Interpolation of bilinear operators between quasi-Banach spaces

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Abstract

We study interpolation, generated by an abstract method of means, of bilinear operators between quasi-Banach spaces. It is shown that under suitable conditions on the type of these spaces and the boundedness of the classical convolution operator between the corresponding quasi-Banach sequence spaces, bilinear interpolation is possible. Applications to the classical real method spaces, Calderón-Lozanovsky spaces, and Lorentz-Zygmund spaces are presented.

1 Introduction

Motivated by applications in harmonic analysis, we are interested in interpolation of bilinear operators defined on products of quasi-Banach spaces. The main aim of this paper is to prove interpolation theorems for bilinear operators on quasi-Banach spaces generated by certain interpolation methods. We study a problem for the abstract method of means as well as for the real interpolation method.

Let us briefly outline the content of the paper. In Section 2 we establish notation and recall basic facts concerning quasi-Banach spaces and interpolation. In Section 3 we introduce a notion of special type of convexity for bilinear operators between quasi-Banach couples and we prove a bilinear interpolation theorem using the method of means, under the condition that the associated convolution operator is bounded on the parameter spaces involved in the construction of these methods. In view of a remarkable result of Kalton [13], the convexity parameters of the bilinear operators that take values in so called natural quasi-Banach spaces, are nicely determined by the types of the domains of the quasi-Banach spaces. We also prove continuous inclusions between spaces generated by the method of means and the Calderón-Lozanovsky method applied to certain classes of couples of Banach lattices satisfying the upper or lower lattice estimates. We give applications in the context of weighted Orlicz spaces. In particular, we obtain that the method of means determined by the corresponding weighted quasi-Banach spaces ℓ_{p_0} and ℓ_{p_1} and any couple of weighted quasi-Banach lattices $(L_{p_0}(w_0), L_{p_1}(w_1))$ coincide, up to equivalence of norms, with the Calderón-Lozanovsky space $\varphi(L_{p_0}(w_0), L_{p_1}(w_1))$.

In Section 4 we discuss applications of these results in the context of interpolation between quasi-Banach spaces generated by the real method of interpolation as well as to Calderón Lozanovsky spaces that include, in particular, Orlicz spaces.

*The author is supported by the National Science Foundation under grant DMS 0099881.
†The author is supported by KBN Grant 1 P03A 013 26.

In Section 5 we discuss abstract K or J interpolation method of bilinear operators between quasi-Banach spaces satisfying weaker convexity type conditions. These results are used for special weighted sequence spaces for which the classical convolution operator is bounded. As a consequence, bilinear interpolation theorems for Lorentz-Zygmund spaces are obtained.

2 Definitions and notation

A *quasi-norm* $\|\cdot\|$ defined on a vector space X (over real or complex field $\mathbb{K}$) is a map $X \to \mathbb{R}_+$ such that

(i) $\|x\| > 0$ for $x \neq 0$,

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$ for $\alpha \in \mathbb{K}$, $x \in X$,

(iii) $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$,

where C is a constant independent of x, y.

Let $0 < p \leq 1$. We call $\|\cdot\|$ a *p-norm* if we also have

(iv) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$.

A quasi-Banach space X is said to be *p-normable*, $0 < p \leq 1$, if there exists an equivalent p-norm $\|\cdot\|_*$ on X and a constant C' such that

$$\|x_1 + ... + x_n\|_* \leq C' \,(\|x_1\|_*^p + ... + \|x_n\|_*^p)^{1/p}.$$

for all $x_1, ..., x_n \in X$. An 1-normable space is simply called normable. While clearly any p-normable space is a quasi-normed space, a theorem of Aoki and Rolewicz (see [14]) asserts that any quasi-normed space X has an equivalent p-norm, where p satisfies $C = 2^{1/p-1}$ with C is as in (iii), defined by

$$\|x\| = \inf \big(\sum_k \|x_k\|_X^p \big)^{1/p},$$

where the infimum is taken over all finite sequences $\{x_k\} \subset X$ satisfying $\sum_k x_k = x$.

If $\|\cdot\|$ is a quasi-norm (resp., p-norm) on X defining a complete metrizable topology, then X is called a *quasi-Banach space* (resp., p-Banach space).

We shall use standard notation and notions from interpolation theory, as presented, e.g., in [2], [3]. Throughout this paper we will let $(\Omega, \mu) = (\Omega, \Sigma, \mu)$ be a complete σ-finite measure space and $L_0(\mu)$ will denote, as usual, the space of equivalence classes of real valued measurable functions on Ω, equipped with the topology of convergence (in the measure μ) on sets of finite measure. By a quasi-Banach lattice on Ω we mean a quasi-Banach space X which is a subspace of $L_0(\mu)$ such that there exists $u \in X$ with $u > 0$ and if $|f| \leq |g|$ a.e., where $g \in X$ and $f \in L_0(\mu)$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$. A quasi-Banach lattice X is said to be *maximal* if its unit ball $B_X = \{x;\, \|x\| \leq 1\}$ is a closed subset in $L_0(\mu)$.

In the special case when $\Omega = \mathbb{Z}$ is the set of integers and μ is the counting measure then a quasi-Banach lattice E on Ω is called a quasi-Banach sequence space on $\mathbb{Z}$, and in this case we denote by E' a Köthe dual space of E.

If X is a quasi-Banach lattice on (Ω) and $w \in L^0(\mu)$ with $w > 0$ a.e., we define the weighted quasi-Banach lattice $X(w)$ by $\|x\|_{X(w)} = \|x\, w\|_X$.

Given $0 < p < \infty$ and a quasi-Banach lattice X let X^p denote the *p-convexification* of X. Here X^p consists of all x such that $|x|^p \in X$ and is equipped with the quasi-norm $\|x\| = \||x|^p\|^{1/p}$.

Let $0 < t < \infty$ and $\overline{A} = (A_0, A_1)$ be a couple of quasi-Banach spaces. We equip $A_0 + A_1$ (resp., $A_0 \cap A_1$) with the quasi-norm $K(1, a)$ (resp. $J(1, a)$) where $K(t, a) = K(t, a; \overline{A})$ and $J(t, a) = J(t, a; \overline{A})$ are the functionals of J. Peetre, defined by

$$K(t, a; \overline{A}) = \inf \{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1};\ a = a_0 + a_1\}$$

and

$$J(t, a; \overline{A}) = \max\{\|a\|_{A_0}, t\,\|a\|_{A_1}\}.$$

It is easy to see that if A_j is p_j-normable $(j = 0, 1)$ then both $A_0 + A_1$ and $A_0 \cap A_1$ are p-normable with $p = \min\{p_0, p_1\}$.

If $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ are couples of quasi-Banach spaces, we let $\mathcal{L}(\overline{X}, \overline{Y})$ be the quasi-Banach space of all linear operators $T : \overline{X} \to \overline{Y}$ (which means, as usual, that $T : X_0 + X_1 \to Y_0 + Y_1$ is linear and the restrictions $T|_{X_j}$ are bounded operators from X_j to Y_j for $j = 0,\ 1$). The space is equipped with the quasi-norm $\|T\|_{\overline{X}\to\overline{Y}} := \max\{\|T\|_{X_0\to Y_0}, \|T\|_{X_1\to Y_1}\}$.

We will deal with vector-valued quasi-Banach sequence spaces. Let E be a quasi-Banach sequence lattice on $\mathbb{Z}$ and let X be a quasi-Banach space. The vector sequence $x = \{x_n\}_{n\in\mathbb{Z}}$ in X is called strongly E-summable if the corresponding scalar sequence $\{\|x_n\|_X\}_{n\in\mathbb{Z}}$ is in E. We denote by $E(X)$ the set of all such sequences in X. This is a quasi-Banach space under pointwise operations, and a natural quasi-norm on it is given by $\|x\|_{E(X)} := \|\{\|x_n\|_X\}\|_E$. It is easy to check that if E is p-Banach and X is q-Banach space then $E(X)$ is r-Banach with $r = \min\{p, q\}$.

Let $\overline{X}$ be a quasi-Banach couple. A couple $\overline{E} = (E_0, E_1)$ of quasi-Banach sequence lattices on $\mathbb{Z}$ is said to be a *parameter of the method of means on* $\overline{X}$ if $E_0 \cap E_1 \hookrightarrow \ell_p$ for some $0 < p \le 1$ such that the quasi-Banach space $X_0 + X_1$ is p-normable. Throughout the paper, for such $\overline{E} = (E_0, E_1)$ and $\overline{X}$, the space denoted by $J_{\overline{E}}(\overline{X}) = J_{E_0,E_1}(\overline{X}) = \overline{X}_{E_0,E_1}$ is built by the method of means consisting of all $x \in X_0 + X_1$ which can be represented in the form

$$x = \sum_{n\in\mathbb{Z}} u_n \qquad \text{(convergence in } X_0 + X_1)$$

with $\{u_n\} \in E_0(X_0) \cap E_1(X_1)$. We note that $J_{\overline{E}}(\overline{X})$ is a quasi-Banach space under the quasi-norm

$$\|x\| := \inf \max\big\{\|\{u_n\}\|_{E_0(X_0)}, \|\{u_n\}\|_{E_1(X_1)}\big\},$$

where the infimum is taken over all the above representations of x.

In fact, the continuous inclusion $E_0 \cap E_1 \hookrightarrow \ell_p$ implies that there exists a constant $C > 0$ such that

$$\sum_{n\in\mathbb{Z}} \|u_n\|^p_{X_0+X_1} \le C\,\|\{u_n\}\|_{E_0(X_0)\cap E_1(X_1)}$$

for any $\{u_n\} \in E_0(X_0) \cap E_1(X_1)$. Since $X_0 + X_1$ is p-normable, we conclude that the linear map $\mathcal{J}$ defined by $\mathcal{J}(\{x_n\}) := \sum_{n\in\mathbb{Z}} x_n$ is continuous from $E_0(X_0) \cap E_1(X_1)$ into $X_0 + X_1$. Thus the quotient space

$$\big(E_0(X_0) \cap E_1(X_1)\big)/\ker(\mathcal{J})$$

is a quasi-Banach space. Since it is isometrically isomorphic to $J_{\overline{E}}(\overline{X})$, we conclude that $J_{\overline{E}}(\overline{X})$ is also a quasi-Banach space.

3 Main results

In this section we prove interpolation theorems for bilinear operators between spaces generated by the method of means. An operator T defined on a product of two quasi-Banach spaces $X \times Y$

and taking values in another quasi-Banach space Z is called *bilinear* if it is linear in each of the two variables, and bounded, i.e., there is a constant C_0 such that for all $x \in X$ and $y \in Y$ we have

$$\|T(x,y)\|_Z \le C_0 \|x\|_X \|y\|_Y.$$

The smallest C_0 so that the above inequality holds for all $x \in X$ and $y \in Y$ is called the norm of B and will be denoted by $\|T\|_{X\times Y\to Z}$.

Inspired by a remarkable result of Kalton [13] we introduce the following terminology: we say that a bilinear operator $T : X \times Y \to Z$ between quasi-Banach spaces is said to be *s-bilinear convex* ($0 < s \le 1$) if there exists a constant $C > 0$ such that for all finite sequences $\{x_j\}_{j=1}^n \subset X$ and $\{y_j\}_{j=1}^n \subset Y$, we have

$$\big\| \sum_{j=1}^n T(x_j, y_j)\big\|_Z \le C \|T\|_{X\times Y\to Z} \big(\sum_{j=1}^n \|x_j\|_X^s \|y_j\|_Y^s \big)^{1/s}.$$

In the case when $s = 1$, we say, in short, that B is bilinear convex.

The triple (X, Y, Z) of quasi-Banach spaces is said to be *s-bilinear* (resp., *bilinear*) *admissible* whenever there exists $C = C(X, Y, Z) > 0$ such that any bilinear operator $T : X \times Y \to Z$ is s-bilinear convex (resp., bilinear convex).

In view of the result of Kalton ([13], p. 311), it follows that if X is a quasi-Banach space of type p, Y is a quasi-Banach of type q, and Z is a natural quasi-Banach space, then the triple (X, Y, Z) is s-bilinear admissible where $1/s = 1/p + 1/q$.

Let us recall that a quasi-Banach space X is *of type* p ($0 < p \le 2$) if there exists a constant $C > 0$ so that

$$\big[\mathcal{E}\big(\big\|\sum_{k=1}^n \varepsilon_k x_k\big\|^p\big)\big]^{1/p} \le C \,\big(\sum_{k=1}^n \|x_k\|^p\big)^{1/p},$$

where $\{\varepsilon_k\}$ is any sequence of independent Bernoulli random variables with

$$P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = 1/2.$$

It is well-known that for $0 < p < 1$, X is of type p if (and only if) X is p-normable; if $p > 1$ and X is of type p, then X is a Banach space (see [14], p. 99 and p. 107).

A quasi-Banach space is called *natural* [15] if it is isomorphic to a subspace of an L-convex quasi-Banach lattice. A quasi-Banach lattice X is said to be *L-convex* if there exists $0 < \varepsilon < 1$ so that if $u \in X$, with $\|u\| = 1$ and $0 \le x_k \le u$ $(1 \le k \le n)$ satisfy $(x_1 + ... + x_n)/n \ge (1 - \delta)u$, then $\max_{1\le k\le n} \|x_k\| \ge \varepsilon$.

The following lemma will be useful in the proof of the main result of this section.

Lemma 3.1. *Let X be a p-normable quasi-Banach space and let $\{x_{k,m}\}$ be an infinite matrix in X with $k, m \in \mathbb{Z}$. Assume that the series $\sum_{k\in\mathbb{Z}} x_{k,m-k}$ is unconditionally convergent for every $m \in \mathbb{Z}$ and $\sum_{m\in\mathbb{Z}} \big\|\sum_{k\in\mathbb{Z}} x_{k,m-k}\big\|_X^p < \infty$. Then the double limit $\lim\limits_{M,N\to\infty} \sum_{|k|\le M}\sum_{|j|\le N} x_{k,j}$ exists in X and*

$$\lim_{M,N\to\infty} \sum_{|k|\le M}\sum_{|j|\le N} x_{k,j} = \sum_{m\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} x_{k,m-k}\Big).$$

Proof. Let $u_m := \sum_{k\in\mathbb{Z}} x_{k,m-k}$ for $m \in \mathbb{Z}$. Fix $\varepsilon > 0$. Since $\sum_{m\in\mathbb{Z}} \|u_m\|_X^p < \infty$ and the series $\sum_{k\in\mathbb{Z}} x_{k,m-k}$ converges unconditionally, there exists $m_0 \in \mathbb{N}$ such that

$$\big\| \sum_{|m|\le m_0} u_m - \sum_{m\in\mathbb{Z}} \big(\sum_{k\in\mathbb{Z}} x_{k,m-k}\big)\big\|_X^p < \varepsilon/2$$

and

$$\sum_{|m|>m_0} \big\| \sum_{k\in F_m} x_{k,m-k}\big\|_X^p < \varepsilon/4,$$

where F_m is any finite subset of $\mathbb{Z}$. Furthermore, there exists $k_0 \in \mathbb{N}$ such that for any $k_m \ge k_0$ with $|m| \le m_0$, we have

$$\sum_{|m|\le m_0} \big\| \sum_{|k|>k_m} x_{k,m-k}\big\|_X < \varepsilon/4.$$

Let M and N be positive integers with $M > k_0$ and $N > m_0 + k_0$. We define two subsets A and B of $\mathbb{Z}\times\mathbb{Z}$ by setting

$$A = \big\{(k,j)\,;\ |k| \le M,\ |j| \le N\big\}$$

and

$$B = \big\{(k,j)\,;\ |k| \le k_0,\ |j+k| \le m_0\big\}.$$

For $m \in \mathbb{Z}$ we let $F_m = \big\{k \in \mathbb{Z}\,;\ (k, m-k) \in A\big\}$ and $k_m = \max\big\{k \in \mathbb{Z}\,;\ k \in F_m\big\}$. Since $B \subset A$, we conclude that $k_m \ge k_0$, whenever $|m| \le m_0$. This implies that the two last inequalities hold for F_m and k_m just defined. It is easy to verify that

$$\sum_{|k|\le M}\sum_{|j|\le N} x_{k,j} = \sum_{|m|\le m_0} u_m \ - \sum_{|m|\le m_0}\sum_{|k|>k_m} x_{k,m-k} + \sum_{|m|>m_0}\sum_{k\in F_m} x_{k,m-k}\,.$$

Combining this identity with the above three norm inequalities, we deduce

$$\big\|\sum_{|k|\le M}\sum_{|j|\le N} x_{k,j} - \sum_{m\in\mathbb{Z}}\big(\sum_{k\in\mathbb{Z}} x_{k,m-k}\big)\big\|_X < \varepsilon.$$

for $M > k_0$ and $N > m_0 + k_0$. This proves the assertion. □

Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ and $\overline{Z} = (Z_0, Z_1)$ be quasi-Banach couples. We will say that $T = (T_0, T_1)$ is a bilinear operator from $\overline{X}\times\overline{Y}$ into $\overline{Z}$, and write $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ if $T_j : X_j \times Y_j \to Z_j$ is a bounded bilinear operator ($j = 0, 1$) and $T_0(x, y) = T_1(x, y)$ for any $x \in X_0 \cap X_1$ and $y \in Y_0 \cap Y_1$. If additionally X, Y and Z are intermediate quasi-Banach spaces with respect to $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, respectively, then we say that $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ extends to a bilinear operator from $X \times Y$ into Z provided that T_0 has a bilinear extension from $X \times Y$ into Z.

Note that any $(T_0, T_1) \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ defines a bilinear operator T^0 (resp., T^1) which will be called in the sequel a *natural bilinear extension* of (T_0, T_1) from $(X_0 + X_1) \times (Y_0 \cap Y_1)$ into $Z_0 + Z_1$ (resp., $(X_0 \cap X_1) \times (Y_0 + Y_1) \to Z_0 + Z_1$) by

$$T^0(x, y) := T_0(x_0, y) + T_1(x_1, y)$$

for any $x = x_0 + x_1$ and $y \in Y_0 \cap Y_1$ (resp., $T^1(x, y) := T_0(x, y_0) + T_1(x, y_1)$ for any $x \in X_0 \cap X_1$ and $y = y_0 + y_1 \in Y_0 + Y_1$ with $y_0 \in Y_0$ and $y_1 \in Y_1$. It is easy to see that T^0 (resp., T^1) does not depend on the representations of $x \in X_0 + X_1$ (resp., $y \in Y_0 + Y_1$).

An operator $T = (T_0, T_1) \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ is said to be (s_0, s_1)-*bilinear convex* $(0 < s_0, s_1 \le 1)$ if there exists a constant $C > 0$ such that for any finite sequences $\{x_j\} \subset X_0\cap X_1$ and $\{y_j\} \subset Y_0\cap Y_1$ the following holds for $k = 0, 1$

$$\big\|\sum_j T_k(x_j, y_j)\big\|_{Z_k} \le C\,\|T_k\|_{X_k\times Y_k\to Z_k}\,\big(\sum_j \|x_j\|_{X_k}^{s_k}\|y_j\|_{Y_k}^{s_k}\big)^{1/s_k}.$$

In the case when $s_0 = s_1$ (resp., $s_0 = s_1 = 1$), we say, in short, that T is s-bilinear convex (resp., bilinear convex).

Throughout the paper we will consider the convolution operator τ of sequences defined for $x = \{\xi_k\}_{-\infty}^{\infty}$ and $y = \{\eta_k\}_{-\infty}^{\infty}$ by

$$\tau(x, y)_n = \sum_{k\in\mathbb{Z}} \xi_k\, \eta_{n-k}, \quad n \in \mathbb{Z}.$$

We now state the main theorem of this section:

Theorem 3.1. *Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$, and $\overline{Z} = (Z_0, Z_1)$ be quasi-Banach spaces and let $T = (T_0, T_1) \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ be (s_0, s_1)-bilinear convex. Assume that E_j, F_j and G_j are quasi-Banach sequence spaces on $\mathbb{Z}$ such that $(E_0^{s_0}, E_1^{s_1})$, $(F_0^{s_0}, F_1^{s_1})$, and $(G_0^{s_0}, G_1^{s_1})$ are parameters of the method of means on $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, respectively. If the convolution operator τ is bounded from $E_j \times F_j$ to G_j $(j = 0, 1)$, then T extends to a bilinear operator $\widehat{T}$ from $\overline{X}_{E_0^{s_0}, E_1^{s_1}} \times \overline{Y}_{F_0^{s_0}, F_1^{s_1}}$ into $\overline{Z}_{G_0^{s_0}, G_1^{s_1}}$ with the norm estimate*

$$\|\widehat{T}\| \le \max_{j=0,1} \left[C_j\, \|\tau\|_{E_j\times F_j\to G_j}\, \|T_j\|_{X_j\times Y_j\to Z_j} \right]$$

for some $C_j > 0$ $(j = 0, 1)$.

Proof. Let $x \in X := \overline{X}_{E_0^{s_0}, E_1^{s_1}}$ and $y \in Y := \overline{Y}_{F_0^{s_0}, F_1^{s_1}}$. For $\varepsilon > 0$ pick $\{u_k\} \in E_0^{s_0}(X_0) \cap E_1^{s_1}(X_1)$ and $\{v_k\} \in F_0^{s_0}(Y_0) \cap F_1^{s_1}(Y_1)$ such that

$$x = \sum_{k\in\mathbb{Z}} u_k \ \text{(convergence in } X_0 + X_1), \quad y = \sum_{k\in\mathbb{Z}} v_k \ \ \text{(convergence in } Y_0 + Y_1)$$

and

$$\|\{u_k\}\|_{E_j^{s_j}(X_j)} \le (1 + \varepsilon)\|x\|_X, \quad \|\{v_k\}\|_{F_j^{s_j}(Y_j)} \le (1 + \varepsilon)\|y\|_Y.$$

Since the convolution operator τ is bounded from $E_j \times F_j \to G_j$ and $T = (T_0, T_1)$ is (s_0, s_1)-bilinear convex, we immediately deduce that if $S := T_0$, then the series

$$\sum_{k\in\mathbb{Z}} S(u_k, v_{m-k})$$

converges unconditionally in both spaces Z_0 and Z_1 for all $m \in \mathbb{Z}$, and thus also in $Z_0 + Z_1$. Furthermore if we set $z_m := \sum_{k\in\mathbb{Z}} S(u_k, v_{m-k})$ for $m \in \mathbb{Z}$, we obtain for some $C_j > 0$

$$\begin{aligned}
\|\{\|z_m\|_{Z_j}\}\|_{G_j{}^{s_j}} &\le C_j\, \|T_j\|_{X_j\times Y_j\to Z_j} \left\|\tau(\{\|u_k\|_{X_0}^{s_j}\}, \{\|v_k\|_{Y_j}^{s_j}\})\right\|_{G_j}^{1/s_j} \\
&\le C_j\, \|T_j\|_{X_j\times Y_j\to Z_j}\, \|\tau\|_{E_j\times F_j\to G_j} \|\{u_k\}\|_{E^{s_j}(X_j)}\, \|\{v_k\}\|_{F^{s_j}(Y_j)} \\
&\le (1 + \varepsilon)^2\, C_j\, \|\tau\|_{E_j\times F_j\to G_j}\, \|T_j\|_{X_j\times Y_j\to Z_j}\, \|x\|_X\, \|y\|_Y.
\end{aligned}$$

These calculations show that the sequence $\{z_m\}_{m\in\mathbb{Z}}$ lies in $G_0^{s_0}(Z_0) \cap G_1^{s_1}(Z_1)$ and

$$\|\{z_m\}\|_{G_0^{s_0}(Z_0)\cap G_1^{s_1}(Z_1)} \le (1 + \varepsilon)^2 \max_{j=0,1} [C_j\, \|T_j\|_{X_j\times Y_j\to Z_j}\, \|\tau\|_{E_j\times F_j\to G_j}]\, \|x\|_X\, \|y\|_Y.$$

Our hypothesis that (G^{s_0}, G^{s_1}) is a parameter of the method of means on (Z_0, Z_1) implies that for some $0 < p \le 1$

$$\sum_{m\in\mathbb{Z}} \|z_m\|_{Z_0+Z_1}^p < \infty.$$

Applying Lemma 3.1, we deduce that the double limit

$$\widehat{T}(x,y) := \lim_{m,n\to\infty} \sum_{|k|\le m} \sum_{|j|\le n} S(u_k, v_j)$$

exists in $Z_0 + Z_1$ and

$$\widehat{T}(x,y) = \sum_{m\in\mathbb{Z}} z_m \ \text{(convergence in } Z_0 + Z_1).$$

Combining the above remarks yield $\widehat{T}(x,y) \in Z := \overline{Z}_{G_0^{s_0}, G_1^{s_1}}$ with

$$\|\widehat{T}(x,y)\|_Z \le C(1+\varepsilon)^2 \, \|x\|_X \, \|y\|_Y,$$

where $C = \max_{j=0,1} [C_j \, \|T_j\|_{X_j\times Y_j\to Z_j}] \, \|\tau\|_{E_j\times F_j\to G_j}$.

Since ε is arbitrary, to onclude it is enough to show that $\widehat{T}$ defines a required bilinear extension of T. To see this recall that the natural extensions $T^0 : (X_0 + X_1) \times (Y_0 \cap Y_0) \to Z_0 + Z_1$ and $T^1 : (X_0 \cap X_1) \times (Y_0 + Y_0) \to Z_0 + Z_1$ are bilinear operators. This implies that the following limits exist in $Z_0 + Z_1$ for all $k, j \in \mathbb{Z}$

$$\lim_{m\to\infty} \sum_{|k|\le m} S(u_k, v_j) = T^0(x, v_j),$$

and

$$\lim_{n\to\infty} \sum_{|j|\le n} S(u_k, v_j) = T^1(u_k, y).$$

Combining this with the fact that double limit $z := \lim_{m,n\to\infty} \sum_{|k|\le m} \sum_{|j|\le n} S(u_k, v_j)$ exists in $Z_0 + Z_1$, we easily obtain

$$z = \lim_{n\to\infty} \sum_{|j|\le n} \Big(\lim_{m\to\infty} \sum_{|k|\le m} S(u_k, v_j) \Big) = \lim_{n\to\infty} T^0\big(x, \sum_{|j|\le n} v_j\big)$$

and

$$z = \lim_{m\to\infty} \sum_{|k|\le m} \Big(\lim_{n\to\infty} \sum_{|j|\le n} S(u_k, v_j) \Big) = \lim_{m\to\infty} T^1\big(\sum_{|k|\le m} u_k, y\big).$$

This shows that the double limit

$$\widehat{T}(x,y) := \lim_{m,n\to\infty} \sum_{|k|\le m} \sum_{|j|\le n} S(u_k, v_j)$$

is independent of the representations of $x = \sum_{k\in\mathbb{Z}} u_k$ and $y = \sum_{k\in\mathbb{Z}} v_k$. Therefore, we conclude that $\widehat{T}$ is a bilinear operator from $X \times Y$ into Z. Since $\widehat{T}$ is an extension of T, the proof is complete. □

From the point of view of applications the following corollary is of independent interest.

Corollary 3.1. *Assume $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ are couples of Banach spaces of type 2 and $\overline{Z} = (Z_0, Z_1)$ is a couple of natural quasi-Banach spaces. Let (E_0, E_1), (F_0, F_1) and (G_0, G_1) be parameters of the method of means on $\overline{X}$, $\overline{Y}$, and $\overline{Z}$, respectively. If the convolution operator τ is bounded from $E_j \times F_j$ to G_j, for $j = 0, 1$, then any $T = (T_0, T_1) \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ extends to a bilinear operator $\widehat{T}$ from $\overline{X}_{E_0,E_1} \times \overline{Y}_{F_0,F_1}$ into $\overline{Z}_{G_0,G_1}$ which satisfies the norm estimate*

$$\|\widehat{T}\| \le \max_{j=0,1} \big[C_j \, \|\tau\|_{E_j\times F_j\to G_j} \, \|T_j\|_{X_j\times Y_j\to Z_j} \big]$$

for some $C_j = C(X_j, Y_j, Z_j) > 0$ $(j = 0, 1)$.

Proof. Using Kalton's [13] result, we conclude that both triples (X_0, Y_0, Z_0) and (X_1, Y_1, Z_1) are bilinear admissible, thus Theorem 3.1 applies. □

We conclude this section by giving applications to methods of means generated by weighted quasi-Banach sequence spaces determined by quasi-concave functions. Recall that a positive function ρ on $(0, \infty)$ is said to be *quasi-concave* if ρ is non-decreasing and the function $t \mapsto \rho(t)/t$ is non-increasing. A quasi-concave function ρ is called a *quasi-power* if $s_\rho(t) = o(\max\{1, t\})$ as $t \to 0$ and $t \to \infty$, where $s_\rho(t) := \sup_{u>0} \big(\rho(tu)/\rho(u)\big)$ for $t > 0$.

Lemma 3.2. *Let ρ be a quasi-power function and let Φ_0, Φ_1 be quasi-Banach sequence spaces on $\mathbb{Z}$ such that $\Phi_j \hookrightarrow \ell_\infty$ $(j = 0, 1)$. Then the following statements are true for $E_0 = \Phi_0(1/\rho(q^n))$ and $E_1 = \Phi_1(q^n/\rho(q^n))$ and any $q > 1$:*

(i) *$E_0 \cap E_1 \hookrightarrow \ell_r$ for any $r > 0$.*

(ii) *$\overline{X}_{E_0,E_1}$ is a quasi-Banach space for any quasi-Banach couple $\overline{X}$.*

(iii) *If $\rho(t) = t^\theta$, $0 < \theta < 1$ and $\Phi_j = \ell_{p_j}$, $0 < p_j \le \infty$, then for any $q > 1$ and any quasi-Banach space $\overline{X}$*

$$\overline{X}_{E_0,E_1} = \overline{X}_{\theta,p},$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

Proof. (i). Since ρ is a quasi-power function, it follows (see, e.g., [17], p. 80–81)

$$\Big(\int_0^\infty \big(\min\big\{1, \frac{s}{t}\big\}\rho(s)\big)^r \frac{ds}{s}\Big)^{1/r} \asymp \rho(t)$$

for any $r > 0$. In particular, this implies that for any $q > 1$

$$C(r) := \Big(\sum_{n\in\mathbb{Z}} \big(\min\{1, q^{-n}\}\rho(q^n)\big)^r\Big)^{1/r} < \infty.$$

Thus if $\Phi_j \hookrightarrow \ell_\infty$ for $j = 0, 1$, it follows that

$$E_0 \cap E_1 \hookrightarrow \ell_\infty(1/\rho(q^n)) \cap \ell_\infty(q^n/\rho(q^n)) = \ell_\infty(\max\{1/\rho(q^n), q^n/\rho(q^n)\}).$$

Consequently there exists a constant $K > 0$ such that for any $\|\{\xi_n\}\|_{E_0\cap E_1} \le 1$, we have

$$|\xi_n| \le K\, \min\{\rho(q^n), \rho(q^n)/q^n\}$$

for all $n \in \mathbb{Z}$. Thus $\|\{\xi_n\}\|_{\ell_r} \le C(r)K$, i.e., $E_0 \cap E_1 \hookrightarrow \ell_r$.

Clearly that (ii) follows by (i) and the remarks in Section 2.

(iii). It is shown in [27] that if $1/p = (1-\theta)/p_0 + \theta/p_1$, then the formula

$$\overline{X}_{E_0,E_1} = \overline{X}_{\theta,p},$$

holds with $q = e$. It is easy to see that the proof works for any $q > 1$. □

Theorem 3.2. *Assume that* $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ *and* $\overline{Z} = (Z_0, Z_1)$ *are quasi-Banach spaces and* $T = (T_0, T_1) \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ *is* (s_0, s_1)*-bilinear convex. If* $p_j, q_j, r_j \in [1, \infty)$ *and* $1/r_j = 1/p_j + 1/q_j - 1$ *for* $j = 0, 1$ *and* $1/p = (1-\theta)/(s_0 p_0) + \theta/(s_1 p_1)$, $1/q = (1-\theta)/(s_0 q_0) + \theta/(s_1 q_1)$, $1/r = (1-\theta)/(s_0 r_0) + \theta/(s_1 r_1)$ *and* $0 < \theta < 1$, *then* T *extends to a bilinear operator* $\widehat{T}$ *from* $\overline{X}_{\theta,p} \times \overline{Y}_{\theta,q}$ *into* $\overline{Z}_{\theta,r}$ *which satisfies the norm estimate*

$$\|\widehat{T}\| \leq \max_{j=0,1} C_j \, \|T_j\|_{X_j \times Y_j \to Z_j},$$

for some constant $C_j > 0$ $(j = 0, 1)$.

Proof. The hypothesis on the indices imply by Young's theorem that the convolution operator is bounded from $\ell_{p_j} \times \ell_{q_j}$ into ℓ_{r_j}, and thus also from $\ell_{p_j}(a^{j\theta}) \times \ell_{q_j}(a^{j\theta})$ into $\ell_{r_j}(a^{j\theta})$ for any $a > 0$, $j = 0, 1$. Applying Theorem 3.1 and Lemma 3.2, the required conclusion follows. $\square$

We note that the triple (L_p, L_q, Z) is s-bilinear admissible for any natural quasi-Banach space Z, when $1/s = 1/u + 1/v$, with $(u, v) = (p, q)$ whenever $0 < p, q \leq 2$, $(u, v) = (2, 2)$ whenever $2 \leq p, q < \infty$, and $(u, v) = (p, 2)$ whenever $0 < p \leq 2 \leq q < \infty$. Thus the obtained results may be applied to many bilinear operators such as bilinear multipliers. Recall that a bounded measurable function σ on $\mathbb{R}^n \times \mathbb{R}^n$ gives rise to a bilinear operator W_σ defined by

$$W_\sigma(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi, \eta) \widehat{f}(\xi) \, \widehat{g}(\eta) e^{2\pi i \langle x, \xi + \eta \rangle} \, d\xi d\eta$$

where f, g are Schwartz functions and $\langle , \rangle$ denotes the inner product in $\mathbb{R}^n$. In this case σ is called the symbol of W_σ.

The study of such bilinear multiplier operators was initiated by Coifman and Meyer. A theorem of them [6] says that if $1 < p, q < \infty$, $1/r = 1/p + 1/q$ and the function σ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfies

$$|\partial_\xi^\alpha \, \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha| - |\beta|},$$

for sufficiently large multi-indices α and β, then W_σ extends to a bilinear operator from $L_p(\mathbb{R}^n) \times L_q(\mathbb{R}^n)$ into $L_{r,\infty}(\mathbb{R}^n)$ whenever $r \geq 1$. Here as usual $L_{r,\infty}$ denotes the space weak L_r. This result was later extended to the range $1 > r \geq 1/2$ by Grafakos and Torres [8] and Kenig and Stein [16]. Multipliers that satisfy the Marcinkiewicz condition were studied by Grafakos and Kalton [9]. The first significant boundedness results concerning non-smooth symbols were proved by Lacey and Thiele [18], [19] who established that W_σ, with $\sigma(\xi, \eta) = \text{sign}(\xi + \alpha\eta)$, $\alpha \in \mathbb{R} \setminus \{0, 1\}$ has a bounded extension from $L_p(\mathbb{R}^n) \times L_q(\mathbb{R}^n)$ to $L_r(\mathbb{R}^n))$ when $r > 2/3$. Extensions of this result were subsequently obtained by Gilbert and Nahmod [7]. Bilinear operators can also be defined on quasi-Banach spaces, such as the Hardy spaces H_p; see for instance [10] for the action of bilinear Calderón-Zygmund operators on real Hardy spaces.

We discuss here only a general application.

Theorem 3.3. *Let* $p_j, q_j, r_j \in [1, \infty)$ *and* $1/r_j = 1/p_j + 1/q_j - 1$ *for* $j = 0, 1$ *and let* $1/p = (1-\theta)/p_0 + \theta/(2p_1)$, $1/q = (1-\theta)/q_0 + \theta/(2q_1)$, $1/r = (1-\theta)/r_0 + \theta/(2r_1)$ *and* $0 < \theta < 1$. *If* $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ *are Banach spaces such that both* X_0 *and* Y_0 *are of type* 2 *and* $\overline{Z} = (Z_0, Z_1)$ *is a couple of natural quasi-Banach spaces, then any* $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ *extends to a bilinear operator* $\widehat{T}$ *from* $\overline{X}_{\theta,p} \times \overline{Y}_{\theta,q}$ *into* $\overline{Z}_{\theta,r}$ *which satisfies the norm estimate*

$$\|\widehat{T}\| \leq C \max_{j=0,1} \|T_j\|_{X_j \times Y_j \to Z_j}$$

for some constant $C > 0$.

Proof. Using the aforementioned result of Kalton's, we conclude that the triple (X_0, Y_0, Z_0) is admissible and (X_1, Y_1, Z_1) is 1/2-admissible, thus Theorem 3.1 applies. □

Using the well-known results on interpolation by the real method method between L_p spaces (see, e.g., [2], Theorem 5.3.1) and the facts that any L_p-space with $2 \leq p < \infty$ is of type 2 and any $L_{q,\infty}$ with $0 < q < \infty$ is L-convex, we obtain the following corollary for Lorentz spaces:

Corollary 3.2. *Let $p_j, q_j, r_j \in [1, \infty)$ and $1/r_j = 1/p_j + 1/q_j - 1$ for $j = 0, 1$ and also let $1/p = (1-\theta)/p_0 + \theta/2p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$ and $0 < \theta < 1$. If $2 \leq u_j, v_j < \infty$ $(j = 0, 1)$ and $0 < t_0, t_1 \leq \infty$, then any bilinear operator $T : (L_{u_0}, L_{u_1}) \times (L_{v_0}, L_{v_1}) \to (L_{t_0,\infty}, L_{t_1,\infty})$ has a bounded extension from $L_{u,p} \times L_{v,q}$ into $L_{t,r}$, where $1/u = (1-\theta)/u_0 + \theta/u_1$, $1/v = (1-\theta)/v_0 + \theta/v_1$ and $1/t = (1-\theta)/t_0 + \theta/t_1$, when $t_0 \neq t_1$, $u_0 \neq u_1$, and $v_0 \neq v_1$.*

Corollary 3.2 yields, in particular, bounds for the bilinear Hilbert transforms and bilinear Calderón-Zygmund operators from products of Lorentz spaces into another Lorentz space.

4 Applications to Calderón-Lozanovsky spaces

In this section we prove a bilinear interpolation theorem for Calderón-Lozanovsky spaces. We show that under certain geometric conditions continuous inclusions hold between the method of means spaces and the Calderón-Lozanovsky spaces. In the case of quasi-Banach couples of weighted L_p-spaces, we obtain equalities of these spaces. Certain results in this direction for Banach spaces were shown in [24]. Following these ideas we extend some of these results for quasi-Banach lattices (see, Theorem 4.1).

Throughout this paper we denote by $\mathcal{P}$ (resp., $\mathcal{U}$) the set of all functions $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that are positive (resp., concave), non-decreasing in each variable, and homogeneous of degree one (that is, $\varphi(\lambda s, \lambda t) = \lambda \varphi(s, t)$ for all $\lambda, s, t \geq 0$).

Let $\varphi \in \mathcal{U}$ and $\overline{X} = (X_0, X_1)$ be a couple of quasi-Banach spaces on a measure space (Ω, μ). Following Calderón [4] and Lozanovsky [21], we define the space $\varphi(\overline{X}) = \varphi(X_0, X_1)$ of all $x \in L_0(\mu)$ such that $|x| = \varphi(|x_0|, |x_1|)$ for some $x_j \in X_j$, $j = 0, 1$. We note that $\varphi(\overline{X})$ is a quasi-Banach (resp., Banach whenever $\overline{X}$ is a Banach couple) lattice equipped with the quasi-norm (resp., norm)

$$\|x\| = \inf \big\{ \max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\};\ |x| = \varphi(|x_0|, |x_1|)\ x_j \in X_j,\ j = 0, 1 \big\}.$$

In particular, if we take $\varphi(s, t) = s^{1-\theta} t^{\theta}$, $0 < \theta < 1$, we obtain in this way the spaces $X_0^{1-\theta} X_1^{\theta}$ introduced by Calderón [4]. The properties of the Banach lattice $\varphi(\overline{X})$ have been studied in Lozanovsky (see [21] and references given therein).

Following Kalton [13], a quasi-Banach lattice X is said to be *p-convex*, $0 < p < \infty$, respectively *q-concave*, $0 < q < \infty$, if there exists a constant $C > 0$ such that

$$\big\|\big(\sum_{k=1}^{n} |x_k|^p\big)^{1/p}\big\| \leq C \big(\sum_{k=1}^{n} \|x_k\|^p\big)^{1/p}$$

respectively,

$$\big(\sum_{k=1}^{n} \|x_k\|^q\big)^{1/q} \leq C \big\|\big(\sum_{k=1}^{n} |x_k|^q\big)^{1/q}\big\|$$

for every choice of elements $x_1, ..., x_n \in X$.

A quasi-Banach lattice X is said to be satisfy *an upper p-estimate*, $0<p<\infty$, respectively *a lower q-estimate*, $0<q<\infty$, if there exists a constant $C>0$ such that for any choice of elements

$$\big\| \sup_{1\le k\le n} |x_k| \,\big\| \le C\,\big(\sum_{k=1}^n \|x_k\|^p\big)^{1/p},$$

respectively

$$\big(\sum_{k=1}^n \|x_k\|^q\big)^{1/q} \le C\,\big\| \sum_{k=1}^n |x_k| \,\big\|.$$

It is clear that if X is maximal, then the notion of an upper p-estimation is equivalent to the condition that

$$\big\| \sup_{k\ge 1} |x_k| \,\big\| \le C\,\big(\sum_{k=1}^\infty \|x_k\|^p\big)^{1/p}$$

holds for all disjointly supported infinite sequences.

We note that p-convexity implies p-normability and this in turn yields an upper p-estimate. For $p=1$, 1-convexity is equivalent to normability (as a Banach lattice).

In the sequel, a function $\varphi\in\mathcal{P}$ is said to be a quasi-power provided that the function $t\mapsto\varphi(1,t)$ is a quasi-power. It follows by Lemma 3.2 that if φ is a quasi-power, then for any $0<p_0,p_1\le\infty$ the couple $(E_0,E_1)=(\ell_{p_0}(1/\varphi(1,2^n)),\ell_{p_1}(2^n/\varphi(1,2^n)))$ is a parameter of the method of means on any couple (X_0,X_1) of quasi-Banach spaces. The space $(X_0,X_1)_{E_0,E_1}$ is denoted by $\varphi(X_0,X_1)_{p_0,p_1}$.

If X is an intermediate quasi-Banach space with respect to a quasi-Banach couple $\overline{X}=(X_0,X_1)$, we define its Gagliardo completion X^c to be the space of all limits in X_0+X_1 of sequences $\{x_n\}$ that are bounded in X, equipped with the quasi-norm

$$\|x\|_{X^c}=\inf_{\{x_n\}}\sup_{n\ge 1}\|x_n\|_X,$$

where $\{x_n\}\subset X$ has the same meaning as above. It is easy to check that if X is a maximal quasi-Banach lattice on a measure space, then its Gagliardo completion X^c equals to X.

We have the following result:

Theorem 4.1. *Assume that (X_0,X_1) is a couple of quasi-Banach lattices on a measure space (Ω,μ). Then the following continuous inclusions hold for any quasi-power function $\varphi\in\mathcal{U}$:*

(i) *If X_j satisfy an upper p_j-estimate $(j=0,1)$, then*

$$\varphi(X_0,X_1)_{p_0,p_1}\hookrightarrow\varphi(X_0,X_1)^c.$$

(ii) *If X_j is maximal and satisfy an upper p_j-estimate $(j=0,1)$, then*

$$\varphi(X_0,X_1)_{p_0,p_1}\hookrightarrow\varphi(X_0,X_1).$$

(iii) *If X_j satisfy a lower q_j-estimate $(j=0,1)$, then*

$$\varphi(X_0,X_1)\hookrightarrow\varphi(X_0,X_1)_{q_0,q_1}.$$

Proof. (i). Let $x \in \varphi(X_0, X_1)_{p_0,p_1}$ with $\|x\| < 1$. Then

$$x = \lim_{m,n\to\infty} \sum_{k=-n}^{m} u_k \quad \text{(convergence in } X_0 + X_1)$$

with $\|\{u_n/\varphi(1, 2^n)\}\|_{\ell_{p_0}(X_0)} \le 1$ and $\|\{2^n u_n/\varphi(1, 2^n)\}\|_{\ell_{p_1}(X_1)} \le 1$. Since φ is a quasi-power, the series $\sum_{n\in\mathbb{Z}} u_n$ is r-absolutely convergent in $X_0 + X_1$ for some $0 < r \le 1$. Thus, in particular, $\{|u_n(\omega)|\} \in \ell_1$ for almost all $\omega \in \Omega$. Since X_j satisfy an upper p_j-estimate ($j = 0, 1$), there are positive constants C_0 and C_1 such that for

$$x_n^0 := \sup_{|k|\le n} \frac{|u_k|}{\varphi(1, 2^k)} \in X_0$$

and

$$x_n^1 := \sup_{|k|\le n} \frac{2^k |u_k|}{\varphi(1, 2^k)} \in X_1$$

and we have $\|x_n^0\|_{X_0} \le C_0$ and $\|x_n^1\|_{X_1} \le C_1$ for all $n \in \mathbb{N}$.

We apply Carlson's inequality (see [12], Corollary 3.1) which states that for any quasi-power function $\varphi \in \mathcal{U}$ there exists a constant $C > 0$ such that for any finite positive sequence $\{a_n\}$ the following inequality holds with $\rho = \varphi(1, \cdot)$:

$$\sum_k a_k \le C\, \varphi\Big(\sup_k \frac{a_k}{\varphi(1, 2^k)}, \sup_k \frac{2^k\, a_k}{\varphi(1, 2^k)}\Big).$$

Combining this inequality with the above estimates yields

$$\Big|\sum_{k=-n}^{n} u_k\Big| \le C\varphi(x_n^0, x_n^1),$$

i.e., $\sum_{k=-n}^{n} u_k \in \varphi(X_0, X_1)$ with $\big\|\sum_{k=-n}^{n} u_k\big\| \le C \max_{j=0,1} C_j$ for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \Big\|\sum_{k=-n}^{n} u_k - x\Big\|_{X_0+X_1} = 0$$

the proof of (i) is complete.

(ii). The result follows by a minor modification of the proof of (i).

(iii). Let $0 \le x \in \varphi(\overline{X})$ and $\|x\|_{\varphi(\overline{X})} < 1$. Then $x = \varphi(x_0, x_1)$ for some $0 \le x_j \in X_j$ such that $\|x_j\|_{X_j} < 1$, $j = 0, 1$. Since φ is a quasi-power function, it follows that the support of x is contained in the intersection of the supports of x_0 and x_1. Hence, without loss of generality, we may suppose that x, x_0, x_1 are not equal to zero on the domain Ω.

Define for any $k \in \mathbb{Z}$,

$$A_k = \big\{\omega \in \Omega;\ 2^k \le x_1(\omega)/x_0(\omega) < 2^{k+1}\big\}$$

and

$$y_k = x\chi_{A_k},\ \ u_k = x_0\chi_{A_k},\ \ v_k = x_1\chi_{A_k}.$$

Clearly $y_k \in X_0 \cap X_1$. Is is easily seen that for any $k \in \mathbb{Z}$, we have

$$y_k \le 2\varphi(1, 2^k)\, u_k$$

and

$$y_k \le \frac{\varphi(1,2^k)}{2^k}\, v_k.$$

This implies that for any positive integer n the following estimates hold

$$0 \le \sum_{k\le -n} y_k \le \varphi(1,2^{-n}) \sum_{k\le -n} \frac{y_k}{\varphi(1,2^k)} \le 2\varphi(1,2^{-n})\, x_0,$$

$$0 \le \sum_{k\ge n} y_k \le \frac{\varphi(1,2^n)}{2^n} \sum_{k\ge n} \frac{2^k y_k}{\varphi(1,2^k)} \le \varphi(2^{-n},1)\, x_1.$$

Combining these estimates, we obtain

$$\begin{aligned}\Big\|x - \sum_{k=-M}^{N} y_k\Big\|_{X_0+X_1} &\le C\Big(\Big\|\sum_{k=-\infty}^{-M-1} y_k\Big\|_{X_0} + \Big\|\sum_{k=N+1}^{\infty} y_k\Big\|_{X_1}\Big) \\ &\le 2C\,\varphi(1,2^{-M-1}) + 2C\,\varphi(2^{-N-1},1)\end{aligned}$$

for any positive integers M and N. Since φ is quasi-power, the right hand of the above inequality approaches 0 whenever $M, N \to \infty$ This implies that the series $\sum_{k\in\mathbb{Z}} y_n$ converges to x in $X_0 + X_1$. Furthermore, by the fact that $\{A_n\}$ is a sequence of pairwise disjoint measurable subsets whose union is equal to Ω, we have

$$\sum_k \frac{|y_k|}{\varphi(1,2^k)} \le \sum_k 2u_k \le 2x_0$$

and

$$\sum_k \frac{2^k|y_k|}{\varphi(1,2^k)} \le \sum_k v_k \le x_1.$$

Now assume that X_j satisfy a lower q_j-estimate ($j = 0, 1$). Combining the above inequalities yields

$$\{x_k/\varphi(1,2^k)\} \in \ell_{q_0}(X_0) \ \text{ and } \ \{2^k x_k/\varphi(1,2^k)\} \in \ell_{q_1}(X_1).$$

Consequently $x \in \varphi(X_0, X_1)_{q_0,q_1}$. □

Since any L_p-space is maximal and satisfies both a lower p-estimate and an upper p-estimate for any $0 < p < \infty$, the following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.1. *If $\varphi \in \mathcal{U}$ is a quasi-power function, then for any $0 < p_0, p_1 < \infty$ and weights w_0 and w_1*

$$\varphi(L_{p_0}(w_0), L_{p_1}(w_1))_{p_0,p_1} = \varphi(L_{p_0}(w_0), L_{p_1}(w_1)).$$

It is well known (see, e.g., [26]) that for any $\varphi \in \mathcal{U}$ and any couple $(L_{p_0}(w_0), L_{p_1}(w_1))$ on (Ω, μ) with $0 < p_0 < p_1 \le \infty$, the Calderón-Lozanovsky space $\varphi(L_{p_0}(w_0), L_{p_1}(w_1))$ coincides up to equivalence of norms with the generalized Orlicz space of all $f \in L^0(\mu)$ such that

$$\int_\Omega M\big(w_1^{1/p_1} w_0^{-1/p_0})^q\, |f|/\lambda\big)\, (w_0/w_1)^q\, d\mu$$

for some $\lambda > 0$. Here $1/q = 1/p_0 - 1/p_1$ and M is an Orlicz function such that $M^{-1}(t) \asymp \varphi(t^{1/p_0}, t^{1/p_1})$ for $t > 0$.

We conclude this section by showing a particular application of Theorems 4.1 and 3.1 to bilinear operators on Orlicz spaces. For others results we refer to [23].

Theorem 4.2. *Let $\varphi_0, \varphi_1, \varphi \in \mathcal{U}$ be quasi-power functions such that $\varphi(1, st) \geq C\,\varphi_0(1, s)\,\varphi_1(1, t)$ for some $C > 0$ and all $s, t > 0$. If $1 \leq p_j, q_j < \infty$, $1 \leq r_j \leq \infty$ $(j = 0, 1)$ are such that $1/r_j = 1/p_j + 1/q_j - 1$, then any operator $T : (L_{p_0}(u_0), L_{p_1}(u_1)) \times (L_{q_0}(v_0), L_{q_1}(v_1)) \to (L_{r_0}(w_0), L_{r_1}(w_1))$ extends to a bounded bilinear operator from $\varphi_0(L_{p_0}(u_0), L_{p_1}(u_1)) \times \varphi_1(L_{q_0}(v_0), L_{q_1}(v_1))$ into $\varphi(L_{r_0}(w_1), L_{r_1}(w_1))$.*

Proof. It is easy to see that if the convolution operator τ is bounded from $E \times F$ into G, then it is bounded from $E(u) \times F(v)$ into $G(w)$ whenever there exists a constant $C > 0$ such that the sequences $u = \{u_k\}$, $v = \{v_k\}$ and $w = \{w_k\}$ satisfy the condition $w_n \leq C u_{n-k}\, v_k$ for some $C > 0$ and all $k, n \in \mathbb{Z}$. Our hypothesis implies that the convolution operator is bounded from $\ell_{p_j} \times \ell_{q_j}$ into ℓ_{r_j} $(j = 0, 1)$, and thus Theorem 3.1 and Corollary 4.1 apply. □

5 Bilinear interpolation between J and K-method spaces

In this section we show that bilinear interpolation is possible under weaker assumptions on quasi-Banach couples. We recall that if $\overline{X} = (X_0, X_1)$ is a couple of quasi-Banach spaces and E is a parameter of the K-method (i.e., E is a quasi-Banach sequence space on $\mathbb{Z}$ such that $\{\min(1, 2^n)\} \in E$), then the *K-method space* is a quasi-Banach space $K_E(\overline{X})$ (denoted also by $\overline{X}_E$) consists of all $x \in X_0 + X_1$ such that $\{K(2^n, x; \overline{X})\} \in E$. The space is equipped with the quasi-norm $\|x\| := \|\{K(2^n, x; \overline{X})\}\|_E$.

In what follows the method of means $J_{\overline{E}}(\overline{X})$ (on a quasi-Banach couple $\overline{X}$) generated by a couple $\overline{E} = (E, E(2^n))$ is called *J-method space* (on $\overline{X}$) and is denoted by $J_E(\overline{X})$ and E is called a parameter of the J-method on $\overline{X}$ (resp., a parameter of J-method if it is a parameter of J-method on any quasi-Banach couple $\overline{X}$). For the study of abstract J and K Banach method spaces we refer to [5] and [3].

In the spirit of the previous terminology we introduce the following: Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$, and (Z_0, Z_1) be couples of quasi-Banach spaces. We say that a bilinear operator $T = (T_0, T_1) : \overline{X} \times \overline{Y} \to \overline{Z}$ is *right* (resp., *left*) *s-convex* $(0 < s \leq 1)$ if there exists a constant $C > 0$ such that, for any $x \in X_0 \cap X_1$ and any finite sequence $\{y_j\} \subset Y_0 \cap Y_1$ (resp., any $y \in Y_0 \cap Y_1$ and any finite sequence $\{x_j\}_{j=1}^n \subset X_0 \cap X_1$) we have

$$\Big\|\sum_{j=1}^{n} T_0(x, y_j)\Big\|_{Z_0} \leq C\|T_0\|_{X_0 \times Y_0 \to Z_0}\, \|x\|_{X_0} \Big(\sum_{j=1}^{n} \|y_j\|_{Y_0}^s\Big)^{1/s}$$

(resp.,

$$\Big\|\sum_{j=1}^{n} T_1(x_j, y)\Big\|_{Z_1} \leq C\,\|T_1\|_{X_1 \times Y_1 \to Z_1}\, \|y\|_{Y_1} \Big(\sum_{j=1}^{n} \|x_j\|_{X_1}^s\Big)^{1/s}).$$

In the case when $s = 1$, we simply say that T is right (resp., left) convex.

Clearly if $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ is (s, s)-convex, then it is right and left s-convex. In particular if $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ are couples of Banach spaces of type 2 and $\overline{Z}$ is any couple of natural quasi-Banach spaces, then any operator $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ is convex.

If a quasi-Banach space X is intermediate with respect to a quasi-Banach couple (X_0, X_1), the closed convex hull of $X_0 \cap X_1$ in X is denoted by X°.

The following theorem is an extension of the classical Lions-Peetre result (see, e.g., Bergh and Löfstrom [2], Theorem 4.4.1 and Exercise 3.13.5, or Lions and Peetre [20], Zafran [28] and Astashkin [1]).

Theorem 5.1. *Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ and $\overline{Z} = (Z_0, Z_1)$ be quasi-Banach space. Assume that E^s is a parameter of the J-method space on $\overline{X}$ and both F^s, G^s with $0 < s < 1$ are parameters of the K-method such that the convolution operator τ is bounded from $E \times F$ into G. Then the following statements are valid*:

(i) *If $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ is left s-convex, then it extends to a bilinear operator from $J_{E^s}(\overline{X}) \times K_{F^s}(\overline{Y})^\circ$ into $K_{E^s}(\overline{Z})$.*

(ii) *If $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ is right s-convex, then it extends to a bilinear operator from $J_{E^s}(\overline{X}) \times K_{F^s}(\overline{Y})^\circ$ into $K_{G^s}(\overline{Z})$.*

Proof. (i). Fix $\varepsilon > 0$, $x \in J_{E^s}(\overline{X})$ and $y \in Y_0 \cap Y_1$. Then there exists $\{u_k\} \subset X_0 \cap X_1$ such that $\{J(2^k, u_k; \overline{X})\} \in E$ with

$$\|x\|_{J_E(\overline{X})} \leq (1 + \varepsilon) \, \|\{J(2^k, u_k; \overline{X})\}\|_E.$$

Let $T^0 : (X_0 + X_1) \times (Y_0 \cap Y_1) \to Z_0 + Z_1$ be a natural bilinear extension of T. Since T is s-convex, there exists a constant $C > 0$ such that for any $n \in \mathbb{Z}$, $y_0 \in Y_0$ and $y_1 \in Y_1$ with $y_0 + y_1 = y$, we have with $\|T\| := \max_{j=0,1} \|T_j\|_{X_j \times Y_j \to Z_j}$

$$\begin{aligned} K(2^n, T^0(x, y); \overline{Z})^s &\leq \|T^0(x, y_0)\|_{Z_0}^s + 2^{sn} \, \|T^0(x, y_1)\|_{Z_1}^s \\ &\leq C \, \|T\| \Big(\sum_{k \in \mathbb{Z}} \|u_k\|_{X_0}^s \, \|y_0\|_{Y_0}^s + 2^{sn} \, \|u_k\|_{X_1}^s \, \|y_1\|_{Y_1}^s \Big) \\ &\leq 2^{1-s} \, C \, \|T\| \Big(\sum_{k \in \mathbb{Z}} J(2^k, u_k; \overline{X})^s \, (\|y_0\|_{Y_0} + 2^{n-k} \, \|u_k\|_{Y_1})^s \Big). \end{aligned}$$

Taking the infimum over all decompositions $y = y_0 + y_1$, we obtain

$$K(2^n, T^0(x, y); \overline{Z})^s \leq 2^{1-s} \, C \, \|T\| \sum_{k \in \mathbb{Z}} J(2^k, u_k; \overline{X})^s \, K(2^{n-k}, y; \overline{Y})^s.$$

Combining these relations with the fact that the convolution operator τ is bounded from $E \times F$ into G and ε is arbitrary we obtain

$$\|T^0(x, y)\|_{K_{G^s}(\overline{Z})} \leq 2^{1-s} \, C \, \|T\| \, \|\tau\|_{E \times F \to G} \, \|x\|_{J_{E^s}(\overline{X})} \, \|y\|_{K_{F^s}(\overline{X})}$$

This concludes the proof of (i). Using a natural bilinear extension $T^1 : (X_0 \cap X_1) \times (Y_0 + Y_1) \to Z_0 + Z_1$, we prove (ii) in a similar way. □

From the point of view of applications in the above theorem the case $E = F = G$ seems interesting. Let us remark that from the proof of Lemma 3.2, it follows that for any quasi-power function ρ and $0 < p \leq \infty$ the weighted quasi-Banach sequence space $E = \ell_p(1/\rho(2^n))$ is a parameter of both the J and K-methods of interpolation. Moreover, we have $J_E(\overline{X}) = K_E(\overline{X})$ for any quasi-Banach couple $\overline{X} = (X_0, X_1)$ (see [11], [22]), and therefore we write $\overline{X}_{\rho,p}$ instead of $J_E(\overline{X})$ or $K_E(\overline{X})$. It is easy to check that $X_0 \cap X_1$ is dense in $\overline{X}_{\rho,p}$ whenever $0 < p < \infty$. Combining these remarks with Theorem 5.1, we obtain immediately the following:

Corollary 5.1. *If $1 \leq p, q, r \leq \infty$ satisfy $1/r = 1/p + 1/q - 1$, then any left or right convex operator $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ extends to a bilinear operator from $\overline{X}_{\theta,p} \times \overline{Y}_{\theta,q}$ into $\overline{Z}_{\theta,r}$ for any $0 < \theta < 1$.*

Corollary 5.2. *If $0 < p \leq \infty$ and ρ is a quasi-power function such that*

$$C(\rho) := \sup_{n \in \mathbb{Z}} \frac{1}{\rho(2^n)} \left\| \left\{ \rho(2^k)\, \rho(2^{n-k}) \right\}_k \right\|_{(\ell_p)'} < \infty,$$

then any left or right convex operator $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$ extends to a bilinear operator from $\overline{X}_{\rho,p} \times \overline{Y}_{\rho,p}$ into $\overline{Z}_{\rho,p}$.

Proof. The proof closely follows the proof of the result of [1] for the Banach case. Let $E := \ell_p(1/\rho(2^n))$, $0 < p \leq \infty$. Fix $x = \{\xi_n\} \in E$ and $y = \{\eta_n\} \in E$. Then we have

$$\begin{aligned} \left|\tau(x,y)_n\right| &\leq \sum_{k \in \mathbb{Z}} \left| \frac{\xi_k}{\rho(2^k)} \frac{\eta_{n-k}}{\rho(2^{n-k})} \right| \rho(2^k)\, \rho(2^{n-k})) \\ &\leq \left\| \left\{ \frac{\xi_k}{\rho(2^k)} \frac{\eta_{n-k}}{\rho(2^{n-k}} \right\}_k \right\|_{\ell_p} \left\| \left\{ \rho(2^k)\, \rho(2^{n-k}) \right\}_k \right\|_{(\ell_p)'}. \end{aligned}$$

This implies that

$$\|\tau(x,y)\|_E \leq C(\rho)\, \|x\|_E\, \|y\|_E.$$

Therefore the convolution operator τ is bounded from $E \times E$ into E, and Theorem 5.1 applies. □

We conclude the paper by discussing some applications to Lorentz-Zygmund spaces. Let (Ω, μ) be a measure space. Let $0 < p \leq \infty$, $0 < q \leq \infty$, and $\gamma \in \mathbb{R}$. Recall that the Lorentz-Zygmund space $L_{p,q}(\log L)^\gamma$ is defined as the space of all functions that satisfy

$$\|f\|_{p,q,\gamma} := \Big(\int_0^{\mu(\Omega)} \big(t^{1/p}(1 + |\log t|)^\gamma f^*(t)\big)^q \frac{dt}{t} \Big)^{1/q} < \infty$$

for $0 < q < \infty$ and

$$\|f\|_{p,\infty,\gamma} := \sup_{0 < t < \mu(\Omega)} \big(t^{1/p}(1 + |\log t|)^\gamma\, f^*(t)\big) < \infty$$

whenever $q = \infty$. This space coincides with the classical Lorentz space $L_{p,q}$ if $\gamma = 0$.

In the next and final result all considered couples are defined on any finite measure space.

Theorem 5.2. *Assume that $2 \leq p_0 < p_1 < \infty$, $2 \leq q_0 < q_1 < \infty$, $0 < r_0 < r_1 \leq \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$ with $0 < \theta < 1$. If $T : (L_{p_0}, L_{p_1}) \times (L_{q_0}, L_{q_1}) \to (L_{r_0}, L_{r_1})$, then T has a bounded extension from $L_{p,s}(\log L)^\gamma \times L_{q,s}(\log L)^\gamma$ to $L_{r,s}(\log L)^\gamma$ for any $\gamma < -1$ and $0 < s \leq \infty$.*

Proof. It is easy to check that if $f(t) = t^\alpha(1 + |\log t|)^\gamma$, where $0 < \alpha < \infty$, $\gamma \in \mathbb{R}$, then $s_f(t) = t^\alpha(1 + |\log t|)^{|\gamma|}$. Thus f is a quasi-power function whenever $0 < \alpha < 1$, $\gamma \in \mathbb{R}$. Further it is well known (see, e.g., [25]) that if $0 < v, u_0, u_1 \leq \infty$, $u_0 \neq u_1$ and $f(t) = t^\theta(1 + |\log t|)^{-\gamma}$ ($0 < \theta < 1$, $\gamma \in \mathbb{R}$), then

$$(L_{u_0}, L_{u_1})_{f,v} = L_{u,v}(\log L)^\gamma$$

where $1/u = (1-\theta)/u_0 + \theta/u_1$.

Now, following [1] we define a function ψ by $\psi(t) = t^a \ln^c(C_1/t)$ for $0 < t \le 1$ and $\psi(t) = t^b \ln^d(C_2\, t)$ for $t > 1$, where $0 < a < b < 1$, $c > 1$, $d > 1$ and $C_1 > e^{c/a}$, $C_2 > e^d d/(1-b)$. Then ψ is a quasi-power function and for $0 < p \le \infty$ the function ρ defined by $\rho(t) = t/\psi(t)$ satisfies

$$C(\rho) = \sup_{n\in\mathbb{Z}} \frac{1}{\rho(2^n)} \left\| \left\{ \rho(2^k)\, \rho(2^{n-k}) \right\}_k \right\|_{(\ell_p)'} < \infty.$$

Observe that if $\overline{A} = (A_0, A_1)$ is a quasi-Banach space such that $A_1 \hookrightarrow A_0$, the K functional is constant for $t > 1$. This easily implies that for any quasi-power function ρ and $0 < p \le \infty$ the real method space $(A_0, A_1)_{\rho,p}$ consists of all $a \in A_0$ equipped with the quasi-norm

$$\|a\| = \Big(\int_0^1 \Big(\frac{K(t, a; \overline{A})}{\rho(t)} \Big)^p \frac{dt}{t} \Big)^{1/p}.$$

Now, fix $0 < \theta < 1$ and $\gamma < -1$. Taking $a = 1 - \theta$ and $c = -\gamma$, we conclude that the real method space $(A_0, A_1)_{\rho,p}$ generated by a quasi-power function $\rho = t/\psi(t)$ defined above depends only on ρ restricted to $(0,1)$. Clearly on the interval $(0,1)$ the function ρ is equivalent to $f(t) = t^\theta (1 + |\log t|)^\gamma$, thus using the interpolation formula of Merucci [25] and the fact that our hypothesis $2 \le p_0 < \infty$, $2 \le q_0 < \infty$ implies by Kalton's result [13] (by L_{p_j} and L_{q_j}, $j = 0, 1$ are of type 2) that T is bilinear convex, we may apply Corollary 5.2 to conclude the proof of the theorem. □

References

[1] S.V. Astashkin *On interpolation of bilinear operators by the real method of interpolation*, Mat. Zametki **52** (1992), 15–24 (Russian).

[2] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin 1976.

[3] Yu.A. Brudnyi and N.Ya. Krugljak, *Interpolation Functors and Interpolation Spaces I*, North-Holland, Amsterdam, 1991.

[4] A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.

[5] M. Cwikel and J. Peetre, *Abstract K and J spaces,* J. Math. Pures Appl. **60** (1981), 1–50.

[6] R.R. Coifman and Y. Meyer, *Commutateurs d'intègrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[7] J. Gilbert and A. Nahmod, *Bilinear operators with non-smooth symbols*, J. Fourier Anal. Appl. **7** (2001), 437–469.

[8] L. Grafakos and R. Torres, *Multilinear Calderón Zygmund theory*, Adv. in Math. **40** (1996), 344–351.

[9] L. Grafakos and N.J. Kalton, *The Marcinkiewicz multiplier condition for bilinear operators*, Studia Math. **146** (2001), 115–156.

[10] L. Grafakos and N.J. Kalton, *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math. **52** (2001), 169–179.

[11] J. Gustavsson, *A function parameter in connetion with interpolation of Banach spaces*, Math. Scand. **42** (1978), 289–305.

[12] J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces*, Studia Math. **60** (1977), 33–59.

[13] N.J. Kalton, *Convexity conditions for non-locally convex lattices*, Glasgow Math. J. **25** (1984), 141–152.

[14] N.J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Math. Soc. Lecture Notes **89**, Cambridge University Press, 1985.

[15] N.J. Kalton, *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math. **84** (1986), 297–324.

[16] C. Kenig and E.M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), 1–15.

[17] S.G. Krein, Yu.I. Petunin and E.M. Semenov, *Interpolation of Linear Operators,* Nauka, Moscow, 1978 (in Russian); English transl.: Amer. Math. Soc., Providence, 1982.

[18] M.T. Lacey and C.M. Thiele, *L^p bounds for the bilinear Hilbert transform, $p > 2$*, Ann. of Math. **146** (1997) 693–724.

[19] M.T. Lacey and C. M. Thiele, *On Calderón's conjecture*, Ann. of Math. **149** (1999) 475–496.

[20] J.L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math. **19** (1964), 5–68.

[21] G.Ya. Lozanovsky, *On some Banach lattices IV*, Sibirsk. Mat. Z. **14** (1973), 140–155 (in Russian); English transl.: Siberian. Math. J. **14** (1973), 97–108.

[22] M. Mastyło, *On interpolation of some quasi-Banach spaces*, J. Math. Anal. Appl. **147** (1990), 403–419.

[23] M. Mastyło, *On interpolation of bilinear operators*, J. Funct. Anal., **214** (2004), 260–283.

[24] M. Mastyło, *Interpolation methods of means and orbits*, Studia Math., to appear.

[25] C. Merucci, *Application of interpolation with function parameter to Lorentz, Sobolev and Besov spaces*, In: Interpolation Spaces and Allied Topics in Analysis, Lecture Notes in Math. **1070** (1984), 183–201.

[26] P. Nilsson, *Interpolation of Banach lattices*, Studia Math. **82** (1985), 135–154.

[27] Y. Sagher, *Interpolation of r-Banach spaces*, Studia Math. **26** (1966), 45–70.

[28] M. Zafran, *A multilinear interpolation theorem*, Studia Math. **62** (1978), 107–124.

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