BILINEAR FOURIER INTEGRAL OPERATORS

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Abstract. We study the boundedness of bilinear Fourier integral operators on products of Lebesgue spaces. These operators are obtained from the class of bilinear pseudodifferential operators of Coifman and Meyer via the introduction of an oscillatory factor containing a real-valued phase of five variables $\Phi(x, y_1, y_2, \xi_1, \xi_2)$ which is jointly homogeneous in the phase variables (ξ_1, ξ_2). For symbols of order zero supported away from the axes and the antidiagonal, we show that boundedness holds in the local-L^2 case. Stronger conclusions are obtained for more restricted classes of symbols and phases.

1. introduction

We initiate the study of a class of operators that extend the classical Fourier integral operators to the bilinear setting. The results in this work are of introductory nature but they indicate that there is probably a rich and extensive underlying theory that awaits to be developed. The present work only touches on certain aspects of the theory.

The results of this article extend known results concerning bilinear pseudodifferential operators; these operators have been introduced and extensively studied by Coifman and Meyer [CM1], [CM2], [CM3]. They have the form

$$P_\sigma(f_1, f_2)(x) = \int_{\mathbf{R}^{2n}} \sigma(x, \xi_1, \xi_2)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)e^{2\pi i x\cdot(\xi_1+\xi_2)}\, d\xi_1 d\xi_2\,, \tag{1}$$

where f_1, f_2 are smooth functions with compact support on $\mathbf{R}^n$ and σ is symbol of $3n$ real variables, usually taken to be in some Hörmander class. Here $\widehat{f}$ denotes the Fourier transform of the function f defined by $\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$. A classical theorem of Coifman and Meyer [CM3] states that if σ is a symbol in the Hörmander class S^0 uniformly in x, then the operator P_σ admits a bounded extension on products of Lebesgue spaces whose indices are related as in Hölder's inequality. An extension of this theorem to Lebesgue spaces with indices $p < 1$ including some endpoint cases was obtained by Grafakos and Torres [GT1] and in some special cases by Kenig and Stein [KS].

A bilinear pseudodifferential operator can also be written in the form

$$P_\sigma(f_1, f_2)(x) = \int_{\mathbf{R}^{4n}} e^{2\pi i((x-y_1)\cdot\xi_1+(x-y_2)\cdot\xi_2)}\sigma(x, \xi_1, \xi_2)f_1(y_1)f_2(y_2)\, dy_1dy_2d\xi_1d\xi_2\,, \tag{2}$$

2000 Mathematics Subject Classification: 42B99.

Key Words: Multilinear operators, Fourier integral operators.

The first author acknowledges the support of the NSF under grant DMS-0900946. The second author is supported by grant Prin 2007 *Analisi Armonica*. The second author would like to acknowledge the support of the MU Mathematics Department Miller Fund.

where f_1, f_2 are smooth functions with compact support. Written in this form, we may allow the symbol σ to also depend (smoothly) on the variables y_1 and y_2. This extra dependence does not present any difficulties in the theory; in fact the aforementioned Coifman-Meyer bilinear multiplier theorem is also valid for symbols of the form $\sigma(x, y_1, y_2, \xi_1, \xi_2)$ that depend smoothly and have compact support in the variables y_1, y_2. The results in this article are of local nature and for this reason the symbols we consider indeed have compact support in the variables x, y_1, y_2.

Looking at the bilinear pseudodifferential operator written in the form (2), it is only a matter of introducing an appropriate oscillatory factor to create a bilinear Fourier integral operator. To set the framework for this theory, we first recall some definitions.

We assume that we are given a smooth function $b(x, y_1, y_2, \xi_1, \xi_2)$, a real number m, and a compact subset $Q \subset \mathbf{R}^n$ such that b is supported in $Q \times Q \times Q$ in the first three variables and all multiindices γ, γ_1, γ_2, α_1, α_2 in $(\mathbf{Z}^+)^n$, there exists a constant $C = C_{|\gamma|,|\gamma_1|,|\gamma_2||\alpha_1|,|\alpha_2|}$ such that

$$|\partial_x^\gamma \partial_{y_1}^{\gamma_1} \partial_{y_2}^{\gamma_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b(x, y_1, y_2, \xi_1, \xi_2)| \le C(1 + |\xi_1| + |\xi_2|)^{m-|\alpha_1|-|\alpha_2|}$$

for all $(x, y_1, y_2) \in Q \times Q \times Q$ and $\xi_1, \xi_2 \in \mathbf{R}^n$. Such functions are called *Hörmander symbols* of order m. In this article, we often use the notation $\vec{\xi}$ for the pair $(\xi_1, \xi_2) \in \mathbf{R}^n \times \mathbf{R}^n$.

We are concerned with bilinear Fourier integral operators (FIO) of the form

$$\mathcal{F}(f_1, f_2)(x) = \int_{\mathbf{R}^{4n}} e^{i\Phi(x,\vec{y},\vec{\xi})} b(x, \vec{y}, \vec{\xi}) f_1(y_1) f_2(y_2)\, d\vec{y}\, d\vec{\xi}, \qquad x \in \mathbf{R}^n, \tag{3}$$

where b is a symbol of Hörmander type and Φ is a real-valued phase that satisfies some nondegeneracy conditions. In this work, we focus attention to phases in *reduced form*

$$\Phi(x, \vec{y}, \vec{\xi}) = (x - y_1) \cdot \xi_1 + (x - y_2) \cdot \xi_2 + \psi(x, \xi_1, \xi_2) \tag{4}$$

where $\psi(x, \xi_1, \xi_2)$ is smooth function on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}) \times (\mathbf{R}^n \setminus \{0\})$ and is homogeneous of degree 1 jointly in the variables (ξ_1, ξ_2).

Setting $\varphi(x, \vec{\xi}) = x \cdot (\xi_1 + \xi_2) + \psi(x, \vec{\xi})$, the nondegeneracy conditions required in this article can be formulated as follows:

$$\det(\varphi_{x,\xi_1}) \ne 0 \tag{5}$$

and

$$\det(\varphi_{x,\xi_2}) \ne 0 \tag{6}$$

on the support of the symbol.

We end this section by providing a motivation for the study of the topic of bilinear FIOs. Inspired by certain restriction problems, we consider the issue of restricting solutions of certain hyperbolic PDEs along subspaces of half the spacial dimension. We present a typical problem that may arise in the case of the wave equation on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$.

Consider the wave equation on $\mathbf{R}^{2n} \times \mathbf{R}$ with coordinates (x,t), where $x = (x', x'')$, $x', x'' \in \mathbf{R}^n$ and $t \in \mathbf{R}$

$$\sum_{j=1}^{2n} \frac{\partial^2 u}{\partial x_j^2} = \frac{\partial^2 u}{\partial t^2}, \qquad u(x,0) = f_0(x')g_0(x''), \qquad \frac{\partial u}{\partial t}(x,0) = f_1(x')g_1(x'').$$

For each fixed t, the solution $u(x,t)$ can be written as a sum of Fourier integral operators with phases $\Phi_\pm = (x'-y')\cdot\xi' + (x''-y'')\cdot\xi'' \pm t\sqrt{|\xi'|^2 + |\xi''|^2}$, where $\xi = (\xi', \xi'') \in \mathbf{R}^n \times \mathbf{R}^n$ is the dual variable of (x', x''). When one considers the restriction of the solution $u(x', x'', t)$ along the diagonal $x' = x''$, one obtains two bilinear FIOs with phases Φ_+ and Φ_- acting on the pairs of functions (f_0, g_0) and (f_1, g_1). To determine if this restriction lies in $L^p(\mathbf{R}^n)$, it is natural to investigate the boundedness of these FIOs when the initial data f_0, g_0, f_1, g_1 lie $L^{p_j}(\mathbf{R}^n)$.

2. The main results

For bilinear operators T that map $L^{p_1} \times L^{p_2} \to L^p$ with $1/p_1 + 1/p_2 = 1/p$, the local-L^2 case is the situation where $2 \le p_1, p_2, p' \le \infty$. In this case, the trilinear form

$$(f_1, f_2, f_3) \mapsto \langle T(f_1, f_2), f_3 \rangle$$

is bounded by $\|T\| \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p'}}$ and the functions f_1, f_2, f_3 are locally in L^2. The Banach case is the situation where the indices satisfy $1 \le p_1, p_2, p \le \infty$, while the quasi-Banach case is the most general situation where the index p is allowed to be less than 1 (but greater than or equal to 1/2). We now state our main results that concern the local-L^2 case, the Banach, and quasi-Banach space cases under different assumptions on the associated symbols. In the rest of the paper $|\xi_1| \approx |\xi_2|$ means $c < |\xi_1|/|\xi_2| < c^{-1}$ for some $c > 0$.

Theorem 2.1. *(Local-L^2 case) Let $\mathcal{F}$ be a bilinear FIO with phase satisfying* (5) *and* (6) *and symbol of order zero which is compactly supported in the first three variables and whose last two variables are supported in a conical set U of the form $|\xi_1| \approx |\xi_2| \approx |\xi_1 + \xi_2|$ such that for $(\xi_1, \xi_2) \in U$ we have*

$$c_0^{-1}|\vec{\xi}| \le |\nabla_x \Phi(x, \vec{y}, \vec{\xi})| \le c_0 |\vec{\xi}| \tag{7}$$

for all $x, y_1, y_2 \in \mathbf{R}^n$. Then the bilinear FIO $\mathcal{F}$ maps $L^{p_1}(\mathbf{R}^n) \times L^{p_2}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ whenever $2 \le p_1, p_2, p' \le \infty$.

Corollary 2.2. *Suppose that the function ψ in* (4) *is is independent of x (such as in the case of the wave equation phases $(x - y_1)\cdot\xi_1 + (x - y_2)\cdot\xi_2 \pm \sqrt{|\xi_1|^2 + |\xi_2|^2}$). Let $\mathcal{F}$ be the associated bilinear FIO having a symbol of order zero which is compactly supported in the first three variables and whose last two variables (ξ_1, ξ_2) are supported away from the antidiagonal, i.e., in a conical set U of the form $|\xi_1| \approx |\xi_2| \approx |\xi_1 + \xi_2|$. Then the bilinear FIO $\mathcal{F}$ maps $L^{p_1}(\mathbf{R}^n) \times L^{p_2}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ whenever $2 \le p_1, p_2, p' \le \infty$.*

Proposition 2.3. *(Banach case) Let $\mathcal{F}$ be a bilinear FIO with phase satisfying* (5) *and* (6) *and symbol of order m that is compactly supported in the first three variables. Then $\mathcal{F}$ maps $L^1(\mathbf{R}^n) \times L^\infty(\mathbf{R}^n) \to L^1(\mathbf{R}^n)$ and $L^\infty(\mathbf{R}^n) \times L^1(\mathbf{R}^n) \to L^1(\mathbf{R}^n)$ whenever*

$$m < -\frac{2n-1}{2}\,.$$

Proposition 2.4. *(Quasi-Banach case) Let $\mathcal{F}$ be a BFIO with phase satisfying* (5) *and* (6) *and symbol that is compactly supported in the first three variables. Assume that the phase Φ of $\mathcal{F}$ can be written as sum $\Phi = \Phi_1 + \Phi_2$ of two non-degenerate linear phases, see* (19)*. If the order of the symbol is $-(n-1)(1/p-1)$ and $1 < p_1, p_2 < 2$, then $\mathcal{F}$ maps $L^{p_1}(\mathbf{R}^n) \times L^{p_2}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, where $1/p = 1/p_1 + 1/p_2$.*

3. Preliminary lemmas

The following lemmas contain straighforward extensions of the standard TT^* lemma and of Schur's lemma in the context of bilinear operators.

Given a bilinear operator T, the adjoints T^{*1} and T^{*2} are defined by the relations

$$\langle T^{*1}(f_1, f_2),\, g\rangle = \langle f_1,\, T(g, f_2)\rangle \qquad \text{and} \quad \langle T^{*2}(f_1, f_2),\, g\rangle = \langle f_2,\, T(f_1, g)\rangle$$

for all functions f, g, h in a dense subclass of the domains of the operators.

Lemma 3.1. *Let T be a bilinear operator and let $1 \le p, p_1, p_2 \le \infty$.*
(a) We have that $T:\, L^{p_1} \times L^{p_2} \to L^2$ with norm at most A if and only if

$$\|T^{*1}(T(h_1, h_2), h_3)\|_{L^{p_1'}} \le A^2 \|h_1\|_{L^{p_1}} \|h_2\|_{L^{p_2}} \|h_3\|_{L^{p_2}} \tag{8}$$

and this happens if and only if

$$\|T^{*2}(h_1, T(h_2, h_3))\|_{L^{p_2'}} \le A^2 \|h_1\|_{L^{p_1}} \|h_2\|_{L^{p_1}} \|h_3\|_{L^{p_2}} \tag{9}$$

for all functions h_1, h_2, h_3 in the appropriate domains.
(b) We have that $T:\, L^{p_1} \times L^2 \to L^p$ with norm at most A if and only if

$$\|T(h_1, T^{*2}(h_2, h_3))\|_{L^p} \le A^2 \|h_1\|_{L^{p_1}} \|h_2\|_{L^{p_2}} \|h_3\|_{L^{p'}} \tag{10}$$

for all functions h_1, h_2, h_3 in the appropriate domains.
(c) We have that $T:\, L^2 \times L^{p_2} \to L^p$ with norm at most A if and only if

$$\|T(T^{*1}(h_1, h_2), h_3))\|_{L^p} \le A^2 \|h_1\|_{L^{p'}} \|h_2\|_{L^{p_2}} \|h_3\|_{L^{p_2}} \tag{11}$$

for all functions h_1, h_2, h_3 in the appropriate domains.

Proof. In all of these assertions, the boundedness of the operator easily implies statements (8)–(11). Conversely, we focus on case (c), since the other cases are similar.

As a consequence of (11) we have that

$$\big|\langle T(T^{*1}(h_1, h_2), h_3)),\, \|h_1\|_{L^{p'}}^{-1} h_1\rangle\big| \le A^2 \|h_1\|_{L^{p'}} \|h_2\|_{L^{p_2}} \|h_3\|_{L^{p_2}}$$

and taking $h_2 = h_3$ we have

$$\big|\langle T(T^{*1}(h_1, h_2), h_2)),\, h_1\rangle\big| \le A^2 \|h_1\|_{L^{p'}}^2 \|h_2\|_{L^{p_2}}^2$$

from which it follows that

$$\left|\langle T^{*1}(h_1,h_2)), T^{*1}(h_1,h_2)\rangle\right| \le A^2 \|h_1\|_{L^{p'}}^2 \|h_2\|_{L^{p_2}}^2$$

and thus T^{*1} maps $L^{p'} \times L^{p_2}$ to L^2. Hence T maps $L^2 \times L^{p_2}$ to L^p. □

The preceding result can also be formulated in a straightforward way for m-linear operators. The following extension of Schur's lemma to the m-linear setting appeared in [BBPR] and [GT2]. It will be useful to us when $m = 2, 3$.

Lemma 3.2. *Let $K(y_0, y_1 \dots, y_m)$ be a function on $\mathbf{R}^{(m+1)n}$ such that for all $0 \le i \le m$ we have*

$$\sup_{y_i \in \mathbf{R}^n} \int_{\mathbf{R}^{mn}} |K(y_0, y_1, \dots, y_m)|\, dy_0 \dots \widehat{dy_i} \dots dy_m = A_i < \infty\,,$$

where $\widehat{dy_i}$ is indicating that the integration variable dy_i is missing. Then the m-linear operator

$$\mathcal{T}(f_1, \dots, f_m)(x) = \int_{\mathbf{R}^{mn}} K(x, y_1 \dots, y_m) f_1(y_1) \dots f_m(y_m)\, dy_1 \dots\, dy_m$$

maps $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ with bound

$$A_0^{1/p'} A_1^{1/p_1} \dots A_m^{1/p_m}$$

whenever $1/p_1 + \cdots + 1/p_m = 1/p$ where $1 \le p_1, \dots, p_m, p \le \infty$.

Proof. We provide the easy proof when $m = 3$. Taking a function f_0 in $L^{p'}$, we calculate $\|\mathcal{T}(f_1, f_2, f_3)\|_{L^p}$ via duality as follows:

$$\begin{aligned}
\Big| \int_{\mathbf{R}^n} \mathcal{T}(f_1,f_2,f_3)(y_0) f_0(y_0)\, dy_0 \Big| \;\le\; & \left(\int_{\mathbf{R}^{4n}} |K(y_0,y_1,y_2,y_3)||f_0(y_0)|^{p'}\, dy_0\, dy_1\, dy_2\, dy_3 \right)^{1/p'} \\
& \left(\int_{\mathbf{R}^{4n}} |K(y_0,y_1,y_2,y_3)||f_1(y_1)|^{p_1}\, dy_0\, dy_1\, dy_2\, dy_3 \right)^{1/p_1} \\
& \left(\int_{\mathbf{R}^{4n}} |K(y_0,y_1,y_2,y_3)||f_2(y_2)|^{p_2}\, dy_0\, dy_1\, dy_2\, dy_3 \right)^{1/p_2} \\
& \left(\int_{\mathbf{R}^{4n}} |K(y_0,y_1,y_2,y_3)||f_3(y_3)|^{p_3}\, dy_0\, dy_1\, dy_2\, dy_3 \right)^{1/p_3} \\
\le\; & A_0^{1/p'} \|f_0\|_{L^{p'}} A_1^{1/p_1} \|f_1\|_{L^{p_1}} A_2^{1/p_2} \|f_2\|_{L^{p_2}} A_3^{1/p_3} \|f_3\|_{L^{p_3}}
\end{aligned}$$

in view of Hölder's inequality with respect to the measure $|K(y_0, y_1, y_2, y_3)| dy_0 dy_1 dy_2 dy_3$ and of the hypotheses on $\mathcal{T}$. □

4. The local-L^2 case

In this section we prove Theorem 2.1. Set

$$\varphi(x,\xi\,) = x\cdot\xi_1 + x\cdot\xi_2 + \psi(x,\vec{\xi}\,)\,.$$

We consider a bilinear FIO $\mathcal{F}$ given by

$$\mathcal{F}(f_1,f_2)(x) = \int_{\mathbf{R}^{4n}} e^{i(\varphi(x,\vec{\eta}\,)-\vec{y}\cdot\vec{\eta}\,)} b(x,\vec{y},\vec{\eta}\,) f_1(y_1) f_2(y_2)\, d\vec{y}\, d\vec{\eta}\,, \tag{12}$$

where $b(x,\vec{y},\vec{\eta}\,)$ is a Hörmander symbol of order 0 which has compact support in the variables x, y_1, y_2. We study mapping properties of the operator (12) originally defined for smooth functions with compact support f_1, f_2.

We point out that, as it is standard in this theory of Fourier integral operators, the above definition must be interpreted in a weak sense in general. For, we can write

$$e^{i(\varphi(x,\vec{\eta}\,)-\vec{y}\cdot\vec{\eta}\,)} = \frac{1}{(1+|\vec{\eta}\,|^2)^N}(I-\Delta_{\vec{y}})^N e^{i(\varphi(x,\vec{\eta}\,)-\vec{y}\cdot\vec{\eta}\,)}\,,$$

and then integrate by parts in $\vec{y}$ the integral in (12) to obtain an equivalent form that converges absolutely.

We pick a nonnegative smooth function β on the real line supported in the interval $[7/8, 2]$ equal to one on $[1, 7/4]$ and a function β_0 supported in $[0, 2]$ such that

$$\beta_0(t) + \sum_{k=1}^{\infty} \beta(2^{-k}t) = 1$$

for all $t \geq 0$. For notational convenience we set $\beta_k(t) = \beta(2^{-k}t)$.

We decompose the bilinear FIO accordingly

$$\begin{aligned}
\mathcal{F}(f_1,f_2)(x) &= \int_{\mathbf{R}^{4n}} e^{i(\varphi(x,\vec{\eta}\,)-\vec{y}\cdot\vec{\eta}\,)} b(x,\vec{y},\vec{\eta}\,) f_1(y_1) f_2(y_2)\, d\vec{y}\, d\vec{\eta} \\
&= \sum_{k=0}^{\infty}\sum_{k'=0}^{\infty} \int_{\mathbf{R}^{4n}} e^{i(\varphi(x,\vec{\eta}\,)-\vec{y}\cdot\vec{\eta}\,)} b_{k,k'}(x,\vec{y},\vec{\eta}\,) f_1(y_1) f_2(y_2)\, d\vec{y}\, d\vec{\eta} \\
&= \sum_{k=0}^{\infty}\sum_{k'=0}^{\infty} \mathcal{F}^{k,k'}(f_1,f_2)(x)\,,
\end{aligned}$$

where $b_{k,k'}(x,\vec{y},\vec{\eta}\,) = \beta_k(|\eta_1|)\beta_{k'}(|\eta_2|) b(x,\vec{y},\vec{\eta}\,)$. The case $k = k' = 0$ is trivial as the symbol of the corresponding operator is a smooth function with compact support in all variables. The same comment applies to all terms of the form $|k|, |k'| \leq c_0$ for some $c_0 > 0$.

Recall that the symbol b is supported in the conical region $|\eta_1| \approx |\eta_2| \approx |\eta_1 + \eta_2|$. This assumption translates to a condition relating k and k' as follows: $|k - k'| < c$ for some constant $c > 0$.This reduces the double sum in k and k' above to essentially one sum where k is arbitrary and k' is within a fixed distance from k. Matters therefore reduce to the study of the bilinear FIO

$$F_k(f_1,f_2)(x)=\int_{\mathbf{R}^{4n}} f_1(y_1)f_2(y_2)b(x,\vec{y},\vec{\eta})\gamma_1(2^{-k}|\eta_1|)\gamma_2(2^{-k}|\eta_2|)e^{i(\varphi(x,\vec{\eta})-\vec{y}\cdot\vec{\eta})}\,d\vec{y}\,d\vec{\eta} \quad (13)$$

where γ_1 and γ_2 are smooth functions with compact support that do not contain the origin. (These functions are the same as the previously defined β.)

We first obtain an orthogonality lemma saying that the uniform boundedness of the F_k's implies the boundedness of their sum.

Lemma 4.1. *Let F_k be as in (13) and let p_1,p_2,p be indices that satisfy $2\le p_1,p_2,p'\le\infty$ and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p'}=1$. Suppose there exists a constant $A<\infty$ such that for all f_1,f_2 in C_0^∞ (i.e., smooth functions with compact support) we have*

$$\sup_{k\ge 1}\|F_k(f_1,f_2)\|_{L^p(\mathbf{R}^n)}\le A\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}$$

Then

$$\sum_{k=1}^\infty F_k(f_1,f_2)$$

is also bounded from $L^{p_1}\times L^{p_2}\to L^p$.

Proof. We define Littlewood-Paley operators Δ_m by setting $\Delta_m(f)\widehat{\ }(\xi)=\widehat{f}(\xi)\psi(2^{-m}\xi)$, for $m\ge 1$ and $\Delta_0(f)\widehat{\ }(\xi)=\widehat{f}(\xi)\psi_0(\xi)$, where ψ is a smooth function that is supported in an annulus that does not contain the origin in $\mathbf{R}^n$ and is equal to one on a smaller such annulus, while ψ_0 is smooth and equal to one on ball containing the origin and supported in a bigger ball. We pick ψ such that $\sum_{m=0}^\infty\psi_m(\xi)=1$, where $\psi_m(\xi)=\psi(2^{-m}\xi)$.

Inspired by the work of [Se], we introduce the decomposition

$$F_k(f_1,f_2)=\sum_{m=0}^\infty\sum_{j_1=0}^\infty\sum_{j_2=0}^\infty\Delta_mF_k(\Delta_{j_1}f_1,\Delta_{j_2}f_2)\,.$$

The key observation is that when the indices m,j_1,j_2 are near the index k, then we may exploit orthogonality, while when they are away from k there is decay in all variables involved. We precisely quantify this statement. We note that

$$\Delta_mF_k(\Delta_{j_1}f_1,\Delta_{j_2}f_2)(x)=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}K_{m,k,j_1,j_2}(x,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1dy_2\,,$$

where

$$K_{m,k,j_1,j_2}(x,y_1,y_2)=\frac{1}{(2\pi)^{8n}}\int_{(\mathbf{R}^n)^8}e^{i(\theta\cdot(x-u)+\vec{\eta}\cdot(\vec{z}-\vec{y})+\varphi(u,\vec{\xi})-\vec{z}\cdot\vec{\xi})}\beta(2^{-k}\xi_1)\beta(2^{-k}\xi_2)$$
$$\times\,\psi(2^{-m}|\theta|)\psi(2^{-j_1}|\eta_1|)\psi(2^{-j_2}|\eta_2|)b(u,\vec{z},\vec{\xi})\,du\,d\vec{z}\,d\theta\,d\vec{\eta}\,d\vec{\xi}.$$

We consider first the case near the diagonal, i.e., the case where $m=k+c$, $j_1=k+c_1$, $j_2=k+c_2$, where c,c_1,c_2 are integer constants that satisfy $\max(|c|,|c_1|,|c_2|)\le C_0$ for some $C_0>0$. There are finitely many such terms and we fix one such choice of c,c_1,c_2.

Suppose first that $2 \le p_1, p_2, p' < \infty$. Let us set $L_k(f_1, f_2) = \Delta_{k+c} T_{2^k}(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2)$ and start with $f_1 \in L^{p_1}$, $f_2 \in L^{p_2}$ and $h \in L^{p'}$. Then, inspired by [GL], we may write

$$\begin{aligned}
\big|\langle \sum_k L_k(f_1, f_2), h\rangle\big| &= \big| \sum_k \langle F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2), \Delta_{k+c} h\rangle\big| \\
&\le \int_{\mathbf{R}^n} \Big(\sum_k |F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2)|^2\Big)^{1/2} \Big(\sum_k |\Delta_{k+c} h|^2\Big)^{1/2} dx \\
&\le \Big\|\Big(\sum_k |F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2)|^2\Big)^{1/2}\Big\|_{L^p} \Big\|\Big(\sum_k |\Delta_k h|^2\Big)^{1/2}\Big\|_{L^{p'}} \\
&\le C_p \Big\|\Big(\sum_k |F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2)|^2\Big)^{1/2}\Big\|_{L^p} \big\|h\big\|_{L^{p'}},
\end{aligned}$$

where the last inequality follows from the Littlewood-Paley theorem. It will suffice to estimate the L^p norm of the previous square function above. We have

$$\begin{aligned}
\Big\|\Big(\sum_k |F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2)|^2\Big)^{1/2}\Big\|_{L^p}^p &\le \int_{\mathbf{R}} \sum_k |F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2)|^p \, dx \\
&= \sum_k \|F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2)\|_{L^p}^p \\
&\le A^p \sum_k \|\Delta_{k+c_1} f_1\|_{L^{p_1}}^p \|\Delta_{k+c_2} f_2\|_{L^{p_2}}^p
\end{aligned}$$

where we used the fact $p \le 2$ and the uniform boundedness of the operators F_k. Now applying Hölder's inequality for sequences and using the embeddings $\ell^2 \subset \ell^{p_1} \bigcap \ell^{p_2}$ (since $p_1, p_2 \ge 2$) we obtain the following

$$\begin{aligned}
&\sum_k \|\Delta_{k+c_1} f_1\|_{L^{p_1}}^p \|\Delta_{k+c_2} f_2\|_{L^{p_2}}^p \\
&\le \Big(\sum_k \|\Delta_{k+c_1} f_1\|_{L^{p_1}}^{p_1}\Big)^{p/p_1} \Big(\sum_k \|\Delta_{k+c_2} f_2\|_{L^{p_2}}^{p_2}\Big)^{p/p_2} \\
&= \Big(\int_{\mathbf{R}^n} \sum_k |\Delta_{k+c_1} f_1|^{p_1} \, dx\Big)^{p/p_1} \Big(\int_{\mathbf{R}^n} \sum_k |\Delta_{k+c_2} f_2|^{p_2} \, dx\Big)^{p/p_2} \\
&\le \Big(\int_{\mathbf{R}^n} \big(\sum_k |\Delta_{k+c_1} f_1|^2\big)^{p_1/2} \, dx\Big)^{p/p_1} \Big(\int_{\mathbf{R}^n} \big(\sum_k |\Delta_{k+c_2} f_2|^2\big)^{p_2/2} \, dx\Big)^{p/p_2} \\
&= \Big\|\Big(\sum_k |\Delta_{k+c_1} f_1|^2\Big)^{1/2}\Big\|_{L^{p_1}}^p \Big\|\Big(\sum_k |\Delta_{k+c_2} f_2|^2\Big)^{1/2}\Big\|_{L^{p_2}}^p \\
&\le C_p^2 \|f_1\|_{L^{p_1}}^p \|f_2\|_{L^{p_2}}^p,
\end{aligned}$$

by the Littlewood-Paley theorem.

We are still considering the case near the diagonal but we now suppose that the point (p_1, p_2, p) in $[2,\infty]^2 \times [1,2]$ is one of the "vertices" $(2,2,1)$, $(2,\infty,2)$, or $(\infty,2,2)$ of the local-L^2 triangle. When $(p_1,p_2,p)=(2,2,1)$ we argue as follows:

$$\begin{aligned}
\Big\| \sum_k \Delta_{k+c} F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2) \Big\|_{L^1} &\le \sum_k \big\| \Delta_{k+c} F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2) \big\|_{L^1} \\
&\le C \sum_k \big\| F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2) \big\|_{L^1} \\
&\le C\, A \sum_k \big\| \Delta_{k+c_1} f_1 \big\|_{L^2} \big\| \Delta_{k+c_2} f_2 \big\|_{L^2} \\
&\le C\, A \Big(\sum_k \big\| \Delta_{k+c_1} f_1 \big\|_{L^2}^2 \Big)^{\frac12} \Big(\sum_k \big\| \Delta_{k+c_2} f_2 \big\|_{L^2}^2 \Big)^{\frac12} \\
&\le C'\, A \, \|f_1\|_{L^2} \|f_2\|_{L^2} \,.
\end{aligned}$$

When $(p_1,p_2,p)=(2,\infty,2)$ there is a similar argument. Using the orthogonality of the Δ_{k+c}'s on the Fourier transform side, we have

$$\begin{aligned}
\Big\| \sum_k \Delta_{k+c} F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2) \Big\|_{L^2}^2 &\le C \sum_k \big\| \Delta_{k+c} F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2) \big\|_{L^2}^2 \\
&\le C' \sum_k \big\| F_k(\Delta_{k+c_1} f_1, \Delta_{k+c_2} f_2) \big\|_{L^2}^2 \\
&\le C'\, A^2 \sum_k \big\| \Delta_{k+c_1} f_1 \big\|_{L^2}^2 \big\| \Delta_{k+c_2} f_2 \big\|_{L^\infty}^2 \\
&\le C''\, A^2 \, \|f_1\|_{L^2}^2 \|f_2\|_{L^\infty}^2 \,.
\end{aligned}$$

The situation $(p_1,p_2,p)=(\infty,2,2)$ is symmetric with $(p_1,p_2,p)=(2,\infty,2)$.

We now consider the case where $\max(|k-m|,|k-j_1|,|k-j_2|) \ge C_0$. We look at the expression defining K_{m,k,j_1,j_2} and we consider the phase of the exponential in it which is

$$\tilde\Phi(u,z_1,z_2,\xi_1,\xi_2) = \varphi(u,\vec\xi) + \theta\cdot(x-u) + \vec\eta\cdot(\vec z - \vec y) - \vec z \cdot \vec \xi .$$

Note that $\nabla_{u,\vec z}\,\tilde\Phi(u,\vec\xi)$ is equal to the following vector in $(\mathbf{R}^n)^3$

$$V = \Big(\nabla_u \varphi(u,\vec\xi) - \theta, \vec\eta - \vec\xi \Big) .$$

We claim that

$$|V| \ge c' \max(2^k, 2^m, 2^{j_1}, 2^{j_2}) \,. \tag{14}$$

Indeed, recall that $|\eta_1| \approx 2^{j_1}$, $|\eta_2| \approx 2^{j_2}$, $|\xi_1| \approx 2^k$, $|\xi_2| \approx 2^k$, and $|\theta| \approx 2^m$. Moreover, by in view of (7) we have $|\nabla_u \varphi(u,\vec\xi)| \approx |\vec\xi| \approx 2^k$. Thus

$$|V| \gtrsim |\nabla_u\varphi(u,\vec\xi) - \theta| + |\xi_1 - \eta_1| + |\xi_2 - \eta_2| \gtrsim \max(2^k, 2^m, 2^{j_1}, 2^{j_2})$$

whenever $\max(|k-m|,|k-j_1|,|k-j_2|) \ge C_0$.

Using this estimate for V and integrating by parts in K_{m,k,j_1,j_2} with respect to the variables $u, \vec{z}$ we obtain the pointwise bound

$$|K_{m,k,j_1,j_2}(x,y_1,y_2)| \le C_M \max(2^k, 2^m, 2^{j_1}, 2^{j_2})^{-M}$$

for any integer M, whenever $\max(|k-m|, |k-j_1|, |k-j_2|) \ge C_0$.

Let Q be a cube centered at the origin in $\mathbf{R}^n$ which contains the support of b in each of the first three variables. When one of x, y_1, y_2 is not in Q, then we can also integrate by parts with respect to the corresponding variables θ, η_1, η_2 in the integral defining K_{m,k,j_1,j_2} to obtain the extra decay $(1+|x|)^{-M'}$, $(1+|y_1|)^{-M'}$, or $(1+|y_2|)^{-M'}$, respectively for $\big|K_{m,k,j_1,j_2}(x,y_1,y_2)\big|$. These extra factors can also be inserted when some of x, y_1, y_2 are in Q since in this case they are comparable to constants.

Combining these observations, we conclude the following estimates for all $x, y_1, y_2 \in \mathbf{R}^n$

$$\Big|K_{m,k,j_1,j_2}(x,y_1,y_2)\Big| \le C_{M,M'} \frac{\max(2^k, 2^m, 2^{j_1}, 2^{j_2})^{-M}}{(1+|x|)^{M'}(1+|y_1|)^{M'}(1+|y_2|)^{M'}} \,.$$

It follows from these estimates via the bilinear Schur lemma (Lemma 3.2) that a bilinear operator with kernel K_{m,k,j_1,j_2} is bounded from $L^{p_1} \times L^{p_2} \to L^p$ for all $1 \le p_1, p_2, p \le \infty$ with norm at most $\max(2^k, 2^m, 2^{j_1}, 2^{j_2})^{-M}$.

These estimates show that

$$\sum_{k,m,j_1,j_2} \Big\| \Delta_m F_k(\Delta_{j_1} f_1, \Delta_{j_2} f_2) \Big\|_{L^p} \le C \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \,,$$

where the sum is over the indices that satisfy $\max(|k-m|, |k-j_1|, |k-j_2|) \ge C_0$.

This concludes the proof of the lemma. □

It remains to show that the F_k's are bounded in the local-L^2 case uniformly in k. We introduce a slightly different notation by setting $\lambda = 2^k$ in (13) and also we define an operator T_λ by setting

$$T_\lambda(f_1, f_2) = \lambda^{2n} F_k(f_1, f_2) \,.$$

Obviously, the uniform boundedness of the F_k's from $L^{p_1} \times L^{p_2}$ to L^p is equivalent to boundedness of T_λ from $L^{p_1} \times L^{p_2}$ to L^p with norm that decays like λ^{-2n} for λ large. This is the assertion of the next lemma which is proved via a bilinear adaptation of the classical T^*T argument.

Lemma 4.2. *Let $a = a(x, y_1, y_2, \xi_1, \xi_2)$ be a smooth function on $\mathbf{R}^{5n}$ whose (ξ_1, ξ_2) support is contained in the set $\{(\xi_1, \xi_2) \in \mathbf{R}^n \times \mathbf{R}^n : |\xi_1| \approx |\xi_1 + \xi_2| \approx |\xi_2| \approx c\}$[1] for some $c > 0$. Let Φ be a non-degenerate phase function and consider the bilinear operator T_λ as*

$$T_\lambda(f_1, f_2)(x) = \int_{\mathbf{R}^{4n}} a(x, \vec{y}, \vec{\xi}) e^{i\lambda\Phi(x,\vec{y},\vec{\xi})} f_1(y_1) f_2(y_2)\, d\vec{y}\, d\vec{\xi} \,.$$

Then, if $2 \le p_1, p_2, p' \le \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p'} = 1$ there exists a constant $C > 0$ such that

$$\|T_\lambda(f_1, f_2)\|_p \le C\lambda^{-2n} \|f_1\|_{p_1} \|f_2\|_{p_2} \,,$$

[1] In the published version of this paper the condition $|\xi_1| \approx |\xi_1 + \xi_2|$ was mistakenly omitted

for λ sufficiently large.

Proof. Recall the assumption that the phase function

$$\Phi(x,y_1,y_2,\xi_1,\xi_2)=\varphi(x,\xi_1,\xi_2)-x\cdot y_1-x\cdot y_2$$

is non-degenerate, that is $\det(\varphi_{x,\xi_1})\neq 0$ and $\det(\varphi_{x,\xi_2})\neq 0$ on $\operatorname{supp} a$, and φ is homogenous of degree 1 in $\vec{\xi}$. We consider the trilinear operator

$$T_\lambda\big(T_\lambda^{*1}(f_1,f_2),f_3\big)=\int_{\mathbf{R}^{3n}} f_1(z_1)f_2(z_2)f_3(z_3)K(x,z_1,z_2,z_3)\,dz_1\,dz_2\,dz_3\,,$$

where

$$K(x,z_1,z_2,z_3)=\int_{\mathbf{R}^{5n}} e^{i\lambda[\Phi(x,y,z_3,\vec{\xi})-\Phi(z_1,y,z_2,\vec{\zeta})]}a(x,y,z_3,\vec{\xi}\,)\bar{a}(z_1,y,z_2,\vec{\zeta}\,)\,d\vec{\xi}\,d\vec{\zeta}\,dy\,. \tag{15}$$

Then,

$$\begin{aligned}
&\nabla_{(y,\xi_1,\xi_2,\zeta_1,\zeta_2)}\big[\Phi(x,y,z_3,\xi_1,\xi_2)-\Phi(z_1,y,z_2,\zeta_1,\zeta_2)\big]\\
&=\nabla_{(y,\xi_1,\xi_2,\zeta_1,\zeta_2)}\big[\varphi(x,\xi_1,\xi_2)-y\cdot\xi_1-z_3\cdot\xi_2-\big(\varphi(z_1,\zeta_1,\zeta_2)-y\cdot\zeta_1-z_2\cdot\zeta_2\big)\big]\\
&=\Big(\zeta_1-\xi_1,\ \varphi_{\xi_1}(x,\xi_1,\xi_2)-y,\ \varphi_{\xi_2}(x,\xi_1,\xi_2)-z_3,\ y-\varphi_{\zeta_1}(z_1,\zeta_1,\zeta_2),\ z_2-\varphi_{\zeta_2}(z_1,\zeta_1,\zeta_2)\Big)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big|\nabla_{(y,\xi_1,\xi_2,\zeta_1,\zeta_2)}\big[\Phi(x,y,z_3,\xi_1,\xi_2)-\Phi(z_1,y,z_2,\zeta_1,\zeta_2)\big]\Big|\\
&\approx|\xi_1-\zeta_1|+\big|\varphi_{\xi_1}(x,\xi_1,\xi_2)-y\big|+\big|\varphi_{\xi_2}(x,\xi_1,\xi_2)-z_3\big|+\big|\varphi_{\zeta_1}(z_1,\zeta_1,\zeta_2)-y\big|\\
&\qquad+\big|\varphi_{\zeta_2}(z_1,\zeta_1,\zeta_2)-z_2\big|\,.
\end{aligned}$$

Integrating by parts in (15) in all variables we obtain that for all $N>0$ there exists a constant C_N such that $\big|K(x,z_1,z_2,z_3)\big|$ is less than or equal to

$$\begin{aligned}
C_N\int_E &\frac{1}{(1+\lambda|\xi_1-\zeta_1|)^N}\cdot\frac{1}{(1+\lambda|\varphi_{\xi_1}(x,\xi_1,\xi_2)-y|)^N}\cdot\frac{1}{(1+\lambda|\varphi_{\xi_2}(x,\xi_1,\xi_2)-z_3|)^N}\\
&\times\frac{1}{(1+\lambda|\varphi_{\zeta_1}(z_1,\zeta_1,\zeta_2)-y|)^N}\cdot\frac{1}{(1+\lambda|\varphi_{\zeta_2}(z_1,\zeta_1,\zeta_2)-z_2|)^N}\,dy\,d\vec{\xi}\,d\vec{\zeta}\,,
\end{aligned}\tag{16}$$

where $E=\{(y,\xi_1,\xi_2,\zeta_1,\zeta_2)\in\mathbf{R}^{5n}:\,|y|\le C_1,|\vec{\xi}\,|,|\vec{\zeta}\,|\approx C_2\}$.

We need to show that the kernel K satisfies the hypotheses of Lemma 3.2. We first consider the integral $\int_{\mathbf{R}^{3n}}|K(x,z_1,z_2,z_3)|\,dxdz_1dz_2$. We integrate first in the variable z_2 in (16) and we obtain a factor of λ^{-n}. Then we integrate in z_1 by making the change of variables $z_1'=\lambda\varphi_{\zeta_1}(z_1,\zeta_1,\zeta_2)-\lambda y$ and using the fact that $\det\varphi_{z_1,\zeta_1}\neq 0$ on the support of a. This provides another factor of λ^{-n}. Then we integrate in y which provides another factor of λ^{-n} and finally the integral in ξ_1 will also yield a factor of λ^{-n}. The remaining integrals are over compact regions and the final result is that

$$\int_{\mathbf{R}^{3n}}|K(x,z_1,z_2,z_3)|\,dx\,dz_1\,dz_2\le C\,\lambda^{-4n}$$

with C independent of λ. A similar calculation yields that

$$\int_{\mathbf{R}^{3n}} |K(x,z_1,z_2,z_3)|\,dx\,dz_1\,dz_3 \le C\,\lambda^{-4n}\,.$$

We now consider the integral $\int_{\mathbf{R}^{3n}} |K(x,z_1,z_2,z_3)|\,dx\,dz_2\,dz_3$. We integrate (16) with respect to the variables z_2, z_3, y, ξ_1 in this order to obtain a factor of λ^{-4n} and the remaining integrals are over compact regions. We deduce the estimate

$$\int_{\mathbf{R}^{3n}} |K(x,z_1,z_2,z_3)|\,dx\,dz_2\,dz_3 \le C\,\lambda^{-4n}$$

with C independent of λ. Analogously, we obtain the estimate

$$\int_{\mathbf{R}^{3n}} |K(x,z_1,z_2,z_3)|\,dz_1\,dz_2\,dz_3 \le C\,\lambda^{-4n}\,.$$

Thus, the kernel K of the trilinear operator $\mathcal{T}_\lambda(f_1,f_2,f_3) = T_\lambda\big(T_\lambda^{*1}(f_1,f_2),f_3\big)$ satisfies the hypotheses Lemma 3.2 with constants $A_0 = A_1 = A_2 = A_3 = C\,\lambda^{-4n}$; thus the conclusion of Lemma 3.2 holds for $\mathcal{T}_\lambda$ and in particular we obtain that

$$\|\mathcal{T}_\lambda\|_{L^{r_1}\times L^{r_2}\times L^{r_3}\to L^r} \le C\,\lambda^{-4n}$$

whenever $1/r_1 + 1/r_2 + 1/r_3 = 1/r$ and $1 \le r_1, r_2, r_3 \le \infty$. Taking (r_1, r_2, r_3, r) to be either (p_1, p_2, p_2, p_1'), or (p_1, p_2, p', p) or (p', p_2, p_2, p) we deduce that $\mathcal{T}_\lambda$ satisfies conditions (8), (9), and (10) of Lemma 3.1. We conclude that T_λ maps $L^{p_1}\times L^{p_2}$ to L^p with norm at most a constant multiple of λ^{-2n} when $1/p_1 + 1/p_2 = 1/p$, $2 \le p', p_2 \le \infty$, and one of the indices p_1, p_2, p is equal to 2. This region consists of the three sides of the local-L^2 triangle $1/p_1 + 1/p_2 = 1/p$, $2 \le p', p_2 \le \infty$. Boundedness for the points in the interior of the triangle follows by interpolation (that also yields the required bound on the norm). □

5. Proof of Proposition 2.3

Let $\Psi \in \mathcal{C}_0^\infty(\mathbf{R}^{2n})$, with $\operatorname{supp}\Psi \subseteq \{\vec{\xi} : 2^{-1} \le |\vec{\xi}| \le 4\}$ and such that $\Psi_0(\vec{\xi}) + \sum_{j=1}^\infty \Psi(2^{-j}\vec{\xi}) = 1$, where Ψ_0 is in $\mathcal{C}_0^\infty(\mathbf{R}^{2n})$ with support near the origin.

Next, for each j select a set of unit vectors $\{\vec{\xi}_j^\nu\}$ of cardinality $c2^{j(2n-1)/2}$ such that $|\vec{\xi}_j^\nu - \vec{\xi}_j^{\nu'}| \approx 2^{-j/2}$ and such that the union of the balls of radii $2^{-j/2}$ centered at the $\vec{\xi}_j^\nu$ covers the unit sphere in $\mathbf{R}^{2n}$. Let $\{\chi_j^\nu\}$ be a partition of unity on the unit sphere subordinate to this covering. Extend these functions to all of $\mathbf{R}^{2n}\setminus\{(0,0)\}$ as functions homogenous of degree 0.

We now write

$$b(x,\vec{y},\vec{\xi}) = b_0(x,\vec{y},\vec{\xi}) + \sum_{j=1}^\infty \sum_{\nu=1}^{c2^{j(2n-1)/2}} b_j^\nu(x,\vec{y},\vec{\xi})$$

where $b_0(x,\vec{y},\vec{\xi}) = b(x,\vec{y},\vec{\xi})\Psi_0(\vec{\xi})$ and $b_j^\nu(x,\vec{y},\vec{\xi}) = b(x,\vec{y},\vec{\xi})\chi_j^\nu(\vec{\xi})\Psi(2^{-j}\vec{\xi})$. Moreover, we define

$$K_j^\nu(x,\vec{y}) = \int_{\mathbf{R}^{2n}} e^{i(\varphi(x,\vec{\xi})-\vec{y}\cdot\vec{\xi})} b_j^\nu(x,\vec{y},\vec{\xi})\,d\vec{y}\,.$$

Via this decomposition we express $\mathcal{F} = \mathcal{F}_0 + \sum_j \sum_\nu \mathcal{F}_j^\nu$, where $\mathcal{F}_j^\nu$ is the bilinear integral operator with kernel K_j^ν:

$$\mathcal{F}_j^\nu(f_1, f_2)(x) = \int_{\mathbf{R}^{2n}} K_j^\nu(x, \vec{y}\,) f_1(y_1) f_2(y_2)\, d\vec{y}\,.$$

Write

$$\varphi(x, \vec{\xi}\,) - \vec{y} \cdot \vec{\xi} = \varphi_{\xi_1}(x, \vec{\xi}_j^{\,\nu}\,) \cdot \xi_1 - y_1 \cdot \xi_1 + \varphi_{\xi_2}(x, \vec{\xi}\,) \cdot \xi_2 - y_2 \cdot \xi_2 + H_j^\nu(x, \vec{\xi}\,)\,,$$

where

$$H_j^\nu(x, \vec{\xi}\,) = \varphi(x, \vec{\xi}\,) - \varphi_{\vec{\xi}}(x, \vec{\xi}_j^{\,\nu}\,) \cdot \vec{\xi}\,.$$

We introduce the differential operator

$$L = I + 2^{2j} \partial^2_{\vec{\xi}_j^{\,\nu}} + 2^j \Delta_{(\vec{\xi}_j^{\,\nu})'}$$

where $(\vec{\xi}_j^{\,\nu})'$ denotes a $(2n-1)$-dimensional set of coordinates orthogonal to $\vec{\xi}_j^{\,\nu}$. We have

Lemma 5.1. *If $b \in S^m$, then for all $N > 0$ there exists $C > 0$ such that*

$$\left| L^N \left(e^{iH_j^\nu(x,\vec{\xi}\,)} b_j^\nu(x, \vec{y}, \vec{\xi}\,) \right) \right| \le C 2^{jm}\,. \tag{17}$$

Proof. In order to prove the estimate, notice that, since b_j^ν lies in S^m and is localized in the set $|\vec{\xi}\,| \approx 2^j$, the worst case is when all the derivatives fall on the exponential factor. Notice however that

$$|b_j^\nu(x, \vec{y}, \vec{\xi}\,)| \le C 2^{jm}\,.$$

The lemma will follow if we prove that $\left| L^N e^{iH_j^\nu(x,\vec{\xi}\,)} \right| \le C_N$ for all $N > 0$, which is a consequence of the estimates below:

(i) $\left| \partial^k_{\vec{\xi}_j^{\,\nu}} H_j^\nu(x, \vec{\xi}\,) \right| \le C 2^{-jk}$;

(ii) $\left| \nabla^{k'}_{(\vec{\xi}_j^{\,\nu})'} H_j^\nu(x, \vec{\xi}\,) \right| \le C 2^{-jk'/2}$;

for $\vec{\xi}$ in the support of b_j^ν, for all $0 \le k, k' \le N$. Here, and in what follows, we denote by $\vec{\theta}'$ the projection of the vector $\vec{\theta}$ of $\mathbf{R}^{2n}$ onto the subspace orthogonal to $\vec{\xi}^{\,\nu}$.

The estimates (i) and (ii) follow from the fact that H is homogeneous of degree 1 in $\vec{\xi}$ and that $|\vec{\xi}\,| \approx 2^j$, as in [St] p. 407. □

Next we estimate the kernel K_j^ν integrating by parts. Set

$$A(x, \vec{y}, \vec{\xi}\,) = \varphi_{\xi_1}(x, \vec{\xi}_j^{\,\nu}\,) \cdot \xi_1 - y_1 \cdot \xi_1 + \varphi_{\xi_2}(x, \vec{\xi}\,) \cdot \xi_2 - y_2 \cdot \xi_2\,,$$

then

$$|K_j^\nu(x, \vec{y}\,)| \le \frac{1}{\left(1 + 2^{2j} |\vec{\xi}_j^{\,\nu} \cdot \nabla_{\vec{\xi}} A|^2 + 2^j |(\nabla_{\vec{\xi}} A)'|^2\right)^N} \int_{\mathbf{R}^{2n}} \left| L^N \left(e^{iH_j^\nu(x,\vec{\xi}\,)} b_j^\nu(x, \vec{y}, \vec{\xi}\,) \right) \right| d\vec{\xi}\,.$$

From this it follows that $|K_j^\nu(x,\vec{y}\,)|$ is controlled by

$$\frac{C\,2^{jm}2^{j+j(2n-1)/2}}{\big(1+2^{2j}|\vec{\xi}_j^{\,\nu}\cdot\big(\varphi_{\xi_1}(x,\vec{\xi}_j^{\,\nu}\,)-y_1,\varphi_{\xi_2}(x,\vec{\xi}_j^{\,\nu}\,)-y_2\big)|^2+2^j|\big(\varphi_{\xi_1}(x,\vec{\xi}_j^{\,\nu}\,)-y_1,\varphi_{\xi_2}(x,\vec{\xi}_j^{\,\nu}\,)-y_2\big)'|^2\big)^N}.$$

In order to prove that $T=\sum_j\sum_\nu \mathcal{F}_j^\nu$ is bounded $T:L^1\times L^\infty\to L^1$ it suffices to show that

$$\sum_j\sum_\nu\Big[\sup_{y_1}\iint|K_j^\nu(x,y_1,y_2)|\,dxdy_2\Big]<\infty\,. \tag{18}$$

Perfoming the changes of variables $u=\varphi_1(x,\vec{\xi}_j^{\,\nu})$, $v=\varphi_2(x,\vec{\xi}_j^{\,\nu})-y_2$ we see that

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|K_j^\nu(x,y_1,y_2)|\,dxdy_2\\
&\le C\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{2^{jm}2^{j(n+\frac12)}}{\big(1+2^{2j}|\vec{\xi}_j^{\,\nu}\cdot(u-y_1,v)|^2+2^j|(u-y_1,v)'|^2\big)^N}\,dudv\\
&=C2^{jm}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{1}{\big(1+|(u,v)|^2\big)^N}\,dudv\\
&=C2^{jm}\,.
\end{aligned}$$

Therefore, the norm of $\mathcal{F}:L^1\times L^\infty\to L^1$ is bounded by a constant times

$$\sum_j\sum_\nu\Big[\sup_{y_1}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|K_j^\nu(x,y_1,y_2)|\,dxdy_2\Big]\le C\sum_j\sum_\nu 2^{jm}\le C\sum_j 2^{j(m+n-1/2)}\le C$$

as long as $m+n-1/2<0$, that is $m<-(2n-1)/2$. □

6. Proof of Proposition 2.4

We consider a bilinear FIO with a Hörmander symbol $\sigma(x,y_1,y_2,\xi_1,\xi_2)$ whose phase has the form

$$\Phi(x,\vec{y},\vec{\xi}\,)=[\phi_1(x,\xi_1)-y_1\cdot\xi_1]+[\phi_2(x,\xi_2)-y_2\cdot\xi_2]\,, \tag{19}$$

where each each expression inside the square brackets is a non-degenerate linear phase; that is, ϕ_1 and ϕ_2 are C^∞ functions real on $\mathbf{R}^n\times\mathbf{R}^n\setminus\{(0,0)\}$, homogeneous of degree 1 in ξ_1 and ξ_2, resp., and they satisfy the non-degeneracy conditions

$$\det(\phi_j)_{x\xi_j}\neq 0\,,\qquad\qquad j=1,2$$

on the support of the symbol, which are equivalent to (5) and (6) for $\phi=\phi_1+\phi_2$.

In this case the associated bilinear FIO has the form

$$T_\sigma(f_1,f_2)(x)=\int_{\mathbf{R}^{4n}}\sigma(x,y_1,y_2,\xi_1,\xi_2)f_1(y_1)f_2(y_2)e^{i(x-y_1)\cdot\xi_1}e^{i(x-y_2)\cdot\xi_2}e^{i\phi_1(x,\xi_1)}e^{i\phi(x,\xi_2)}\,d\vec{y}\,d\vec{\xi}\,.$$

We assume that the symbol σ has compact support in the variables x,y_1,y_2. Denote the support by Q, that is, the function $(x,y_1,y_2)\mapsto\sigma(x,y_1,y_2,\xi_1,\xi_2)$ is supported in the set

$Q\times Q\times Q$. We cover the set Q by a finite collection of balls of radius 2 and we introduce a smooth partition of unity subordinate to this collection of balls. We may therefore write the symbol $\sigma(x,y_1,y_2,\xi_1,\xi_2)$ as a finite sum of symbols $\sigma_\rho(x,y_1,y_2,\xi_1,\xi_2)$, where each σ_ρ is supported in a ball of radius 2 in the variables y_1 and y_2. We fix such a ρ and by a translation we may assume that σ_ρ is supported in the ball of radius 2 centered at the origin in the variables y_1 and y_2. For notational convenience we set $\sigma_\rho=\sigma$ in the argument below.

We introduce a smooth function ζ on $\mathbf{R}^{2n}$ whose support in contained in the annulus $1/2<|\vec{\xi}\,|<2$ such that

$$\zeta_0(\vec{\xi}\,)+\sum_{j=1}^{\infty}\zeta(2^{-j}\vec{\xi}\,)=1$$

for some smooth function ζ_0 supported in a ball centered at the origin.

We set $\sigma_0(x,y_1,y_2,\xi_1,\xi_2)=\sigma(x,y_1,y_2,\xi_1,\xi_2)\zeta_0(\vec{\xi}\,)$, and for $j\geq 1$ set $\sigma_j(x,y_1,y_2,\xi_1,\xi_2)=\sigma(x,y_1,y_2,\xi_1,\xi_2)\zeta(2^{-j}\vec{\xi}\,)$. We split the symbol σ as

$$\sigma=\sigma_0+\sum_{j=1}^{\infty}\sigma_j$$

and this introduces a decomposition of the bilinear FIO

$$T_\sigma=T_{\sigma_0}+\sum_{j=1}^{\infty}T_{\sigma_j}\,.$$

As T_{σ_0} has a symbol that is compactly supported in all variables, one trivially obtains

$$|T_{\sigma_0}(f_1,f_2)(x)|\leq C\|f_1\|_{L^1(Q)}\|f_2\|_{L^1(Q)}\chi_Q(x)$$

Consequently, T_{σ_0} maps $L^{p_1}\times L^{p_2}\to L^p$ for any $1\leq p_1,p_2\leq\infty$ with $1/p=1/p_1+1/p_2$.

We focus therefore our attention to the sum of the operators T_{σ_j}. Fix a $j\geq 1$ for the moment. For every $x\in\mathbf{R}^n$, the function

$$(y_1,y_2,\xi_1,\xi_2)\mapsto\sigma_j(x,y_1,y_2,2^j\xi_1,2^j\xi_2)\tag{20}$$

is supported in $B(0,2)^4$, where $B(0,2)$ is the ball of radius 2 centered at the origin in $\mathbf{R}^n$. Since $B(0,2)$ is contained in $[-\pi,\pi]^n$, by expanding the function in (20) in Fourier series over $[-\pi,\pi]^{4n}$ (as in the work of Coifman and Meyer [CM1], [CM2]) we obtain that for all $y_1,y_2,\xi_1,\xi_2\in\mathbf{R}^n$ the function $\sigma_j(x,y_1,y_2,2^j\xi_1,2^j\xi_2)$ is equal to

$$\sum_{\ell_1\in\mathbf{Z}^n}\sum_{\ell_2\in\mathbf{Z}^n}\sum_{k_1\in\mathbf{Z}^n}\sum_{k_2\in\mathbf{Z}^n}c^j_{\ell_1,\ell_2,k_1,k_2}(x)e^{i(\ell_1\cdot y_1+\ell_2\cdot y_2+k_1\cdot\xi_1+k_2\cdot\xi_2)}\eta(y_1)\eta(y_2)\eta(\xi_1)\eta(\xi_2)\,,$$

where η is a smooth function on $\mathbf{R}^n$ equal to 1 on the square $[-5/2,5/2]^n$ (and thus on the ball $B(0,2)$) and vanishing outside the square $[-\pi,\pi]^n$. The coefficient $c^j_{\ell_1,\ell_2,k_1,k_2}(x)$ of the Fourier series expansion is equal to

$$\frac{1}{(2\pi)^{4n}}\int_{[-\pi,\pi]^{4n}}\sigma(x,y_1,y_2,2^j\xi_1,2^j\xi_2)\zeta(\xi_1,\xi_2)e^{-i(\ell_1\cdot y_1+\ell_2\cdot y_2+k_1\cdot\xi_1+k_2\cdot\xi_2)}\,d\vec{y}\,d\vec{\xi}\,.$$

To estimate $c^j_{\ell_1,\ell_2,k_1,k_2}(x)$ we integrate by parts and we use the fact that σ is a Hörmander symbol of order m to obtain the estimate

$$\begin{aligned}|c^j_{\ell_1,\ell_2,k_1,k_2}(x)| &\le \sum_{r=0}^{4N} \frac{C_{N,r}\, 2^{jr}(1+2^j|\xi_1|+2^j|\xi_2|)^{m-r}\chi_{1/4<|\xi_1|^2+|\xi_2|^2<4}}{(1+|\ell_1|^2)^N(1+|\ell_2|^2)^N(1+|k_1|^2)^N(1+|k_1|^2)^N}\\ &\le \frac{C_N\, 2^{jm}}{(1+|\ell_1|^2)^N(1+|\ell_2|^2)^N(1+|k_1|^2)^N(1+|k_1|^2)^N}\,.\end{aligned}$$

We define

$$\tilde c^{j,N}_{\ell_1,\ell_2,k_1,k_2}(x) = 2^{-jm} c^j_{\ell_1,\ell_2,k_1,k_2}(x)(1+|\ell_1|^2)^N(1+|\ell_2|^2)^N(1+|k_1|^2)^N(1+|k_1|^2)^N$$

and we note that

$$|\tilde c^{j,N}_{\ell_1,\ell_2,k_1,k_2}(x)| \le C_N \chi_Q(x)\,. \tag{21}$$

We introduce modulation operators $M_\ell(g)(x) = g(x)e^{i\ell\cdot x}$ and a smooth function with compact support $a(x)$ which is bounded by 1 in absolute value and is equal to 1 on the set Q. Using the above decomposition, we express

$$\sum_{j=1}^\infty T_{\sigma_j}(f_1,f_2) = \sum_{1,2,3,4}\sum_{j=1}^\infty 2^{jm}\,\tilde c^{j,N}_{\ell_1,\ell_2,k_1,k_2}(x)\,F^1_j(M_{\ell_1}(f_1\eta))(x)\,F^2_j(M_{\ell_2}(f_2\eta))(x)\,,$$

where

$$\sum_{1,2,3,4} = \sum_{k_1\in\mathbf{Z}^n}(1+|k_1|^2)^{-N}\sum_{k_2\in\mathbf{Z}^n}(1+|k_2|^2)^{-N}\sum_{l_1\in\mathbf{Z}^n}(1+|l_1|^2)^{-N}\sum_{l_2\in\mathbf{Z}^n}(1+|l_2|^2)^{-N}\,,$$

F^1_j and F^2_j are FIOs with non-degenerate phases $-y_1\cdot\xi_1+\phi_1(x,\xi_1)$, $-y_2\cdot\xi_2+\phi_2(x,\xi_2)$ and symbols $a(x)e^{i2^{-j}k_1\cdot\xi_1}\eta(2^{-j}\xi_1)$, $a(x)e^{i2^{-j}k_2\cdot\xi_2}\eta(2^{-j}\xi_2)$, respectively.

We now fix indices $1<p_1,p_2<2$ and $1/2<p<1$ where $1/p=1/p_1+1/p_2$. To obtain the required estimate for the L^p quasi-norm of the operator $\sum_{j=1}^\infty T_{\sigma_j}(f_1,f_2)$, due to the rapid convergence of the sums in $\sum_{1,2,3,4}$, it suffices to obtain the same estimate the L^p quasi-norm of the expression

$$\Big\|\sum_{j=1}^\infty 2^{jm}\tilde c^{j,N}_{\ell_1,\ell_2,k_1,k_2}F^1_j(M_{\ell_1}(f_1\eta))F^2_j(M_{\ell_2}(f_2\eta))\Big\|_{L^p}\,.$$

Setting $m=m_1+m_2$, for some $m_1,m_2<0$ and using (21), we control the preceding expression by

$$C_N\Big\|\Big(\sum_{j=1}^\infty|2^{jm_1}F^1_j(M_{\ell_1}(f_1\eta))|^2\Big)^{1/2}\Big(\sum_{j=1}^\infty|2^{jm_2}F^2_j(M_{\ell_2}(f_2\eta))|^2\Big)^{1/2}\Big\|_{L^p}$$

and this is at the most

$$C_N\Big\|\Big(\sum_{j=1}^\infty|2^{jm_1}F^1_j(M_{\ell_1}(f_1\eta))|^2\Big)^{1/2}\Big\|_{L^{p_1}}\,\Big\|\Big(\sum_{j=1}^\infty|2^{jm_2}F^2_j(M_{\ell_2}(f_2\eta))|^2\Big)^{1/2}\Big\|_{L^{p_2}} \tag{22}$$

via Hölder's inequality ($1/p=1/p_1+1/p_2$).

To control each of these terms, we make use of the following lemma:

Lemma 6.1. *Let $m_1 < 0$. Then we have the estimate*

$$\Big\| \big(\sum_{j=1}^{\infty} |2^{jm_1} F_j^1(g)|^2 \big)^{1/2} \Big\|_{L^{p_1}} \leq C_r (1 + |k_1|)^b \|g\|_{L^{p_1}} \tag{23}$$

whenever $\frac{1}{p_1} - \frac{1}{2} = -\frac{m_1}{n-1}$ and $1 < p_1 < 2$. Here b is a positive constant that depends only on p_1 and n.

Proof. By Khinchine's inequality matters reduce to estimating

$$\Big\| \sum_{j=1}^{\infty} \varepsilon_j 2^{jm_1} F_j^1(g) \Big\|_{L^{p_1}}$$

where $\varepsilon_j = \pm 1$. This is a FIO with the following symbol of order $m_1 < 0$

$$(x, \xi_1) \mapsto a(x) \sum_{j=1}^{\infty} \varepsilon_j 2^{jm_1} e^{i 2^{-j} k_1 \cdot \xi_1} \eta(2^{-j} \xi_1) \,.$$

A careful examination of the proof of Theorem 2.2 in [SSS] shows that the constant depends only on finitely many derivatives of the symbol and thus it grows at most polynomially in $|k_1|$. By interpolation the same assertion is valid on L^{p_1} for $p_1 \in (1, 2)$ and thus the claimed estimate (23) folllows. □

Using this lemma for F_j^1 and F_j^2 and choosing N large enough (say bigger than $(b+n)/p$) we obtain that expression (22) is bounded by a constant multiple of

$$(1 + |k_1|)^b (1 + |k_2|)^b \|M_{\ell_1}(f_1 \eta)\|_{L^{p_1}} \|M_{\ell_2}(f_2 \eta)\|_{L^{p_2}} \,.$$

Consequently, we deduce the estimate

$$\Big\| \sum_{j=1}^{\infty} T_{\sigma_j}(f_1, f_2) \Big\|_{L^p} \leq C \, \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \,,$$

where

$$\frac{1}{p} - 1 = \left(\frac{1}{p_1} - \frac{1}{2} \right) + \left(\frac{1}{p_2} - \frac{1}{2} \right) = -\frac{m_1}{n-1} - \frac{m_2}{n-1} = -\frac{m}{n-1}$$

when $1 < p_1, p_2 < 2$. This completes the proof of Proposition 2.4

Remark 6.2. *Proposition 2.4 can be extended to the range $1 \leq p_1, p_2 < 2$, when L^{p_i} is replaced by the local Hardy space h^1 whenever $p_i = 1$.*

Indeed, Corollary 2.3 in [SSS] says that FIOs with symbols of order $-(n-1)/2$ map h^1 to h^1, in particular they map h^1 to L^1 (and an examination of the proof there indicates that the constant depends on finitely many derivatives of the symbol). Consequently, one has the estimate

$$\Big\| \big(\sum_{j=1}^{\infty} |2^{jm_1} F_j^1(g)|^2 \big)^{1/2} \Big\|_{L^1} \leq C_r (1 + |k_1|)^b \|g\|_{h^1} \tag{24}$$

when $m_1 = -(n-1)/2$. Taking $g = M_{\ell_1}(f_1\eta)$ and noting that h^1 preserves multiplications by smooth bumps and modulations, we obtain the required conclusion. □

The authors would like to acknowledge the recent work of Bernicot and Germain [BG] on bilinear oscillatory integrals which seems to mildly overlap with our present work.

References

[BBPR] D. Bekollé, A. Bonami, M. Peloso, and F. Ricci, *Boundedness of Bergman projections on tube domains over light cones*, Math. Z. **237** (2001), 31–59.

[BG] F. Bernicot, P. Germain, *Bilinear Oscillatory integrals and boundedness for new bilinear multipliers*, Advances in Math., to appear.

[CM1] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[CM2] ______, *Commutateurs d' intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[CM3] ______, *Au délà des opérateurs pseudo-différentiels*, Astérisque No. 57, Société Mathématique de France, 1979.

[D] J. J. Duistermaat, *Fourier integral operators*, Progress in Mathematics, 130, Birkhäuser, Boston 1996.

[E] G. I. Eskin, *Degenerate elliptic pseudo-differential operators of principal type* (Russian) Mat. Sbornik **82** (124) (1970), 585–628; English translation, Math. USSR Sbornik **11** (1970), 539–582.

[GL] L. Grafakos and X. Li, *The disc as a bilinear multiplier,* Amer. J. Math. **128** (2006), 91–119.

[GT1] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory,* Adv. in Math. **165** (2002), 124–164.

[GT2] ______, *A multilinear Schur test and multiplier operators,* Jour. Funct. Anal. **187** (2001), 1–24.

[GrS] A. Grigis, J. Sjöstrand, *Microlocal analysis for differential operators. An introduction*, London Math. Soc. Lecture Note Series, 196, Cambridge University Press, Cambridge, 1994.

[He] C. S. Herz, *Fourier transforms related to convex sets*, Ann. Math. **75** (1962), 81–92.

[Hö1] L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.

[Hö2] ______, *The analysis of linear partial differential operators, IV*, Grundlehren der Math. Wissenschaften 275, Springer-Verlag, Berlin, 1994.

[KS] C. E. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Letters **6** (1999), 1–15.

[L] W. Littman, *$L^p - L^q$-estimates for singular integral operators*, Proc. Symp. Pure and Appl. Math. A.M.S. **23** (1973), 479–481.

[Se] A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), 685–745.

[SSS] A. Seeger, C. D. Sogge, E. M. Stein, *Regularity properties of Fourier integral operators*, Ann. Math. **134** (1991), 231–251.

[St] E. M. Stein, *Harmonic Analysis, Real-variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton 1993.

[Ta] M. Taylor, *Pseudodifferential Operators*, Princeton Univ. Press, Princeton 1981.

[Tr] F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators*, Vol. II, Plenum Press, New York 1982.

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