Convolution Calderón-Zygmund singular integral operators with rough kernels

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Abstract

A survey of known results in the theory of convolution type Calderón-Zygmund singular integral operators with rough kernels is given. Some recent progress is discussed. A list of remaining open questions is presented.

1 Introduction

Throughout this article, Ω will be a complex-valued integrable function over the sphere $\mathbf{S}^{n-1}$, with mean value zero with respect to surface measure. Define a tempered distribution K_Ω on $\mathbf{R}^n$ by setting

$$K_\Omega(f) = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\Omega(x/|x|)}{|x|^n} f(x)\,dx = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(x/|x|)}{|x|^n} f(x)\,dx, \tag{1}$$

for f in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$. The limit in (1) can be easily shown to exist for any f C^1 function on $\mathbf{R}^n$ which satisfies $|f(x)| \le C|x|^{-\delta}$ for some $C, \delta > 0$ and all $|x|$ large.

We will denote by T_Ω the operator given by convolution with Ω initially defined on the set of Schwartz functions $\mathcal{S}(\mathbf{R}^n)$. The operators T_Ω were introduced by Calderón and Zygmund in [1] and today are referred to as Calderón-Zygmund singular integral operators (of convolution type).

In this article we shall be concerned with the following questions: What conditions on Ω imply L^p boundedness for T_Ω and other related operators? It is a classical result, that if Ω has some smoothness on $\mathbf{S}^{n-1}$, say Lipschitz of order $\alpha > 0$, then T_Ω is a bounded operator

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on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. In fact, for such Ω's we have that K_Ω satisfies Hörmander's condition

$$\int_{|x|\geq 2|y|} |K_\Omega(x-y) - K_\Omega(x)|\, dx \leq B, \tag{2}$$

for some $B = B(n, \Omega) > 0$. Condition (2) implies that T_Ω is of weak type $(1,1)$, a property which will be discussed in section 4. This property, together with the L^2 boundedness of Ω (which follows from a Fourier transform calculation), implies that T_Ω is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. See [19] for details.

In 1956 Calderón and Zygmund [2] introduced the method of rotations. The idea is the following: If Ω is an odd function on $\mathbf{S}^{n-1}$, then it is easy to see that

$$(T_\Omega f)(x) = \frac{1}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta)(H_\theta f)(x)\, d\theta, \tag{3}$$

where $H_\theta f$ is the directional Hilbert transform of f in the direction $\theta \in \mathbf{S}^{n-1}$, defined by

$$(H_\theta f)(x) = \text{p.v.}\, \frac{1}{\pi}\int_{\mathbf{R}^1} \frac{f(x-t\theta)}{t}\, dt = \frac{1}{\pi} T_{\delta_\theta - \delta_{-\theta}}, \tag{4}$$

where δ_a is Dirac mass at a. (Of course $\Omega = \delta_\theta - \delta_{-\theta}$ is not in L^1, but we can extend the definition of T_Ω for Ω bounded Borel measures on $\mathbf{S}^{n-1}$.) Using a rotation, it is easy to show that $H_\theta f$ maps $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ with the same norm as the usual Hilbert transform from $L^p(\mathbf{R}^1) \to L^p(\mathbf{R}^1)$. It follows from (3) that T_Ω maps $L^p(\mathbf{R}^n)$ into itself for any Ω odd in $L^1(\mathbf{S}^{n-1})$.

In the same paper [2], Calderón and Zygmund proved that if

$$\int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \operatorname{Log}^+ |\Omega(\theta)|\, d\theta < \infty, \tag{5}$$

then T_Ω is a bounded operator on L^p, $1 < p < \infty$. In view of the previous discussion about odd kernels, condition (5) is only relevant to even Ω's.

The general question along these lines is the following:

Question 1. *Let Ω be an integrable even function on $\mathbf{S}^{n-1}$ with integral zero. Given a $1 < p < \infty$, find a necessary and sufficient condition on Ω such that T_Ω extends to a bounded operator from $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$.*

It is likely that such a condition will depend on the parameter p.

2 L^2 boundedness

L^2 is a good starting point to study boundedness of the operators T_Ω on L^p spaces. We begin with the following natural question: If Ω is merely an L^1 function with integral zero, is T_Ω a bounded operator on $L^2(\mathbf{R}^n)$?

The answer is known to be negative. More precisely, an example constructed by M. Weiss and A. Zygmund gives a dramatic answer to this question:

Theorem 1. *(M. Weiss and A. Zygmund [21]) Let $\phi(u)$ be a non-negative increasing (non-necessarily strictly) function defined for $u \geq 0$ which satisfies:*

$$\lim_{u\to\infty} \frac{\phi(u)}{u \log u} = 0.$$

Then there exists an Ω in $L^1(\mathbf{S}^{n-1})$ with integral zero which satisfies

$$\int_{\mathbf{S}^{n-1}} \phi(|\Omega(\theta)|)\, d\theta < +\infty,$$

and a continuous $f \in L^p(\mathbf{R}^n)$ for all $1 \leq p \leq \infty$, which tends to zero at infinity such that

$$\limsup_{\varepsilon\to 0} \left| \int\limits_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy \right| = +\infty$$

for almost all x in $\mathbf{R}^n$.

In particular, taking $\phi(u) = u$, we conclude that there exists an Ω in $L^1(\mathbf{S}^{n-1})$ such that T_Ω is not a bounded operator on all L^p spaces. Taking $\phi(u) = u(\log u)^{1-\varepsilon}$ we obtain that $\Omega \in L\text{Log}^{1-\varepsilon}L$ is not a strong enough condition to imply L^p boundedness for T_Ω.

However, the question is far from over. We know precisely when a convolution operator maps $L^2(\mathbf{R}^n)$ into itself. This happens exactly when the Fourier transform of the convolving distribution is a bounded function. Let us compute the Fourier transform of the distribution K_Ω. Fix f in the Schwartz class. We have

$$\begin{aligned}
\widehat{K_\Omega}(f) &= \int_{\mathbf{R}^n} K_\Omega(x) \widehat{f}(x)\, dx \\
&= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbf{R}^n} f(y) \left[\int\limits_{\varepsilon \leq |x| \leq N} \frac{\Omega(x/|x|)}{|x|^n} e^{-2\pi i y\cdot x} dx \right] dy \qquad (6)\\
&= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbf{R}^n} f(y) \left[\int_{\mathbf{S}^{n-1}} \Omega(\theta) \left\{ \int_{r=\varepsilon/|y|}^{N/|y|} e^{-2\pi i r\, y'\cdot\theta} \frac{dr}{r} \right\} d\theta \right] dy
\end{aligned}$$

where $y' = y/|y|$. It can be shown (see [19] for details) that the expression inside the curly brackets above converges pointwise to

$$\frac{\pi i}{2} \mathrm{sgn}(\theta \cdot y') + \log \frac{1}{|\theta \cdot y'|}\,.$$

Therefore, if we assume that

$$\sup_{y' \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \log \frac{1}{|\theta \cdot y'|}\, d\theta < +\infty, \qquad (7)$$

it is an easy consequence of the Lebesgue dominated convergence theorem that $\widehat{K_\Omega}$ is the bounded function:

$$\widehat{K_\Omega}(y) = \int_{\mathbf{S}^{n-1}} \Omega(\theta) \left[\frac{\pi i}{2} \mathrm{sgn}(\theta \cdot y') + \log \frac{1}{|\theta \cdot y'|} \right] d\theta. \tag{8}$$

More generally, it can be seen from the calculations above that $\widehat{K_\Omega}$ is a function in $L^\infty(\mathbf{R}^n)$ if and only if the limit of the bracketed expression in (6) exists and is equal to a bounded function, i.e.

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int\limits_{\varepsilon \le |x| \le N} \frac{\Omega(x/|x|)}{|x|^n} e^{-2\pi i y \cdot x} dx = m(y) \in L^\infty(\mathbf{R}^n). \tag{9}$$

Condition (7), even though not equivalent to (9) contains most of its essence.

An easy consequence of the above is the following

Theorem 2. *Suppose that Ω satisfies (7) or more generally (9). Then T_Ω extends to an operator bounded from $L^2(\mathbf{R}^n)$ into itself. In fact condition (9) is equivalent to the L^2 boundedness of T_Ω.*

Exercise. Use Young's inequality in the context of Orlicz spaces to prove directly that condition (5) implies condition (7).

3 L^p boundedness, $1 < p < \infty$

It is well known that if a convolution operator maps $L^p \to L^p$ then by duality it also maps $L^{p'} \to L^{p'}$ with the same norm. ($p' = p/(p-1)$ throughout this paper.) It follows that it maps $L^2 \to L^2$ by interpolation. Since condition (9) is equivalent to L^2 boundedness, it is unlikely to expect that condition (9) would imply that T_Ω is L^p bounded. Condition (7) is slightly weaker, and we can pose the following question:

Question 2. *Let Ω be an integrable function on $\mathbf{S}^{n-1}$ with integral zero satisfying condition (7). Does it follow that T_Ω is a bounded operator on $L^p(\mathbf{R}^n)$ for some $p \neq 2$?*

A weaker question is answered in Theorem 4.

Let us denote by $H^1(\mathbf{S}^{n-1})$ the 1-Hardy space on the sphere in the sense of Coifman and Weiss [6]. It is a known result that functions Ω on $\mathbf{S}^{n-1}$ which satisfy (5) are in $H^1(\mathbf{S}^{n-1})$. It is natural to ask whether T_Ω is L^p bounded when $\Omega \in H^1(\mathbf{S}^{n-1})$. With the aid of a theorem in [3] and with a bit of work one can show that the condition $\Omega \in H^1(\mathbf{S}^{n-1})$ is equivalent to

$$\frac{\Omega(x/|x|)}{|x|^n} \chi_{1/2 \le |x| \le 2} \in H^1(\mathbf{R}^n), \tag{10}$$

where $H^1(\mathbf{R}^n)$ denotes the Hardy space on $\mathbf{R}^n$. See [18] for details.

We now investigate connections between condition (10) and L^2 boundedness. Take Ω to be an even function in this discussion. Using polar coordinates and the fact that Ω has mean value zero, it is easy to see that

$$\log 4 \int\limits_{\mathbf{S}^{n-1}} \Omega(\theta) \log \frac{1}{|\theta \cdot \xi|} d\theta = \int\limits_{1/2 \le |x| \le 2} \frac{|\Omega(x/|x|)|}{|x|^n} \log \frac{1}{|x \cdot \xi|} dx, \tag{11}$$

where both integrals in (11) are finite for almost all $\xi \in \mathbf{R}^n$ by an easy application of Fubini's theorem. The H^1-BMO duality now gives

$$\left| \int\limits_{1/2\leq|x|\leq 2} \frac{\Omega(x)}{|x|^n} \log \frac{1}{|x\cdot\xi|} dx \right| \leq \left\| \log \frac{1}{|x\cdot\xi|} \right\|_{BMO(dx)} \left\| \frac{\Omega(x)}{|x|^n} \chi_{1/2\leq|x|\leq 2} \right\|_{H^1(dx)} . \tag{12}$$

Since the BMO norm is invariant under rotations, it is easy to see the BMO norms of the functions $x \to -\log|x\cdot\xi|$ are uniformly bounded in ξ. It follows from (11) and (12) that

$$\sup_{|\xi|=1} \left| \int_{\mathbf{S}^{n-1}} \Omega(\theta) \ln \frac{1}{|\theta\cdot\xi|}\, d\theta \right| \leq C \left\| \frac{\Omega(x)}{|x|^n} \chi_{1/2\leq|x|\leq 2} \right\|_{H^1(dx)} . \tag{13}$$

Since Ω is even, the left hand side of (13) is equal to $\|\widehat{K_\Omega}\|_{L^\infty}$ in view of (8). We conclude that T_Ω is L^2 bounded, and hence condition (10) implies L^2 boundedness.

We now show that the $H^1(\mathbf{S}^{n-1})$ condition implies that L^p boundedness for T_Ω for $1 < p < \infty$. The theorem below was independently discovered by Connett [7] and Ricci and Weiss [14]. See also [6] for a proof in dimension $n = 2$. The proof we give below uses the equivalent hypothesis (10).

Theorem 3. *(W. Connett, F. Ricci and G. Weiss) Let Ω be an integrable function on $\mathbf{S}^{n-1}$ with mean value zero which satisfies condition (10). Then T_Ω extends to a bounded operator from $L^p(\mathbf{R}^n)$ into itself for $1 < p < \infty$.*

Proof. As discussed before, it suffices to consider Ω even. Denote by R_j the j^{th} Riesz transform given by convolution with p.v. $\dfrac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \dfrac{x_j}{|x|^{n+1}}$. Since

$$I = \sum_{i=1}^n R_j^2,$$

it follows that

$$T = \sum_{i=1}^n R_j T_j, \tag{14}$$

where $T_j = R_j T$. Observe that T_j is well defined as an operator on L^2. Let V_j be the kernel of T_j. Since T has an even kernel and R_j has an odd kernel, T_j has an odd kernel K_j which is also homogeneous of degree $-n$. Write

$$K_j(x) = R_j \left(\text{p.v.} \frac{\Omega(\,\cdot\,)}{|\cdot|^n} \right)(x) = \frac{V_j(x/|x|)}{|x|^n},$$

where V_j is an odd distribution on the sphere. ($V_j(x/|x|)$ denotes the distribution $\phi \to \langle V_j, \phi(x/|x|)\rangle$ on $\mathbf{R}^n$). We will show that V_j is a function satisfying

$$\int_{\mathbf{S}^{n-1}} |V_j(\theta)| d\theta < \infty. \tag{15}$$

To prove (15) write $K_j = K_j^0 + K_j^1 + K_j^\infty$, where

$$K_j^0 = R_j \left(\text{p.v.} \frac{\Omega(\,\cdot\,)}{|\,\cdot\,|^n} \chi_{|\,\cdot\,|<\frac{1}{2}} \right), \quad K_j^1 = R_j \left(\frac{\Omega(\,\cdot\,)}{|\,\cdot\,|^n} \chi_{\frac{1}{2}\le|\,\cdot\,|\le 2} \right), \quad K_j^\infty = R_j \left(\frac{\Omega(\,\cdot\,)}{|\,\cdot\,|^n} \chi_{2<|\,\cdot\,|} \right). \tag{16}$$

Fix x in the annulus $3/4 \le |x| \le 3/2$. Then

$$\begin{aligned} \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \left| K_j^0(x) \right| &= \left| \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<\frac{1}{2}} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n} dy \right| \\ &= \left| \int_{|y|<\frac{1}{2}} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) \frac{\Omega(y)}{|y|^n} dy \right| \\ &= \left| \int_0^{\frac{1}{2}} \int_{\mathbf{S}^{n-1}} \theta \cdot \nabla \left(\frac{x_j}{|x|^{n+1}} \right) (x - \rho\,\theta\, t_{x,\rho\theta})\, \Omega(\theta)\, d\theta d\rho \right| \\ &\le \frac{1}{2} \|\Omega\|_{L^1} \max_{1/4\le|x|\le 7/4} \left| \nabla \left(\frac{x_j}{|x|^{n+1}} \right) \right| = C\|\Omega\|_{L^1}, \end{aligned}$$

for some $t_{x,\rho\theta} \in [0,1]$. Similarly,

$$\begin{aligned} \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \left| K_j^\infty(x) \right| &= \left| \int_{|y|>2} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n} dy \right| \\ &\le \int_{|y|>2} \frac{1}{|x-y|^n} \frac{|\Omega(y)|}{|y|^n}\, dy \\ &\le \int_{|y|>2} \frac{4^n}{|y|^{2n}} |\Omega(y)|\, dy = C\|\Omega\|_{L^1}, \end{aligned}$$

for $3/4 \le |x| \le 3/2$. Finally, K_j^1 is in $L^1(\mathbf{R}^n)$ since by assumption $(\Omega(x/|x|)/|x|^n)\,\chi_{1/2\le|x|\le 2}$ is in the Hardy space $H^1(\mathbf{R}^n)$. See [20] p. 114.

It follows that K_j is integrable over the annulus $3/4 \le |x| \le 3/2$. Therefore $V_j(x/|x|)/|x|^n$ has to be integrable over a sphere $a\mathbf{S}^{n-1}$, for some $3/4 \le a \le 3/2$. By homogeneity V_j is integrable over $\mathbf{S}^{n-1}$. Therefore $T_j = T_{V_j}$ and by identity (3) for $\Omega = V_j$ we deduce that $T_j = T_{V_j}$ is bounded on L^p. (14) now gives that T is bounded on L^p. □

Remark. In the proof of Theorem 3, we showed that condition (10) implies that V_j is integrable over $\mathbf{S}^{n-1}$. In fact, the converse is also true. It is shown in [14] that $V_j \in L^1(\mathbf{S}^{n-1})$ for all $j = 1, \dots, n$ if and only if $\Omega \in H^1(\mathbf{S}^{n-1})$. Moreover, condition $\Omega \in H^1(\mathbf{S}^{n-1})$ is equivalent to condition (10) as shown in [18]. Therefore all these three conditions on Ω are equivalent and they all imply that T_Ω is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$.

We end this section with a another sufficient condition on Ω that implies L^p boundedness for T_Ω. The theorem below is proved based on ideas developed in [9]. Littlewood-Paley decomposition and a bootstrapping argument are used in conjunction with the logarithmic decay at infinity of the Fourier transform of the expression in (10). For a proof we refer the reader to [12].

Theorem 4. *Let $\alpha > 0$. Let Ω be an even function in $L^1(\mathbf{S}^{n-1})$ with mean value zero which satisfies:*

$$\sup_{y' \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \left(\log \frac{1}{|\theta \cdot y'|} \right)^{1+\alpha} d\theta < +\infty. \tag{17}$$

Then T_Ω extends to a bounded operator from $L^p(\mathbf{R}^n)$ into itself for $(2+\alpha)/(1+\alpha) < p < 2+\alpha$.

Remark. It follows that if condition (17) holds for every $\alpha > 0$, then T_Ω maps $L^p \to L^p$ for all $1 < p < \infty$. It is natural to ask how condition (17) for all $\alpha > 0$ compares with condition (5) or even the condition $\Omega \in H^1(\mathbf{S}^{n-1})$. The authors have constructed examples of functions Ω which satisfy condition (17) for all $\alpha > 0$ but do not satisfy the H^1 condition (10). See [12] for details. Conversely, the function

$$\Omega(\theta) = \sum_{k=2}^{\infty} \frac{e^{ik\theta}}{(\log k)^2}$$

is in $H^1(\mathbf{S}^1)$ but it behaves like $\theta^{-1}\log^{-2}(\theta^{-1})$ as $\theta \to 0+$ and therefore it fails to satisfy condition (17) for any $\alpha > 0$. See [22] p. 189 for a justification of this.

4 The L^1 theory

We now turn to questions regarding the behavior of T_Ω on $L^1(\mathbf{R}^n)$. T_Ω is said to be of weak type $(1,1)$ if there is a constant $C = C(n,\Omega) > 0$ such that for all $f \in L^1(\mathbf{R}^n)$ we have

$$|\{x : \ |(T_\Omega f)(x)| > \alpha\}| \leq C\|f\|_{L^1}/\alpha.$$

The question of weak type $(1,1)$ boundedness of T_Ω for Ω rough has puzzled many authors who obtained partial results. An important question along these lines was whether a condition bearing on the size of Ω alone sufficed for the weak type $(1,1)$ boundedness of T_Ω. The answer turned out to be positive. See M. Christ [4] and S. Hofmann [13] for the case $\Omega \in L^q(\mathbf{S}^1)$, $q > 1$, and M. Christ and J.-L. Rubio de Francia [5] for $\Omega \in L\mathrm{Log}^+L(\mathbf{S}^1)$. The latter authors were able to extend their result to all dimensions $n \leq 7$ (unpublished). Finally A. Seeger [15] proved that T_Ω is weak type $(1,1)$ bounded when $\Omega \in L\mathrm{Log}^+L(\mathbf{S}^{n-1})$ in all dimensions.

Theorem 5. *Let Ω be in $L^1(\mathbf{S}^{n-1})$ with integral zero. Suppose that Ω satisfies condition (5). Then T_Ω can be extended to an operator of weak type $(1,1)$.*

At this point it is natural to ask whether the method of rotations can be used to show that T_Ω is of weak type $(1,1)$. This is known to be false. The following question is therefore more difficult than its L^p counterpart:

Question 3. *Let Ω be an integrable odd function on $\mathbf{S}^{n-1}$. Is T_Ω of weak type $(1,1)$?*

Outside the context of odd functions, the general question for weak type $(1,1)$ which is analogous to Question 1 can be phrased as follows:

Question 4. *Let Ω be an integrable function on $\mathbf{S}^{n-1}$ with integral zero. Find a necessary and sufficient condition on Ω such that the associated operator T_Ω is of weak type $(1,1)$.*

In the context of question 4 posed above, it is not as natural to assume that $\Omega \in L^1(\mathbf{S}^{n-1})$, as it is to assume that Ω is a general distribution on the sphere. The reason for that it is sometimes easier to handle finite sums of Dirac masses than general L^1 functions. In this case, it is conceivably easier to handle a finite sum of directional Hilbert transforms than a general T_Ω with $\Omega \in L^1(\mathbf{S}^{n-1})$. Furthermore, one sees from (8) that certain distributions Ω give rise to bounded operators on L^2.

Question 5. *Let Ω be a distribution on $\mathbf{S}^{n-1}$ with mean value zero. Find a necessary and sufficient condition on Ω such that the associated operator T_Ω is of weak type $(1,1)$. Likewise for T_Ω to be bounded on L^p.*

Obtaining weak type $(1,1)$ bounds is usually a more difficult task than proving L^p boundedness, for, the latter bounds follow from the weak type $(1,1)$ bounds by interpolation. In some occasions a more natural aspect of the L^1 theory is to prove that the operator in question is bounded from the Hardy space H^1 to L^1.

It is fairly easy to check that if K_Ω possesses a certain amount of smoothness then T_Ω extends to a bounded operator from $H^1 \to L^1$. Here is a precise statement.

Theorem 6. *Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$ has mean value zero and assume that K_Ω satisfies (2) and Ω satisfies (9). Then T_Ω extends to a bounded operator from $H^1 \to L^1$.*

Proof. The proof is standard. Fix an atom a_Q and prove that $\|T(a_Q)\|_{L^1} \le C$ with C independent of Q. For $x \in 2Q$ use the L^2 estimate (which is follows from (9)) and Hölder's inequality. For $x \notin 2Q$ subtract $K(x)a_Q(x)$ from $T(a_Q)(x)$ and then use condition (2). □

Even though $H^1 \to L^1$ boundedness holds for Ω smooth enough, it may fail for Ω rough. A good starting point to study $H^1 \to L^1$ boundedness is the directional Hilbert transform. Consider the unit vector $e_1 = (1,0)$ in $\mathbf{R}^2$ and the operator H^{e_1}. Let f be the H^1 function in $\mathbf{R}^2$ defined by $f(x_1,x_2) = \chi_{|x|<1,x_2>0} - \chi_{|x|<1,x_2<0}$. Then it is easy to see that $|(H^{e_1}f)(x)| \ge C|x|^{-1}$ when $|x| \ge 2$ and $|x_2| \le 1/2$. It follows that $H^{e_1}f$ cannot be in $L^1(\mathbf{R}^2)$.

Other examples can be found in [8]. Below, we give the an example communicated to us by M. Christ.

Example. *(M. Christ) There exists an Ω in $L^2(\mathbf{S}^1)$ such that T_Ω does not map $H^1(\mathbf{R}^2)$ to $L^1(\mathbf{R}^2)$.*

For $x \in \mathbf{R}^2$, let $\theta_x = \mathrm{Arg}x$ denote the argument of x. Choose a lacunary sequence $\lambda_j \ge 2^j$ whose properties will be specified later and let a_j be a square summable sequence also to be chosen later. Define

$$\Omega(x) = \sum_{j=1}^{\infty} a_j e^{i\lambda_j\theta_x}.$$

We have that Ω is in $L^2(\mathbf{S}^1)$ and it has mean value zero. Now take f to be a C^∞ and radial atom which is supported in the unit disc in $\mathbf{R}^2$. Fix $x \in \mathbf{R}^2$ satisfying $1/2 \le |x_2|/|x_1| \le 2$ in the annulus $\lambda_j\mu_j \le |x| \le 2\lambda_j\mu_j$ for some $j \ge 1$. When we write $O(\,\cdot\,)$, we are tacitly implying that the constants involved in the bounds are independent of the λ_j's and x but may depend on f and the other parameters. For $1 \le k \le j$ we calculate

$$
\begin{aligned}
&\left(f * \frac{e^{i\lambda k\theta_y}}{|y|^2}\right)(x) \\
=&\frac{1}{|x|^2}\iint\limits_{|y|\le 1} f(y)e^{i\lambda k\theta_{x-y}}dy + \iint\limits_{|y|\le 1} f(y)e^{i\lambda k\theta_{x-y}}\left(\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right)dy \\
=&\frac{1}{|x|^2}\int_0^1 f(\rho(1,0))\rho\left[\int_0^{2\pi} e^{i\lambda k \mathrm{Arg}(x-\rho\, e^{i\phi})}d\phi\right]\,d\rho + O\left(\frac{1}{|x|^3}\right) \\
=&\frac{1}{|x|^2}\int_0^1 f(\rho(1,0))\rho e^{i\lambda k\theta_x}\left[\int_0^{2\pi} e^{i\lambda k(\mathrm{Arg}(x-\rho\, e^{i\phi})-\theta_x)}d\phi\right]\,d\rho + O\left(\frac{1}{|x|^3}\right) \\
=&\frac{1}{|x|^2}\int_0^1 f(\rho(1,0))\rho e^{i\lambda k\theta_x}\int_0^{2\pi}\Bigg[1+\sum_{m=1}^{6}\frac{(i\lambda_k)^m}{m!}(g_\phi(\rho)-g_\phi(0))^m + \\
&\qquad + O\left(\lambda_k^7(g_\phi(\rho)-g_\phi(0))^7\right)\Bigg]d\phi\, d\rho + O\left(\frac{1}{|x|^3}\right),
\end{aligned} \tag{18}
$$

where $g_\phi(\rho) = \arctan[(x_2 - \rho\sin\phi)/(x_1 - \rho\cos\phi)]$. The mean value theorem and an easy estimate give that

$$
g_\phi(\rho) - g_\phi(0) = \frac{\rho}{|x|^2}(-x_1\sin\phi + x_2\cos\phi) + O\left(\frac{1}{|x|^2}\right).
$$

Plugging in the estimate above in (18), calculating, and integrating with respect to ϕ, we obtain that

$$
\begin{aligned}
&\left(f * \frac{e^{i\lambda k\theta_y}}{|y|^2}\right)(x) \\
=&\frac{1}{|x|^2}\int_0^1 f(\rho(1,0))\rho e^{i\lambda k\theta_x}\left[1-4\frac{\lambda_k^2\rho^2}{|x|^2}+c_4\frac{\lambda_k^4\rho^4}{|x|^4}+c_6\frac{\lambda_k^6\rho^6}{|x|^6}+O\left(\frac{\lambda_k^7}{|x|^7}\right)\right]\,d\rho + O\left(\frac{1}{|x|^3}\right).
\end{aligned}
$$

Since f is an atom we have that $\int_0^1 f(\rho(1,0))\rho\,d\rho = 0$. At this point we select f such that $c_f = \int_0^1 f(\rho(1,0))\rho^3\,d\rho \ne 0$, but $\int_0^1 f(\rho(1,0))\rho^5\,d\rho = \int_0^1 f(\rho(1,0))\rho^7\,d\rho = 0$. It follows that

$$
\left(f * \frac{e^{i\lambda k\theta_y}}{|y|^2}\right)(x) = -4c_f\frac{\lambda_k^2}{|x|^4}e^{i\lambda k\theta_x} + O\left(\frac{\lambda_k^7}{|x|^9}\right) + O\left(\frac{1}{|x|^3}\right)
$$

and therefore

$$
\left(f * \sum_{k=1}^{j} a_k\frac{e^{i\lambda k\theta_y}}{|y|^2}\right)(x) = -4c_f a_j\frac{\lambda_j^2}{|x|^4}e^{i\lambda_j\theta_x} + O\left(\frac{\lambda_{j-1}^2}{\lambda_j^4\mu_j^4}\right) + O\left(\frac{\lambda_j^7}{\lambda_j^9\mu_j^9}\right) + O\left(\frac{j}{\lambda_j^3\mu_j^3}\right).
$$

For fixed x as above, let $I_{x,\rho}$ be the set of all $\phi \in [0, 2\pi]$ with $|x - \rho e^{i\phi}| \leq 1$. We have

$$\begin{aligned}
&\left| \left(f * \sum_{k=j+1}^{\infty} a_k \frac{e^{\lambda k \theta_y}}{|y|^2} \right)(x) \right| \\
=&\left| \int_{|x|-1}^{|x|+1} \int_{I_{x,\rho}} f(x - \rho e^{i\phi}) \sum_{k=j+1}^{\infty} a_k e^{i\lambda_k \phi} d\phi \, \frac{d\rho}{\rho} \right| \\
=&\left| \int_{|x|-1}^{|x|+1} \int_{I_{x,\rho}} \frac{d}{d\phi}\big(f(x - \rho e^{i\phi})\big) \sum_{k=j+1}^{\infty} a_k \frac{e^{i\lambda_k \phi}}{i\lambda_k} d\phi \, \frac{d\rho}{\rho} \right| \\
\leq& \int_{|x|-1}^{|x|+1} \left\| \frac{d}{d\phi}\big(f(x - \rho e^{i\phi})\big) \right\|_{L^2(d\phi)} \left\| \sum_{k=j+1}^{\infty} a_k \frac{e^{i\lambda_k \phi}}{i\lambda_k} \right\|_{L^2(d\phi)} \frac{d\rho}{\rho} \\
\leq& \int_{|x|-1}^{|x|+1} \|\nabla f\|_{L^\infty} \left(\sum_{k=j+1}^{\infty} \frac{1}{\lambda_k^2} \right)^{1/2} d\rho \\
=&O\left(\frac{1}{\lambda_{j+1}} \right).
\end{aligned}$$

Combining this result with the one obtained above for the remaining terms we obtain that

$$|(T_\Omega f)(x)| \geq \frac{c_f |a_j|}{\mu_j^4 \lambda_j^2} - C \left[\frac{\lambda_{j-1}^2}{\lambda_j^4 \mu_j^4} + \frac{1}{\lambda_j^2 \mu_j^9} + \frac{j}{\lambda_j^3 \mu_j^3} + \frac{1}{\lambda_{j+1}} \right]$$

for x satisfying $\lambda_j \mu_j \leq |x| \leq 2\lambda_j \mu_j$ and $1/2 \leq |x_2|/|x_1| \leq 2$. Estimate the L^1 norm of $T_\Omega f$ from below by

$$\|T_\Omega f\|_{L^1} \geq c_f \frac{\pi}{10} \sum_{j=1}^{\infty} \frac{|a_j|}{\mu_j^2} - C \sum_{j=1}^{\infty} \left[\frac{\lambda_{j-1}^2}{\lambda_j^2 \mu_j^2} + \frac{1}{\mu_j^7} + \frac{j}{\lambda_j \mu_j} + \frac{\lambda_j^2 \mu_j^2}{\lambda_{j+1}} \right]. \tag{19}$$

Choose now $\mu_j = j^{1/6}$ and $a_j = j^{-5/8}$. Select also a lacunary sequence λ_j such that $\sum_{j=1}^{\infty} \lambda_j^2 \mu_j^2 / \lambda_{j+1} < \infty$.

This choice of numbers makes the expression in (19) equal to infinity.

Remark. Observe that the proof above gives that for a radial C^∞ function f supported in the unit disc we have

$$\left(f * \frac{e^{i\lambda\theta_y}}{|y|^2} \right)(x) = c_0 \frac{1}{|x|^2} e^{i\lambda\theta_x} + c_1 \frac{\lambda^2}{|x|^4} e^{i\lambda\theta_x} + O\left(\frac{\lambda^4}{|x|^6} \right) + O\left(\frac{1}{|x|^3} \right)$$

with bounds independent of λ for $|x| \geq \lambda$ satisfying $|x_2|/|x_1| \sim 1$. The constants c_0 and c_1 are multiples of the integral and of the first moment of f respectively.

Remark. M. Christ has informed us that his example can be modified so that $\Omega \in L^\infty$.

Remark. A fundamental result of J. Daly and K. Phillips [8] says that if T_Ω maps $H^1(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$, then the function Ω has to be in $H^1(\mathbf{S}^{n-1})$. Using this theorem we conclude that for *every* $\Omega \in L^1(\mathbf{S}^{n-1}) - H^1(\mathbf{S}^{n-1})$ we have that the corresponding operator T_Ω does not map H^1 to L^1.

Remark. Recently A. Seeger and T. Tao [16] have shown that the best possible result on H^1 is that T_Ω maps H^1 to $L^{1,q}$ for $q \geq 2$, where denotes the Lorentz space. This means that for some Ω, T_Ω does not map H^1 into $L^{1,q}$ for $q < 2$.

Techniques from the L^1 theory can be used to answer some questions about the L^p theory.
Question 6. Give an example of an $\Omega \in L^1(\mathbf{S}^{n-1})$ such that T_Ω is bounded on some L^q but not on some other L^p.

If Ω is allowed to be a distribution on the sphere, such an Ω is shown to exist by abstract methods. To be more precise, let us introduce the following Banach spaces of distributions on the sphere.

$$\mathcal{S}_p = \{\Omega\colon\ \Omega \text{ distribution on } \mathbf{S}^{n-1} \text{ and } \|\Omega\|_{\mathcal{S}_p} = \|T_\Omega\|_{L^p\to L^p} < \infty\}.$$

By duality and interpolation we see that $\mathcal{S}_p = \mathcal{S}_{p'}$ and $\mathcal{S}_p \subseteq \mathcal{S}_q$, where $1 < p < q \leq 2$. What is not immediately clear here is that $\mathcal{S}_p, \mathcal{S}_q$ are different spaces.

Theorem 7. *(M. Christ) We have* $\mathcal{S}_p \subsetneq \mathcal{S}_q$ *whenever* $1 < p < q \leq 2$.

Proof. It suffices to prove the theorem in dimension $n = 2$. For $x \in \mathbf{R}^2$, let us denote by $\theta_z = \text{Arg}x$ the argument of x. Consider the operators $T_N = T_{\Omega_N}$ where $\Omega_N(x) = e^{iN\theta_x}$ for $N = 1, 2 \ldots.$ According to ([8], Section 3)

$$\widehat{\left(\frac{e^{iN\theta_x}}{|x|^2}\right)}(\xi) = \frac{2\pi i(i\text{sgn}(N))^{N+1}}{N} e^{iN\theta_\xi}.$$

Therefore $\|T_N\|_{L^2\to L^2} \leq CN^{-1}$. Next, we show that $\|T_N\|_{H^1\to L^1} \leq C$. By dilation invariance, it suffices to consider f to be an atom supported in the unit ball and $\|f\|_\infty \leq 1$. For $|x| \leq N$ we use the Cauchy-Schwartz inequality to deduce that

$$\int_{|x|\leq N} |T_N f(x)|\, dx \leq CN\|T_N f\|_{L^2} \leq CNN^{-1}\|f\|_{L^2} = C.$$

For $|x| \geq N$ we have

$$T_N f(x) = \int_{|y|\leq 1} \frac{e^{iN\text{Arg}(x-y)}}{|x-y|^2} f(y)dy = \int_{|y|\leq 1} \left(\frac{e^{iN\text{Arg}(x-y)}}{|x-y|^2} - \frac{e^{iN\text{Arg}(x)}}{|x|^2}\right) f(y)dy.$$

Now use that, for $|x| \geq N \geq 2$ we have $|\text{Arg}(x-y) - \text{Arg}(x)| \leq C/|x|$, to obtain

$$\begin{aligned}\int_{|x|\geq N} |T_N f(x)|dx &\leq \int_{|y|\leq 1} |f(y)|dy \left(\int_{|x|\geq N} \frac{\left|e^{iN(\text{Arg}(x-y)-\text{Arg}(x))} - 1\right|}{|x|^2}dx + \int_{|x|\geq N} \frac{C}{|x|^3}dx\right)\\ &\leq C\left(\int_{|x|\geq N} \frac{N}{|x|^3}dx + C\right) = C.\end{aligned}$$

Therefore $\|T_N\|_{H^1\to L^1} \leq C$. By interpolation we see that for every $1 < p < 2$

$$\|T_N\|_{L^p\to L^p} \leq CN^{2/p-2}.$$

On the other hand, as we saw in the remark after the previous example, for suitable f the following is true

$$T_N f(x) = c_f \frac{e^{iN\operatorname{Arg}(x)}}{|x|^2} + O\left(\frac{N^2}{|x|^4}\right) + O\left(\frac{1}{|x|^3}\right), \qquad \text{for} \quad |x| \geq N \text{ near the diagonal,}$$

with bounds independent of N. Therefore for $|x| \geq N$ and N very large, the first two terms above are the dominant ones and hence

$$\|T_N f\|_{L^p} \geq C_1 \left(\int_{|x|\geq N} \frac{1}{|x|^{2p}} dx \right)^{1/p} = C_2 N^{2/p-2}.$$

From this we conclude that

$$\|\Omega_N\|_{\mathcal{S}_p} \sim C N^{2/p-2} \qquad \text{for } N \text{ large.}$$

Suppose now that $\mathcal{S}_p = \mathcal{S}_q$. Then by the open mapping theorem we must have that

$$c\|\Omega\|_{\mathcal{S}_p} \leq \|\Omega\|_{\mathcal{S}_q} \qquad \text{for every } T \text{ in } \mathcal{S}_q.$$

In particular, for every N large

$$C N^{2/p-2} \leq c\|\Omega_N\|_{\mathcal{S}_p} \leq \|\Omega_N\|_{\mathcal{S}_q} \leq C N^{2/q-2},$$

which is a contradiction as $N \to \infty$ since $p < q$. $\square$

5 Another H^1 condition in dimension 2

The following fact is well known. If f is supported in a ball B in $\mathbf{R}^n$, f is in $L^p(B)$ for some $1 < p \leq \infty$ (or more generally in $L\mathrm{Log}^+L\,(B)$), and f has mean value zero, then f is in the Hardy space $H^1(\mathbf{R}^n)$. It is also known that $L\mathrm{Log}^+L(B)$ cannot be replaced by $L^1(B)$ nor $L\mathrm{Log}^{1-\varepsilon}L(B)$ in this context. The following question is therefore naturally raised:

Question 7. *Let B be a ball in $\mathbf{R}^n$. Find a condition bearing on the size of a function such that for all f supported in B we have*

$$\int_B f(x)\,dx = 0 \quad \textit{and} \quad \textit{(size condition on } f\textit{)} \quad \Longleftrightarrow \quad f \in H^1(\mathbf{R}^n).$$

In dimension 1 an interesting answer was given in [18]. The condition discovered by the author reflects more the oscillation/variation of the function than its size. We state the result below:

Theorem 8. *Let f be supported in $[0,1]$, be integrable, and have integral zero. Define*

$$m_f(y) = \int_0^1 f(x) \log \frac{1}{|x-y|} dx, \tag{20}$$

for $A_f \subset [0,1]$, where A_f a full measure subset of $[0,1]$, on which the integral giving $m_f(y)$ converges absolutely. Then $f \in H^1(\mathbf{R}^1)$ if and only if m_f is a function of bounded variation on A_f. Quantitatively speaking, there exists a constant $C > 0$ such that for all f supported in $[0,1]$ with integral zero, we have

$$\|f\|_{H^1} \le \mathrm{Var}_{A_f}(m_f) + C\|f\|_{L^1},$$
$$\mathrm{Var}_{A_f}(m_f) \le C\|f\|_{H^1}.$$

Remark. The Variation of m_f over A_f is defined as

$$\mathrm{Var}_{A_f}(m_f) = \sup_P\{\sum_{j=1}^n |m_f(x_j) - m_f(x_{j-1})| : \ P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}, \ x_j \in A_f\}.$$

Let us now try to explain Theorem 8 along some heuristic lines. Recall the following: A function is in $H^1(\mathbf{R}^1)$ if and only if its Hilbert transform is in $L^1(\mathbf{R}^1)$. Theorem 8 states that m_f is of total variation if and only if Hf is integrable. Formally speaking, to find the derivative of the function m_f we differentiate under the integral sign to obtain the Hilbert transform of the function f. Of course this argument cannot be justified for a general $f \in L^1$ since Hf is not necessarily given in a form of a convergent integral. (Hf can be written as a convergent integral for smooth enough f.) However, m_f is defined almost everywhere and the condition that m_f has finite variation makes sense for all integrable functions f. Theorem 8 is first proved for step functions and then by approximation is extended to general functions. The extension to general functions is a little delicate because of the convergence problems indicated above.

We now use the result in Theorem 8 to state an alternative characterization of the "H^1 condition" for singular integrals in $\mathbf{R}^2$. We have the following:

Theorem 9. *Let $\Omega \in L^1(\mathbf{S}^1)$ have mean value zero. If there exists a $G \subset \mathbf{S}^1$ with measure $|G| = 2\pi$ such that*

$$\mathrm{Var}_G(m_\Omega) < +\infty, \tag{21}$$

where

$$m_\Omega(\xi) = \int_{\mathbf{S}^1} \Omega(\theta) \log \frac{1}{|\theta \cdot \xi|} d\theta,$$

then T_Ω maps $L^p(\mathbf{R}^2)$ into itself for $1 < p < \infty$.

Compare condition (21) to condition (7) which is essentially required for L^2 boundedness of T_Ω.

The idea of the proof of Theorem 9 is straightforward. In view of Theorem 8 and via a simple transference argument from the interval to the circle, we obtain that condition (21) is equivalent to the condition that $\Omega \in H^1(\mathbf{S}^1)$. As observed before, this condition is equivalent to (10). Now Theorem 3 gives the desired conclusion. We refer the reader to [18] for details.

6 Maximal functions and maximal singular integrals

In this section we discuss two operators related to T_Ω, the maximal function M_Ω and the maximal singular integral T^*_Ω. First consider the maximal function

$$(M_\Omega f)(x) = \sup_{r>0} r^{-n} \int_{|y|\le r} |f(x-y)||\Omega(y/|y|)|\, dy,$$

where Ω is in $L^1(\mathbf{S}^{n-1})$. The following theorem is a straightforward consequence of the method of rotations See [20] p. 72.

Theorem 10. *If Ω is in $L^1(\mathbf{S}^{n-1})$ then M_Ω maps $L^p(\mathbf{R}^n)$ into itself for $1 < p \le \infty$.*

Note that M_Ω is a positive operator and no mean value property is imposed on Ω.

It is reasonable to ask if there is an L^1 theory for M_Ω. Again the main question here is whether a condition bearing only on the size of Ω suffices for the weak type $(1,1)$ property. This question was first answered positively by M. Christ ($\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$, $n = 2$), and later by M. Christ and J.-L. Rubio de Francia ($\Omega \in L\mathrm{Log}^+L(\mathbf{S}^{n-1})$ all n).

Theorem 11. *(M. Christ and J.-L. Rubio de Francia) If Ω satisfies (5), then M_Ω is of weak type $(1,1)$.*

The question still left open is the following:

Question 8. Is M_Ω weak type $(1,1)$ bounded when Ω is merely in $L^1(\mathbf{S}^{n-1})$?

It is fairly easy to check that the singular integral operator with kernel K_Ω shares the same mapping properties as its truncated version having kernel $\Omega(x/|x|)|x|^{-n}\chi_{|x|>\varepsilon}$. Obtaining estimates for the supremum of the truncated singular integrals allows us to conclude that the principal value integral in (1) is almost everywhere convergent for $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$.

For Ω an integrable function of the sphere with mean value zero, define

$$(T^*_\Omega f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy \right|.$$

We call this operator the maximal singular integral operator associated with T_Ω. The L^p boundedness of T^*_Ω for Ω in $L\mathrm{Log}^+L$ is due to Calderón and Zygmund [2]. T^*_Ω is also L^p bounded for $\Omega \in H^1(\mathbf{S}^{n-1})$. The theorem below was proved by the authors and independently by Fan and Pan [10] in a more general context. The proof given combines ideas from [2] and from the proof of Theorem 3.

Theorem 12. *Let Ω be an integrable function on $\mathbf{S}^{n-1}$ with mean value zero which satisfies condition (10). Then T^*_Ω extends to a bounded operator from $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Proof. For a unit vector $\theta \in \mathbf{S}^{n-1}$ define

$$(M_\theta f)(x) = \sup_{a>0} \frac{1}{2a} \int_{-a}^{a} |f(x-r\theta)|\, dr. \tag{22}$$

$$(H^*_\theta f)(x) = \sup_{\varepsilon>0} \left| \int_{|r|>\varepsilon} \frac{f(x-r\theta)}{r}\, dr \right| \tag{23}$$

It is a classical result that for some $C_p > 0$ and all f we have

$$\sup_{|\theta|=1} \|M_\theta f\|_{L^p} + \sup_{|\theta|=1} \|H_\theta^* f\|_{L^p} \le C_p \|f\|_{L^p}.$$

For Ω odd, the method of rotations gives

$$|(T_\Omega^* f)(x)| \le \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| (H_\theta^* f)(x) d\theta$$

and therefore $\|T_\Omega^* f\|_{L^p} \le C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p}$. Let us now consider the case when Ω is even.

Fix Φ to be a smooth radial function such that $\Phi(x) = 0$ for $|x| < 1/4$ and $\Phi(x) = 1$ for $|x| > 1/2$. We have that

$$(T_\Omega^\varepsilon f)(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \Phi\left(\frac{x-y}{\varepsilon}\right) f(y) dy - \int_{|x-y|<\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} \Phi\left(\frac{x-y}{\varepsilon}\right) f(y) dy,$$

where we extended Ω to be a homogeneous of degree zero function on $\mathbf{R}^n$. Since the pointwise estimate

$$\begin{aligned} \sup_{\varepsilon>0} \left| \int_{|x-y|<\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} \Phi\left(\frac{x-y}{\varepsilon}\right) f(y) dy \right| &\le C \sup_{\varepsilon>0} \int_{\varepsilon/4<|x-y|<\varepsilon} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \\ &\le C \sup_{\varepsilon>0} \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \frac{1}{\varepsilon} \int_{\varepsilon/4}^{\varepsilon} |f(x-r\theta)| dr d\theta \le C \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| (M_\theta f)(x) d\theta \end{aligned}$$

is valid, and the last term above maps $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ for $1 < p < \infty$, it suffices to obtain an L^p bound for the smoothly truncated maximal singular integral operator

$$(\widetilde{T}_\Omega^* f)(x) = \sup_{\varepsilon>0} |(\widetilde{T}_\Omega^\varepsilon f)(x)|, \quad \text{where} \quad (\widetilde{T}_\Omega^\varepsilon f)(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \Phi\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy.$$

As usually, we denote by R_j the j^{th} Riesz transform. Let $U_j = R_j\left(\Omega(\,\cdot\,)/|\,\cdot\,|^n\right)$.

We can write $U_j(x)$ as $V_j(x/|x|)/|x|^n$, where V_j is an odd distribution on $\mathbf{S}^{n-1}$. Here we use the fact that Ω is even and that R_j has an odd kernel.

It turns out that the fact $\Omega \in H^1(\mathbf{S}^{n-1})$ is equivalent to the fact that V_j are integrable functions on $\mathbf{S}^{n-1}$ for all $j = 1, \ldots, n$. See [14]. Here we only that $V_j \in L^1(\mathbf{S}^{n-1})$ a fact proved in Theorem 3.

Also let $\widetilde{V}_j(x) = R_j\left(\Omega(\,\cdot\,)\Phi(\,\cdot\,)/|\,\cdot\,|^n\right)$. Then

$$(\widetilde{T}_\Omega^\varepsilon f)(x) = \sum_{j=1}^n R_j\left(\frac{\Omega(\,\cdot\,)}{|\,\cdot\,|^n} \Phi\left(\frac{\cdot}{\varepsilon}\right)\right) * R_j(f) = \sum_{j=1}^n \frac{1}{\varepsilon^n} \widetilde{V}_j\left(\frac{\cdot}{\varepsilon}\right) * R_j(f) \tag{24}$$

We shall need the following lemma whose proof is postponed until the end of this section (see also [2], p.299).

Lemma 1. *There exist G_j, homogeneous of degree 0, integrable on $\mathbf{S}^{n-1}$ functions, such that*

$$|\widetilde{V}_j(x)| \le G_j(x) \quad \text{for every} \quad |x| \le 1,$$
$$|\widetilde{V}_j(x) - U_j(x)| \le C\|\Omega\|_{L^1(\mathbf{S}^{n-1})}|x|^{-n-1} \quad \text{for every} \quad |x| > 1.$$

Using Lemma 1 and (24), we obtain

$$|(\widetilde{T}_\Omega^\varepsilon f)(x)| = \left|\sum_{j=1}^n \frac{1}{\varepsilon^n}\int \widetilde{V}_j\left(\frac{x-y}{\varepsilon}\right)(R_jf)(y)dy\right| \le \tag{25}$$

$$\le A_1(f,\varepsilon) + A_2(f,\varepsilon) + A_3(f,\varepsilon), \tag{26}$$

where

$$A_1(f,\varepsilon) = \sum_{j=1}^n \left| \int\limits_{|x-y|>\varepsilon} U_j(x-y)(R_jf)(y)dy \right|,$$

$$A_2(f,\varepsilon) = C\sum_{j=1}^n \varepsilon \int\limits_{|x-y|>\varepsilon} \frac{|R_jf(y)|}{|x-y|^{n+1}}dy,$$

$$A_3(f,\varepsilon) = \sum_{j=1}^n \frac{1}{\varepsilon^n}\int\limits_{|x-y|<\varepsilon} |G_j(x-y)||(R_jf)(y)|dy.$$

First we observe that the $\sup_{\varepsilon>0}|A_1(f,\varepsilon)|$ is controlled by a sum of maximal singular integral operators associated with odd integrable kernels applied to the Riesz transforms of f, hence this term is bounded on L^p.

The j^{th} term in $A_2(f,\varepsilon)$ is controlled by

$$\begin{aligned}
\varepsilon \int\limits_{|x-y|>\varepsilon} \frac{|(R_jf)(y)|}{|x-y|^{n+1}}dy &\le \varepsilon \int_{\mathbf{S}^{n-1}}\int_\varepsilon^\infty \frac{1}{r^2}|R_jf(x-r\theta)|drd\theta \\
&\le C\int_{\mathbf{S}^{n-1}}\sum_{k=0}^\infty 2^{-k}\frac{1}{2^k\varepsilon}\int_{2^k\varepsilon}^{2^{k+1}\varepsilon}|(R_jf)(x-r\theta)|drd\theta \\
&\le C\int_{\mathbf{S}^{n-1}} M_\theta(R_jf)(x)d\theta,
\end{aligned}$$

hence

$$\|\sup_{\varepsilon>0}|A_2(f,\varepsilon)|\|_{L^p} \le C\sum_{j=1}^n \sup_\theta \|M_\theta(R_jf)\|_{L^p} \le C_p\|f\|_{L^p}.$$

Finally,

$$\begin{aligned}
\frac{1}{\varepsilon^n}\int\limits_{|x-y|<\varepsilon}|G_j(x-y)||R_jf(y)|dy &\le \int_{\mathbf{S}^{n-1}}|G_j(\theta)|\frac{1}{\varepsilon}\int\limits_0^\varepsilon |R_jf(x-r\theta)|drd\theta \\
&\le \int_{\mathbf{S}^{n-1}}|G_j(\theta)|M_\theta(R_jf)(x)d\theta,
\end{aligned}$$

which implies that

$$\|\sup_{\varepsilon>0}|A_3(f,\varepsilon)|\|_{L^p} \le C\sum_{j=1}^{n}\sup_{\theta}\|M_\theta(R_jf)\|_{L^p} \le C_p\|f\|_{L^p}.$$

Theorem 12 is now proved and we turn and we turn our attention to the proof of Lemma 1 left open. □

Proof. If $|x|>1$ since $\Phi(y)=1$ for every $|y|>1/2$, we have

$$\begin{aligned}|\widetilde{V}_j(x)-U_j(x)| &\le \left|\int \frac{x_j-y_j}{|x-y|^{n+1}}(\Phi(y)-1)\frac{\Omega(y)}{|y|^n}dy\right| \\ &\le C\int\limits_{|y|<1/2}\frac{|\Omega(y)|}{|y|^n}\left|\frac{x_j-y_j}{|x-y|^{n+1}}-\frac{x_j}{|x|^{n+1}}\right|dy \le \frac{C}{|x|^{n+1}}\int\limits_{|y|<1/2}\frac{|\Omega(y)|}{|y|^{n-1}}dy \\ &\le C\|\Omega\|_{L^1(\mathbf{S}^{n-1})}|x|^{-n-1}.\end{aligned}$$

For the case $|x|<1$, notice first that if $|x|<1/8$, then $|\widetilde{V}_j(x)|\le C\|\Omega\|_{L^1}$, because the singularity is away from x. If $1/8\le|x|\le1$, then

$$\begin{aligned}|\widetilde{V}_j(x)-\Phi(x)U_j(x)| &\le \int\limits_{|y|>2}\frac{|\Omega(y)|}{|y|^n}|\Phi(y)-\Phi(x)|\frac{|x_j-y_j|}{|x-y|^{n+1}}dy \\ &+ \int\limits_{1/16<|y|<2}\frac{|\Omega(y)|}{|y|^n}|\Phi(y)-\Phi(x)|\frac{|x_j-y_j|}{|x-y|^{n+1}}dy \\ &+ \int\limits_{0<|y|<1/16}\frac{|\Omega(y)|}{|y|^n}|\Phi(y)-\Phi(x)|\left|\frac{|x_j-y_j|}{|x-y|^{n+1}}-\frac{x_j}{|x|^{n+1}}\right|dy \\ &= P_1(x)+P_2(x)+P_3(x).\end{aligned}$$

The first term is easy:

$$P_1(x)\le C\int\limits_{|y|>2}\frac{|\Omega(y)|}{|y|^{2n}}dy\le C\|\Omega\|_{L^1}.$$

For the second term $P_2(x)$, we use that Φ is a Lipshitz function to obtain

$$\begin{aligned}P_2(x)\le C\int\limits_{1/16<|y|<2}\frac{|\Omega(y)|}{|y|^n|y-x|^{n-1}}dy &\le \int\frac{|y|^{1/2}|\Omega(y)|}{|y|^n|y-x|^{n-1}}dy \\ &\le C|x|^{n-3/2}\int\frac{|y|^{1/2}|\Omega(y)|}{|y|^n|y-x|^{n-1}}dy.\end{aligned}$$

For the third term use the elementary inequality

$$\left|\frac{x_j-y_j}{|x-y|^{n+1}}-\frac{x_j}{|x|^{n+1}}\right|\le C|y|$$

to get

$$P_3(x) \le C \int\limits_{0<|y|<1/16} \frac{|\Omega(y)|}{|y|^{n-1}} dy \le C\|\Omega\|_{L^1}.$$

Therefore choose G_j to be

$$G_j(x) = C\left[|V_j(x)| + \|\Omega\|_{L^1} + |x|^{n-3/2} \int \frac{|\Omega(y)|}{|y|^{n-1/2}|y-x|^{n-1}} dy\right].$$

This proves the Lemma. □

We note that condition (17) also implies L^p boundedness for T^*_Ω for a certain range of p's depending on α. We refer the reader to [12] for details.

For $\Omega \in L^1(\mathbf{S}^{n-1})$, let us define three operators

$$(M^*_\Omega f)(x) = \int\limits_{\mathbf{S}^{n-1}} |\Omega(\theta)|(M_\theta f)(x)\, d\theta, \tag{27}$$

$$(H_\Omega f)(x) = \int\limits_{\mathbf{S}^{n-1}} |\Omega(\theta)||(H_\theta f)(x)|\, d\theta, \tag{28}$$

$$(H^*_\Omega f)(x) = \int\limits_{\mathbf{S}^{n-1}} |\Omega(\theta)|(H^*_\theta f)(x)\, d\theta, \tag{29}$$

where M_θ and H^*_θ are given in (22) and (23), and H_θ in (4). As observed in the proof of the previous theorem, L^p estimates for the operators M^*_Ω, H_Ω, and H^*_Ω were useful in establishing L^p bounds for the operators M_Ω, T_Ω, and T^*_Ω. One may wonder whether (27), (28), and (29) are also of weak type $(1,1)$. The answer turns out to be false. In fact with $\Omega = 1$, there is an example of R. Fefferman [11] which says that M^*_Ω is not of weak type $(1,1)$. Examples can also be given to show that H_Ω and H^*_Ω are also not of weak type $(1,1)$.

The L^1 theory of T^*_Ω for Ω rough is still open as of this writing. The basic question here is whether T_Ω is of weak type $(1,1)$ if Ω does not possess any smoothness. The following problem was posed by A. Seeger.

Question 9. Is T^*_Ω of weak type $(1,1)$ when Ω is in $L^\infty(\mathbf{S}^{n-1})$?

We end this exposition with three tables of known and open questions regarding the operators M_Ω, T_Ω and T^*_Ω.

Tables 1 and 2 refer to general functions Ω, while table 3 refers to odd functions.

	$\Omega \in L^q(\mathbf{S}^{n-1})$	$\Omega \in \mathrm{Llog}^+\mathrm{L}(\mathbf{S}^{n-1})$	(17) $\forall \alpha > 0$	$\Omega \in H^1(\mathbf{S}^{n-1})$	$\Omega \in L^1(\mathbf{S}^{n-1})$
M_Ω	Yes, trivial	Yes, trivial	Yes, trivial	Yes, trivial	Yes, trivial
T_Ω	Yes, [2]	Yes, [2]	Yes, [12]	Yes, [14], [7]	No, [21]
T^*_Ω	Yes, [2]	Yes, [2]	Yes, [12]	Yes, [10][1]	No, [21]

[1] independently proved by the authors (Theorem 12)

Table 1: Boundedness from $L^p \to L^p$ for $1 < p < \infty$, $(1 < q \leq \infty)$.

	$\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$	$\Omega \in \mathrm{Llog}^+\mathrm{L}(\mathbf{S}^{n-1})$	$\Omega \in H^1(\mathbf{S}^{n-1})$	$\Omega \in L^1(\mathbf{S}^{n-1})$
M_Ω	[4] (n=2)	[5] (all n)	irrelevant	open[2]
T_Ω	[13] (n=2), [5][3] ($n \leq 7$)	[5][3] ($n \leq 7$), [15] (all n)	open	No, [21][2]
T^*_Ω	open	open	open	No, [21][2]

[2] true for the subspace of weak L^1 consisting of radial functions [17]
[3] only the case $n = 2$ is given in this reference

Table 2: Weak type $(1, 1)$ boundedness

	$L^p \to L^p,\ 1 < p < \infty$	$L^1 \to$ weak L^1
M_Ω	Yes, by method of rotations	open
T_Ω	Yes, by method of rotations	open
T^*_Ω	Yes, by method of rotations	open

Table 3: The case Ω is an odd function in $L^1(\mathbf{S}^{n-1})$.

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