# BEST BOUNDS FOR THE HILBERT TRANSFORM ON $L^p(\mathbb{R}^1)$; A CORRIGENDUM

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Abstract. We fix a point left unexplained in [1] by a small modification of the proof.

In the text after (5) in [1] for $R > 100$ define $C_R$ as the circle centered at $iR$ with radius $R' = R - 1/R$, but define a (smaller) lower arc $C_R^L = \left\{ iR + R'e^{i\phi} : \frac{5\pi}{4} \le \phi \le \frac{7\pi}{4} \right\}$ and a (larger) upper arc $C_R^U = C_R \setminus C_R^L$. Then (6) holds as stated in [1] because of the subharmonicity of $g(u(z), v(z))$. Observe that the following stronger version of (7) holds for some constant $C_f$ (depending only on the Schwartz function $f$):

$$|u(x+iy) + iv(x+iy)| \le \frac{C_f}{1 + |x| + |y|}.$$

This is routine to check for $|x| \ge 2A$, where $[-A, A]$ contains the support of f. Also, for $|x| \le 2A$ we have

$$\begin{aligned} |u(x+iy) + iv(x+iy)| &= \frac{1}{\pi} \left| \int_{-\infty}^{\infty} \frac{f(x-t)}{t+iy} dt \right| \\ &= \frac{1}{\pi} \left| \int_{-(A+|x|)}^{A+|x|} \frac{f(x-t) - f(x)}{t+iy} + f(x) \frac{t - iy}{t^2 + y^2} \, dt \right| \\ &\le C' \left( \int_{|t| \le 3A} \frac{|t|}{|t| + |y|} dt + 2 \tan^{-1} \frac{3A}{|y|} \right) \\ &\le \frac{C''}{1 + |y|} \le \frac{C_f}{1 + |x| + |y|}. \end{aligned}$$

We conclude from this that

$$(*) \qquad |g(u(x+iy), v(x+iy))| \le |u(x+iy) + iv(x+iy)|^p \le \left( \frac{C_f}{1 + |x| + |y|} \right)^p,$$

which clearly implies the following versions of (8) and (9) in [1]:

$$R' \, |g(u(iR), v(iR))| \le R' \frac{C}{(1+R)^p} \to 0 \qquad \text{as } R \to \infty,$$

$$\left| \int_{C_R^U} g(u(z), v(z)) ds \right| \le R' \frac{C}{(1+R)^p} \to 0 \qquad \text{as } R \to \infty.$$

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In view of (6), (8), and (9) in order to obtain (3), it will suffice to show that the integral of $g(u(z), v(z))$ over $C_R^L$ tends to the left hand side of (3). This will be a consequence of the Lebesgue dominated convergence theorem combined with the fact that $C_R^L$ is strictly smaller than half circle. Using parametric equations, the integral $\int_{C_R^L} g(u(z), v(z))ds$ is equal to

$$\int_{\frac{-R'\sqrt{2}}{2}}^{\frac{R'\sqrt{2}}{2}} \frac{g\left(u\left(x+iR-iR'\sqrt{1-\frac{x^2}{R'^2}}\right), v\left(x+iR-iR'\sqrt{1-\frac{x^2}{R'^2}}\right)\right)}{\sqrt{1-\frac{x^2}{R'^2}}}\, dx\,.$$

In view of $(*)$, for all $R > 100$, the preceding integrand is bounded by the integrable function $C_f\sqrt{2}(1+|x|)^{-p}$, since $\sqrt{1-\frac{x^2}{R'^2}}$ is bounded below by $\frac{1}{\sqrt{2}}$ in the range of integration. The Lebesgue dominated convergence theorem, therefore, implies that the previously displayed expressions converge to

$$\int_{-\infty}^{\infty} g\left(u\left((x+i0, v\left(x+i0\right)\right)\right) dx = \int_{-\infty}^{\infty} \mathrm{Re}\left[(|f(x)| + iHf(x))^p\right]\, dx,$$

as $R \to \infty$, since for almost all $x$ we have

$$\lim_{z\to x} g(u(z), v(z)) = g(u(x+i0), v(x+i0)) = \mathrm{Re}\left[(|f(x)| + iHf(x))^p\right].$$

This proves (3) and completes the proof.

## References

[1] L. Grafakos Best bounds for the Hilbert transform on $L^p(R^1)$, Math. Res. Let. **4** (1997), no. 4, 469–471.

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