Interpolation of multilinear operators acting between quasi-Banach spaces

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Abstract

We show that multilinear interpolation can be lifted to multilinear operators from spaces generated by the minimal methods to spaces generated by the maximal methods of interpolation defined on a class of couples of compatible p-Banach spaces. We also prove mutlilinear interpolation theorem for operators on Calderón-Lozanovskii spaces between L_p-spaces with $0 < p \leq 1$. As an application we obtain interpolation theorems for multilinear operators on quasi-Banach Orlicz spaces.

1 Introduction

In the study of many problems which appear in various areas of analysis it is essential to know whether important operators are bounded between certain quasi-Banach spaces. Motivated in particular by applications in harmonic analysis, we are interested in proving new abstract multilinear interpolation theorems for multilinear operators between quasi-Banach spaces. Based on ideas from the theory of operators between Banach spaces, we use the universal method of interpolation defined on proper classes of quasi-Banach spaces. It should be pointed out that in general the interpolation methods used in the case of Banach spaces do not apply in the setting of quasi-Banach spaces. The main reason is that the topological dual spaces of quasi-Banach spaces could be trivial and the same may be true for spaces of continuous linear operators between spaces from a wide class of quasi-Banach spaces.

We introduce relevant notation and we recall some definitions. Let $(X, \|\cdot\|)$ be a quasi-normed space. A quasi-norm induces locally bounded topology. A complete quasi-normed space is called quasi-Banach space. If in addition we have for some $0 < p \leq 1$

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \qquad x, y \in X,$$

2010*Mathematics Subject Classification*: 46B70, 46M35.

Key words and phrases: Interpolation spaces, quasi-Banach spaces, multilinear operators, functor Calderón-Lozanovskii spaces, Orlicz spaces.

The first named author was supported by the NSF under grant DMS 0900946. The second named author was supported by the National Science Centre (NCN), Poland, grant no. 2011/01/B/ST1/06243.

then X is said to be a *p-Banach space* (or if $p = 1$ a Banach space). A theorem of Aoki and Rolewicz (see [6]) states that every quasi-Banach space is p-normed for some $p \in (0, 1]$.

Throughout the paper we use the standard notion from the Banach space theory and interpolation theory. We refer to [2] and [3] for the fundamentals of interpolation theory that will be of use.

A pair $\overline{A} = (A_0, A_1)$ of quasi-Banach (p-Banach) spaces is called a quasi-Banach (p-Banach couple) if A_0 and A_1 are both algebraically and topologically embedded in some Hausdorff topological vector space. For a quasi-Banach (p-Banach) couple $\overline{A} = (A_0, A_1)$ we define quasi-Banach spaces $A_0 \cap A_1$ and $A_0 + A_1$ equipped with the natural norms. A quasi-Banach space A is called intermediate with respect to $\overline{A}$ provided $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$, where $\hookrightarrow$ denotes the continuous inclusion map.

Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be quasi-Banach couples. We denote by $\mathcal{L}(\overline{A}, \overline{B})$ the Banach space of all linear operators $T\colon A_0 + A_1 \to B_0 + B_1$ such that the restrictions of T to A_i are bounded operators from A_i to B_i for $i = 0, 1$. We equip $\mathcal{L}(\overline{A}, \overline{B})$ with the quasi-norm

$$\|T\|_{\overline{A}\to\overline{B}} = \max\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\}.$$

Let $\overline{A}$ and $\overline{X}$ be quasi-Banach couples. Following Aronszajn and Gagliardo [1], the *orbit* of an element $a \in A_0 + A_1$ in $\overline{X}$ is the quasi-Banach space $O_{\overline{A}}(a, \overline{X}) = \{Ta;\, T \in \mathcal{L}(\overline{A}, \overline{X})\}$ equipped with the norm

$$\|x\| = \inf\{\|T\|_{\overline{A}\to\overline{X}};\, Ta = x\}.$$

If we assume that $A_0 + A_1$ has a total dual space, then $X_0 \cap X_1 \hookrightarrow O_{\overline{A}}(a, \overline{X})$ and so $F(\cdot) := O_{\overline{A}}(a, \cdot)$ is an *exact interpolation functor*, i.e., for any quasi-Banach couples $\overline{X}$ and $\overline{Y}$ and every operator $T \in \mathcal{L}(\overline{X}, \overline{Y})$ we have $T\colon F(\overline{X}) \to F(\overline{Y})$ with

$$\|T\|_{F(\overline{X})\to F(\overline{Y})} \le \|T\|_{\overline{X}\to\overline{Y}}$$

Fix $0 < p \le 1$. Let $\overline{A} = (A_0, A_1)$ be a quasi-Banach couple such that $A_0 + A_1$ has a total dual space, and let A be an intermediate quasi-Banach space with respect to $\overline{A}$. For any quasi-Banach space $\overline{X}$ we define the *p-interpolation orbit space* $G^{A}_{\overline{A},p}(\overline{X})$ as the space of all $x \in X_0 + X_1$ such that

$$x = \sum_{n=1}^{\infty} T_n a_n \qquad (\text{convergence in } X_0 + X_1),$$

where $T_n \in \mathcal{L}(\overline{A}, \overline{X})$, $a_n \in A$, and $\sum_{n=1}^{\infty}(\|T_n\|_{\overline{A}\to\overline{X}}\|a_n\|_A)^p < \infty$. We set

$$\|x\|_{G_p} = \inf\Big(\sum_{n=1}^{\infty}(\|T_n\|_{\overline{A}\to\overline{X}}\|a_n\|_A)^p\Big)^{1/p},$$

where the infimum is taken over all admissible representations of x as above.

We notice here that if $\overline{X}$ is a p-Banach couple, then $G^{A}_{\overline{A},p}(\overline{X})$ is a p-Banach space, intermediate with respect to $\overline{X}$. Moreover we have

$$G^{A}_{\overline{A},p}(\overline{X}) \hookrightarrow F(\overline{X})$$

for any interpolation functor F such that $F(\overline{X})$ is p-normed space and $A \hookrightarrow F(\overline{A})$ (see [8, Proposition 2.1]).

Suppose we are given a quasi-Banach couple $\overline{B}$ and an intermediate quasi-Banach space B with respect to the couple $\overline{B}$. Following [1], we define for any Banach couple $\overline{X}$ the space $H^{B}_{\overline{B}}(\overline{X})$ of all $x \in X_0 + X_1$ such that $\sup_{\|T\|_{\overline{X}\to\overline{B}}\leq 1} \|Tx\|_B < \infty$. The quasi-norm in $H^{B}_{\overline{B}}(\overline{X})$ is given by

$$\|x\|_{H^{B}_{\overline{B}}(\overline{X})} = \sup\big\{\|Tx\|_B;\ \|T\|_{\overline{X}\to\overline{B}} \leq 1\big\}.$$

Note that if F is an interpolation method, then $F(\overline{X}) \hookrightarrow H^{B}_{\overline{B}}(\overline{X})$ for any quasi-Banach couple provided $F(\overline{B}) \hookrightarrow B$. This property, according to Aronszajn and Gagliardo [1] motivates calling $H^{B}_{\overline{B}}$ the maximal interpolation functor.

2 Main results

For each $m \in \mathbb{N}$ the product $X_1 \times \cdots \times X_m = \prod_{i=1}^{m} X_i$ of Banach spaces is equipped with the norm $\|(x_1, \ldots, x_m)\| = \max_{1\leq i\leq m} \|x_i\|_{X_i}$. We denote by $\mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ the quasi-Banach space of all m-linear bounded operators defined on $X_1 \times \cdots \times X_m$ with values in a quasi-Banach space Y, equipped with the quasi-norm

$$\|T\| = \sup\{\|T(x_1, \ldots, x_m)\|_Y;\ \|x_1\|_{X_1} \leq 1, \ldots, \|x_m\|_{X_m} \leq 1\}.$$

As in the case where $m = 1$, we write $\mathcal{L}(X_1, Y)$ instead of $\mathcal{L}_1(X_1, Y)$.

Let $\overline{Y} = (Y_0, Y_1)$ and $\overline{X}_i = (X_{0i}, X_{1i})$ for each $1 \leq i \leq m$ be couples of quasi-Banach spaces. If an operator $T \in \mathcal{L}_m(\Pi_{i=1}^{m}(X_{0i} + X_{1i}), Y_0 + Y_1)$ is such the restriction of T is bounded from $X_{j1} \times \cdots \times X_{jm}$ to Y_j for $j = 0, 1$, then we write $T \in \mathcal{L}_m(\Pi_{i=1}^{m}\overline{X}_i, \overline{Y})$.

Assume that X_i are quasi-Banach spaces intermediate with respect to $\overline{X}_i$ for $1 \leq i \leq m$ and Y is a quasi-Banach intermediate with respect to $\overline{Y}$. If there exists a finite constant $C > 0$ such that for every $T \in \mathcal{L}_m(\Pi_{i=1}^{m}\overline{X}_i, \overline{Y})$, the restriction of T is bounded from $X_1 \times \cdots \times X_m$ to Y with $\|T\| \leq C$, then $X_1, \ldots, X_m$ and Y are called C-multilinear interpolation spaces with respect to $(\overline{X}_1, \ldots, \overline{X}_m)$ and $\overline{Y}$ (we write for short $(X_1, \ldots, X_m; Y) \in \mathcal{M}int_C(\overline{X}_1, \ldots, \overline{X}_m; \overline{Y})$).

Our first result is the following.

Theorem 2.1. *Let $G^{A_i}_{\overline{A}_i,p}$ be a p-interpolation orbit for each $1 \leq i \leq m$ and let $H^{B}_{\overline{B}}$ be a maximal interpolation method. Assume that $(A_1, \ldots, A_m; B) \in \mathcal{M}int_C(\overline{A}_1, \ldots, \overline{A}_m; \overline{B})$, then for any p-Banach couples $\overline{X}_1$,...,$\overline{X}_m$ and any quasi-Banach couple $\overline{Y}$, we have*

$$\big(G^{A_1}_{\overline{A}_1,p}(\overline{X}_1), \ldots, G^{A_m}_{\overline{A}_m,p}(\overline{X}_m); H^{B}_{\overline{B}}(\overline{Y})\big) \in \mathcal{M}int_C(\overline{X}_1, \ldots, \overline{X}_m; \overline{Y}).$$

Proof. Let $X_i := G^{A_i}_{A_i,p}(\overline{X}_i)$ for each $1 \leq i \leq m$ and $Y := H^{B}_{\overline{B}}(\overline{Y})$. For $j = 0, 1$ fix $T \in \mathcal{L}_m(\Pi_{i=1}^m (X_{0i} + X_{1i}), Y_0 + Y_1)$ such that $\|T\|_{\mathcal{L}_m(X_{j1},\dots,X_{jn};Y_j)} \leq 1$. Assume that $x_i \in X_i$ and $x_i = S_i a_i$ with $a_i \in A_i$, where $S_i\colon \overline{A}_i \to \overline{X}_i$. For a given $R\colon \overline{Y} \to \overline{B}$ with $\|R\|_{\overline{Y}\to\overline{B}} \leq 1$ define an operator $U_R\colon \Pi_{i=1}^m (A_{0i} + A_{1i}) \to B_0 + B_1$ by setting

$$U_R(v_1, \dots, v_m) = RT(S_1 v_1, \dots, S_m v_m), \qquad (v_1, \dots, v_m) \in \Pi_{i=1}^m (A_{0i} + A_{1i}).$$

For $j = 0, 1$ we have

$$\|U_R(u_1, \dots, u_m)\|_{B_j} \leq \|R\|_{\overline{Y}\to\overline{B}} \|T\|_{\mathcal{L}_m(X_{j1},\dots,X_{jn};Y_j)} \prod_{i=1}^m \|S_i\|_{\overline{A}_i \to \overline{X}_i} \|v_i\|_{A_{ji}}$$

and so $U_R \in \mathcal{L}_m(\overline{A}_1 \times \cdots \times A_m, \overline{B})$ with and its norm satisfies

$$\|U_R\| \leq \prod_{i=1}^m \|S_i\|_{\overline{A}_i \to \overline{X}_i}.$$

Our hypothesis gives that $U_R \in \mathcal{L}_m(A_1, \dots, A_m, B)$ and

$$\|U_R\|_{\mathcal{L}_m(A_1,\dots,A_m,B)} \leq C \prod_{i=1}^m \|S_i\|_{\overline{A}_i \to \overline{X}_i}.$$

Consequently, we obtain

$$\begin{aligned} \|T(x_1, \dots, x_m)\|_Y &\leq \sup\{\|RT(S_1 v_1, \dots, S_m v_m)\|_B;\ \|R\|_{\overline{Y}\to\overline{B}} \leq 1\} \\ &= \sup\{\|U_R(a_1, \dots, a_m)\|_B;\ \|R\|_{\overline{Y}\to\overline{B}} \leq 1\} \\ &\leq C \prod_{i=1}^m \|S_i\|_{\overline{A}_i \to \overline{X}_i} \|a_i\|_{A_i}. \end{aligned} \tag{1}$$

Suppose now that for each $1 \leq i \leq m$

$$x_i = \sum_{j=1}^{\infty} S_{ij} a_{ij} \quad \text{(convergence in } X_{0i} + X_{1i}),$$

where $S_{ij}\colon \overline{A}_i \to \overline{X}_i$, $a_{ij} \in A_i$ are such that

$$\sum_{j=1}^{\infty} (\|S_{ij}\|_{\overline{A}_j \to \overline{X}_j} \|a_{ij}\|_{A_i})^p < \infty.$$

Since $T \in \mathcal{L}_m(\Pi_{i=1}^m (X_{0i} + X_{1i}), Y_0 + Y_1)$ we conclude that

$$T(x_1, \dots, x_m) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_m=1}^{\infty} T(S_{1j_1} a_{1j_1}, \dots, S_{mj_m} a_{mj_m}) \quad \text{(convergence in } Y_0 + Y_1).$$

Estimate (1) yields for each $j_1, \dots, j_m$

$$\|T(S_{1j_1} a_{1j_1}, \dots, S_{mj_m} a_{mj_m})\|_Y \leq C \prod_{i=1}^m \|S_{ij_i}\|_{\overline{A}_{j_i} \to \overline{X}_{j_i}} \|a_{j_i}\|_{A_{j_i}},$$

and so

$$\begin{aligned}
\|T(x_1,\dots,x_m)\|_Y &\le C\Big(\sum_{j_1=1}^{\infty}\cdots\sum_{j_m=1}^{\infty}\|T(S_{1j_1}a_{1j_1},\dots,S_{mj_m}a_{mj_m})\|_Y^p\Big)^{1/p}\\
&\le C\Big(\sum_{j_1=1}^{\infty}\cdots\sum_{j_m=1}^{\infty}\prod_{i=1}^{m}\big(\|S_{ij_i}\|_{\overline{A}_i\to\overline{X}_i}\|a_{ij}\|_{A_i}\big)^p\Big)^{1/p}\\
&= C\prod_{i=1}^{m}\Big(\sum_{j=1}^{\infty}\big(\|S_{ij}\|_{\overline{A}_{j_i}\to\overline{X}_j}\|a_{ij}\|_{A_i}\big)^p\Big)^{1/p}.
\end{aligned}$$

Combining the above estimates, we conclude that $T\in\mathcal{L}_m(X_1\times\cdots\times X_m,Y)$ with $\|T\|\le C$, and this completes proof. $\square$

Our results could be applied to the real methods of interpolation. Let $0<p\le 1$. Following [9], a quasi-Banach space E is said to be (p,J)-nontrivial, if

$$E\hookrightarrow \ell_p+\ell_p(2^{-n}).$$

Let E be a nontrivial $(1,J)$ quasi-Banach lattice. For any quasi-Banach couple (X_0,X_1) we denote by $J_E(\overline{X})$ the space of all $x\in X_0+X_1$, which can be represented in the form

$$x=\sum_{n=-\infty}^{\infty}x_n,\qquad x_n\in X_0\cap X_1\quad(\text{convergence in } X_0+X_1),$$

with $\{J(2^n,x_n;\overline{X})\}\in E$ where $J(t,x;\overline{X})=\max\{\|x\|_{X_0},t\|x\|_{X_1}\}$ for all $x\in X_0\cap X_1$, $t>0$. The space $J_E(\overline{X})$ is said to be a J-space provided it is a quasi-Banach space under the quasi-norm,

$$\|x\|=\inf\|\{J(2^n,x_n;\overline{X})\}\|_E,$$

where the infimum is taken over all representations of $x=\sum_n x_n$ as above.

Theorem 2.2. *For each $1\le i\le m$ let $\overline{X}_i$ be p-Banach couples and let $J_{E_i}(\overline{X}_i)$ be J-spaces generated by quasi-Banach lattices on $\mathbb{Z}$ intermediate between $\overline{\ell}_p=(\ell_p,\ell_p(2^{-n}))$, and let $H_{\overline{B}}^{B}$ be a maximal interpolation method. Assume that $(E_1,\dots,E_m;B)\in\mathcal{M}int_C(\overline{\ell}_p,\dots,\overline{\ell}_p;\overline{B})$. Then for any quasi-Banach couple $\overline{Y}$ we have*

$$\big(J_{E_1}(\overline{X}_1),\dots,J_{E_m}(\overline{X}_m);H_{\overline{B}}^{B}(\overline{Y})\big)\in\mathcal{M}int_C(\overline{X}_1,\dots,\overline{X}_m;\overline{Y}).$$

Proof. It follows from [8, Theorem 3.2] that the continuous inclusion map

$$J_{E_i}(\overline{X}_i)\hookrightarrow G_{\overline{\ell}_p,p}^{E_i}(\overline{X}_i),\qquad 1\le i\le m$$

has norm less or equal than 1. Thus the required statement follows from Theorem 2.1. $\square$

We refer to [5] where interpolation of bilinear operators between quasi-Banach spaces was studied by the method of means.

3 Multilinear interpolation between Orlicz spaces

In what follows we let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a complete σ-finite measure space and let $L^0(\Omega, \mu) = L^0(\mu)$ denote the space of equivalence classes of real valued measurable functions on Ω, equipped with the topology of convergence (in the measure μ) on sets of finite measure. By a quasi-Banach lattice on Ω we mean a quasi-Banach space X which is a subspace of $L^0(\mu)$ such that there exists $u \in X$ with $u > 0$ and if $|f| \leq |g|$ a.e., where $g \in X$ and $f \in L^0(\mu)$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$. A quasi-Banach lattice X is said to be *maximal* if its unit ball $B_X = \{x; \|x\| \leq 1\}$ is a closed subset in $L^0(\mu)$.

In the special case when $\Omega = \mathbb{Z}$ is the set of integers and μ is the counting measure then a quasi-Banach lattice E on Ω is called a quasi-Banach sequence space on $\mathbb{Z}$.

If X is a quasi-Banach lattice on (Ω, μ) and $w \in L^0(\mu)$ with $w > 0$ a.e., we define the weighted quasi-Banach lattice $X(w)$ by $\|x\|_{X(w)} = \|x\, w\|_X$.

Throughout the rest of the paper for given measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ $1 \leq i \leq m$, we let (Ω, Σ, μ) to be a product measure space with $\Omega := \Omega_1 \times \cdots \times \Omega_m$, $\Sigma := \Sigma_1 \times \cdots \times \Sigma_m$ and $\mu := \mu_1 \times \cdots \times \mu_m$ be a product measure space.

We define a map $\bigotimes\colon L^0(\mu_1) \times \cdots \times L^0(\mu_m) \to L^0(\mu)$ by

$$\bigotimes(f_1, \dots, f_m) = f_1 \otimes \cdots \otimes f_m, \qquad (f_1, \dots, f_m) \in L^0(\mu_1) \times \cdots \times L^0(\mu_m),$$

where $f_1 \otimes \cdots \otimes f_m(\omega_1, \dots, \omega_m) = f_1(\omega_1) \cdots f_m(\omega_m)$ for all $(\omega_1, \dots, \omega_m) \in \Omega_1 \times \cdots \times \Omega_m$.

Next we state a useful result in terms of applications.

Theorem 3.1. *Let $0 < p_0, p_1 \leq 1$ and X_i be quasi-Banach lattices on (Ω_i, μ_i) that are intermediate with respect to $(L^{p_0}(\mu_i), L^{p_1}(\mu_i))$ for $1 \leq i \leq m$ and let Y be a quasi-Banach space intermediate with respect to a quasi-Banach couple (Y_0, Y_1). Assume X is a quasi-Banach function lattice on (Ω, μ) such that $\bigotimes\colon X_1 \times \cdots \times X_m \to X$ with $\left\|\bigotimes\right\| \leq C_1$ and $(X, Y) \in int_{C_2}((L^{p_0}(\mu), L^{p_1}(\mu); (Y_0, Y_1))$. Then, with $C = C_1 C_2$, we have*

$$(X_1, \dots, X_m; X) \in \mathcal{M}int_C\big((L^{p_0}(\mu_1), L^{p_1}(\mu_1)) \times \cdots \times (L^{p_0}(\mu_m), L^{p_1}(\mu_m)); (Y_0, Y_1)\big).$$

Proof. In [13] Vogt identifies the tensor product

$$L^p(\mu_1)\widehat{\otimes}_p \cdots \widehat{\otimes}_p L^p(\mu_m) \tag{2}$$

for $m = 2$, however the proof works for each positive integer $m \geq 2$. In our setting this implies that there are continuous linear operators $T_0\colon L^{p_0}(\mu) \to Y_0$ and $T_1\colon L^{p_1}(\mu) \to Y_1$ such that

$$T(f_1, \dots, f_m) = T_0 \bigotimes(f_1, \dots, f_m), \qquad (f_1, \dots, f_m) \in \prod_{i=1}^m L^{p_0}(\mu_i),$$

and

$$T(g_1,\dots,g_m) = T_1\bigotimes(g_1,\dots,g_m), \qquad (g_1,\dots,g_m) \in \prod_{j=1}^{m} L^{p_1}(\mu_i).$$

In particular this yields that for all $(f_1,\dots,f_m) \in \prod_{i=1}^m (L_{p_0}(\mu_i) \cap L^{p_1}(\mu_i))$,

$$T(f_1,\dots,f_m) = T_0(f_1\otimes\cdots\otimes f_m) = T_1(f_1\otimes\cdots\otimes f_m).$$

The density argument (via (2)) implies that $T_0 = T_1$ on $L^{p_0}(\mu)\cap L^{p_1}(\mu)$. Consequently, the operator $\widetilde{T}\colon L^{p_0}(\mu)+L^{p_1}(\mu) \to Y_0+Y_1$ given by $\widetilde{T}(f) = T_0(f_0)+T_1(f_1)$ for any $f = f_0+f_1$ with $f_j \in L_{p_j}(\mu)$ for $j=0,1$ is well defined. Since $\widetilde{T} = T_j$ on $L^{p_j}(\mu)$, it follows

$$\widetilde{T}\colon (L^{p_0}(\mu), L^{p_1}(\mu)) \to (Y_0, Y_1).$$

Hence we have $T = \widetilde{T}\bigotimes$ with

$$T\colon (L^{p_0}(\mu_1), L^{p_1}(\mu_1)) \times\cdots\times (L^{p_m}(\mu_m), L_{p_m}(\mu_m)) \xrightarrow{\bigotimes} (L^{p_0}(\mu), L^{p_1}(\mu)) \xrightarrow{\widetilde{T}} (Y_0, Y_1).$$

Our interpolation hypotheses imply that

$$T\colon X_1\times\cdots\times X_m \xrightarrow{\bigotimes} X \xrightarrow{\widetilde{T}} Y.$$

with $\|T\| \le \|\widetilde{T}\|\,\|\bigotimes\| \le C_1C_2$, and this completes the proof. $\square$

To obtain a variety of applications we recall some interpolation constructions. Let Φ denote the set of non-vanishing functions $\varphi\colon [0,\infty)\times[0,\infty) \to [0,\infty)$, which are non-decreasing in each variable and positively homogeneous of degree one. We define Φ_0 to be a subset of all $\varphi\in\Phi$ such that $\varphi(1,t)\to 0$ and $\varphi(t,1)\to 0$ as $t\to 0$.

We consider the Banach couples $\overline{c}_0 = (c_0, c_0(2^{-n}))$ and $\overline{\ell}_\infty := (\ell_\infty, \ell_\infty(2^{-n}))$ of sequences on $\mathbb{Z}$. If $\varphi\in\Phi$ (resp., $\varphi\in\Phi_0$), then $a_\varphi \in c_0 + c_0(2^{-n})$ (resp., $a_\varphi \in \ell_\infty + \ell_\infty(2^{-n})$).

For any $\varphi\in\Phi$ (resp., $\varphi\in\Phi_0$) and any quasi-Banach couple $\overline{X}$, we denote by $\langle X_0, X_1\rangle_\varphi$ (resp., $\varphi_\ell(\overline{X})$) the interpolation orbit space $O_{\overline{c}_0}(a_\varphi, \overline{X})$ (resp., $O_{\overline{\ell}_\infty}(a_\varphi, \overline{X})$). It was shown in [10], [9] that for a large class of couples of quasi-Banach lattices these spaces are connected with the Calderón-Lozanovskii spaces.

We recall that if $\overline{X} = (X_0, X_1)$ is a couple of quasi-Banach lattices on a measure space (Ω,μ) and $\varphi\in\Phi$, then the Calderón-Lozanovskii space $\varphi(\overline{X}) = \varphi(X_0,X_1)$ consists of all $x\in L^0(\mu)$ such that $|x| = \varphi(|x_0|,|x_1|)$ μ-a.e. for some $x_j\in X_j$ with $\|x_j\|_{X_j}\le 1$, $j=0,1$. The space $\varphi(\overline{X})$ is a quasi-Banach lattice equipped with the quasi-norm [7])

$$\|x\|_{\varphi(\overline{X})} := \inf\big\{\max_{j=0,1}\|x_j\|_{X_j};\ \varphi(|x_0|,|x_1|) = |x|\big\}.$$

If φ is a concave function then $\varphi(X_0,X_1)$ is a Banach lattice provided (X_0,X_1) is a Banach couple (see [7]). In the case when $\varphi(s,t) = s^{1-\theta}t^\theta$, $\varphi(\overline{X})$ is the Calderón space (see [4]).

We note that in analogy with the Banach case the following continuous inclusion

$$\varphi(X_0, X_1) \hookrightarrow \varphi_\ell(X_0, X_1)$$

holds for all couples (X_0, X_1) of quasi-Banach lattices and $\varphi \in \Phi$. It was shown in [10, 11] (see also [9]) that for any $\varphi \in \Phi_0$ the following continuous inclusions hold for a large class $\mathcal{C}$ of couples (X_0, X_1) of quasi-Banach lattices,

$$\varphi_l(X_0, X_1) \hookrightarrow \langle X_0, X_1\rangle_\varphi \hookrightarrow \varphi_\ell(X_0, X_1)^c, \tag{3}$$

where $\varphi(X_0, X_1)$ is a Calderón-Lozanovskii space and $\varphi_\ell(X_0, X_1)^c$ is the Gagliardo completion of $\varphi(X_0, X_1)$ with respect $X_0 + X_1$, i.e., the space of all limits in $X_0 + X_1$ of sequences $\{x_n\}$ that are bounded in $X = \varphi(X_0, X_1)$; this space is equipped with the quasi-norm

$$\|x\|_{X^c} = \inf_{\{x_n\}} \sup_{n\geq 1} \|x_n\|_X,$$

where $\{x_n\} \subset X$ has the same meaning as above.

The class $\mathcal{C}$ contains p-convex quasi-Banach lattices with $1 \leq p < \infty$. Recall that a quasi-Banach lattice X is called p-convex if there exists a constant $C > 0$ such that if $x_1, \dots, x_n \in X$ then

$$\Big\| \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} \Big\|_X \leq C \Big(\sum_{i=1}^n \|x_i\|_X^p\Big)^{1/p}.$$

Notice that if $\varphi \in \Phi$ then $\varphi = \varphi_0 + \varphi_1$, where $\varphi_1(s,t) := as + bt$ for all $s, t \geq 0$ with $a = \lim_{s\to 0+} \varphi(1, s)$, $b = \lim_{s\to 0+} \varphi(s, 1)$. Thus $a_\varphi = a_{\varphi_0} + a_{\varphi_1}$. Clearly that

$$\varphi(\overline{X}) = \varphi_0(\overline{X}) + \varphi_1(\overline{X})$$

for any couple $\overline{X}$ of quasi-Banach lattices. Since $\varphi_0 \in \Phi$, thus combining the above remarks and the mentioned inclusions (3), we can easily deduce that the following continuous inclusions

$$\varphi(X_0, X_1) \hookrightarrow \varphi_\ell(X_0, X_1) \hookrightarrow \varphi(X_0, X_1)^c$$

are true every $\varphi \in \Phi$ and all quasi-Banach couples $(X_0, X_1) \in \mathcal{C}$.

Theorem 3.2. *Let $L^{p_0}_{w_{0i}}$ and $L^{p_1}_{w_{1i}}$ with $0 < p_0, p_1 \leq 1$ be weighted spaces on measure spaces (Ω_i, μ_i) for $1 \leq i \leq m$, and let (Y_0, Y_1) be a couple of quasi-Banach spaces such that Y_0 is p_0-normed and Y_1 is a p_1-normed space. Assume that for some $\varphi_1, \dots, \varphi_m, \varphi \in \Phi$ there exists $C > 0$ such that $\varphi(1, t_1 \cdots t_m) \leq C\varphi_1(1, t_1) \cdots \varphi_m(1, t_m)$ for all $t_1, \dots, t_m \geq 0$. Then any operator in $\mathcal{L}_m((L^{p_0}(\mu_1), L^{p_1}(\mu_1)) \times \cdots \times (L^{p_0}(\mu_m), L^{p_1}(\mu_m)), (Y_0, Y_1))$ lies in*

$$\mathcal{L}_m(\varphi_1(L^{p_0}(\mu_1), L^{p_1}(\mu_1)) \times \cdots \times \varphi_m(L_{p_0}(\mu_m), L_{p_1}(\mu_m)), \varphi_\ell(Y_0, Y_1)).$$

Proof. It is clear that the condition $\varphi(1, t_1\cdots t_m) \le C\varphi_1(1, t_1)\cdots\varphi_m(1, t_m)$ for all $t_1, \ldots, t_m \ge 0$ is equivalent to

$$\varphi(s_1 \cdots s_m, t_1 \cdots t_m) \le C\varphi_1(s_1, t_1) \cdots \varphi_m(s_m, t_m), \quad s_i, t_i > 0, \quad 1 \le i \le m.$$

This easily implies that for any measure space (Ω_i, ν_i) with $1 \le i \le m$ and $\nu := \nu_1 \times \cdots \times \nu_m$,

$$\bigotimes\colon \varphi_1(L^{p_0}(\nu_1), L^{p_1}(\nu_1)) \times \cdots \times \varphi_m(L^{p_0}(\nu_m), L^{p_1}(\nu_m)) \to \varphi(L^{p_0}(\nu), L^{p_1}(\nu))$$

with $\|\bigotimes\| \le C$.

Without loss of the generality we my assume that $1/q := 1/p_0 - 1/p_1 > 0$. Based on ideas of Stein-Weiss [12], for each $1 \le i \le m$ set $\tau_i = (w_{1i}^{1/p_0} w_{0i}^{1/p_1}), \sigma_i = (w_{0i}/\tau_i)^{p_0}$ and $d\nu_i = \sigma_i d\mu_i$. For $f \in L^0(\nu_i)$ define $S_i(f) = \tau_i f$ and note that for each $1 \le i \le m$ and $j = 0, 1$ we have

$$\|S_i f\|_{L^{p_j}(\nu_i)} = \|f\|_{L^{p_j}_{w_{ji}}}, \quad f \in L^{p_j}_{w_{ji}}.$$

This shows that S_i is a positive isometrical isomorphism between quasi-Banach couples $(L^{p_0}_{w_{0i}}, L^{p_1}_{w_{1i}})$ and $(L^{p_0}(\nu_i), L^{p_1}(\nu_i))$ for each $1 \le i \le m$, and so

$$S_i(\varphi_i(L^{p_0}_{w_{0i}}, L^{p_1}_{w_{1i}})) = \varphi_i(L^{p_0}(\nu_i), L^{p_1}(\nu_i)), \quad 1 \le i \le m.$$

This implies that the operator S given by

$$S(f_1, \ldots, f_m) := T(S_1^{-1} f_1, \ldots, S_m^{-1} f_m)$$

for all $(f_1, \ldots, f_m) \in \prod_{i=1}^m (L^{p_0}(\nu_i) + L^{p_1}(\nu_i))$, satisfies

$$S \in \mathcal{L}_m((L^{p_0}(\nu_0), L^{p_1}(\nu_0)) \times \cdots \times ((L^{p_0}(\nu_m), L^{p_1}(\nu_m)), (Y_0, Y_1)).$$

Applying Theorem 3.1, we conclude that S is bounded from $\prod_{i=1}^m \varphi_i(L^{p_i}(\nu_i), L^{p_1}(\nu_i))$ to $\varphi_\ell(Y_0, Y_1)$. The above isometries yield that the operator

$$T\colon \varphi_1(L^{p_0}(\nu_0), L^{p_1}(\nu_0)) \times \cdots \times \varphi_m(L^{p_0}(\nu_m), L^{p_1}(\nu_m)) \to \varphi_\ell(Y_0, Y_1)$$

is bounded, and this completes the proof. □

We give below applications of Theorem 3.2 to quasi-Banach Orlicz spaces. First we recall that if ψ is an Orlicz function (i.e., $\psi\colon [0, \infty) \to [0, \infty)$ is an increasing, continuous such that $\psi(0) = 0$), then the Orlicz space L_ψ on a given measure space (Ω, μ) is defined to be a subspace of $L^0(\mu)$ consisting of all $f \in L^0(\mu)$ such that for some $\lambda > 0$ we have $\int_\Omega \psi(\lambda|f|)\, d\mu < \infty$. We set

$$\|f\|_\psi = \inf\Big\{\lambda > 0;\ \int_\Omega \psi(|f|/\lambda)\, d\mu \le 1\Big\}.$$

If there exists $C > 0$ such that $\psi(t/C) \le \psi(t)/2$ for all $t > 0$, then

$$\|f+g\|_\psi \le C(\|f\|_\psi + \|g\|_\psi), \quad f, g \in L_\psi,$$

and so L_ψ is a quasi-Banach space. In what follows we consider only Orlicz spaces L^ψ generated by Orlicz functions ψ which satisfy the above inequality.

It is well known (see [9], [11, pp. 460-461]) that for any $\varphi \in \Phi$ and any couple $(L^{p_0}_{w_0}, L^{p_1}_{w_1})$ on a measure space (Ω, μ) with $0 < p_0, p_1 \le \infty$ we have

$$\varphi(L^{p_0}_{w_0}, L^{w_1}_{w_1}) = L_M$$

with equivalence of the quasi-norms, where L_M is the generalized Orlicz space generated by the function $M(u,t) = \psi((w_1(t)^{1/p_1} w_0(t)^{-1/p_0})^q)(w_0(t)/w_1(t))^q$ for all $u \ge 0$ and $t \in \Omega$. Here $1/q = 1/p_0 - 1/p_1$ and ψ is an Orlicz function given by $\psi^{-1}(t) = \varphi(t^{1/p_0}, t^{1/p_1})$ for all $t > 0$. L_M is equipped with the quasi-norm

$$\|f\|_\Phi = \inf\Big\{\lambda > 0;\ \int_\Omega M(|f(t)|/\lambda, t)\, d\mu \le 1\Big\}.$$

In particular we have

$$\varphi(L^{p_0}, L^{p_1}) = L_\psi.$$

We note also that a simple calculation shows (see [11, p. 459]) that in the case $0 < p_0 = p_1 = p \le \infty$,

$$\varphi(L^p_{w_0}, L^p_{w_1}) = L^p_{1/\varphi(w_0, w_1)}. \tag{4}$$

Our final result is the following:

Theorem 3.3. *Let $L_\psi(\nu)$ and $L_{\psi_i}(\mu_i)$ for $1 \le i \le m$ be Orlicz spaces. Assume that there is a constant C and $0 < p_0 < p_1 \le 1$ so that $\psi(t_1 \cdots t_m) \ge C\psi_1(t_1)\cdots\psi_m(t_m)$ for all $t_1, \ldots, t_m > 0$ and assume that the functions $t \mapsto \psi(t)/t^{p_0}$, $t \mapsto \psi_i(t)/t^{p_0}$ are non-decreasing and $t \mapsto \psi(t)/t^{p_1}$, $t \mapsto \psi_i(t)/t^{p_1}$ $(1 \le i \le m)$ are non-increasing. Then for every $T \in \mathcal{L}_m((L^{p_0}(\mu_1), L^{p_1}(\mu_1)) \times \cdots \times (L^{p_0}(\mu_m), L^{p_1}(\mu_m)), (L^{p_0}(\nu), L^{p_1}(\nu))$ we have that*

$$T \in \mathcal{L}_m(L_{\psi_1}(\mu_1) \times \cdots \times L_{\psi_m}(\mu_m), L_\psi(\nu)).$$

Proof. First observe that if an Orlicz function ϕ is such that the function $t \mapsto \phi(t)/t^{p_0}$ is non-decreasing and $t \mapsto \phi(t)/t^{p_1}$ is non-increasing, respectively. Then the function $t \mapsto \phi^{-1}(t)/t^{1/p_0}$ and $t \mapsto \phi^{-1}(t)/t^{1/p_1}$ is non-increasing and non-decreasing, respectively. This implies that the function $\rho\colon [0,\infty) \to [0,\infty)$ defined by $\rho(0) = 0$ and

$$\phi^{-1}(t) = t^{1/p_1}\rho(t^{1/p_0 - 1/p_1}), \quad t > 0,$$

is a quasi-concave, i.e., $t \mapsto \rho(t)$ is non-decreasing and $t \mapsto \rho(t)/t$ is non-increasing.

Now define $\varphi \in \Phi$ by $\varphi(s,t) = t\rho(s/t)$ for $t > 0$ and 0 if $t = 0$. Then, we have

$$\phi^{-1}(t) \asymp \varphi(t^{1/p_0}, t^{1/p_1}), \qquad t \geq 0.$$

Thus, it follows by (4) that for any measure space (Ω, ν) we have that

$$L_\phi(\nu) = \varphi(L^{p_0}(\nu), L^{p_1}(\nu)),$$

up to equivalence of quasi-norms.

Combining our hypotheses, we conclude that there exist $\varphi, \varphi_i \in \Phi$ such that

$$L_\psi(\nu) = \varphi(L^{p_0}(\nu), L^{p_1}(\nu)), \qquad L_{\psi_i}(\mu_i) = \varphi_i(L^{p_0}(\mu_i)), L^{p_1}(\mu_i)), \qquad 1 \leq i \leq m$$

with $\varphi(1, t_1 \cdots t_m) \leq C\varphi_1(1, t_1) \cdots \varphi_m(1, t_m)$ for some $C > 0$ and for all $t_1, \ldots, t_m \geq 0$. An application of Theorem 3.2 completes the proof. $\square$

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