ON THE RESTRICTION CONJECTURE

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Abstract. We prove the restriction conjecture for the class of functions consisting of products of radial functions and spherical harmonics $Y(\omega)$, when $Y(\omega)$ is a product of factors of the form $(\sin\omega)^{s-j}P_n^{(s)}(\cos(\omega))$ and $P_n^{(s)}(t)$ is an ultraspherical polynomial.

1. Introduction

The restriction conjecture is a challenging open problem in Fourier analysis. Denoting by

$$\widehat{f}(\zeta) = \int_{\mathbf{R}^d} f(x)e^{-2\pi i\langle x,\,\zeta\rangle}dx$$

the Fourier transform of C_0^∞ function on $\mathbf{R}^d$ and by $\mathbf{S}^{d-1} = \{x \in \mathbf{R}^d : \|x\| = 1\}$ the unit sphere in $\mathbf{R}^d$, the restriction conjecture, (RC henceforth), states that for every $1 \le p < \frac{2d}{d+1}$ and $q \ge \frac{d-1}{d+1}p'$ the following inequality holds

$$\sup_{F\in C_0^\infty(\mathbf{R}^d)} \frac{\|\widehat{F}\|_{L^q(\mathbf{S}^{d-1},\,d\sigma)}}{\|F\|_{L^p(\mathbf{R}^d)}} \le C \tag{1.1}$$

where $d\sigma(\zeta)$ denotes surface measure on $\mathbf{S}^{d-1}$ and $\mathbf{R}^+ = (0,\,\infty)$. Here C is a constant that depends only on p, q, and d, and p' is the dual exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

The conditions on p and q are optimal, (see [10]). The RC has been proved in the case $d = 2$ by C. Fefferman [6], and is still open in the other cases. When $d > 2$ only partial results are known; one of these results is the Stein[1] -Tomas restriction theorem [13], [9] which asserts that the RC holds whenever $1 \le p < \frac{2(d+1)}{d+3}$ and every $q \ge \frac{d-1}{d+1}p'$. See also [10]. When $p = \frac{2(d+1)}{d+3}$ we have $\frac{d-1}{d+1}p' = 2$ and the exponent $q = 2$ plays a crucial role as it allows a reduction of (1.1) to the equivalent "dual" inequality

$$\left\| \int_{\mathbf{S}^{d-1}} \widehat{F}(\zeta)e^{2\pi i\langle x,\,\zeta\rangle}d\sigma(\zeta)\right\|_{L^{p'}(\mathbf{R}^n)} \le C\|F\|_{L^p(\mathbf{R}^n)}$$

via a TT^* technique. The case $q < 2$ cannot be handled with the same technique, and requires more work.

When $\frac{2(d+1)}{d+3} < p < \frac{2d}{d+1}$ we can prove that the ratio in (1.1) is uniformly bounded on special subspaces of $L^p(\mathbf{R}^d)$. For example, it easy to see that (1.1) holds for every $q \le 2$ and

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[1]Stein proved that (1.1) holds for $1 \le p < \frac{4n}{3n+1}$ and $q = 2$ using complex interpolation, (unpublished).

every $p \le \frac{2d}{d+1}$ if $L^p(\mathbf{R}^d)$ is replaced by the Sobolev space $W^{s,p_0}(\mathbf{R}^d)$, where $p_0 = \frac{2(d+1)}{d+3}$, $s \ge \frac{d-1}{d(d+1)}$. By the Sobolev embedding theorem, the latter embeds in $L^p(\mathbf{R}^d)$ for every $p \le \frac{2d}{d+1}$.

Another class of functions for which the conjecture is valid is the class of radial functions. Let $x = r\omega$, with $r = |x|$ and $\omega \in \mathbf{S}^{d-1}$. Let $F(x) = f(|x|) \in C_0^\infty(\mathbf{R}^d)$. The Fourier transform of $F(x)$ is

$$\widehat{F}(\xi) = |\xi|^{-\frac{d}{2}+1} \int_0^{+\infty} f(r) J_{\frac{d}{2}-1}(r|\xi|) r^{\frac{d}{2}} dr = \widetilde{\mathcal{H}}_{\frac{d}{2}-1} f(|\xi|),$$

where $J_\nu(r)$, is the usual Bessel function of the first kind and $\widetilde{\mathcal{H}}_\alpha f(\rho)$ is the Hankel-Fourier-Bessel transform of $f(r)$. See [11] for the definition and properties of the Bessel function and [4] for the definition of the Hankel-Fourier-Bessel transform.

To see the validity of the RC for radial functions, we note that the $L^p(\mathbf{R}^d)$ norm of a radial function $F(x) = f(|x|) \in C_0^\infty(\mathbf{R}^d)$ is

$$\|F\|_{L^p(\mathbf{R}^d)} = |\mathbf{S}^{d-1}|^{\frac{1}{p}} \|f(r) r^{\frac{d-1}{p}}\|_{L^p(\mathbf{R}^+)},$$

where $|\mathbf{S}^{d-1}| = \dfrac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ denotes the measure of the surface of $\mathbf{S}^{d-1}$. We also have

$$\left(\int_{\mathbf{S}^{d-1}} |\widehat{F}(\xi)|^q d\sigma(\xi) \right)^{\frac{1}{q}} = |\mathbf{S}^{d-1}|^{\frac{1}{q}} \left| \int_0^{+\infty} f(r) J_{\frac{d}{2}-1}(r) r^{\frac{d}{2}} dr \right|,$$

and applying Hölder's inequality, we obtain

$$\begin{aligned} \left(\int_{\mathbf{S}^{d-1}} |\widehat{F}(\xi)|^q d\sigma(\xi) \right)^{\frac{1}{q}} &= |\mathbf{S}^{d-1}|^{\frac{1}{q}} \|f(r) r^{\frac{d-1}{p}}\|_{L^p(\mathbf{R}^+)} \|r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r)\|_{L^{p'}(\mathbf{R}^+)} \\ &= |\mathbf{S}^{d-1}|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(\mathbf{R}^d)} \|r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r)\|_{L^{p'}(\mathbf{R}^+)}. \end{aligned}$$

Since $J_{\frac{d}{2}-1}(r)$ is $\mathcal{O}(r^{-\frac{1}{2}})$ when $r \to +\infty$ and is $\mathcal{O}(r^{\frac{d}{2}-1})$ when $r \to 0$, we can easily check that $r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r) \in L^{p'}(\mathbf{R}^+)$ if and only if $p < \frac{2d}{d+1}$.

Note that in this special case, $|\mathbf{S}^{d-1}|^{\frac{1}{q}-\frac{1}{p}} \|r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}\|_{L^{p'}(\mathbf{R}^+)}$ is the best constant for the restriction inequality (1.1), i.e.

$$\sup_{F \text{ radial}} \frac{\|\widehat{F}\|_{L^q(\mathbf{S}^{d-1})}}{\|F\|_p} = |\mathbf{S}^{d-1}|^{\frac{1}{q}-\frac{1}{p}} \|r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r)\|_{L^{p'}(\mathbf{R}^+)}.$$

We also observe that in this case (1.1) holds for every $q < \infty$.

More generally, let $\mathcal{H}_m$ be the subspace of $L^2(S^{d-1})$ spanned by the products of spherical harmonics of degree m, with $m \ge 0$, and radial functions in $C_0^\infty(\mathbf{R}^d)$. If $F(x) = F(r\omega) = r^m f_m(r) Y(\omega) \in \mathcal{H}_m$, where Y is a spherical harmonic, then

$$\widehat{F}(\zeta) = \widehat{F}(\rho\sigma) = \rho^m \widehat{f_m}(\rho) Y(\sigma)\,, \tag{1.2}$$

where

$$\widehat{f_m}(\rho) = i^m \rho^{-\frac{n}{2}+1} \int_0^{+\infty} f_m(r) J_{\frac{d}{2}-1+m}(r\rho) r^{\frac{d}{2}+m} dr = i^m \widetilde{\mathcal{H}}_{\frac{d}{2}-1+m} f_m(\rho).$$

Let n be a nonnegative integer and let $s > -\frac{1}{2}$. We denote by $P_n^{(s)}$ the ultraspherical polynomial of degree n and order s. This is defined by

$$P_n^{(s)}(t) = C_n^s P_n^{s-\frac{1}{2},s-\frac{1}{2}}(t),$$

where $P_n^{(\alpha,\beta)}(t)$ is the usual Jacobi polynomial of degree n on $[-1,1]$ and C_n^s is a constant of normalization. We refer the reader to the Appendix for the value of the constant C_n^s and for the definition of Jacobi polynomials.

The spherical harmonics have an explicit expression in terms of the Jacobi (or ultraspherical) polynomials. Indeed, let $m_0 \geq m_1 \geq \cdots m_{d-2} \geq 0$ be integers and let

$$Y_{(m_k)}(z) = e^{\pm i m_{d-2} z_{d-1}} \Pi_{k=0}^{d-3} (\sin z_{k+1})^{m_{k+1}} P_{m_k - m_{k+1}}^{(m_{k+1} + \frac{d-1-k}{2})} (\cos z_{k+1})\,. \tag{1.3}$$

Then every spherical harmonic $Y_m(\omega)$ of degree $m = m_0 \geq 0$ can be written as a finite linear combination of the $Y_{(m_k)}$'s, (see [5]). This may be proved using a dimension comparison with space of the spherical harmonics of degree m which has dimension

$$\delta_{m,d} = (2m + d - 2)\frac{\Gamma(m + d - 2)}{\Gamma(m + 1)\Gamma(d - 1)}\,.$$

In this paper we consider the following class of functions: products of radial functions in $C_0^\infty(\mathbf{R}^d)$ and spherical harmonics which, in polar coordinates, can be expressed as products of factors of the form of $(\sin z)^{s-j} P_n^{(s)}(\cos z)$. We denote this class of functions by $\mathcal{L}$. It is easy to verify that the space $\mathcal{L}$ is invariant under the action of the Fourier transform. Moreover, one can easily see that the space

$$\text{span}(\mathcal{L}) = \left\{ \sum_{i=1}^{N} r^{m_i} f_i(r) Y_{m_i}(\omega) : \quad N > 0, \quad f_i(r) \in C_0^\infty(\mathbf{R}^+), \text{ and } Y_{m_i}(\omega) \text{ as in } (1.3) \right\}$$

is dense in $L^p(\mathbf{R}^d)$ for every $p \leq 2$. Therefore, the RC is equivalent to the estimate

$$\sup_{\text{span}(\mathcal{L})} \frac{\| \sum_{i=1}^N \widehat{f}_i(1) Y_{m_i} \|_{L^q(\mathbf{S}^{d-1})}}{\| \sum_{i=1}^N r^{m_i} f_i \, Y_{m_i} \|_{L^p(\mathbf{R}^d)}} \leq C, \tag{1.4}$$

where C depends only on p, q, and d (and in particular is independent of N). This provides a strong motivation for the consideration of the class $\mathcal{L}$.

Our main result, Theorem 1.1 below, says that RC holds for the space $\mathcal{L}$, i.e. (1.4) is valid when $N = 1$.

Theorem 1.1. *Let $1 \leq p < \frac{2d}{d+1}$ and let $q = \frac{d-1}{d+1} p'$. Then we have*

$$\sup_{F \in \mathcal{L}} \frac{\|\widehat{F}\|_{L^q(\mathbf{S}^{d-1})}}{\|F\|_{L^p(\mathbf{R}^d)}} \leq C \tag{1.5}$$

where C depends only on p, q, and d.

The basic strategy in proving Theorem 1.1 is the following. Let $F(x) \in \mathcal{L}$. Since $F(r\omega) = r^m f_m(r) Y(\omega)$ and $\widehat{F}(\zeta)$ is as in (1.2), then we have

$$\frac{\|\widehat{F}\|_{L^q(\mathbf{S}^{d-1})}}{\|F\|_{L^p(\mathbf{R}^d)}} = \frac{\|\hat{f}_m(1) Y\|_{L^q(\mathbf{S}^{d-1})}}{\|r^m f_m Y\|_{L^p(\mathbf{R}^d)}}\,. \tag{1.6}$$

We can therefore reduce matters to estimating the ratios of the radial parts and the angular parts separately. Our main task is to obtain the appropriate estimates for these parts. Finally, we show that the combined estimates for

$$\frac{|\widehat{f_m}(1)|}{\|r^m f_m\|_{L^p(\mathbf{R}^+,\, r^{d-1}dr)}} \qquad \text{and} \qquad \frac{\|Y\|_{L^q(S^{d-1})}}{\|Y\|_{L^p(S^{d-1})}}$$

yield (1.5).

2. Four useful propositions

In what follows we will often denote by C a generic constant which is not necessarily the same at each occurrence. The following results are ingredients of the proof of Theorem 1.1.

Proposition 1. *Let $J_\nu(x)$ be the usual Besel function of the first kind with $\nu \geq 0$. Then, $x^\alpha J_\nu(x) \in L^q(\mathbf{R}^+)$ if and only if*

$$-\frac{1}{q} - \nu < \alpha < \frac{1}{2} - \frac{1}{q}, \tag{2.1}$$

and for $\frac{1}{4} - \frac{1}{q} < \alpha < \frac{1}{2} - \frac{1}{q}$, for $2 \leq q < \infty$, and for ν sufficiently large, we have

$$\|x^\alpha J_\nu(x)\|_q \leq A\nu^{\alpha - \frac{1}{2} + \frac{1}{q}}. \tag{2.2}$$

where A depends only on α and q.

Proposition 2. *Let $s \geq j \geq 0$. Then*

$$\sup_{0 \leq z \leq \frac{\pi}{2}} |(\sin z)^{s-j} P_n^{(s)}(\cos z)| \leq (P_n^{(s)}(1))^{\frac{j}{s}} (c_{n,\,s})^{1-\frac{j}{s}} \tag{2.3}$$

where $P_n^{(s)}(1) = \dfrac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}$, and

$$c_{n,\,s} = \begin{cases} \dfrac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma\left(\frac{n}{2}+1\right)\Gamma(s)} & \text{if } n \text{ is even,} \\[2ex] \dfrac{(1+n)\frac{\Gamma\left(\frac{n+1}{2}+s\right)}{\Gamma\left(\frac{n+3}{2}\right)\Gamma(s)}}{\sqrt{(1-s)\,s + (n+s)^2}} & \text{if } n \text{ is odd.} \end{cases} \tag{2.4}$$

Moreover,

$$\sup_{0 \leq z \leq \frac{\pi}{2}} |(\sin z)^{s-j} P_n^{(s)}(\cos z)| \leq e^j \left(1 + \frac{n+1}{2s}\right)^j c_{n,\,s}. \tag{2.5}$$

Proposition 2 is a generalization of Theorem 7.33.2 in [12], where the same result is proved for $j = 0$ and $0 < s < 1$. Note that the inequality (2.3) is sharp in the case $j = s$. Indeed, $P_n^{(s)}(t) \leq P_n^{(s)}(1)$ for every $-1 \leq t \leq 1$, (see (A.7)).

Proposition 3. *Let $n \geq 0$ and let $j \leq s$, with $s \geq 0$. Then*

$$\frac{\sup_{t\in[0,\,1]} |(1-t^2)^{\frac{1}{2}(s-j)} P_n^{(s)}(t)|}{\left(\int_0^1 \left|P_n^{(s)}(t)\right|^2 (1-t^2)^{s+\frac{1}{2}} dt\right)^{\frac{1}{2}}} \leq C(s+n)^{\frac{1}{4}} \tag{2.6}$$

where C is a constant that depends only on j.

The following proposition is as easy consequence of Proposition 3 using complex interpolation.

Proposition 4. *Let $2 \leq r \leq q$ and let $\eta(x)$ be an analytic function on $[2,\,\infty) \times i\mathbf{R}$ which is bounded on $[2,\,\infty]$ and satisfies $\eta(2) \geq -\frac{1}{2}$. Then,*

$$\frac{\left(\int_0^1 \left|P_n^{(s)}(t)\right|^q (1-t^2)^{q\left(\frac{s}{2}-\eta(q)\right)} dt\right)^{\frac{1}{q}}}{\left(\int_0^1 \left|P_n^{(s)}(t)\right|^r (1-t^2)^{r\left(\frac{s}{2}-\eta(r)\right)} dt\right)^{\frac{1}{r}}} \leq C(s+n)^{\frac{1}{2r}-\frac{1}{2q}} \tag{2.7}$$

where C is a constant that depends only on r, q, and $\sup_{x\geq 2} |\eta(x)|$.

It is worthwhile comparing Proposition 4 with Theorem 3 in the recent article of Carbery and Wright [3]. They prove that the following inequality is satisfied for all $0 \leq p \leq q \leq \infty$, $j \in \mathbf{N}$ and $\lambda \geq 1$, and every polynomial on $\mathbf{R}$ of degree at most n.

$$\left(\frac{\int_0^1 |p(t)|^{\frac{q}{n}} (\lambda-t)^{j-1} dt}{\int_0^1 (\lambda-t)^{n-1} dt}\right)^{\frac{1}{q}} \leq \sigma \frac{(jB(j,\ q+1))^{\frac{1}{q}}}{(jB(j,\ r+1))^{\frac{1}{r}}} \left(\frac{\int_0^1 |p(t)|^{\frac{r}{n}} (\lambda-t)^{j-1} dt}{\int_0^1 (\lambda-t)^{n-1} dt}\right)^{\frac{1}{r}} \tag{2.8}$$

where σ is independent of the above parameters and $B(a,\,b)$ is the Beta function. If we let $\lambda = 1$, $\overline{q} = nq$, $\overline{r} = nr$, from (2.8) we obtain

$$\left(\int_0^1 |p(t)|^{\overline{q}} (1-t)^{j-1} dt\right)^{\frac{1}{\overline{q}}} \leq \sigma^n \frac{(B(j,\ n\overline{q}+1))^{\frac{1}{\overline{q}}}}{(B(j,\ n\overline{r}+1))^{\frac{1}{\overline{r}}}} \left(\int_0^1 |p(t)|^{\overline{r}} (1-t)^{j-1} dt\right)^{\frac{1}{\overline{r}}}$$

It is not difficult to show, (see also Lemma 5), that

$$\frac{(B(j,\ n\overline{q}+1))^{\frac{1}{\overline{q}}}}{(B(j,\ n\overline{r}+1))^{\frac{1}{\overline{r}}}} \approx C\Gamma(j)^{\frac{1}{\overline{q}}-\frac{1}{\overline{r}}} (n+1)^{\frac{j}{\overline{r}}-\frac{j}{\overline{q}}}$$

as $n \to \infty$ with the other parameters fixed. Therefore (2.8) is equivalent to

$$\left(\int_0^1 |p(t)|^{\overline{q}} (1-t)^{j-1} dt\right)^{\frac{1}{\overline{q}}} \leq C\sigma^n \Gamma(j)^{\frac{1}{\overline{q}}-\frac{1}{\overline{r}}} (n+1)^{\frac{j}{\overline{r}}-\frac{j}{\overline{q}}} \left(\int_0^1 |p(t)|^{\overline{r}} (1-t)^{j-1} dt\right)^{\frac{1}{\overline{r}}},$$

which is weaker than (2.7) and moreover the constant σ is not explicit.

3. Proof of Proposition 1.

In this section we prove Proposition 1 and we state some facts that we will need in the proof of Theorem 1.1. To prove Proposition 1 we make use of the following precise asymptotics of the Bessel functions for large values of the argument that J.A. Barceló proved in his thesis, (see also [2]).

Theorem (B) *There exists a universal constant $C > 0$ which is such that for all $\nu > \frac{1}{2}$ and for all $r > \nu + \nu^{\frac{1}{3}}$ we have*

$$J_\nu(r) = \sqrt{\frac{2}{\pi}} \frac{\cos\theta(r)}{(r^2-\nu^2)^{\frac{1}{4}}} + h_\nu(r)$$

where

$$\theta(r) = (r^2-\nu^2)^{\frac{1}{2}} - \nu \arccos\left(\frac{\nu}{r}\right) - \frac{\pi}{4},$$

and

$$|h_\nu(r)| \le \begin{cases} C\left(\dfrac{\nu^2}{(r^2-\nu^2)^{\frac{7}{4}}} + \dfrac{1}{r}\right) & \text{if } \nu+\nu^{\frac{1}{3}} \le r \le 2\nu, \\ \dfrac{C}{r} & \text{if } r > 2\nu. \end{cases}$$

Proof. (Proposition 1.) The conditions (2.1) on α are necessary because $J_\nu(x) = \mathcal{O}(x^\nu)$ when $x \to 0$ and is $\mathcal{O}(x^{-\frac{1}{2}})$ when $x \to \infty$.

By Theorem (B) we have

$$\|J_\nu(x)x^\alpha\|_{L^q(2\nu,\,\infty)} \le C\left(\int_{2\nu}^\infty \left(r^{(\alpha-\frac{1}{2})} + r^{(\alpha-1)}\right)^q dr\right)^{\frac{1}{q}}.$$

Condition (2.1) on α guarantees that the integral above converges. Thus,

$$\|J_\nu(x)x^\alpha\|_{L^q(2\nu,\,\infty)} \le C\nu^{\alpha-\frac{1}{2}+\frac{1}{q}},$$

which is the required estimate.

We use again Theorem (B) in the interval $(\nu+\nu^{\frac{1}{3}},\ 2\nu)$. We obtain,

$$\begin{aligned} &\|J_\nu(x)x^\alpha\|_{L^q(\nu+\nu^{\frac{1}{3}},\ 2\nu)} \\ \le& C\left(\|r^\alpha(r^2-\nu^2)^{-\frac{1}{4}}\|_{L^q(\nu+\nu^{\frac{1}{3}},\ 2\nu)} + \nu^2\|r^\alpha(r^2-\nu^2)^{-\frac{7}{4}}\|_{L^q(\nu+\nu^{\frac{1}{3}},\ 2\nu)} + \|r^{\alpha-1}\|_{L^q(\nu+\nu^{\frac{1}{3}},\ 2\nu)}\right) \\ \le& C\nu^{\alpha+\frac{1}{q}-\frac{1}{2}}\left(\|s^\alpha(s^2-1)^{-\frac{1}{4}}\|_{L^q(1+\nu^{-\frac{2}{3}},\ 2)} + \|s^\alpha(s^2-1)^{-\frac{7}{4}}\|_{L^q(1+\nu^{-\frac{2}{3}},\ 2)} + \nu^{-\frac{1}{2}}\right) \\ \le& C\nu^{\alpha+\frac{1}{q}-\frac{1}{2}}. \end{aligned}$$

We are left with estimating the norm of $r^\alpha J_\nu(r)$ in the interval $(0,\ \nu+\nu^{\frac{1}{3}})$.

It is a well known fact, (see e.g. [7]) that there is a constant $C > 0$ such that for all $\nu \ge 0$ and all $r \ge 0$ we have $|J_\nu(r)| < C\nu^{-\frac{1}{3}}$. Furthermore, $J_\nu(r)$ is increasing and is $|J_\nu(r)| < C\nu^{-\frac{1}{2}}$ in the interval $[0,\ \nu-\nu^{\frac{1}{3}}]$. The latter can be easily proved using the following estimate, (see [14], pg. 255),

$$J_\nu(\nu x) \le \frac{e^{-\nu f(x)}}{(1-x^2)^{\frac{1}{4}}\sqrt{2\pi\nu}}$$

where $0 \le x < 1$, and

$$f(x) = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}.$$

Therefore,

$$\|J_\nu(x)x^\alpha\|_{L^q(\nu-\nu^{\frac{1}{3}},\,\nu+\nu^{\frac{1}{3}})} \le C\nu^{-\frac{1}{3}}\frac{(\nu+\nu^{\frac{1}{3}})^{\alpha+\frac{1}{q}} - (\nu-\nu^{\frac{1}{3}})^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}} \le C\nu^{\frac{1}{3}(\alpha+\frac{1}{q}-1)},$$

which is better than what we need. Indeed,

$$\frac{1}{3}(\alpha+\frac{1}{q}-1) \le \alpha + \frac{1}{q} - \frac{1}{2} \iff \alpha \ge \frac{1}{4} - \frac{1}{q},$$

as required.

Since $|J_\nu(r)| < C\nu^{-\frac{1}{2}}$ for all $r \le \nu - \nu^{\frac{1}{3}}$, the estimate claimed in Proposition 1 easily follows. □

Remark. Proposition 1 can also be proved as a corollary of Proposition 4.1 in [4].

We now let $F(x) = F(r\omega) = r^m f_m(r)Y_m(\omega) \in \mathcal{L}$, and we recall that $\widehat{F}(\zeta) = \widehat{F}(\rho\sigma) = \rho^m \widehat{f}_m(\rho)Y_m(\sigma)$, where

$$\widehat{f_m}(\rho) = i^m\rho^{-\frac{n}{2}+1}\int_0^{+\infty} f_m(r)J_{\frac{d}{2}-1+m}(r\rho)r^{\frac{d}{2}+m}\,dr.$$

In order to prove (1.1) for a function F in $\mathcal{L}$, we shall prove that, for every $1 \le p < \frac{2d}{d+1}$ and $q \ge \frac{d-1}{d+1}p'$, the ratio

$$\frac{\|\widehat{F}\|_{L^q(\mathbf{S}^{d-1},\ d\sigma)}}{\|F\|_{L^p(\mathbf{R}^d)}} = \frac{\left|\int_0^{+\infty} f(r)J_{\frac{d}{2}-1+m}(r)r^{\frac{d}{2}+m}dr\right|}{\left(\int_0^{+\infty}|f(r)|^p r^{d-1+mp}dr\right)^{\frac{1}{p}}}\frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1},\ d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},\ d\sigma)}} \tag{3.1}$$

is bounded by a constant that depends only on p, q and d. Then (3.1) will be a consequence of the following lemmas.

Lemma 1. *Let $1 \le p < \frac{2d}{d+1}$ and let $f(r) \in C_0^\infty(0, +\infty)$. Then,*

$$\frac{\left|\int_0^{+\infty} f_m(r)J_{\frac{d}{2}-1+m}(r)r^{\frac{d}{2}+m}dr\right|}{\left(\int_0^{+\infty}|f_m(r)|^p r^{d-1+mp}dr\right)^{\frac{1}{p}}} \le Cm^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}}. \tag{3.2}$$

Lemma 2. *Let $p \le q \le 2$. Let $Y_m(\omega)$ be a spherical harmonics which, in polar coordinates, can be expressed as the product of factors of the form of $(\sin z)^{s-j}P_n^{(s)}(\cos z)$, (see section 2). Then,*

$$\frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1},\ d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},\ d\sigma)}} \le Cm^{(d-2)(\frac{1}{2p}-\frac{1}{2q})}. \tag{3.3}$$

When Lemmas 1 and 2 are proved, then Theorem 1.1 easily follows. Indeed, let $p \le \frac{2d}{d-1}$ and $q = \frac{d-1}{d+1}p'$. By (3.2) and (3.3) the right-hand side of (3.1) is at most

$$Cm^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}+(d-1)(\frac{1}{2p}-\frac{1}{2q})},$$

and the conditions on p and q guarantee that the exponent of m above is equal to $-\frac{d-1}{d+1}\frac{1}{q} = -\frac{1}{p'}$ which is nonpositive.

Proof. (Lemma 1). By Hölder's inequality,

$$\begin{aligned} &\left|\int_0^{+\infty} f(r)J_{\frac{d}{2}-1+m}(r)r^{\frac{d}{2}+m}dr\right| \\ \le &\left(\int_0^{+\infty}|f(r)|^p r^{mp+d-1}dr\right)^{\frac{1}{p}} \|J_{\frac{d}{2}-1+m}r^{\frac{d}{2}-\frac{d-1}{p}}\|_{L^{p'}(\mathbf{R}^+)}. \end{aligned} \tag{3.4}$$

By Proposition 1, the $L^{p'}$ norm in (3.4) is finite if and only if $p < \frac{2d}{d+1}$ and is at most a constant multiple of the quantity $m^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}}$. □

The proof of Lemma 2 utilizes Proposition 2 and will be given in section 7.

4. Some more lemmas

The proof of Proposition 2 relies on Lemmas 3, 4 and 5 stated and proved below.

Lemma 3. *Let* $0 < j \le s$. *The relative extrema of* $(\sin z)^{s-j}P_n^{(s)}(\cos z)$ *in the interval* $[0, \frac{\pi}{2}]$ *are increasing whenever*

$$z \le z_{j,n}^s = \frac{1}{2}\arccos\left(\frac{j-3j^2+j^3+jn^2-s+2js+2jns+s^2-js^2}{j\,(n^2-j^2+2ns+s^2)}\right) \tag{4.1}$$

and decreasing otherwise. The relative extrema of $(\sin z)^s P_n^{(s)}(\cos z)$ *in the interval* $[0, \frac{\pi}{2}]$ *are increasing whenever* $0 \le s \le 1$ *and decreasing otherwise.*

Proof. Let

$$\psi_j(z) = (n+s)^2 + j^2 + \frac{j(j-1)+s(s-1)}{\sin^2 z}.$$

Since $y_{0,n}^s(z) = (\sin z)^s P_n^{(s)}(\cos z)$ satisfies the differential equation

$$u'' + \psi_0(z)u = 0,$$

(see the Appendix), it is not difficult to prove that $y_{j,n}^s(z)$ satisfies the differential equation

$$v'' + 2jv'\cot z + \psi_j(z)v = 0. \tag{4.2}$$

Let

$$f(z) = (y_{j,n}^s(z))^2 + \frac{(\frac{d}{dz}\,y_{j,n}^s(z))^2}{\psi_j(z)}.$$

Then,

$$f'(z) = 2\frac{d}{dz}y_{j,n}^s(z)\left(y_{j,n}^s(z) + \frac{\frac{d^2}{d^2z}y_{j,n}^s(z)\psi_j(z) - \frac{1}{2}\frac{d}{dz}y_{j,n}^s(z)\psi_j'(z)}{\psi_j^2(z)}\right),$$

and by (4.2)

$$f'(z) = -2g(z)\frac{(\cot z)(\csc z)^2\left(\frac{d}{dz}y^s_{j,n}(z)\right)^2}{\left(n^2 - j^2 + 2ns + s^2 + (j-s)\left(-1+j+s\right)(\csc z)^2\right)^2},$$

where

$$g(z) = j^3 - 3j^2 + s(s-1) + j(1+n^2+2s+2ns-s^2) + j(j^2-(n+s)^2)\cos(2z).$$

$f(z)$ is increasing if and only if $g(z) \le 0$. If $j=0$ then $g(z) = s(s-1)$, and therefore the sequence of the relative extrema of $y^{(s)}_{0,n}(z)$ is increasing if $s \ge 1$ and decreasing if $0 \le s \le 1$.

If $j \ne 0$, then $f(z)$ is increasing if and only if $z \le z^s_{j,n}$, where $z^s_{j,n}$ is defined as in (4.1). Since $f(z) = (y^s_{j,n}(z))^2$ at the critical points of $y^s_{j,n}(z)$, then the theorem is proved. □

The next ingredient of our proof is a theorem of Sturm type.

Lemma 4. *Let $H(z)$ be continuous on (z_1, z_2). Suppose that $u(z)$ satisfies $u'' + H(z)u = 0$ and that $H(z) \ge N > 0$ on (z_1, z_2). Then, $u(z)$ has a zero on every subinterval of $(z_1,\ z_2)$ of length $\frac{\pi}{\sqrt{N}}$.*

Proof. It is an easy consequence of Theorem 1.82.1 in [12], (see also [8]). □

We will also need the following easy Lemma.

Lemma 5. *Let $-x < y < \infty$, with $x > 0$. The function*

$$x \to \frac{\Gamma(x)\, x^y}{\Gamma(x+y)}$$

is an increasing function of x.

Proof. Let $f(x) = \frac{\Gamma(x)x^y}{\Gamma(x+y)}$. To prove that $f(x)$ is increasing we prove that

$$\ln f(x) = y\ln x + \ln(\Gamma(x))) - \ln(\Gamma(x+y))$$

is increasing, that is, its derivative is positive.

We recall that the logarithmic derivative of $\Gamma(z)$ is

$$\frac{\Gamma'(z)}{\Gamma(z)} = \gamma - \frac{1}{z} - \sum_{m=1}^{\infty}\left(\frac{1}{z+m} - \frac{1}{m}\right)$$

where γ is Euler's constant. Therefore,

$$(\ln f(x))' = \frac{y}{x} - \sum_{m=0}^{\infty}\frac{1}{x+m} - \frac{1}{x+y+m}.$$

The sum above is

$$\le \int_0^{\infty}\left(\frac{1}{x+\zeta} - \frac{1}{x+y+\zeta}\right)d\zeta = \ln\left(\frac{x+y}{x}\right).$$

Therefore,

$$(\ln f(x))' \ge \frac{y}{x} - \ln(1+\frac{y}{x}) > 0$$

as required. □

An immediate consequence of Lemma 5 is that $\dfrac{\Gamma(x)\,x^y}{\Gamma(x+y)} \le \lim_{x\to\infty} f(x) = 1$, by Stirling's formula, while, if $x \ge x_0$, $f(x) \ge f(x_0)$. Therefore,

$$\frac{\Gamma(x+y)}{\Gamma(x)} \le x^y x_0^{-y} \frac{\Gamma(x_0+y)}{\Gamma(x_0)}. \tag{4.3}$$

5. Proof of Proposition 2

Let $y^s_{j,n}(z) = (\sin z)^{s-j} P_n^{(s)}(\cos z)$, and let $c_{n,\,s}$ be as defined in (2.4).

In what follows we will assume that n is even, since the proof in the other case is similar. We first consider the case $j = 0$. By complex interpolation we can extend the result to the general case. Indeed, the function $y^s_{j,n}(z)$ depends analytically on j. If $j = s$ then $\|y^s_{s,\,n}\|_\infty = P_n^{(s)}(1) = \frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}$, (see the Appendix). If we prove that $\|y^s_{0,n}\|_\infty = c_{n,\,s}$, then

$$\|y^s_{j,n}\|_\infty \le (c^s_n)^{1-\frac{j}{s}} (P_n^{(s)}(1))^{\frac{j}{s}}$$

which is (2.3). We now prove (2.5). ¿From the inequality above follows that

$$\|y^s_{j,n}\|_\infty \le c_{n,\,s} \left(\frac{P_n^{(s)}(1)}{c_{n,\,s}} \right)^{\frac{j}{s}} = c_{n,\,s} \left(\frac{\sqrt{\pi}\Gamma(\frac{1+n}{2}+s)}{\Gamma(\frac{1+n}{2})\Gamma(\frac12+s)} \right)^{\frac{j}{s}}.$$

Let $t = \frac{1+n}{2}$ for the sake of simplicity. We prove that

$$\left(\frac{\sqrt{\pi}\Gamma(t+s)}{\Gamma(t)\Gamma(\frac12+s)} \right)^{\frac{1}{s}} \le e\left(1+\frac{t}{s}\right). \tag{5.1}$$

Let

$$g(t,\,s) = \frac{\sqrt{\pi}s^s\Gamma(t+s)}{e^s(t+s)^s\Gamma(t)\Gamma(\frac12+s)}.$$

We aim to prove that $g(t,\,s) \le 1$ for every $t \ge \frac12$ and $s \ge 0$.

By Lemma 5, $t \to g(t,\,s)$ is increasing. That can be easily seen if we let $x = s+t$ and $s = y$. Therefore

$$g(t,\,s) \le \sqrt{\pi} \frac{s^s}{e^s\Gamma(s+\frac12)} \lim_{t\to\infty} \frac{\Gamma(t+s)}{(t+s)^s\Gamma(t)}.$$

By Stirling's formula,

$$\frac{\Gamma(t+s)}{(t+s)^s\Gamma(t)} \sim \frac{\left(\frac{s+t}{e}\right)^{t+s-\frac12}}{(t+s)^s\left(\frac{t}{e}\right)^{t-\frac12}} = e^{-s}\left(1+\frac{s}{t}\right)^{t-\frac12},$$

and thus

$$\lim_{t\to\infty} \frac{\Gamma(t+s)}{(t+s)^s\Gamma(t)} = 1.$$

Therefore,

$$g(t,\,s) \le \sqrt{\pi}\frac{s^s}{e^s\Gamma(s+\frac12)}. \tag{5.2}$$

Let $h(s)$ be the function on the right-hand side of (5.2). We prove that $h(s)$ is decreasing, and therefore that $g(t,\, s) \le h(0) = 1$ as required.

It is enough to prove that, for every $s > 0$, $h(s+1) \ge h(s)$, or equivalently that

$$\frac{h(s+1)}{h(s)} = \frac{(s+1)^{s+1}}{e\, s^s (s+\frac{1}{2})} = \frac{1}{e}\left(1+\frac{1}{s}\right)^s \frac{s+1}{s+\frac{1}{2}} \le 1.$$

which can easily seen too be the case.

To prove Proposition 2 in the case $j = 0$ we use induction on n. Assume $s > 1$, since the case $s < 1$ is known (see [12]).

The case $n = 0$ is easy to check. Indeed, $P_0^{(s)}(t) \equiv 1$, and the right-hand side of (2.5) is also equal to one.

We now assume that the result is true for $n-1$ and we prove that it is also true for n.

We recall that we have set $y^s_{j,n}(t) = (\sin t)^{s-j} P_n^{(s)}(t)$ and that $(P_n^{(s)}(t))' = 2sP_{n-1}^{(s+1)}(t)$, (see the Appendix). Thus,

$$(y^s_{j,n})'(z) = (s-j)(\cos z) P_n^{(s)}(\cos z)\ (\sin z)^{s-1-j} - 2sP_{n-1}^{(s+1)}(\cos z)(\sin z)^{s+1-j}.$$

Therefore, the following equation is satisfied by the critical points of $y^s_{j,n}(z)$.

$$y^s_{j,n}(z) = \frac{2s}{s-j}(\tan z)\ y^{s+1}_{j,n-1}(z).$$

When $j = 0$, $y^s_{j,n}(z)$ satisfies

$$y^s_{0,n}(z) = 2(\tan z)\ y^{s+1}_{j,n-1}(z). \tag{5.3}$$

Let z^s_n be the point at which $y^s_{0,n}(z)$ attains its maximum. By Lemma 3, the sequence of the relative extrema of $y^s_{0,n}(z)$ is decreasing, and therefore z^s_n is the smallest critical point of $y^s_{0,n}(z)$ in the interval $[0, \frac{\pi}{2}]$.

To estimate z^s_n we use the Lemma 4. We recall that $y^s_{0,n}(z)$ satisfies the differential equation (A.5), with

$$\psi_0(z) = (n+s)^2 + \frac{s(s-1)}{(\sin z)^2} \ge (n+s)^2 + s(s-1).$$

By Lemma 4, $y^s_{0,n}(z)$ has a zero in each interval $[\epsilon,\ \xi(s,n)+\epsilon]$ for every $\epsilon > 0$, where we have let $\xi(s,\, n) = \frac{\pi}{\sqrt{(n+s)^2+s(s-1)}}$. Since $y^s_{0,n}(z)$ vanishes at $z = 0$, then $\frac{d}{dz} y^s_{0,n}(z)$ vanishes at least once in $(0,\ \xi+\epsilon]$. Therefore, $z^s_n \le \xi$, and

$$\tan(z^s_n) = \frac{\sin(z^s_n)}{\cos(z^s_n)} \le \frac{z^s_n}{\sqrt{1-(z^s_n)^2}} = \frac{\pi}{\sqrt{n^2-\pi^2-s+2ns+2s^2}}.$$

If $\|y^{s+1}_{0,n-1}\|_\infty \le c_{n-1,\, s+1}$, then by (5.3) and and the estimate above,

$$\|y^s_{0,n}\|_\infty = |y^s_{0,n}(z^s_n)| \le \frac{2\pi c_{n-1,s+1}}{\sqrt{n^2-\pi^2-s+2ns+2s^2}},$$

and the right-hand side of the inequality above is $\le c_{n,\, s}$ if

$$h(n,s) = \frac{2\pi c_{n-1,s+1}}{c_{n,\, s}\sqrt{n^2-\pi^2-s+2ns+2s^2}} \le 1.$$

Recalling that $n-1$ is odd, since we have assumed that n is even, after easy simplifications we can write

$$h(n,s)=\frac{\pi n(n+2s)}{s\sqrt{n^2-s+2ns}\,\sqrt{n^2-\pi^2-s+2ns+2s^2}}, \tag{5.4}$$

which is easily seen to be at most 1.

6. Proof of Lemma 2

We show that Lemma 2 is a consequence of Proposition 3. The proof of Proposition 3 will be given in section 7.

Let Y_m be as in Lemma 2. We shall prove (3.3), that is: for every $1\le p<q\le 2$ and every $d\ge 2$,

$$\frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1},\,d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},\,d\sigma)}}\le Cm^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})}. \tag{6.1}$$

First of all we observe that it suffices to prove Lemma 2 when $q=2$. Indeed, say that $\frac{1}{q}=\frac{\alpha}{p}+\frac{1-\alpha}{2}$, where $\alpha=\frac{\frac{1}{q}-\frac{1}{2}}{\frac{1}{p}-\frac{1}{2}}$. By the Riesz-Thorin convexity theorem,

$$\|Y_m\|_{L^q(\mathbf{S}^{d-1},\,d\sigma)}\le\|Y_m\|^{\alpha}_{L^p(\mathbf{S}^{d-1},\,d\sigma)}\|Y_m\|^{1-\alpha}_{L^2(\mathbf{S}^{d-1},\,d\sigma)},$$

and if (2.6) holds when $q=2$, then

$$\|Y_m\|_{L^q(\mathbf{S}^{d-1},\,d\sigma)}\le\left(Cm^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\right)^{1-\alpha}\|Y_m\|_{L^p(\mathbf{S}^{d-1},\,d\sigma)}=C^{1-\alpha}m^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{q}\right)},$$

i.e. it holds for all other $q\le 2$. Then, we observe that in order to prove (6.1) for $q=2$ it suffices to prove that

$$\frac{\|Y_m\|_{L^{p'}(\mathbf{S}^{d-1},\,d\sigma)}}{\|Y_m\|_{L^2(\mathbf{S}^{d-1},\,d\sigma)}}\le Cm^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}, \tag{6.2}$$

where p' is the dual exponent of p. Indeed, we observe that

$$\|Y_m\|^2_{L^2(\mathbf{S}^{d-1},\,d\sigma)}\le\|Y_m\|_{L^{p'}(\mathbf{S}^{d-1},\,d\sigma)}\|Y_m\|_{L^p(\mathbf{S}^{d-1},\,d\sigma)},,$$

by Hölder's inequality. Therefore,

$$\frac{\|Y_m\|_{L^2(\mathbf{S}^{d-1},\,d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},\,d\sigma)}}\le\frac{\|Y_m\|_{L^{p'}(\mathbf{S}^{d-1},\,d\sigma)}}{\|Y_m\|_{L^2(\mathbf{S}^{d-1},\,d\sigma)}},$$

and if (6.2) holds, then (6.1) also holds with $q=2$. Finally, we can use Riesz-Thorin convexity theorem once more to reduce the proof of (6.2) to the case $p'=\infty$. We shall therefore prove that

$$\frac{\|Y_m\|_{L^\infty(\mathbf{S}^{d-1},\,d\sigma)}}{\|Y_m\|_{L^2(\mathbf{S}^{d-1},\,d\sigma)}}\le Cm^{\frac{d-2}{4}}. \tag{6.3}$$

We now recall that Y_m is as in Lemma 2, that is, as in (1.3). If we use spherical coordinates, (6.3) can be rewritten as

$$\frac{\sup\limits_{\substack{z_{k+1}\in[0,\,\pi]\\ k=0,\dots,d-3}} \prod_{k=0}^{d-3} |P_{m_k-m_{k+1}}^{(m_{k+1}+\frac{d-k-1}{2})}(\cos z_{k+1})|(\sin z_{k+1})^{m_{k+1}}}{\left(\int_0^{\pi}\cdots\int_0^{\pi}\prod_{k=0}^{d-3}|P_{m_k-m_{k+1}}^{(m_{k+1}+\frac{d-k-1}{2})}(\cos z_{k+1})|^2(\sin z_{k+1})^{2m_{k+1}+d-2-k}dz_1\cdots dz_{d-2}\right)^{\frac{1}{2}}}$$

$$=\prod_{k=0}^{d-3}\frac{\sup\limits_{z_{k+1}\in[0,\,\pi]}|P_{m_k-m_{k+1}}^{(m_{k+1}+\frac{d-k-1}{2})}(\cos z_{k+1})|(\sin z_{k+1})^{m_{k+1}}}{\left(\int_0^{\pi}|P_{m_k-m_{k+1}}^{(m_{k+1}+\frac{d-k-1}{2})}(\cos z_{k+1})|^2(\sin z_{k+1})^{2m_{k+1}+d-2-k}dz_{k+1}\right)^{\frac{1}{2}}}\leq Cm^{\frac{d-2}{4}}.$$

Thus, (6.3) follows if we can prove that, for every $0\leq k\leq d-3$,

$$\frac{\sup\limits_{z_{k+1}\in[0,\,\pi]}|P_{m_k-m_{k+1}}^{(m_{k+1}+\frac{d-k-1}{2})}(\cos z_{k+1})|(\sin z_{k+1})^{m_{k+1}}}{\left(\int_0^{\pi}|P_{m_k-m_{k+1}}^{(m_{k+1}+\frac{d-k-1}{2})}(\cos z_{k+1})|^2(\sin z_{k+1})^{2m_{k+1}+d-2-k}dz_{k+1}\right)^{\frac{1}{2}}}\leq Cm^{\frac{1}{4}}. \tag{6.4}$$

To simplify notation, we will let $z=z_{k+1}$, $n=m_k-m_{k+1}$, $s=m_{k+1}+\frac{d-k-1}{2}$, and $j=\frac{d-2-k}{2}$.

We also observe that we can integrate over the interval $(0,\,\frac{\pi}{2})$ since the ultraspherical polynomials are either even or odd. With the new formalism, the inequality that we shall prove is

$$\frac{\sup\limits_{z\in[0,\,\frac{\pi}{2}]}|P_n^{(s)}(\cos z)|(\sin z)^{s-j}}{\left(\int_0^{\frac{\pi}{2}}|P_n^{(s)}(\cos z)|^2(\sin z)^{2s}dz\right)^{\frac{1}{2}}}\leq Cm^{\frac{1}{4}}. \tag{6.5}$$

A change of variables shows that (6.5) is equivalent to (2.6), which will be proved in the next section.

7. Proof of Proposition 3

As observed at the end of the previous section, (2.6) is equivalent to (6.5). We therefore concentrate our attention to the proof of (6.5). We divide the proof of the inequality (6.5) into four steps.

Step 1. In what follows we will often denote by $I_{j,n}^s$ the ratio on the left-hand side of (6.5) and we will let $\|f\|_p=\|f\|_{L^p(0,\frac{\pi}{2})}$.

The L^2 norm of $(\sin z)^s P_n^{(s)}(\cos z)$ is

$$q_{n,s} = \left(\frac{\pi 2^{1-2s}\Gamma(n+2s)}{(n+s)(\Gamma(s))^2\Gamma(n+1)} \right)^{\frac{1}{2}}, \tag{7.1}$$

(see the Appendix).

By Proposition 2 and (7.1) we obtain,

$$I_{j,n}^s = \frac{\sup\limits_{z\in[0,\frac{\pi}{2}]} \left|(\sin z)^{(s-j)} P_n^{(s)}(\cos z)\right|^2}{\int_0^{\frac{\pi}{2}} |(\sin z)^s P_n^{(s)}(\cos z)|^2 dz} \le e^{2j}\left(1+\frac{n}{s}\right)^{2j}\left(\frac{c_{n,s}}{q_{n,s}}\right)^2.$$

We will assume that n is even, since the proof in the other case is similar. Thus,

$$\left(\frac{c_{n,s}}{q_{n,s}}\right)^2 = \frac{(n+s)\,\Gamma(\frac{1+n}{2})\Gamma(\frac{n}{2}+s)}{\pi\Gamma(1+\frac{n}{2})\Gamma(\frac{1+n}{2}+s)}.$$

By Lemma 5, $\dfrac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{1+n}{2}+s)} \le \left(\dfrac{n}{2}+s\right)^{-\frac{1}{2}}$ and $\dfrac{\Gamma(\frac{1+n}{2})}{\Gamma(1+\frac{n}{2})} \le \left(\dfrac{1+n}{2}\right)^{-\frac{1}{2}}$. We obtain

$$\left(\frac{c_{n,s}}{q_{n,s}}\right)^2 \le \frac{2(n+s)}{\pi(n+1)^{\frac{1}{2}}(n+2s)^{\frac{1}{2}}}$$

Therefore

$$I_{j,n}^s \le e^{2j}\left(1+\frac{n}{s}\right)^{2j}\frac{2(n+s)}{\pi(n+1)^{\frac{1}{2}}(n+2s)^{\frac{1}{2}}}. \tag{7.2}$$

If $n \le \alpha s$ for some fixed $\alpha > 1$, then

$$I_{j,n}^s \le e^{2j}\,(1+\alpha)^{2j}\sup_{n\le\alpha s}\frac{2(n+s)}{\pi(n+1)^{\frac{1}{2}}(n+2s)^{\frac{1}{2}}}$$

One can easily verify that $(n+1)(n+2s) \ge (n+\sqrt{s})^2$; therefore

$$I_{j,n}^s \le e^{2j}\,(1+\alpha)^{2j}\,\frac{2(n+s)}{\pi(n+\sqrt{s})} \le \frac{4}{\pi}e^{2j}\,(1+\alpha)^{2j}\,s^{\frac{1}{2}},$$

which is what we shall prove.

In the next step we will show that we can always reduce matters to this case.

Step 2. In the proof of Lemma 3 we have observed that the following equation is satisfied by the critical points of $y_n^s(t)$.

$$y_{j,n}^s(z) = \frac{2s}{s-j}(\tan z)\; y_{j,n-1}^{s+1}(z).$$

By Lemma 3, the relative extrema of $y_{j,n}^s(z)$ in the interval $[0,\frac{\pi}{2}]$ are increasing whenever

$$z \le z_{j,n}^s = \frac{1}{2}\arccos\left(\frac{j-3j^2+j^3+jn^2-s+2js+2jns+s^2-js^2}{j\,(-j^2+n^2+2ns+s^2)}\right)$$

and decreasing otherwise.

Therefore, $y^s_{j,n}(z)$ attains its maximum at one of the two critical points that immediately follow or precede $z^s_{j,n}$. Let $\overline{z}$ be such point. Then,

$$\sup_{z\in[0,\,\frac{\pi}{2}]} y^s_{j,n}(z) \leq \frac{2s}{s-j}(\tan\overline{z}) \sup_{z\in[0,\,\frac{\pi}{2}]} y^{s+1}_{j,n-1}(z).$$

In the next steps we will prove that there exists $\alpha > \alpha_j$, where

$$\alpha_j = \frac{4\pi}{\sqrt{2j-1}},$$

such that the following inequalities hold whenever $j > \frac{1}{2}$, $s \geq j$ and $n \geq \alpha s$:

a) $\overline{z} \leq 2z^s_{j,n}$,

b) $\frac{2s}{s-j}\tan\overline{z} \leq 1$.

That will be enough to conclude the proof of the Theorem. Indeed, from a) and b) follow that

$$\sup_{0\leq z\leq\frac{\pi}{2}} y^s_{j,n}(z) \leq \sup_{0\leq z\leq\frac{\pi}{2}} y^{s+k}_{j,n-k}(z)$$

for every k which is such that $(n-k+1) \geq \alpha(s+k-1)$. If we let $k = \left[\frac{n-\alpha s}{\alpha+1}\right]$, we have $(n-k+1) \geq \alpha(s+k-1)$ and $n-k \leq \alpha(s+k)$.

By Step 1,

$$I^{s+k}_{j,n-k} \leq Ce^{2j}\left(1+j^2\right)^{2j}(s+k)^{\frac{1}{2}} \leq Ce^{2j}\left(1+j^2\right)^{2j}(s+n)^{\frac{1}{2}},$$

where C depends only on j, which is what we required.

Step 3. In proving a) we suppose that $\overline{z} \geq z^s_{j,n}$, since the other case is trivial.

We recall that $\overline{z}$ is the first critical point of $y^s_{j,n}(z)$ in the interval $[z^s_{j,n},\ \frac{\pi}{2}]$. By Lemma 4, the function $y^s_{j,n}(z)$ has at least a zero in the interval $[z^s_{j,n},\ \sigma(s,\,n)+z^s_{j,n}]$, and at least two zeroes in $[z^s_{j,n},\ 2\sigma(s,\,n)+z^s_{j,n}]$, where $\sigma(s,\,n) = \frac{\pi}{\sqrt{(n+s)^2+s(s-1)}}$. By Rolle's theorem, $y^s_{j,n}(z)$ has at least one critical point in $[z^s_{j,n},\ 2\sigma(s,\,n)+z^s_{j,n}]$, and thus $\overline{z} \leq 2\sigma(s,\,n)+z^s_{j,n}$.

We prove that $2\sigma(s,\,n) \leq z^s_{j,n}$ whenever $n \geq \alpha_j s$ and $s \geq j$. To this aim it is sufficient to prove that

$$\begin{aligned}(4\sigma(s,\,n))^2 &\leq \sin(2z^s_{j,n})^2\\ &= 1-\left(\frac{j-3j^2+j^3+jn^2+(2j+2jn-1)s+s^2(1-j)}{j\left(n^2-j^2+2ns+s^2\right)}\right)^2\\ &= 2u(s,n)-(u(s,n))^2\end{aligned}$$

where we have let

$$u(s,n) = \frac{(2j-1)\left(s^2-s-j^2-j\right)}{j\left((n+s)^2-j^2\right)}.$$

Thus, we shall prove that

$$(u(s,n))^2 - 2u(s,n) + (4\sigma(s,\,n))^2 \leq 0$$

or equivalently that

$$u(s,n) \leq 1+\sqrt{1-(4\sigma(s,\,n))^2}.$$

We observe that $u(s,n) \le u(s,0) \le \lim_{s\to\infty} u(0,s) = 2 - \frac{1}{j}$ since $u(s,n)$ is increasing with respect to s and decreasing with respect to n. Thus, we prove that

$$2 - \frac{1}{j} \le 1 + \sqrt{1 + (4\sigma(s,\,n))^2},$$

or

$$(4\sigma(s,\,n))^2 \le \frac{2}{j} - \frac{1}{j^2}$$

whenever $n \ge \alpha_j s$ and $s \ge j$. But

$$(4\sigma(s,\,n))^2 \le \frac{16\pi^2}{(1+\alpha_j)^2 s^2} \le \frac{16\pi^2}{(1+\alpha_j)^2 j^2},$$

and so the claim readily follows.

Step 4. We now prove b). We shall prove that there exists $\alpha \ge \alpha_j$ which is such that $\tan(2z^s_{j,n}) \le \frac{s-j}{2s}$ whenever $n \ge \alpha s$, $s > j$ and $j > \frac{1}{2}$. That is equivalent to $(2z^s_{j,n}) \le \arctan\left(\frac{s-j}{2s}\right)$, or to

$$\cos(2z^s_{j,n}) \ge \cos\left(\arctan\left(\frac{s-j}{2s}\right)\right) = \frac{1}{\left(1+\left(\frac{s-j}{2s}\right)^2\right)^{\frac{1}{2}}}, \tag{7.3}$$

since $t \to \cos t$ is a decreasing function in $[0,\, \frac{\pi}{2}]$.

The function on the right-hand side of (7.3) is an increasing function of s and its supremum is $\frac{2}{\sqrt{5}}$. Therefore, it is sufficient to prove that

$$\cos(2z^s_{j,n}) = \frac{j - 3\,j^2 + j^3 + j\,n^2 - s + 2\,j\,s + 2\,j\,n\,s + s^2 - j\,s^2}{j\,\left(-j^2 + n^2 + 2\,n\,s + s^2\right)} \ge \frac{2}{\sqrt{5}}.$$

Let $\cos(2z^s_{j,n})) = A(s,\,n)$. It is easy to see that $n \to A(s,\,n)$ is increasing, and therefore that $A(s,\,n) \ge A(s, \alpha s)$. We now prove that for some $\alpha > \alpha_j$

$$A(s,\alpha s) - \frac{2}{\sqrt{5}} = \frac{-5j + 15j^2 - 5j^3 - 2\sqrt{5}j^3 + (5-10j)\,s + \psi(\alpha,j)s^2}{5j\,(j^2 - s^2 - 2\alpha s^2 - \alpha^2 s^2)} \ge 0$$

where $\psi(\alpha,\,j) = \left(-5 + 5j + 2\sqrt{5}j - 10\alpha j + 4\sqrt{5}\alpha j - 5\alpha^2 j + 2\sqrt{5}\alpha^2 j\right)$, whenever $s \ge j > \frac{1}{2}$. But this is easily seen to be satisfied.

8. Appendix

We collect here the definitions and the identities that we have used throughout this paper related to Jacobi polynomials and Bessel functions. Our main reference is the classical book of Szegö [12], but the formulas listed here can also be found in many other standard textbooks on special functions, (see also [1]).

Let $\alpha, \beta \in \mathbf{R}$. The Jacobi polynomials of degree n and order (α,β) are

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\left(\frac{d^n}{dx}\right)(1-x)^{\alpha+n}(1+x)^{\beta+n}. \tag{A.1}$$

They are a complete orthogonal system in $L^2([-1,\,1],\ (1-x)^\alpha(1+x)^\beta dx)$.

When $\alpha = \beta$ the Jacobi polynomials take the name of ultraspherical, or Gegenbauer polynomials and are denoted by $P_n^{(s)}(x)$, The following is the customary notation and normalization.

$$P_n^{(s)}(t) = C_n^s P_n^{s-\frac{1}{2},s-\frac{1}{2}}(t), \qquad s > -\frac{1}{2}, \tag{A.2}$$

where $C_n^s = \dfrac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)} \dfrac{\Gamma(n+2s)}{\Gamma(n+s+\frac{1}{2})}$. We can easily see that $P_n^{(s)}(x) \equiv 1$ when $n = 0$ and $P_n^{(s)}(x) = 2sx$ when $n = 1$; furthermore,

$$P_n^{(s)}(-x) = (-1)^n P_n^{(s)}(x). \tag{A.3}$$

$P_n^{(s)}(t)$ satisfies the following differential equation:

$$(1-x^2)y'' - (2s+1)xy' + n(n+2s)y = 0, \tag{A.4}$$

and $(\sin t)^s P_n^{(s)}(\cos t)$ satisfies the differential equation:

$$u'' + \left((n+s)^2 + \frac{s(s-1)}{(\sin z)^2}\right) u = 0. \tag{A.5}$$

We also recall that

$$\frac{d}{dt} P_n^{(s)}(t) = 2s P_{n-1}^{(s+1)}(x), \tag{A.6}$$

and

$$\sup_{-1 \le x \le 1} |P_n^{(s)}(x)| = P_n^{(s)}(1) = \frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)} \tag{A.7}$$

and that

$$q_{n,s} = \left(\int_0^{\frac{\pi}{2}} (\sin t)^2 \left| P_n^{(s)}(\cos t) \right|^2 dt \right)^{\frac{1}{2}} = \left(\frac{\pi 2^{1-2s}\Gamma(n+2s)}{(n+s)(\Gamma(s))^2\Gamma(n+1)} \right)^{\frac{1}{2}}. \tag{A.8}$$

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