SHARP INEQUALITIES FOR MAXIMAL FUNCTIONS ASSOCIATED WITH GENERAL MEASURES

L. Grafakos*
Department of Mathematics, University of Missouri,
Columbia, MO 65203, U.S.A.
(e-mail: loukas@math.missouri.edu)
and
J. Kinnunen
Department of Mathematics,
P.O. Box 4, FIN-00014 University of Helsinki, Finland
(e-mail: Juha.Kinnunen@Helsinki.Fi)

Abstract. Sharp weak type $(1,1)$ and L^p estimates in dimension one are obtained for uncentered maximal functions associated with Borel measures which do not necessarily satisfy a doubling condition. In higher dimensions uncentered maximal functions fail to satisfy such estimates. Analogous results for centered maximal functions are given in all dimensions.

1. Introduction

Let μ be a nonnegative Borel measure on $\mathbf{R}^n$ and let $f\colon \mathbf{R}^n \to [0,\infty]$ be a μ-locally integrable function. The uncentered maximal function of f with respect to μ is defined by

$$\widetilde{\mathcal{M}}f(x) = \sup_B \frac{1}{\mu(B)} \int_B f\,d\mu, \tag{1.1}$$

where the supremum is taken over all closed balls B containing x. Let $B(x,r)$ denote the closed ball with center x and radius $r > 0$. The centered maximal function of f with respect to μ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f\,d\mu, \tag{1.2}$$

1991 *Mathematics Subject Classification.* 42B25.

*Research partially supported by the National Science Foundation and by the University of Missouri Research Board.

with the interpretation that the integral averages in (1.1) and (1.2) are equal to $f(x)$ if $\mu(B) = 0$ or $\mu(B(x,r)) = 0$.

If μ is Lebesgue measure, these definitions give the usual uncentered and centered Hardy–Littlewood maximal operators. It is a classical result, see [**9**, p. 13], that if μ satisfies a doubling condition,

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \qquad \text{for all } x \in \mathbf{R}^n \text{ and } r > 0, \tag{1.3}$$

both of these operators are of weak type (1,1) and they map $L^p(\mathbf{R}^n, \mu)$, $p > 1$, into itself.

Omitting the doubling requirement, it is still true that $\mathcal{M}$ maps $L^p(\mathbf{R}^n, \mu)$, $p > 1$, into itself, but the corresponding result for $\widetilde{\mathcal{M}}$ is false if $n \geq 2$. An example indicating this statement is given in section 3. Examples showing that $\widetilde{\mathcal{M}}$ is not of weak type (1,1) if $n \geq 2$ can be found in [**8**].

It is a geometrical phenomenon, however, that such counterexamples do not exist in dimension one. In fact in dimension one, $\widetilde{\mathcal{M}}$ maps $L^p(\mathbf{R}^1, \mu)$, $p > 1$, into itself without the doubling assumption about μ, see [**2**] and [**8**]. This is a consequence of a special covering argument available only on the real line. In this article we give sharp L^p and weak type (1,1) estimates for $\widetilde{\mathcal{M}}$ with constants independent of μ.

In higher dimensions we obtain an improvement of the known estimate

$$\mu(\{\mathcal{M}f > \lambda\}) \leq \frac{c_n}{\lambda} \int_{\{\mathcal{M}f > \lambda\}} f \, d\mu, \qquad \lambda > 0, \tag{1.4}$$

where c_n is the Besicovitch constant. See section 3 for details.

2. The one-dimensional case

On $\mathbf{R}^1$, fix a nonnegative Borel measure μ. The inequality below was first proved in [**7**] when μ is the usual Lebesgue measure. The proof given there is different and doesn't generalize to this context.

Theorem 2.1. *For any $\lambda > 0$ and any μ-locally integrable function $f\colon \mathbf{R}^1 \to [0,\infty]$ we have*

$$\mu(\{\widetilde{\mathcal{M}}f > \lambda\}) + \mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{\{\widetilde{\mathcal{M}}f>\lambda\}} f\,d\mu + \frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu. \tag{2.1}$$

Proof. Fix $\lambda > 0$ and denote $E_\lambda = \{\widetilde{\mathcal{M}}f > \lambda\}$. If $\mu(\{f > \lambda\}) = \infty$, then by Chebyshev's inequality the right side of (2.1) is infinity and there is nothing to prove. Hence we may assume that $\mu(\{f > \lambda\}) < \infty$. For every $x \in E_\lambda$ there is an interval I_x containing x such that

$$\frac{1}{\mu(I_x)}\int_{I_x} f\,d\mu > \lambda. \tag{2.2}$$

By Lindelöf's theorem there is a countable subcollection I_j, $j = 1, 2, \dots$, such that

$$\bigcup_{j=1}^{\infty} I_j = \bigcup_{x\in E_\lambda} I_x.$$

Let $\mathcal{I} = \{I_j\colon j = 1, 2, ..., N\}$ and write

$$F^N = \bigcup_{I\in\mathcal{I}} I.$$

By Lemma 4.4 in [**6**] we obtain two subcollections $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{I}$ so that the intervals in each of these are pairwise disjoint and that

$$F^N = \bigcup_{i=1}^{2}\bigcup_{I\in\mathcal{I}_i} I.$$

We denote $F_i = \bigcup_{I\in\mathcal{I}_i} I$, $i = 1, 2$. Since the intervals in $\mathcal{I}_i$, $i = 1, 2$, are pairwise disjoint and (2.2) holds we obtain

$$\mu(F_i) = \sum_{I\in\mathcal{I}_i}\mu(I) < \frac{1}{\lambda}\sum_{I\in\mathcal{I}_i}\int_I f\,d\mu = \frac{1}{\lambda}\int_{F_i} f\,d\mu \qquad \text{for } i = 1, 2. \tag{2.3}$$

Therefore

$$\begin{aligned}\mu(F^N) + \mu(F_1 \cap F_2) &= \mu(F_1) + \mu(F_2) \\ &< \frac{1}{\lambda}\int_{F_1} f\,d\mu + \frac{1}{\lambda}\int_{F_2} f\,d\mu \\ &= \frac{1}{\lambda}\int_{F^N} f\,d\mu + \frac{1}{\lambda}\int_{F_1\cap F_2} f\,d\mu.\end{aligned} \tag{2.4}$$

For any μ-measurable set E such that $\mu(E) < \infty$ we have

$$\frac{1}{\lambda}\int_E f\,d\mu + \mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu + \mu(E). \tag{2.5}$$

To see this, we observe that

$$\begin{aligned}\int_E (f-\lambda)\,d\mu &= \int_{\{f\le\lambda\}\cap E} (f-\lambda)\,d\mu + \int_{\{f>\lambda\}\cap E} (f-\lambda)\,d\mu \\ &\le \int_{\{f>\lambda\}} (f-\lambda)\,d\mu.\end{aligned}$$

Using (2.4) and (2.5) we deduce that

$$\mu(F^N) + \mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{F^N} f\,d\mu + \frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu.$$

Since F^N is an increasing sequence of $\mu-$measurable sets whose union is E_λ, inequality (2.1) follows by letting $N \to \infty$.

Remarks 2.2. (1) Inequality (2.1) is stronger than the standard weak type (1,1) estimate obtained, for example, in [**2**]. In particular, estimate (2.1) implies that $\widetilde{\mathcal{M}}$ is of weak type (1,1) with constant 2.

(2) Equality can actually occur in (2.1). For instance this is the case when f even, symmetrically decreasing about the origin and μ is Lebesgue measure, see [**7**].

Now we show that the sharp weak type estimate (2.1) implies a sharp version of the Hardy–Littlewood Theorem.

Corollary 2.3. *Let $1 < p < \infty$ and let A_p be the unique positive solution of the equation*

$$(p-1)\,x^p - p\,x^{p-1} - 1 = 0. \tag{2.6}$$

Then

$$\|\widetilde{\mathcal{M}}f\|_{p,\mu} \leq A_p \|f\|_{p,\mu}. \tag{2.7}$$

Proof. We may suppose that f is not zero μ-almost everywhere and that $f \in L^p(\mathbf{R}^1, \mu)$ since otherwise there is nothing to prove. Fubini's theorem and (2.1) imply that

$$\begin{aligned}\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}}f)^p \, d\mu + \int_{\mathbf{R}^1} f^p \, d\mu &= p\int_0^\infty \lambda^{p-1}\mu(\{\widetilde{\mathcal{M}}f > \lambda\})\, d\lambda + p\int_0^\infty \lambda^{p-1}\mu(\{f > \lambda\})\, d\lambda \\ &\leq p\int_0^\infty \lambda^{p-2}\int_{\{\widetilde{\mathcal{M}}f>\lambda\}} f\, d\mu\, d\lambda + p\int_0^\infty \lambda^{p-2}\int_{\{f>\lambda\}} f\, d\mu\, d\lambda \\ &= \frac{p}{p-1}\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}}f)^{p-1} f\, d\mu + \frac{p}{p-1}\int_{\mathbf{R}^1} f^p\, d\mu\end{aligned}$$

and hence

$$\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}}f)^p \, d\mu \leq \frac{p}{p-1}\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}}f)^{p-1} f\, d\mu + \frac{1}{p-1}\int_{\mathbf{R}^1} f^p\, d\mu.$$

Hölder's inequality gives

$$\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}}f)^{p-1} f\, d\mu \leq \Big(\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}}f)^p\, d\mu\Big)^{(p-1)/p}\Big(\int_{\mathbf{R}^1} f^p\, d\mu\Big)^{1/p}$$

and hence

$$(p-1)\|\widetilde{\mathcal{M}}f\|_{p,\mu}^p \leq p\|\widetilde{\mathcal{M}}f\|_{p,\mu}^{p-1}\|f\|_{p,\mu} + \|f\|_{p,\mu}^p$$

or equivalently

$$(p-1)\Big(\frac{\|\widetilde{\mathcal{M}}f\|_{p,\mu}}{\|f\|_{p,\mu}}\Big)^p - p\Big(\frac{\|\widetilde{\mathcal{M}}f\|_{p,\mu}}{\|f\|_{p,\mu}}\Big)^{p-1} - 1 \leq 0.$$

The claim follows from this inequality.

Remarks 2.4. (1) When μ is Lebesgue measure, then the L^p-bound above is the best possible, see [**7**].

(2) The bound A_p in (2.7) is independent of the measure μ.

We close this section by studying the reverse inequality to (2.1).

Proposition 2.5. *Suppose that $f\colon \mathbf{R}^1 \to [0,\infty]$ is a locally μ-integrable function. Then*

$$\int_{\{\widetilde{\mathcal{M}}f>\lambda\}} f\,d\mu \le \lambda\mu(\{\widetilde{\mathcal{M}}f>\lambda\}) \tag{2.8}$$

for every $\lambda \ge \operatorname{ess\,inf}_{\mathbf{R}^1} \widetilde{\mathcal{M}}f$.

Proof. Let $\lambda > \operatorname{ess\,inf}_{\mathbf{R}^1} \widetilde{\mathcal{M}}f$ and denote $E_\lambda = \{\widetilde{\mathcal{M}}f > \lambda\}$. Then $\mathbf{R}^1 \setminus E_\lambda$ has a positive measure. On the other hand, E_λ is an open set on the real line and hence it is a union of countably many pairwise disjoint open intervals $E_\lambda = \bigcup_{j=1}^\infty I_j$. For an interval I and $\sigma > 0$, let σI be the interval with the same center whose length is multiplied by σ. Since every σI_j intersects $\mathbf{R}^1 \setminus E_\lambda$ when $\sigma > 1$, we see that

$$\frac{1}{\mu(\sigma I_j)} \int_{\sigma I_j} f\,d\mu \le \lambda, \qquad \text{for } j = 1,2,\dots$$

By letting $\sigma \to 1$ we obtain

$$\frac{1}{\mu(I_j)} \int_{I_j} f\,d\mu \le \lambda, \qquad \text{for } j = 1,2,\dots,$$

and hence by summing up we deduce that

$$\int_{E_\lambda} f\,d\mu \le \sum_{j=1}^\infty \int_{I_j} f\,d\mu \le \lambda \sum_{j=1}^\infty \mu(I_j) = \lambda\mu(E_\lambda).$$

This implies that (2.8) is true for every $\lambda \ge \operatorname{ess\,inf}_{\mathbf{R}^1} \widetilde{\mathcal{M}}f$ and the proof is now complete.

Remark 2.6. Suppose that $f \in L^1(\mathbf{R}^1,\mu)$. If $\lambda < \operatorname{ess\,inf}_{\mathbf{R}^1} \widetilde{\mathcal{M}}f$, then $\mu(E_\lambda) = \mu(\mathbf{R}^1)$ and (2.8) holds for every

$$\lambda \ge \frac{1}{\mu(\mathbf{R}^1)} \int_{\mathbf{R}^1} f\,d\mu.$$

In particular, if $\mu(\mathbf{R}^1) = \infty$, then (2.8) holds for every $\lambda > 0$.

3. The higher dimensional case

If $n \geq 2$, the uncentered maximal function associated to a general measure is not bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. To see this, select closed balls $B_1, B_2, \dots$ so that the origin is on the boundary of each ball and such that for every B_i, $i = 1, 2, \dots$, there is a point $x_i \in B_i \setminus \cup_{j \neq i} B_j$. Set

$$\mu = \sum_{i=0}^{\infty} \delta_{x_i},$$

where $x_0 = 0$ and δ_{x_i} denotes Dirac mass at x_i. Let f be the characteristic function of B_1. Clearly $\|f\|_{p,\mu} \leq 2^{1/p}$, but

$$\widetilde{\mathcal{M}} f(x_i) \geq \frac{1}{\mu(B_i)} \int_{B_i} f \, d\mu \geq \frac{1}{2} \qquad \text{for all } i = 1, 2, \dots,$$

and hence

$$\|\widetilde{\mathcal{M}} f\|_{p,\mu} \geq \frac{1}{2} \mu(\mathbf{R}^n)^{1/p} = \infty.$$

A similar counterexample for the strong maximal operator was given in [**4**].

Next we discuss an improvement of (1.4). Here we need the following Besicovitch's covering theorem.

Theorem 3.1. *Suppose that E is a bounded subset of $\mathbf{R}^n$ and that $\mathcal{B}$ is a collection of of closed balls such that each point of E is a center of some ball in $\mathcal{B}$. Then there exists an integer $c_n \geq 2$ (depending only on the dimension) and subcollections $\mathcal{B}_1, \dots, \mathcal{B}_{c_n} \subset \mathcal{B}$ of at most countably many balls such that the balls in each family $\mathcal{B}_i$ are pairwise disjoint and such that*

$$E \subset \bigcup_{i=1}^{c_n} \bigcup_{B \in \mathcal{B}_i} B.$$

For the proof of Besicovitch's covering theorem we refer to [**3**, Theorem 1.1]. Some estimates for the constant c_n are obtained in [**5**].

Theorem 3.2. *For any $\lambda > 0$ and any μ-locally integrable function $f\colon \mathbf{R}^n \to [0,\infty]$ we have*

$$\mu(\{\mathcal{M}f > \lambda\}) + (c_n - 1)\mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{\{\mathcal{M}f>\lambda\}} f\,d\mu + (c_n-1)\frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu. \tag{3.1}$$

Here c_n is the Besicovitch constant.

Proof. We fix $\lambda > 0$ and denote $E_\lambda = \{\mathcal{M}f > \lambda\}$. We may assume that $\mu(E_\lambda) < \infty$, since otherwise by (1.4) the right side of (3.1) is infinity. For every $x \in E_\lambda$ there is a ball $B(x, r_x)$ so that

$$\frac{1}{\mu(B(x,r_x))}\int_{B(x,r_x)} f\,d\mu > \lambda. \tag{3.2}$$

We have that

$$\begin{aligned}\int_{B(x,r_x)} f\,d\mu &= \int_{B(x,r_x)\cap(\mathbf{R}^n\setminus E_\lambda)} f\,d\mu + \int_{B(x,r_x)\cap E_\lambda} f\,d\mu \\ &\le \lambda\mu(B(x,r_x)\cap(\mathbf{R}^n\setminus E_\lambda)) + \int_{B(x,r_x)\cap E_\lambda} f\,d\mu \end{aligned} \tag{3.3}$$

and that

$$\mu(B(x,r_x)) = \mu(B(x,r_x)\cap(\mathbf{R}^n\setminus E_\lambda)) + \mu(B(x,r_x)\cap E_\lambda). \tag{3.4}$$

Combining (3.2), (3.3), and (3.4) we obtain

$$\int_{B(x,r_x)\cap E_\lambda} f\,d\mu > \lambda\mu(B(x,r_x)\cap E_\lambda). \tag{3.5}$$

Let $B_R = B(0,R)$ be a fixed ball and denote $\mathcal{B} = \{B(x,r_x)\colon x \in B_R \cap E_\lambda\}$. By Besicovitch's covering theorem there are subfamilies $\mathcal{B}_1, \dots, \mathcal{B}_{c_n}$, of $\mathcal{B}$ such that each of these subfamilies consists of at most countably many pairwise disjoint balls and that

$$B_R \cap E_\lambda \subset \bigcup_{i=1}^{c_n}\bigcup_{B\in\mathcal{B}_i} B.$$

We denote $F_i = \bigcup_{B\in\mathcal{B}_i} B$, $i = 1, 2, \ldots, c_n$, and $F = \bigcup_{i=1}^{c_n} F_i$. Since the balls in each $\mathcal{B}_i$ are pairwise disjoint, it follows from (3.5) that

$$\mu(F_i \cap E_\lambda) < \frac{1}{\lambda}\int_{F_i\cap E_\lambda} f\, d\mu, \qquad \text{for } i = 1, 2, \ldots, c_n. \tag{3.6}$$

Then we use the elementary fact that for any measure ν we have

$$\sum_{i=1}^{c_n} \nu(F_i \cap E_\lambda) = \nu(F \cap E_\lambda) + \sum_{j=2}^{c_n} \nu(G_j \cap E_\lambda), \tag{3.7}$$

where

$$G_j = \cup_{\{k_1,\ldots,k_j\}\subset\{1,\ldots,c_n\}}(F_{k_1} \cap \cdots \cap F_{k_j}), \qquad j = 2, 3, \ldots, c_n.$$

Using (3.6) and (3.7) we deduce that

$$\mu(F \cap E_\lambda) + \sum_{j=2}^{c_n} \mu(G_j \cap E_\lambda) < \frac{1}{\lambda}\int_{F\cap E_\lambda} f\, d\mu + \frac{1}{\lambda}\sum_{j=2}^{c_n}\int_{G_j\cap E_\lambda} f\, d\mu.$$

Inequality (2.5) then implies that

$$\mu(B_R \cap E_\lambda) + (c_n - 1)\mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{E_\lambda} f\, d\mu + (c_n - 1)\frac{1}{\lambda}\int_{\{f>\lambda\}} f\, d\mu,$$

and by letting $R \to \infty$ we prove the desired conclusion.

As in Corollary 2.3 we obtain an estimate for the constant in the Hardy–Littlewood ∎ Theorem.

Corollary 3.3. *Let $A_{p,n}$ be the unique positive solution of the equation*

$$(p - 1)\, x^p - p\, x^{p-1} - (c_n - 1) = 0, \tag{3.8}$$

where c_n is the Besicovitch constant. Then the estimate

$$\|\mathcal{M}f\|_{p,\mu} \le A_{p,n}\|f\|_{p,\mu}, \tag{3.9}$$

holds.

The constant $A_{p,n}$ given by (3.8) tends to one as p goes to infinity. This shows that it is asymptotically sharp near ∞. However, $A_{p,n}$ grows as $n \to \infty$. It is still unknown to us whether the constant $A_{p,n}$ in (3.9) can be replaced with a constant both independent of the measure μ and of the dimension n.

References

1. K. F. Andersen, *Weighted inequalities for maximal functions associated with general measures*, Trans. Amer. Math. Soc. **326** (1991), 907–920.
2. A. Bernal, *A note on the one-dimensional maximal function*, Proc. Roy. Soc. Edinburgh **111A** (1989), 325–328.
3. M. deGuzmán, *Differentiation of integrals in* $\mathbf{R}^n$, Lecture Notes in Mathematics 481, Springer–Verlag, 1975.
4. R. Fefferman, *Strong differentiation with respect to measures*, Amer. J. Math. **103** (1981), 33–40.
5. Z. Füredi and P.A. Loeb, *On the best constant for the Besicovitch covering theorem*, Proc. Amer. Math. Soc. **121** (1994), 1063–1073.
6. J. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, Academic Press, 1981.
7. L. Grafakos and S. Montgomery–Smith, *Best constants for uncentered maximal functions*, Bull. London Math. Soc. **29** (1996), 60–64.
8. P. Sjögren, *A remark on the maximal function for measures in* $\mathbf{R}^n$, Amer. J. Math. **105** (1983), 1231–1233.
9. E. M. Stein, *Harmonic Analysis: Real-Variable Theory, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.