$a_{jk}(\nu,\rho) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\eta^\beta \big(\sigma(2^{j+3}\xi, 2^{k+3}\eta) \widehat{\phi}(8\xi) \widehat{\phi}(8\eta)\big) \frac{e^{-2\pi i(\langle\xi,\nu\rangle + \langle\eta,\rho\rangle)}}{(-2\pi i\rho)^\beta} d\xi d\eta, \tag{7.11}$$

provided $\nu_1^{\alpha_1} \dots \nu_n^{\alpha_n}$ and $\rho^{\beta_1} \dots \rho_n^{\beta_n}$ are nonzero.

Now using the fact that H is a norm it is easy to see that by choosing an appropriate α or β for each pair $(\nu,\rho) \ne (0,0)$ one obtains the estimate

$$H(A(\nu,\rho)) \le C(N, p_1, p_2)(1 + |\nu| + |\rho|)^{-N} \|\sigma\|_H^{(N)}.$$

If $(\nu,\rho) = (0,0)$ the same estimate follows directly from (7.7).

Finally we turn to (iii). For fixed δ_j, δ'_k with $\sup|\delta_j|, \sup|\delta'_k| \le 1$ let us define $\mu(\xi) = \sum_{j\in\mathbb{Z}} \delta_j \widehat{\phi}_j(\xi)$ and $\upsilon(\eta) = \sum_{k\in\mathbb{Z}} \delta'_k \widehat{\phi}_j(\eta)$. Then it follows from Lemma 5.5 that for any multi-indices α, α' we have

$$\| |\xi|^{|\alpha|+|\alpha'|} \partial_\xi^\alpha \mu(\xi) \partial_\xi^{\alpha'} \sigma(\xi,\eta) \upsilon(\eta) \|_{\mathcal{M}_{p_1,p_2}} \le C(\alpha,\alpha') \|\sigma\|_{\mathcal{M}_{p_1,p_2}}^{(|\alpha'|)}$$

This implies that for fixed N and any α with $|\alpha| = N$ we have

$$\sup_{|\delta_j|\le 1}\sup_{|\delta'_k|\le 1} \||\xi|^N \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \delta_j\delta'_k \partial_\xi^\alpha \sigma_{jk}(\xi,\eta)\|_{\mathcal{M}_{p_1,p_2}} \le C(N)\|\sigma\|^{(N)}_{\mathcal{M}_{p_1,p_2}}. \tag{7.12}$$

We now use either (7.7) if $(\nu,\rho) = (0,0)$ or we refer back to Proposition 6.2 (7.10) or (7.11) according to the values of ν or ρ, when $(\nu,\rho)\ne(0,0)$. For example when $N = |\nu| \ge |\rho|$ and the lth entry of ν has maximal size N, then

$$\begin{aligned}&\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}(\nu,\rho)\widehat{\phi_j}(\xi)\widehat{\phi_k}(\eta)\|_{\mathcal{M}_{p_1,p_2}}\\ \le& C\sup_{|\delta_j|\le 1}\sup_{|\delta'_k|\le 1}\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_k 2^{jN}\frac{\partial^N}{\partial\xi_l^N}\sigma_{jk}(\xi,\eta)\frac{e^{-2\pi i(\langle 2^{-j}\xi,\nu\rangle+\langle 2^{-k}\eta,\rho\rangle)}}{(-2\pi i\nu_l)^N}\|_{\mathcal{M}_{p_1,p_2}}.\end{aligned}$$

Now by Lemma 5.5 we can estimate the last expression side above by

$$C(1+|\nu|+|\rho|)^{2m-N}\sup_{|\delta_j|\le 1}\sup_{|\delta'_k|\le 1}\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta'_k|\xi|^N\frac{\partial^N}{\partial\xi_l^N}\sigma_{jk}(\xi,\eta)\|_{\mathcal{M}_{p_1,p_2}}.$$

Using (7.12) we obtain (iii). □

Let us state the main result of this section.

Theorem 7.2. *Suppose* $0 < p_1, p_2 < \infty$ *and* $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$. *Let* $N = 2n + 1$ *if* $p_0 \ge 1$ *and* $N = n + 2 + [\frac{n}{p_0}]$ *if* $p_0 < 1$. *Then for any* σ C^N-*symbol such that* $\|\sigma\|_H^{(N)} < \infty$ *we have* $\|\sigma\|_{\mathcal{M}_{p_1,p_2}} < \infty$. *Furthermore, there is a constant* $C = C(p_1,p_2)$ *so that* $\|\sigma\|_{\mathcal{M}_{p_1,p_2}} \le C\|\sigma\|_H^{(N)}$.

Proof. This follows directly from Lemma 7.1 and (7.9). Indeed, we have

$$\|\tau^{\nu,\rho}\|_{\mathcal{M}_{p_1,p_2}} \le C(1+|\nu|+|\rho|)^{2m-N}.$$

If $t = \min(p_0,1)$ we have

$$\|\sigma\|_{\mathcal{M}_{p_1,p_2}} \le C(\sum_{\nu\in\mathbb{Z}}\sum_{\rho\in\mathbb{Z}}(1+|\nu|+|\rho|)^{(2m-N)t})^{\frac{1}{t}}\|\sigma\|_H^{(N)}.$$

Since $(N-2m)t > n$ this gives the result. □

We next show that in a certain sense the preceding theorem is best possible.

Theorem 7.3. *Suppose* $1 < p_1, p_2 < \infty$ *and* $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} \le 1$. *Suppose* σ *is a* $C^\infty-$*symbol. Then the following are equivalent:*
(i) $\|\sigma\|^{(N)}_{\mathcal{M}_{p_1,p_2}} < \infty$ *for every* $N \ge 0$.
(ii) $\|\sigma\|_H^{(N)} < \infty$ *for every* $N \ge 0$.

Proof. Assume (i); then it follows from Lemma 7.1 that for any $N > 0$ we have an estimate $H(A(\nu,\rho)) \le C_N(1+|\nu|+|\rho|)^{-N}$. Now it is clear from the definition and from Theorem 6.5 and Lemma 6.3 that we have an estimate

$$\||\xi|^{|\alpha|}\partial_\xi^\alpha \tau^{\nu,\rho}\|_H \le C_\alpha(1+|\nu|)^{|\alpha|}H(A(\nu,\rho)).$$

Hence we can deduce easily that

$$\||\xi|^\alpha \partial_\xi^\alpha \sigma\|_H < \infty$$

for each multi-index α. Repeating the same reasoning with the second variable η gives (ii).

Now assume (ii). Then for any multi-index α one can see easily by differentiation that for any pair of multi-indices α,β we have that (ii) is satisfied by the symbols $|\xi|^{|\alpha|}\partial_\xi^\alpha\sigma$ and $|\eta|^{|\beta|}\partial_\eta^\beta\sigma$ in place of σ. Applying Theorem 7.2 gives (i). □

Now let us recast Theorem 7.2 in terms of estimates on the symbol σ using the results of Section 3.

Theorem 7.4. *Suppose $0 < p_1, p_2 < \infty$ and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $N = 2n+1$ if $p_0 \ge 1$ and $N = n+2+[\frac{n}{p_0}]$ if $p_0 < 1$. Suppose $\theta > 1$ Suppose σ is a C^N-symbol such that for any pair of multi-indices α,β with $0 \le |\alpha| \le N$ and $0 \le |\beta| \le N$ there exist constants C_α, C_β, with*

$$|\xi|^{|\alpha|}|\partial_\xi^\alpha\sigma(\xi,\eta)| \le C_\alpha(\log(1+|\log\tfrac{|\xi|}{|\eta|}|))^{-\theta} \tag{7.13}$$

$$|\eta|^{|\beta|}|\partial_\eta^\beta\sigma(\xi,\eta)| \le C_\beta(\log(1+|\log\tfrac{|\xi|}{|\eta|}|))^{-\theta}. \tag{7.14}$$

Then $\|\sigma\|_{\mathcal{M}_{p_1,p_2}} < \infty$.

Remark. We have already seen that in (7.2) that this is false when $0 < \theta < \frac{1}{2}$. However the arguments of Section 3 shows that we can improve (7.13) and (7.14) somewhat. For example we can replace $\big(\log(1+|\log\frac{|\xi|}{|\eta|}|)\big)^{-\theta}$ where $\theta > 1$ by $\big(\log(1+|\log\frac{|\xi|}{|\eta|}|)\big)^{-1}\big(\log(1+\log(1+|\log\frac{|\xi|}{|\eta|}|)\big)^{-\gamma}$ where $\gamma > 1$.

Proof. This follows immediately from Theorem 7.2 and Theorem 3.4 which yields the estimate

$$H(A) \le C\sup_{j,k}\frac{|a_{jk}|}{w_{|j-k|+1}}$$

with $w_k = \log(1+k)^{-\theta}$. □

It is possible to "mix and match" the estimates in Section 3: for example, in the following theorem we remove the conditions for $|\alpha|, |\beta| = 0$ but insist on a stronger condition for $|\alpha| = |\beta| = 1$:

Theorem 7.5. *Suppose $0 < p_1, p_2 < \infty$ and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $N = 2n+1$ if $p_0 \geq 1$ and $N = n + 2 + [\frac{n}{p_0}]$ if $p_0 < 1$. Suppose $\theta > 1$ Suppose σ is a C^N-symbol which satisfies conditions (7.13) and (7.14) for $2 \leq |\alpha|, |\beta| \leq N$ and if $|\alpha| = |\beta| = 1$*

$$|\xi|^{|\alpha|}|\partial_\xi^\alpha \sigma(\xi,\eta)| \leq C_\alpha (1 + |\log \tfrac{|\xi|}{|\eta|}|)^{-\theta} \tag{7.15}$$

$$|\eta|^{|\beta|}|\partial_\eta^\beta \sigma(\xi,\eta)| \leq C_\beta (1 + |\log \tfrac{|\xi|}{|\eta|}|)^{-\theta}. \tag{7.16}$$

Then $\|\sigma\|_{\mathcal{M}_{p_1,p_2}} < \infty$.

Proof. It is only necessary to show that $\|\sigma\|_H < \infty$. Note first that Proposition 3.1 can be used to give the estimate for any infinite matrix:

$$H(A) \leq C\big(\|A\|_\infty + \sup_j \sum_{k<j} |a_{j,k} - a_{j,k+1}| + \sup_k \sum_{j<k} |a_{j,k} - a_{j+1,k}|\big).$$

Now suppose $1 \leq |\xi|, |\eta| \leq 2$. Then if $k < j$,

$$|\sigma(2^j\xi, 2^k\eta) - \sigma(2^j\xi, 2^{k+1}\eta)| \leq Ck^{-\theta}$$

by (7.16). Combining with a similar estimate from (7.15) gives the theorem. □

We conclude this section with a theorem of the type of Theorem 7.2 for operators on L_1.

Theorem 7.6. *Suppose $N = 2n+3$ and that σ is a C^N-symbol with $\|\sigma\|_H^{(N)} < \infty$; then $W_\sigma : L_1 \times L_1 \to L_{\frac{1}{2},\infty}$ is bounded.*

Proof. Let Q be the cube $\{x : \max_k |x_k| \leq 1\}$ and consider the bilinear operator $W_{\sigma,Q}(f,g) = \chi_Q W_\sigma(f,g)$. We will show that if $r < \frac{1}{2}$ is such that $n + 2 + [\frac{n}{2r}] = N$, then $W_{\sigma,Q} : L_1(2Q) \times L_1(2Q) \to L_r(Q)$ is bounded and $\|W_{\sigma,Q}\| \leq C\|\sigma\|_H^{(N)}$ where C is a constant depending only on dimension.

Suppose that $f, g \in \mathcal{S}$ are functions with support contained in $2Q$ and such that $\int f(x)\,dx = \int g(x)\,dx = 0$. Then $f, g \in H_{2r}$ with $\|f\|_{H_{2r}} \leq C\|f\|_{L_1}$ and $\|g\|_{H_{2r}} \leq C\|g\|_{L_1}$. Applying Theorem 7.2 we obtain that

$$\|W_\sigma(f,g)\|_{L_r} \leq C\|\sigma\|_H^{(N)}\|f\|_{L_1}\|g\|_{L_1} \tag{7.17}$$

where C is an absolute constant. It follows that W_σ extends unambiguously to any $f, g \in L_1(2Q)$ with $\int f(x)\,dx = \int g(x)\,dx = 0$ and (7.17) holds.

Next fix $\psi \in \mathcal{S}$ so that $\int \psi(x)\,dx = 1$ and ψ has support contained in Q. Now for any $f, g \in L_1(3Q)$ let $f_0 = f - (\int f(x)\,dx)\psi$ and $g_0 = g - (\int g(x)\,dx)\psi$. Then (7.17) gives

$$\|W_{\sigma,Q}(f_0,g_0)\|_{L_r} \leq C\|\sigma\|_H^{(N)}\|f\|_{L_1}\|g\|_{L_1}.$$

We also note that $\|W_{\sigma,Q}(\psi,\psi)\|_{L_r} \le C\|\sigma\|_H^{(N)}$. Now consider the linear map $Tf = W_\sigma(f,\psi)$. Since $\psi \in L_2$ we have that, if $\frac{1}{s} = \frac{1}{2r} + \frac{1}{2}$, $T : H_{2r} \to L_s$ is bounded with norm controlled by $C\|\sigma\|_H^{(N)}$ (again using Theorem 7.2.) Hence since $r < s$,

$$\|W_{\sigma,Q}(f_0,\psi)\|_{L_r} \le C\|\sigma\|_H^{(N)}\|f\|_{L_1}.$$

Similarly

$$\|W_{\sigma,Q}(\psi,g_0)\|_{L_r} \le C\|\sigma\|_H^{(N)}\|g\|_{L_1}.$$

Combining these estimates gives

$$\|W_{\sigma,Q}(f,g)\|_{L_r} \le C\|\sigma\|_H^{(N)}\|f\|_{L_1}\|g\|_{L_1}. \tag{7.18}$$

We now use a Nikishin type argument as earlier in Lemma 2.3. Suppose $(f_j)_{j=1}^J$ and $(g_j)_{j=1}^J$ satisfy $\|f_j\|_{L_1}, \|g_j\|_{L_1} \le 1$ and that $\sum_{j=1}^J |b_j|^{1/2} = 1$. Then if $(\epsilon_j)_{j=1}^J$ and $(\epsilon'_j)_{j=1}^J$ are two independent sequences of Bernoulli random variables we have

$$\big(\mathbb{E}(\|\sum_{j=1}^J\sum_{k=1}^J \epsilon_j\epsilon'_k|b_j|^{\frac12}|b_k|^{\frac12}W_{\sigma,Q}(f_j,g_k)\|_{L_r}^r)\big)^{\frac1r} \le C\|\sigma\|_H^{(N)}.$$

Again by using the result of Bonami [1], we obtain an estimate

$$\|(\sum_{j=1}^J\sum_{k=1}^J |b_j||b_k||W_{\sigma,Q}(f_j,g_k)|^2)^{1/2}\|_{L_r} \le C\|\sigma\|_H^{(N)}.$$

Extracting the diagonal gives

$$\big\|\max_{1\le j\le J}|b_j||W_{\sigma,Q}(f_j,g_j)|\big\|_{L_r} \le C\|\sigma\|_H^{(N)}.$$

We now use [17] as before. There is a weight function $w \in L_1(Q)$ with $w \ge 0$ a.e. and $\int w(x)\,dx = 1$ so that for any $f,g \in L_1(3Q)$ with $\|f\|_{L_1}, \|g\|_{L_1} \le 1$ and any measurable $E \subset Q$ we have

$$\left(\int_E |W_\sigma(f,g)|^r dx\right)^{\frac1r} \le C\|\sigma\|_H^{(N)}\left(\int_E w(x)\,dx\right)^{\frac1r-2}.$$

Now suppose f,g are supported in Q and $\lambda > 0$. Let $E = \{x \in Q : |W_\sigma(f,g)| > \lambda$. Then the above equation yields

$$\lambda|E|^{\frac1r} \le C\|\sigma\|_H^{(N)}\left(\int_E w(x)\,dx\right)^{\frac1r-2}. \tag{7.19}$$

On the other hand if we apply (7.19) to $f_t(x) = f(x-t)$ where $t \in Q$ and note that $W_\sigma(f_t,g) = (W_\sigma(f,g))_t$ we also obtain that

$$\lambda|E\cap(Q+t)|^{\frac1r} \le C\|\sigma\|_H^{(N)}\left(\int_E w(x-t)\,dx\right)^{\frac1r-2}.$$

Raising to the power $(\frac{1}{r}-2)^{-1}$ and averaging gives:

$$\lambda|E|^{\frac{1}{r}} \le C\|\sigma\|_H^{(N)}|E|^{\frac{1}{r}-2}.$$

Thus $W_{\sigma,Q}$ maps $L_1(2Q)\times L_1(2Q)$ into $L_{\frac{1}{2},\infty}(Q)$ with norm at most $C\|\sigma\|_H^{(N)}$.

Now let $\lambda>1$. If we define $\sigma_\lambda(\xi,\eta)=\sigma(\lambda^{-1}\xi,\lambda^{-1}\eta)$, then we have $\|\sigma_\lambda\|_H^{(N)}=\|\sigma_\lambda\|_H^{(N)}$ and we can apply this result to σ_λ. Notice that $W_{\sigma_\lambda}(f,g)(x)=W_\sigma(f_\lambda,g_\lambda)(\lambda x)$ where $f_\lambda(x)=f(\lambda x)$ and $g_\lambda(x)=g(\lambda x)$. This implies that for any $\lambda>0$ we have the estimate

$$\|\chi_{\lambda Q}W_\sigma(f,g)\|_{L_{\frac{1}{2},\infty}} \le C\|\sigma\|_H^{(N)}\|f\|_{L_1}\|g\|_{L_1}$$

for f,g supported in λQ. Letting $\lambda\to\infty$ gives the result. □

8. Discussion on paraproducts

Paraproducts are bilinear operators of the type σ_A for some specific upper (or lower) triangular matrices A of zeros and ones. Paraproducts are important tools which have been used in several occasions in harmonic analysis, such as in the proof of the $T1$ theorem of David and Journé [6]. We define the lower and upper paraproducts as the bilinear operators Π_L and Π_U with symbols

$$\tau_L(\xi,\eta)=\sum_{j\in\mathbb{Z}}\sum_{k\le j-3}\widehat{\phi_j}(\xi)\widehat{\phi_k}(\eta)$$

and

$$\tau_U(\xi,\eta)=\sum_{k\in\mathbb{Z}}\sum_{j\le k-3}\widehat{\phi_j}(\xi)\widehat{\phi_k}(\eta)$$

respectively. It is easy to see that $\|\tau_L\|_{\mathcal{M}_{p_1,p_2}},\|\tau_U\|_{\mathcal{M}_{p_1,p_2}}<\infty$ for all $0<p_1,p_2<\infty$. This can be deduced in several ways, e.g. from Proposition 5.4 using Lemma 6.3 or directly from Theorem 7.2 and Proposition 3.1. We conclude that for all $0<p,q<\infty$ Π_L maps $H_{p_1}\times H_{p_2}\to H_{p_0}$ when $1/p_1+1/p_2=1/p_0$ and $H_q=L_q$ when $1<q<\infty$. We now turn to some endpoint cases regarding the paraproduct operator Π_L.

Proposition 8.1. *Let $0<q<\infty$. Then the paraproduct operator Π_L is bounded on the following products of spaces.*

1. $BMO\times H_q(\mathbb{R}^n)\to H_q(\mathbb{R}^n)$, *where $H_q=L_q$ when $1<q<\infty$.*
2. $BMO\times H_1(\mathbb{R}^n)\to L_1(\mathbb{R}^n)$.
3. $BMO\times L_\infty(\mathbb{R}^n)\to BMO$.
4. $H_q(\mathbb{R}^n)\times L_\infty(\mathbb{R}^n)\to H_q(\mathbb{R}^n)$, *where $H_q=L_q$ when $1<q<\infty$.*
5. $L_1(\mathbb{R}^n)\times L_\infty(\mathbb{R}^n)\to L_{1,\infty}(\mathbb{R}^n)$.
6. $BMO\times L_1(\mathbb{R}^n)\to L_{1,\infty}(\mathbb{R}^n)$.
7. $L_1(\mathbb{R}^n)\times L_1(\mathbb{R}^n)\to L_{1/2,\infty}(\mathbb{R}^n)$.

Proof. Statement (1) is a classical result on paraproducts when $1 < q < \infty$ and we refer the reader to [19] p. 303 for a proof. Note that for a fixed $f \in BMO$, the map $g \to \Pi_L(f,g)$ is a Calderón-Zygmund singular integral. The extension of (1) to H_q for $q \leq 1$, is consequence of the that if a convolution type singular integral operator maps $L_2 \to L_2$ with bound a multiple of $\|f\|_{BMO}$, then it also maps H_q into itself with bound a multiple of this constant. (2) follows from a similar observation while (3) is a dual statement to (2). To prove (4) set $\widetilde{S}_j g = \sum_{k \leq j-3} \widetilde{\Delta}_k g$. We have that $\Pi_L(f,g) = \sum_{j\in\mathbb{Z}} \widetilde{\Delta}_j f \widetilde{S}_j g$ and the Fourier transform of $\widetilde{\Delta}_j f \widetilde{S}_j g$ is supported in the annulus $2^{j-2} \leq |\xi| \leq 2^{j+2}$. It follows that

$$\|\Pi_L(f,g)\|_{H_q} \leq C \Big\| \Big(\sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f \widetilde{S}_j g|^2 \Big)^{1/2} \Big\|_{L_q} \leq \|f\|_{H_q} \|Mg\|_{L_\infty},$$

where M is the Hardy-Littlewood maximal operator which is certainly bounded on L_∞. To prove (5) we freeze g and look at the linear operator $f \to \Pi_L(f,g)$ whose kernel is $K(x,y) = \sum\limits_{j\in\mathbb{Z}} \phi_j(x-y) S_j(g)(x)$. It is easy to see that

$$|\nabla_y K(x,y)| \leq C \|g\|_{L_\infty} |x-y|^{-n-1}.$$

This estimate together with the fact that the linear operator $f \to \Pi_L(f,g)$ maps $L_2 \to L_2$ gives that $f \to \Pi_L(f,g)$ maps $L_1 \to L_{1,\infty}$ using the Calderón-Zygmund decomposition. This proves (5). To obtain (6) we use (1) (with $q = 2$) and we apply to the Calderón-Zygmund decomposition to the operator $g \to \Pi_L(f,g)$ for fixed $f \in BMO$. Finally (7) is a consequence of Theorem 7.6. □

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