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Boundedness of paraproduct operators on RD-spaces

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Abstract Let $(\mathcal{X}, d, \mu)$ be an RD-space with "dimension" n, namely, a space of homogeneous type in the sense of Coifman and Weiss satisfying a certain reverse doubling condition. Using the Calderón reproducing formula, the authors hereby establish boundedness for paraproduct operators from the product of Hardy spaces $H^p(\mathcal{X}) \times H^q(\mathcal{X})$ to the Hardy space $H^r(\mathcal{X})$, where p, q, $r \in (n/(n+1), \infty)$ satisfy $1/p + 1/q = 1/r$. Certain endpoint estimates are also obtained. In view of the lack of the Fourier transform in this setting, the proofs are based on the derivation of appropriate kernel estimates.

Keywords paraproduct operator, space of homogeneous type, Hardy space

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1 Introduction

The theory of Calderón-Zygmund singular integrals on Euclidean spaces and the real variable methods subsequently developed are of fundamental importance in harmonic analysis and of wide use in partial differential equations, several complex variables, operator theory, potential theory, and other areas.

Paraproduct operators on Euclidean space $\mathbb{R}^n$ (cf. [23, pp. 302–305]) play a crucial role in establishing the original proof of the $T1$-theorem of David and Journé [8]. For locally integrable functions f and g on $\mathbb{R}^n$, the paraproduct operator Π_L is defined by

$$\Pi_L(f,g) := \sum_{j\in\mathbb{Z}} (\Psi_j * f)\,(\Phi_{j-3} * g), \tag{1.1}$$

where Φ is a radial Schwartz function satisfying $\operatorname{supp}\widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\}$ and $\widehat{\Phi}(\xi) = 1$ whenever $|\xi| \leqslant 1/2$, $\Phi_j(\cdot) := 2^{jn}\Phi(2^j\cdot)$ and $\Psi_j := \Phi_j - \Phi_{j-1}$ for all $j \in \mathbb{Z}$. We observe that an alternative but equivalent way of expressing (1.1) is

$$\Pi_L(f,g) := \sum_{j\in\mathbb{Z}} \widetilde{\Psi}_j * ((\Psi_j * f)(\Phi_{j-3} * g)), \tag{1.2}$$

where the Fourier transform of $\widetilde{\Psi}$ is supported in $\{\xi \in \mathbb{R}^n : \frac{1}{16} < |\xi| < 2\}$ and is equal to 1 on the annuals $\frac{1}{8} < |\xi| < \frac{9}{8}$. For all p, q, $r \in (0, \infty)$ satisfying $1/p + 1/q = 1/r$, Grafakos and Kalton [10] proved that

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Π_L is bounded from the product of Hardy spaces $H^p(\mathbb{R}^n) \times H^q(\mathbb{R}^n)$ to the Hardy space $H^r(\mathbb{R}^n)$ (here $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $p \in (1, \infty)$); they also proved several endpoint estimates, when some indices are equal to 1 or ∞. Many of the results in [10] are proved by a variant of the classical method of Coifman and Meyer (cf. [4, 5]), expressing the bilinear multiplier as a sum of products of linear multipliers via Fourier series. The study of paraproducts falls within the general theory of multilinear operators developed by Coifman and Meyer. Parproducts arise in numerous situations involving productlike operations and have been systematically studied in several articles; in addition to the aforementioned articles of Coifman and Meyer, see also [2, 10, 13, 14, 19] and the references therein.

The main aim of this paper is to extend the boundedness results in [10] concerning the paraproduct operators to the setting of the so-called RD-spaces. The lack of Fourier transform techniques in this setting forces the introduction of a new approach in the study of this problem. This approach is based on suitable kernel estimates, inspired by those obtained in [13] for products of three wavelets.

Recall that $\mathcal{X}$ is called an RD-space if it is a space of homogeneous type in the sense of Coifman and Weiss [6, 7] and it also satisfies some reverse doubling condition; see [16, 17] or Definition 2.1 below. Spaces of homogeneous type present a natural setting for the Calderón-Zygmund theory of singular integrals; see, for instance, [7]. It is well-known that structures of spaces of homogeneous type encompass several important examples in harmonic analysis, such as Euclidean spaces with A_∞-weights (of the Muckenhoupt class), Ahlfors n-regular metric measure spaces (namely, $\mu(B(x, r)) \sim r^n$ for all $x \in \mathcal{X}$ and $r > 0$), Lie groups of polynomial growth (see, for instance, [1, 25, 26]), and Carnot-Carathéodory spaces with doubling measures (see [20–22, 24]). All these examples fall under the scope of the study of RD-spaces introduced in [16, 17].

Function spaces on spaces of homogeneous type including RD-spaces have been widely studied. Assume that $\mathcal{X}$ is an RD-space with "dimension" n; see Remark 2.2(i) below. Han et al. [17] established a theory of Triebel-Lizorkin and Besov spaces on $\mathcal{X}$. In [16], Han et al. also established a Littlewood-Paley theory of Hardy spaces $H^p(\mathcal{X})$ for $p \in (n/(n+1), 1]$. These Hardy spaces are further proved to be coincided with some of Triebel-Lizorkin spaces in [17], and the characterizations of these Hardy spaces on RD-spaces via various maximal functions are also given in [11].

It should be mentioned that atomic Hardy spaces $H^p_{\rm at}(\mathcal{X})$ for all $p \in (0, 1]$ on spaces $\mathcal{X}$ of homogeneous type were first introduced by Coifman and Weiss [7]. Moreover, Coifman and Weiss [7] further established a molecular characterization for $H^1_{\rm at}(\mathcal{X})$. Under the assumption that $\mathcal{X}$ is an Ahlfors 1-regular metric measure space, Macías and Segovia [18] obtained the grand maximal function characterization for $H^p_{\rm at}(\mathcal{X})$ with $p \in (1/2, 1]$ via distributions acting on certain spaces of Lipschitz functions; Han [15] established a Lusin-area characterization for $H^p_{\rm at}(\mathcal{X})$ with $p \in (1/2, 1]$; Duong and Yan [9] characterized these atomic Hardy spaces in terms of Lusin area functions associated with certain Poisson semigroups.

Paraproduct operators on RD-spaces naturally appear in the extension of the classical $T1$-theorem of David and Journé for these spaces; see [17, Theorem 5.56]. Paraproducts on RD-spaces have the form

$$P(b, g) = \sum_{j\in\mathbb{Z}} \widetilde{D}_j\,(D_j(b)S_j(g)), \tag{1.3}$$

where $b \in \mathrm{BMO}\,(\mathcal{X})$, g is any "smooth" function, $\{S_j\}_{j\in\mathbb{Z}}$ is *an approximation of the identity* (see Definition 2.3 below), $D_j := S_j - S_{j-1}$ for all $j \in \mathbb{Z}$, and $\{\widetilde{D}_j\}_{j\in\mathbb{Z}}$ is the sequence of operators appearing in the Calderón reproducing formula (see Lemma 3.4 below) which possess similar properties as those of $\{D_j\}_{j\in\mathbb{Z}}$. For any $b \in \mathrm{BMO}\,(\mathcal{X})$, it is proved in [17, Theorem 5.56] that $P(b, \cdot)$ is a singular integral with a standard Calderón-Zygmund kernel (see [17, pp. 166–169]) and bounded on $L^2(\mathcal{X})$.

Motivated by the definitions given in (1.1)–(1.3), we define paraproduct operators on $\mathcal{X}$ as below.

Definition 1.1. *Let* $\epsilon_1 \in (0, 1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$, $\{S_j\}_{j\in\mathbb{Z}}$ *and* $\{A_j\}_{j\in\mathbb{Z}}$ *be two* $(\epsilon_1, \epsilon_2, \epsilon_3)$-*AOTI's. Set* $D_j := S_j - S_{j-1}$ *and* $E_j := A_j - A_{j-1}$ *for* $j \in \mathbb{Z}$. *Let* $\beta, \gamma \in (0, \epsilon)$. *The paraproduct operator* Π *is defined by setting, for all* $f \in (\mathring{\mathcal{G}}^\epsilon_0(\beta, \gamma))'$, $g \in (\mathcal{G}^\epsilon_0(\beta, \gamma))'$ *and* $x \in \mathcal{X}$,

$$\Pi(f, g)(x) := \sum_{j\in\mathbb{Z}} E_j(D_j(f)S_j(g))(x). \tag{1.4}$$

In Definition 1.1, $(\epsilon_1, \epsilon_2, \epsilon_3)$- AOTI is the abbreviation of *the approximation of the identity of order* $(\epsilon_1, \epsilon_2, \epsilon_3)$, where $(\epsilon_1, \epsilon_2, \epsilon_3)$ in some sense measures the "smoothness" of S_k; $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ are the distribution function spaces on $\mathcal{X}$. See Section 2 below for the appropriate definitions.

At this point, the series in (1.4) is formal and we will show its convergence in the proofs of our main results. Moreover, it is easy to see that if $f \in \mathrm{BMO}(\mathcal{X})$ and $g \in L^\infty(\mathcal{X})$, then the series in (1.4) converges in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$.

The main result of this paper concerns boundedness of Π on products of Lebesgue and Hardy spaces and is stated below. For convenience, when $p \in (1,\infty)$, we sometimes denote $L^p(\mathcal{X})$ by $H^p(\mathcal{X})$ since they coincide; see Remark 2.6(i) below.

Theorem 1.2. *Let ϵ be as in Definition* 1.1 *and the paraproduct operator Π be as in* (1.4)*. Assume that p, q, $r \in (n/(n+\epsilon),\infty)$ satisfy $1/p+1/q=1/r$. If $\max\{0, n(1/r-1)\} < \beta$, $\gamma < \epsilon$, then Π is bounded from $H^p(\mathcal{X}) \times H^q(\mathcal{X})$ to $H^r(\mathcal{X})$.*

We describe the main ideas used in the proof of Theorem 1.2 in Section 3 below. By applying the inhomogeneous discrete Calderón-Zygmund reproducing formula (see Lemma 3.6 below) we decompose each $D_j(f)S_j(g)$ into the sum of "smooth" functions and we reduce the estimate for $\|\Pi(f,g)\|_{H^r(\mathcal{X})}$ to that in (3.10) below. To prove (3.10), we respectively apply the homogeneous discrete Calderón-Zygmund reproducing formula (see Lemma 3.4 below) to break up f and the inhomogeneous discrete Calderón-Zygmund reproducing formula (see Lemma 3.6 below) to break up g. Then we use Lemma 3.8 and Proposition 3.2 below to bound these summations with some maximal operators (see (3.23) below). The desired result is obtained via the Fefferman-Stein vector-valued maximal function inequality; see (3.26) below. In addition, we point out that some basic estimates presented in Lemmas 2.7 and 2.8 and Remark 2.9 below are used throughout the proof.

Some endpoint estimates regarding boundedness of paraproduct operators as in (1.4) are also proved in Section 4, via the singular integral theory on RD-spaces.

Theorem 1.3. *Let ϵ be as in Definition* 1.1 *and Π be as in* (1.4)*. Assume that $q \in (n/(n+\epsilon),\infty)$. If $\max\{0, n(1/q-1)\} < \beta$, $\gamma < \epsilon$, then Π satisfies the following boundedness estimates*:

(a) $BMO(\mathcal{X}) \times H^q(\mathcal{X}) \to H^q(\mathcal{X})$, *where* $H^q(\mathcal{X}) = L^q(\mathcal{X})$ *for* $q \in (1,\infty)$;
(b) $BMO(\mathcal{X}) \times H^1(\mathcal{X}) \to L^1(\mathcal{X})$;
(c) $BMO(\mathcal{X}) \times L^\infty(\mathcal{X}) \to BMO(\mathcal{X})$;
(d) $BMO(\mathcal{X}) \times L^1(\mathcal{X}) \to L^{1,\infty}(\mathcal{X})$;
(e) $H^q(\mathcal{X}) \times L^\infty(\mathcal{X}) \to H^q(\mathcal{X})$, *where* $H^q(\mathcal{X}) = L^q(\mathcal{X})$ *for* $q \in (1,\infty)$;
(f) $L^1(\mathcal{X}) \times L^\infty(\mathcal{X}) \to L^{1,\infty}(\mathcal{X})$.

Remark 1.4. Indeed, Theorems 1.2 and 1.3 still hold if in (1.4) we relax the assumptions of E_j, D_j and S_j to the following: E_j satisfies (i) and (ii) of Definition 2.3 and $\int_{\mathcal{X}} E_j(w,x)\,d\mu(w) = 0 = \int_{\mathcal{X}} E_j(x,w)\,d\mu(w)$; D_j and S_j satisfy (i) and (iii) of Definition 2.3 and $\int_{\mathcal{X}} D_j(x,w)\,d\mu(w) = 0$.

To the best of our knowledge, Theorems 1.2 and 1.3 are also new even when $\mathcal{X}$ is an Ahlfors n-regular metric measure space.

The paper is organized as follows. In Section 2, we review notation, the notions of RD-spaces, approximations of the identity, spaces of test functions, the Hardy spaces, and a few basic estimates. Theorems 1.2 and 1.3 are proved in Sections 3 and 4, respectively.

We make some conventions on notation. Let $\mathbb{N} := \{1,2,\ldots\}$, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ and $\mathbb{R}_+ := [0,\infty)$. Denote by C a positive constant independent of the main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ to denote $f \leqslant Cg$. If $f \lesssim g \lesssim f$, we then write $f \sim g$. For any a, $b \in \mathbb{R}$, set $a \wedge b := \min\{a,b\}$.

2 Preliminaries

We begin with recalling the notions of the space of homogeneous type in the sense of Coifman and Weiss [6, 7] and the notion of RD-spaces [16, 17].

Definition 2.1. *Let $(\mathcal{X}, d)$ be a metric space with a regular Borel measure μ such that all balls defined by d have finite and positive measures. For any $x \in \mathcal{X}$ and $r > 0$, set $B(x,r) := \{y \in \mathcal{X} : d(x,y) < r\}$. The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if there exists a constant $C_1 \in [1,\infty)$ such that for all $x \in \mathcal{X}$ and $r > 0$,*

$$\mu(B(x,2r)) \leqslant C_1 \mu(B(x,r)) \quad (\textit{doubling condition}). \tag{2.1}$$

The triple $(\mathcal{X}, d, \mu)$ is called an RD-space if it is a space of homogeneous type and there exist constants $\kappa \in (0,\infty)$ and $C_2 \in (0,1]$ such that for all $x \in \mathcal{X}$, $0 < r < \operatorname{diam}(\mathcal{X})/2$ and $1 \leqslant \lambda < \operatorname{diam}(\mathcal{X})/(2r)$,

$$C_2 \lambda^\kappa \mu(B(x,r)) \leqslant \mu(B(x,\lambda r)), \tag{2.2}$$

where and in what follows, $\operatorname{diam}(\mathcal{X}) := \sup_{x,\, y \in \mathcal{X}} d(x,y)$.

Remark 2.1. (i) For a space $\mathcal{X}$ of homogeneous type, by (2.1), there exist $C_3 \in [1,\infty)$ and $n \in (0,\infty)$ such that for all $x \in \mathcal{X}$, $r > 0$ and $\lambda \geqslant 1$, $\mu(B(x,\lambda r)) \leqslant C_3 \lambda^n \mu(B(x,r))$. Indeed, we can choose $C_3 := C_1$ and $n := \log_2 C_1$. In some sense, n measures the "dimension" of $\mathcal{X}$. When $\mathcal{X}$ is an RD-space, we obviously have $n \in [\kappa, \infty)$.

(ii) It was proved in [17, Remark 1.2] that $\mathcal{X}$ is an RD-space if and only if $\mathcal{X}$ is a space of homogeneous type with the additional property that there exists a constant $a_0 > 1$ such that for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$, $B(x, a_0 r) \setminus B(x,r) = \emptyset$. Consequently, a connected space of homogeneous type is an RD-space. See also [27] for more equivalent characterizations of RD-spaces.

Throughout this paper, we always assume that $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X}) = \infty$, and set $V_\delta(x) := \mu(B(x,\delta))$ and $V(x,y) := \mu(B(x, d(x,y)))$ for all x, $y \in \mathcal{X}$ and $\delta > 0$. It follows from (2.1) that $V(x,y) \sim V(y,x)$. The following approximations of the identity on RD-spaces were introduced in [17], whose existence was proved in [17, Theorem 2.6].

Definition 2.3. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an approximation of the identity of order $(\epsilon_1, \epsilon_2, \epsilon_3)$ (in short, $(\epsilon_1, \epsilon_2, \epsilon_3)$-*AOTI*), if there exists a positive constant C such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x,y)$, the integral kernel of S_k is a measurable function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying*

(i)

$$|S_k(x,y)| \leqslant C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k} + d(x,y))^{\epsilon_2}};$$

(ii)

$$|S_k(x,y) - S_k(x',y)| \leqslant C \frac{[d(x,x')]^{\epsilon_1}}{(2^{-j} + d(x,y))^{\epsilon_1}} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k} + d(x,y))^{\epsilon_2}}$$

for all $d(x,x') \leqslant (2^{-k} + d(x,y))/2$;

(iii) *S_k satisfies* (ii) *with x and y interchanged*;

(iv)

$$\begin{aligned} |[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| &\leqslant C \frac{[d(x,x')]^{\epsilon_1}}{(2^{-k} + d(x,y))^{\epsilon_1}} \frac{[d(y,y')]^{\epsilon_1}}{(2^{-k} + d(x,y))^{\epsilon_1}} \\ &\quad \times \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x,y)} \frac{2^{-k\epsilon_3}}{(2^{-k} + d(x,y))^{\epsilon_3}} \end{aligned}$$

for $d(x,x') \leqslant (2^{-k} + d(x,y))/3$ and $d(y,y') \leqslant (2^{-k} + d(x,y))/3$;

(v) $\int_{\mathcal{X}} S_k(x,w)\, d\mu(w) = 1 = \int_{\mathcal{X}} S_k(w,y)\, d\mu(w)$.

With all the notation as in Definition 2.3, if $\{S_k\}_{k\in\mathbb{Z}}$ is an $(\epsilon_1, \epsilon_2, \epsilon_3)$- AOTI with bounded support, namely, there exists a positive constant C such that $S_k(x,y) = 0$ whenever $d(x,y) \geqslant C 2^{-k}$, then $\{S_k\}_{k\in\mathbb{Z}}$ is an $(\epsilon_1, \epsilon_2', \epsilon_3')$- AOTI for all $\epsilon_2' > 0$ and $\epsilon_3' > 0$. Such an $\{S_k\}_{k\in\mathbb{Z}}$ is called to be an approximation of the identity of order ϵ_1 with bounded support (for short, ϵ_1- AOTI with bounded support); see [17, Definition 2.3] and [17, Theorem 2.6] for the existence of 1- AOTI with bounded support.

The following notion of the space of test functions on $\mathcal{X}$ was given in [17]; see also [16].

Definition 2.4. *Let $x_1 \in \mathcal{X}$, $r \in (0,\infty)$, $\beta \in (0,1]$ and $\gamma \in (0,\infty)$. A function φ on $\mathcal{X}$ is called a test function of type (x_1, r, β, γ) if there exists a positive constant C such that*

(i) $|\varphi(x)| \leqslant C\frac{1}{V_r(x_1)+V(x_1,x)}(\frac{r}{r+d(x_1,x)})^\gamma$ *for all* $x \in \mathcal{X}$;

(ii) $|\varphi(x)-\varphi(y)| \leqslant C(\frac{d(x,y)}{r+d(x_1,x)})^\beta \frac{1}{V_r(x_1)+V(x_1,x)}(\frac{r}{r+d(x_1,x)})^\gamma$ *for all* x, $y \in \mathcal{X}$ *satisfying that* $d(x,y) \leqslant (r+d(x_1,x))/2$.

Denote by $\mathcal{G}(x_1,r,\beta,\gamma)$ the set of all test functions of type (x_1,r,β,γ). If φ lies in $\mathcal{G}(x_1,r,\beta,\gamma)$, its norm is defined by

$$\|\varphi\|_{\mathcal{G}(x_1,\,r,\,\beta,\,\gamma)} := \inf\{C: \text{(i) and (ii) hold}\}.$$

The space $\mathcal{G}(x_1,r,\beta,\gamma)$ is called the space of test functions.

Fix $x_1 \in \mathcal{X}$. Let $\mathcal{G}(\beta,\gamma) := \mathcal{G}(x_1,1,\beta,\gamma)$. It is easy to see that $\mathcal{G}(\beta,\gamma)$ is a Banach space. For any given $\epsilon \in (0,1]$, let $\mathcal{G}_0^\epsilon(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$ when β, $\gamma \in (0,\epsilon]$. Then, it is easy to see that $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ if and only if $\varphi \in \mathcal{G}(\beta,\gamma)$ and there exist $\{\phi_i\}_{i\in\mathbb{N}} \subset \mathcal{G}(\epsilon,\epsilon)$ such that $\lim_{i\to\infty}\|\varphi-\phi_i\|_{\mathcal{G}(\beta,\gamma)} = 0$. If $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, we then define $\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} := \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$. It is easy to see that $\mathcal{G}_0^\epsilon(\beta,\gamma)$ is a Banach space and for the above chosen $\{\phi_i\}_{i\in\mathbb{N}}$, $\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} = \lim_{i\to\infty}\|\phi_i\|_{\mathcal{G}(\beta,\gamma)}$.

Set $\mathring{\mathcal{G}}(x_1,r,\beta,\gamma) := \left\{\varphi \in \mathcal{G}(x_1,r,\beta,\gamma): \int_{\mathcal{X}}\varphi(x)\,d\mu(x) = 0\right\}$. The space $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ is defined to be the completion of $\mathring{\mathcal{G}}(\epsilon,\epsilon)$ in $\mathring{\mathcal{G}}(\beta,\gamma)$ as above, where $\epsilon \in (0,1]$ and β, $\gamma \in (0,\epsilon]$. For $\varphi \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, we define $\|\varphi\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} := \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$.

Denote by $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$, respectively, the set of all bounded linear functionals on $\mathcal{G}_0^\epsilon(\beta,\gamma)$ and $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$. Define $\langle f,\varphi\rangle$ to be the natural pairing of elements $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, or $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and $\varphi \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$.

Now we recall the definition of Hardy space on $\mathcal{X}$; see [17, Definitions 5.8 and 5.14].

Definition 2.5. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0,\epsilon_1\wedge\epsilon_2)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-*AOTI*. For $k \in \mathbb{Z}$, set $D_k := S_k - S_{k-1}$. Let $p \in (n/(n+\epsilon),\infty)$ and*

$$\max\{0, n(1/p-1)\} < \beta,\ \gamma < \epsilon. \tag{2.3}$$

Then the Hardy space $H^p(\mathcal{X})$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ such that

$$\|f\|_{H^p(\mathcal{X})} := \left\|\left\{\sum_{k\in\mathbb{Z}}\left|D_k(f)\right|^2\right\}^{1/2}\right\|_{L^p(\mathcal{X})} < \infty. \tag{2.4}$$

Remark 2.6. (i) When $p \in (1,\infty)$, by [17, Proposition 5.10], the Hardy space $H^p(\mathcal{X})$ coincides with $L^p(\mathcal{X})$ with an equivalence of norms.

(ii) For $p \in (n/(n+\epsilon),1]$, the definition of the Hardy space $H^p(\mathcal{X})$ is independent of the choice of $(\epsilon_1,\epsilon_2,\epsilon_3)$- AOTI's and of the distribution spaces $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ with β and γ satisfying (2.3); see [17, Remark 5.9]. Moreover, it was proved in [27] that the spaces $H^p(\mathcal{X})$ are also independent of the choice of $\epsilon \in (0,1)$.

(iii) With the notation of Definition 2.5, for any $\alpha \in (0,\infty)$ and $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$, we define the nontangential maximal function $\mathcal{M}_\alpha(f)$ of f by setting, for all $x \in \mathcal{X}$,

$$\mathcal{M}_\alpha(f)(x) := \sup_{k\in\mathbb{Z}}\ \sup_{d(x,\,y)\leqslant\alpha 2^{-k}}|S_k(f)(y)|; \tag{2.5}$$

see [11, Definition 2.9]. For any given $p \in (n/(n+\epsilon),\infty)$, it was proved in [11] that for all $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with β, γ satisfying (2.3), the following holds:

$$\|f\|_{H^p(\mathcal{X})} \sim \|\mathcal{M}_\alpha(f)\|_{L^p(\mathcal{X})}. \tag{2.6}$$

Recall that the centered Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$,

$$\mathcal{M}(f)(x) := \sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,\,r)}|f(y)|\,d\mu(y). \tag{2.7}$$

Coifman and Weiss [6, 7] proved that $\mathcal{M}$ is bounded on $L^p(\mathcal{X})$ for $p \in (1,\infty)$, and also bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$.

We conclude this section with some useful estimates (see Lemmas 2.7 and 2.8 below), which are used throughout the whole paper. The proofs of the statements in Lemma 2.7 below are contained in [17, Lemma 2.1] and [16, Lemma 2.1].

Lemma 2.7. *Let $\mathcal{X}$ be an RD-space, $\delta > 0$, $a > 0$, $r > 0$ and $\theta \in (0,1)$.*

(a) *For all $x, y \in \mathcal{X}$ and all $r > 0$, $\mu(B(x, r + d(x,y))) \sim \mu(B(y, r + d(x,y))) \sim V_r(x) + V(x,y) \sim V_r(x) + V_r(y) + V(x,y) \sim V_r(y) + V(x,y)$.*

(b) $\int_{\mathcal{X}} \frac{1}{\mu(B(x,r+d(x,y)))}(\frac{r}{r+d(x,y)})^a \, d\mu(x) < C$ *uniformly in $x \in \mathcal{X}$ and $r > 0$ if $a > 0$.*

(c) $\int_{\mathcal{X}} \frac{1}{\mu(B(x,r+d(x,y)))}(\frac{r}{r+d(x,y)})^a |f(y)| \, d\mu(y) \leqslant C\mathcal{M}(f)(x)$ *uniformly in $f \in L^1_{\rm loc}(\mathcal{X})$ and $x \in \mathcal{X}$.*

The following estimate is established in [17, Lemma 3.2].

Lemma 2.8. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\{S_k\}_{k\in\mathbb{Z}}$ and $\{A_k\}_{k\in\mathbb{Z}}$ be two $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI's. Set $P_k := S_k - S_{k-1}$ and $Q_k := A_k - A_{k-1}$ for all $k \in \mathbb{Z}$. Then for any $\delta \in (0, \epsilon_1 \wedge \epsilon_2)$, there exists a positive constant C, depending on ϵ_1, ϵ_2 and δ, such that the kernel of $P_\ell Q_k$, denoted still by $P_\ell Q_k$, satisfies that for all $k, \ell \in \mathbb{Z}$ and all $x, y \in \mathcal{X}$,*

$$|P_\ell Q_k(x,y)| \leqslant \frac{C2^{-|k-\ell|\delta}}{V_{2^{-(k\wedge\ell)}}(x) + V_{2^{-(k\wedge\ell)}}(y) + V(x,y)} \left(\frac{2^{-(k\wedge\ell)}}{2^{-(k\wedge\ell)} + d(x,y)}\right)^{\epsilon_2}. \tag{2.8}$$

Remark 2.9. (a) Assume that $\{Q_k\}_{k\in\mathbb{Z}}$ satisfy properties (i) and (ii) of Definition 2.3, $\{P_\ell\}_{\ell\in\mathbb{Z}}$ satisfy (i) of Definition 2.3 and $\int_{\mathcal{X}} P_\ell(x,y)\, d\mu(y) = 0$ for all $\ell \geqslant k$. By an argument similar to the proof of [17, Lemma 3.2], we still obtain the estimate (2.8) with $\ell \geqslant k$.

(b) Assume that $\{P_\ell\}_{\ell\in\mathbb{Z}}$ satisfy properties (i) and (iii) of Definition 2.3, $\{Q_k\}_{k\in\mathbb{Z}}$ satisfy (i) of Definition 2.3 and $\int_{\mathcal{X}} Q_k(x,y)\, d\mu(x) = 0$ for all $k \geqslant \ell$. Then by the symmetry and (a) of this remark, we also obtain the estimate (2.8) with $k \geqslant \ell$.

(c) If we only assume that $\{P_\ell\}_{\ell\in\mathbb{Z}}$ and $\{Q_k\}_{k\in\mathbb{Z}}$ satisfy property (i) of Definition 2.3, then the estimate (2.8) holds whenever $k = \ell$.

3 Proof of Theorem 1.2

The whole section is devoted to the proof of Theorem 1.2. First we prove an auxiliary lemma, which is a variant of [13, Lemma 1].

Lemma 3.1. *Let $\epsilon \in (0,\infty)$. Then there exists a positive constant C, depending only on ϵ, C_3 and n, such that for all $w \in \mathcal{X}$, $r \in (0,\infty)$ and $R \in (0,\infty)$,*

$$\begin{aligned}\mathcal{J}(w; r, R) &:= \int_{d(x,y)<R} \frac{1}{V_r(y) + V_r(w) + V(y,w)} \left(\frac{r}{r + d(y,w)}\right)^{\epsilon} d\mu(y) \\ &\leqslant C \max\left\{\left(\frac{R}{r}\right)^{\epsilon}, 1\right\} \frac{V_R(x)}{V_r(x) + V_r(w) + V(x,w)} \left(\frac{r}{r + d(x,w)}\right)^{\epsilon}.\end{aligned} \tag{3.1}$$

Proof. Suppose first that $R \leqslant 10r$. In this case, if $d(x,w) \geqslant 20r$, then for all $y \in B(x,R)$, we have $d(x,y) < 10r \leqslant d(x,w)/2$ and hence $d(y,w) \geqslant d(x,w) - d(x,y) > d(x,w)/2$, which together with Lemma 2.7(a) implies (3.1). If $d(x,w) < 20r$, then by (2.1), we obtain

$$\mathcal{J}(w; r, R) \leqslant \frac{\mu(B(x,R))}{\mu(B(w,r))} \lesssim \frac{\mu(B(x,R))}{\mu(B(w, r + d(x,w)))} \left(\frac{r}{r + d(x,w)}\right)^{\epsilon},$$

which combined with Lemma 2.7(a) yields (3.1).

Now we assume that $R > 10r$. In this case, we set

$$A_1 := \{y \in \mathcal{X} : d(x,y) < R,\ d(y,w) \geqslant d(x,w)\}$$

and $A_2 := \{y \in \mathcal{X} : d(x,y) < R,\ d(y,w) < d(x,w)\}$. Using Lemma 2.7(a), we obtain

$$\int_{A_1} \frac{1}{V_r(y)+V_r(w)+V(y,w)}\left(\frac{r}{r+d(y,w)}\right)^{\epsilon} d\mu(y) \lesssim \frac{V_R(x)}{V_r(x)+V_r(w)+V(x,w)}\left(\frac{r}{r+d(x,w)}\right)^{\epsilon}.$$

To estimate the integral in (3.1) over the set A_2, we consider the following two cases. If $d(x,w) \leqslant 2R$, then by (a) and (b) of Lemma 2.7, we have

$$\int_{A_2} \frac{1}{V_r(y)+V_r(w)+V(y,w)}\left(\frac{r}{r+d(y,w)}\right)^{\epsilon} d\mu(y) \lesssim 1 \lesssim \frac{V_R(x)}{\mu(B(x,r+d(x,w)))}\left(\frac{R}{r+d(x,w)}\right)^{\epsilon},$$

which implies the desired estimate. If $d(x,w) > 2R$, then $d(y,w) > d(x,w)/2$ whenever $y \in A_2$. Thus,

$$\int_{A_2} \frac{1}{V_r(y)+V_r(w)+V(y,w)}\left(\frac{r}{r+d(y,w)}\right)^{\epsilon} d\mu(y) \lesssim \frac{V_R(x)}{\mu(B(x,r+d(x,w)))}\left(\frac{r}{r+d(x,w)}\right)^{\epsilon}.$$

Combining the last three formulae above yields that (3.1) holds when $R > 10r$, which completes the proof of Lemma 3.1.

For the sake of simplicity, in the sequel, we use the following notation: for any given $\epsilon \in (0,\infty)$, $j \in \mathbb{Z}$ and all $x,\ y \in \mathcal{X}$,

$$\mathcal{K}_{\epsilon}(j;\, x, y) := \frac{1}{V_{2^{-j}}(x)+V_{2^{-j}}(y)+V(x,y)}\left(\frac{2^{-j}}{2^{-j}+d(x,y)}\right)^{\epsilon}. \tag{3.2}$$

Using Lemma 3.1 and borrowing some ideas from [13], we obtain the following result.

Proposition 3.2. *Suppose that* $\{G_k\}_{k\in\mathbb{Z}}$, $\{A_k\}_{k\in\mathbb{Z}}$ *and* $\{S_k\}_{k\in\mathbb{Z}}$ *are measurable functions from* $\mathcal{X} \times \mathcal{X} \to \mathbb{C}$ *satisfying that there exist positive constants* C, a *and* σ *such that for all* $k \in \mathbb{Z}$ *and* $x,\ y \in \mathcal{X}$, $G_k(x,y) = 0$ *whenever* $d(x,y) \geqslant a2^{-k}$ *and*

$$|G_k(x,y)| + |A_k(x,y)| + |S_k(x,y)| \leqslant C\,\mathcal{K}_{\sigma}(k;\, x,y). \tag{3.3}$$

Then for any given $\sigma' \in (0,\sigma)$, *there exists a positive constant* $\widetilde{C}$ *depending on* C, a, σ, σ' *and* C_3 *such that for all* $j,k,\ell \in \mathbb{Z}$ *satisfying* $\ell \geqslant \max\{k,j\}$ *and all* $x,\ z,\ w \in \mathcal{X}$,

$$\int_{\mathcal{X}} |G_\ell(x,y)A_k(y,z)S_j(y,w)|\, d\mu(y) \leqslant \widetilde{C}\,\mathcal{K}_{\sigma}(k;\, x,z)\,\mathcal{K}_{\sigma'}(j;\, x,w). \tag{3.4}$$

Proof. By the support condition of G_ℓ, (3.3), Lemma 2.7(a) and (2.1), we obtain

$$\begin{aligned}
&\int_{\mathcal{X}} |G_\ell(x,y)A_k(y,z)S_j(y,w)|\, d\mu(y)\\
&\quad\lesssim \int_{d(x,y)<a2^{-\ell}} \frac{1}{\mu(B(x,2^{-\ell}))}\frac{1}{\mu(B(z,2^{-k}+d(y,z)))}\\
&\qquad\times\left(\frac{2^{-k}}{2^{-k}+d(y,z)}\right)^{\sigma}\frac{1}{\mu(B(w,2^{-j}+d(y,w)))}\left(\frac{2^{-j}}{2^{-j}+d(y,w)}\right)^{\sigma} d\mu(y)\\
&\quad\lesssim \sum_{t=0}^{\infty} 2^{-t\sigma}\frac{1}{\mu(B(x,2^{-\ell}))}\frac{1}{\mu(B(w,2^{t-j}))}\\
&\qquad\times\int_{\substack{d(x,y)<a2^{-\ell}\\ d(y,w)\sim 2^{t-j}}} \frac{1}{\mu(B(z,2^{-k}+d(y,z)))}\left(\frac{2^{-k}}{2^{-k}+d(y,z)}\right)^{\sigma} d\mu(y),
\end{aligned} \tag{3.5}$$

where the notation $d(y,w) \sim 2^{t-j}$ means that $2^{t-j} \leqslant d(y,w) < 2^{t-j+1}$ for $t \geqslant 1$ and $d(y,w) < 2^{-t}$ for $t = 0$.

Fix $t \geqslant 0$. If there exists $y \in \mathcal{X}$ satisfying that $d(x,y) < a2^{-\ell}$ and $d(y,w) < 2^{t-j+1}$, then we have $d(x,w) < (a+2)\max\{2^{-\ell},\, 2^{t-j}\} < (a+2)2^{t-j}$ since $\ell \geqslant j$, which further implies that $2^{-j} + d(x,w) \leqslant$

$(a+3)2^{t-j}$. By this, Lemma 3.1 and Lemma 2.7(a), we obtain that the last formula in (3.5) is bounded by a constant multiple of

$$\begin{aligned}&\sum_{t=0}^{\infty}2^{-t\sigma}\frac{1}{\mu(B(x,2^{-\ell}))}\frac{\mu(B(w,2^{-j}+d(x,w)))}{\mu(B(w,2^{t-j}))}2^{t\sigma'}\\&\quad\times\frac{1}{\mu(B(w,2^{-j}+d(x,w)))}\left(\frac{2^{-j}}{2^{-j}+d(x,w)}\right)^{\sigma'}\\&\quad\times\max\left\{\frac{a2^{-\ell}}{2^{-k}},1\right\}^{\sigma}\frac{\mu(B(x,a2^{-\ell}))}{\mu(B(z,2^{-k}+d(x,z)))}\left(\frac{2^{-k}}{2^{-k}+d(x,z)}\right)^{\sigma},\end{aligned}\tag{3.6}$$

where $\sigma'\in(0,\sigma)$. We now use $2^{-j}+d(x,w)\leqslant(a+3)2^{t-j}$ again, $\sigma'\in(0,\sigma)$, (2.1) and the fact $\ell\geqslant k$ to bound (3.6) by a constant multiple of

$$\sum_{t=0}^{\infty}2^{-t\sigma}2^{t\sigma'}\mathcal{K}_{\sigma'}(j;x,w)\mathcal{K}_{\sigma}(k;x,z)\lesssim\mathcal{K}_{\sigma'}(j;x,w)\mathcal{K}_{\sigma}(k;x,z).\tag{3.7}$$

Combining (3.5), (3.6) and (3.7) yields (3.4). This finishes the proof of Proposition 3.2.

Next we recall the cube constructions on $\mathcal{X}$, which provide an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type; see [3].

Lemma 3.3. *Let $\mathcal{X}$ be a space of homogeneous type. Then there exist a collection $\{Q_\alpha^k\subset\mathcal{X}:k\in\mathbb{Z},\ \alpha\in I_k\}$ of open subsets, where I_k is certain index set, and constants $\delta\in(0,1)$ and C_4, $C_5>0$ such that*

(i) *$\mu(\mathcal{X}\setminus\cup_\alpha Q_\alpha^k)=0$ for each fixed k and $Q_\alpha^k\cap Q_\beta^k=\emptyset$ if $\alpha\neq\beta$;*
(ii) *for any α, β, k, ℓ with $\ell\geqslant k$, either $Q_\beta^\ell\subset Q_\alpha^k$ or $Q_\beta^\ell\cap Q_\alpha^k=\emptyset$;*
(iii) *for each (k,α) and each $\ell<k$, there exists a unique β such that $Q_\alpha^k\subset Q_\beta^\ell$;*
(iv) *$\operatorname{diam}(Q_\alpha^k)\leqslant C_4\delta^k$ and each Q_α^k contains certain ball $B(z_\alpha^k,C_5\delta^k)$, where $z_\alpha^k\in\mathcal{X}$.*

In fact, one can think of Q_α^k as being a dyadic cube with diameter rough δ^k centered at z_α^k. In what follows, for simplicity, we may assume that $\delta=1/2$; see [17].

In the sequel, we use the following notation. For $k\in\mathbb{Z}$ and $\tau\in I_k$, we denote by $Q_\tau^{k,\nu}$, $\nu=1,2,\ldots,N(k,\tau)$, the set of all cubes $Q_{\tau'}^{k+j_0}\subset Q_\tau^k$, where Q_τ^k is the dyadic cube as in Lemma 2.7 and j_0 is a positive integer satisfying

$$2^{-j_0}C_4<1/3.\tag{3.8}$$

Denote by $z_\tau^{k,\nu}$ the "center" of $Q_\tau^{k,\nu}$, and by $y_\tau^{k,\nu}$ any point of $Q_\tau^{k,\nu}$.

The following discrete homogeneous Calderón reproducing formula is proved in [17, Theorem 4.13].

Lemma 3.4. *Let $\epsilon_1\in(0,1]$, $\epsilon_2>0$, $\epsilon_3>0$, $\epsilon\in(0,\epsilon_1\wedge\epsilon_2)$ and let $\{P_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-$\ell_0$-*AOTI. *Set $G_k:=P_k-P_{k-1}$ for $k\in\mathbb{Z}$. Then for any fixed j_0 satisfying* (3.8) *large enough, there exists a family $\{\widetilde{G}_k\}_{k\in\mathbb{Z}}$ of linear operators such that for any fixed $y_\tau^{k,\nu}\in Q_\tau^{k,\nu}$ with $k\in\mathbb{Z}$, $\tau\in I_k$ and $\nu\in\{1,2,\ldots,N(k,\tau)\}$, and all $f\in(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ with β, $\gamma\in(0,\epsilon)$ and $x\in\mathcal{X}$,*

$$f(x)=\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{G}_k(x,y_\tau^{k,\nu})G_k(f)(y_\tau^{k,\nu}),$$

where the series converges in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$. Moreover, the kernels of the operators $\{\widetilde{G}_k\}_{k\in\mathbb{Z}}$ satisfy properties (i) *and* (ii) *of Definition* 2.3 *with ϵ_1 and ϵ_2 replaced by any $\epsilon'\in(\epsilon,\epsilon_1\wedge\epsilon_2)$, and*

$$\int_{\mathcal{X}}\widetilde{G}_k(w,x)\,d\mu(w)=0=\int_{\mathcal{X}}\widetilde{G}_k(x,w)\,d\mu(w)$$

for all $k\in\mathbb{Z}$ and $x\in\mathcal{X}$.

The inhomogeneous discrete Calderón reproducing formula (see [11, Theorem 3.3]) stated in Lemma 3.6 below, and established in [17] in the case $\ell_0=0$, is a main ingredient in the proof of Theorem 1.2.

Definition 3.5. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\ell_0 \in \mathbb{Z}$. A sequence $\{S_k\}_{k=\ell_0}^{\infty}$ of linear operators is said to be an inhomogeneous approximation of the identity of order $(\epsilon_1, \epsilon_2, \epsilon_3)$ (for short, $(\epsilon_1, \epsilon_2, \epsilon_3)$-$\ell_0$-AOTI), if S_k satisfies* (i) *through* (v) *of Definition* 2.3 *for all $k \in \{\ell_0, \ell_0+1, \dots\}$.*

Lemma 3.6. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\ell_0 \in \mathbb{Z}$. Let $\{P_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-$\ell_0$-AOTI. Set $G_{\ell_0} := P_{\ell_0}$ and $G_k := P_k - P_{k-1}$ for $k \geqslant \ell_0 + 1$. Then for any fixed j_0 satisfying* (3.8) *large enough, there exists functions $\widetilde{P}_{\ell_0}(x,y)$ and $\{\widetilde{G}_k(x,y)\}_{k=\ell_0+1}^{\infty}$ such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k \geqslant \ell_0+1$, $\tau \in I_k$ and $\nu \in \{1,2,\dots,N(k,\tau)\}$, and all $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with β, $\gamma \in (0,\epsilon)$ and $x \in \mathcal{X}$,*

$$f(x) = \sum_{\tau\in I_{\ell_0}} \sum_{\nu=1}^{N(\ell_0,\tau)} \left\{ \int_{Q_\tau^{\ell_0,\nu}} \widetilde{P}_{\ell_0}(x,y)\, d\mu(y) \right\} \left\{ \frac{1}{\mu(Q_\tau^{\ell_0,\nu})} \int_{Q_\tau^{\ell_0,\nu}} P_{\ell_0}(f)(w)\, d\mu(w) \right\}$$
$$+ \sum_{k=\ell_0+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{G}_k(x, y_\tau^{k,\nu}) G_k(f)(y_\tau^{k,\nu}),$$

where the series converges in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Moreover, $\widetilde{P}_{\ell_0}$ and $\widetilde{G}_k$ for $k \geqslant \ell_0+1$ satisfy (i) *and* (ii) *of Definition* 2.3 *with ϵ_1 and ϵ_2 replaced by any $\epsilon' \in (\epsilon, \epsilon_1 \wedge \epsilon_2)$, and for all $x \in \mathcal{X}$,*

$$\int_{\mathcal{X}} \widetilde{P}_{\ell_0}(w,x)\, d\mu(w) = 1 = \int_{\mathcal{X}} \widetilde{P}_{\ell_0}(x,w)\, d\mu(w)$$

and

$$\int_{\mathcal{X}} \widetilde{G}_k(w,x)\, d\mu(w) = 0 = \int_{\mathcal{X}} \widetilde{G}_k(x,w)\, d\mu(w)$$

when $k > \ell_0$.

Remark 3.7. The constant C appearing in (i) and (ii) of Definition 2.3 for $\widetilde{P}_{\ell_0}$ and $\{\widetilde{G}_k\}_{k=\ell_0+1}^{\infty}$, depends on j_0 and ϵ', but not on ℓ_0; see [11, Remark 3.4].

The following technical lemma proved in [17, Lemma 5.3] is also crucial in the proof of Theorem 1.2.

Lemma 3.8. *Let $\epsilon > 0$, k', $k \in \mathbb{Z}$, and $y_\tau^{k,\nu}$ be any point in $Q_\tau^{k,\nu}$ for $\tau \in I_k$ and $\nu = 1,2,\dots,N(k,\tau)$. If $r \in (n/(n+\epsilon), 1]$, then there exists a positive constant C depending on r such that for all $a_\tau^{k,\nu} \in \mathbb{C}$ and all $x \in \mathcal{X}$,*

$$\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \frac{1}{V_{2^{-(k'\wedge k)}}(x) + V(x, y_\tau^{k,\nu})} \frac{2^{-(k'\wedge k)\epsilon}}{(2^{-(k'\wedge k)} + d(x, y_\tau^{k,\nu}))^\epsilon} |a_\tau^{k,\nu}|$$
$$\leqslant C 2^{[(k'\wedge k)-k]n(1-1/r)} \left\{ \mathcal{M}\left(\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} |a_\tau^{k,\nu}|^r \chi_{Q_\tau^{k,\nu}} \right)(x) \right\}^{1/r},$$

where C is also independent of k, k', τ and ν.

Proof of Theorem 1.2. Let $\epsilon_1 \in (0,1]$ and $\{P_\ell\}_{\ell\in\mathbb{Z}}$ be an ϵ_1-AOTI with bounded support. Set $G_\ell := P_\ell - P_{\ell-1}$ for all $\ell \in \mathbb{Z}$. For any $N \in \mathbb{N}$, $f \in H^p(\mathcal{X})$ and $g \in H^q(\mathcal{X})$, we consider

$$\Pi_N(f,g) := \sum_{|j|\leqslant N} E_j(D_j(f)S_j(g)).$$

Suppose for the moment that $\Pi_N(f,g)$ is a Cauchy sequence in $H^r(\mathcal{X})$. Then $\Pi_N(f,g)$ converges in $H^r(\mathcal{X})$ as $N \to \infty$ to an element of $H^r(\mathcal{X})$ that we will denote by $\Pi(f,g)$. To prove Theorem 1.2, by (2.4) and Remark 2.6(ii), it suffices to show that for all $N \in \mathbb{N}$,

$$\|\Pi_N(f,g)\|_{H^r(\mathcal{X})} \sim \left\| \left\{ \sum_{\ell\in\mathbb{Z}} |G_\ell(\Pi_N(f,g))|^2 \right\}^{1/2} \right\|_{L^r(\mathcal{X})}$$
$$\sim \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \left| \sum_{|j|\leqslant N} G_\ell E_j(D_j(f)S_j(g)) \right|^2 \right\}^{1/2} \right\|_{L^r(\mathcal{X})}$$

$$\lesssim \|f\|_{H^p(\mathcal{X})}\|g\|_{H^q(\mathcal{X})}. \tag{3.9}$$

Indeed, if (3.9) holds, then we can use Fatou's lemma to pass the limit to $\Pi(f,g)$ and obtain the boundedness of Π from $H^p(\mathcal{X})\times H^q(\mathcal{X})$ to $H^r(\mathcal{X})$.

Now we prove (3.9). For any $j\in\mathbb{Z}$, applying Lemma 3.6, we obtain that for all $x\in\mathcal{X}$,

$$\begin{aligned}(D_j(f)S_j(g))(x) &= \sum_{\tau\in I_j}\sum_{\nu=1}^{N(j,\tau)}\left\{\int_{Q_\tau^{j,\nu}}\widetilde{P}_j(x,z)\,d\mu(z)\right\}\frac{1}{\mu(Q_\tau^{j,\nu})}\int_{Q_\tau^{j,\nu}}P_j(D_j(f)S_j(g))(z)\,d\mu(z)\\ &\quad+\sum_{k=j+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{G}_k(x,y_\tau^{k,\nu})G_k\left(D_j(f)S_j(g)\right)(y_\tau^{k,\nu})\end{aligned}$$

holds in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, where $y_\tau^{k,\nu}$ is an arbitrary fixed point in $Q_\tau^{k,\nu}$, all $\{\widetilde{G}_k\}_{k\geqslant j+1}$ have vanishing property and $\widetilde{P}_j$ with no vanishing property. Consequently, we have

$$\begin{aligned}&\sum_{|j|\leqslant N}G_\ell E_j\left(D_j(f)S_j(g)\right)(x)\\ &\quad=\sum_{|j|\leqslant N}\sum_{\tau\in I_j}\sum_{\nu=1}^{N(j,\tau)}\left\{\int_{Q_\tau^{j,\nu}}G_\ell E_j\widetilde{P}_j(x,z)\,d\mu(z)\right\}\frac{1}{\mu(Q_\tau^{j,\nu})}\int_{Q_\tau^{j,\nu}}P_j(D_j(f)S_j(g))(z)\,d\mu(z)\\ &\quad\quad+\sum_{|j|\leqslant N}\sum_{k=j+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})G_\ell E_j\widetilde{G}_k(x,y_\tau^{k,\nu})G_k\left(D_j(f)S_j(g)\right)(y_\tau^{k,\nu})\\ &\quad=:\mathrm{Z}_1^N+\mathrm{Z}_2^N.\end{aligned}$$

In this way, the proof of (3.9) reduces to the estimate

$$\left\|\left\{\sum_{\ell\in\mathbb{Z}}\left|\mathrm{Z}_1^N\right|^2\right\}^{1/2}\right\|_{L^r(\mathcal{X})}+\left\|\left\{\sum_{\ell\in\mathbb{Z}}\left|\mathrm{Z}_2^N\right|^2\right\}^{1/2}\right\|_{L^r(\mathcal{X})}\lesssim\|f\|_{H^p(\mathcal{X})}\|g\|_{H^q(\mathcal{X})}. \tag{3.10}$$

We first prove that $\|\{\sum_{\ell\in\mathbb{Z}}|\mathrm{Z}_2^N|^2\}^{1/2}\|_{L^r(\mathcal{X})}\lesssim\|f\|_{H^p(\mathcal{X})}\|g\|_{H^q(\mathcal{X})}$. Lemma 3.4 yields that for all $w\in\mathcal{X}$,

$$D_j(f)(w)=\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})D_j\widetilde{G}_{k'}(w,y_{\tau'}^{k',\nu'})G_{k'}(f)(y_{\tau'}^{k',\nu'}). \tag{3.11}$$

Using Lemma 3.6 again we obtain that for all $w\in\mathcal{X}$, we have

$$\begin{aligned}S_j(g)(w)&=\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\left\{\int_{Q_{\tau''}^{j,\nu''}}S_j\widetilde{P}_j(w,z)\,d\mu(z)\right\}\frac{1}{\mu(Q_{\tau''}^{j,\nu''})}\int_{Q_{\tau''}^{j,\nu''}}P_j(g)(z)\,d\mu(z)\\ &\quad+\sum_{k''=j+1}^{\infty}\sum_{\tau''\in I_{k''}}\sum_{\nu''=1}^{N(k'',\tau'')}\mu(Q_{\tau''}^{k'',\nu''})S_j\widetilde{G}_{k''}(w,y_{\tau''}^{k'',\nu''})G_{k''}(g)(y_{\tau''}^{k'',\nu''}).\end{aligned} \tag{3.12}$$

Thus, for all $k\geqslant j+1$, $\tau\in I_k$ and $\nu\in\{1,2,\ldots,N(k,\tau)\}$, in view of (3.11) and (3.12), we may write

$$\begin{aligned}G_k(D_j(f)S_j(g))(y_\tau^{k,\nu})&=\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\mu(Q_{\tau'}^{k',\nu'})G_{k'}(f)(y_{\tau'}^{k',\nu'})\\ &\quad\times\left\{\frac{1}{\mu(Q_{\tau''}^{j,\nu''})}\int_{Q_{\tau''}^{j,\nu''}}P_j(g)(z)\,d\mu(z)\right\}\\ &\quad\times\int_{Q_{\tau''}^{j,\nu''}}G_k(D_j\widetilde{G}_{k'}(\cdot,y_{\tau'}^{k',\nu'})S_j\widetilde{P}_j(\cdot,z))(y_\tau^{k,\nu})\,d\mu(z)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\sum_{k''=j+1}^{\infty}\sum_{\tau''\in I_{k''}}\sum_{\nu''=1}^{N(k'',\tau'')}\mu(Q_{\tau'}^{k',\nu'})\mu(Q_{\tau''}^{k'',\nu''})\\
&\times G_{k'}(f)(y_{\tau'}^{k',\nu'})G_{k''}(g)(y_{\tau''}^{k'',\nu''})G_k(D_j\widetilde{G}_{k'}(\cdot,y_{\tau'}^{k',\nu'})S_j\widetilde{G}_{k''}(\cdot,y_{\tau''}^{k'',\nu''}))(y_\tau^{k,\nu})\\
=&:\mathrm{J}_1+\mathrm{J}_2. \qquad (3.13)
\end{aligned}
$$

Now we estimate J_2. For any given $\sigma\in(0,\epsilon_1\wedge\epsilon_2)$, by (a) and (b) of Remark 2.9, we have

$$
|D_j\widetilde{G}_{k'}(w,y_{\tau'}^{k',\nu'})|\lesssim 2^{-|j-k'|\sigma}\,\mathcal{K}_\sigma(j\wedge k';\,w,y_{\tau'}^{k',\nu'}).
$$

Since $k''>j$, using Remark 2.9(b), we obtain that

$$
|S_j\widetilde{G}_{k''}(w,y_{\tau''}^{k'',\nu''})|\lesssim 2^{-|j-k''|\sigma}\,\mathcal{K}_\sigma(j;\,w,y_{\tau''}^{k'',\nu''}).
$$

From these two estimates and Proposition 3.2, we deduce that for any given $\sigma\in(0,\epsilon_1\wedge\epsilon_2)$,

$$
\begin{aligned}
&|G_k(D_j\widetilde{G}_{k'}(\cdot,y_{\tau'}^{k',\nu'})S_j\widetilde{G}_{k''}(\cdot,y_{\tau''}^{k'',\nu''}))(y_\tau^{k,\nu})|\\
&\quad\lesssim 2^{-|j-k'|\sigma}2^{-|j-k''|\sigma}\,\mathcal{K}_\sigma(j\wedge k';\,y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})\,\mathcal{K}_\sigma(j;\,y_\tau^{k,\nu},y_{\tau''}^{k'',\nu''}), \qquad (3.14)
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\mathrm{J}_2\lesssim&\sum_{k'\in\mathbb{Z}}\sum_{k''=j+1}^{\infty}2^{-|j-k'|\sigma}2^{-|j-k''|\sigma}\\
&\times\Bigg\{\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\,\mathcal{K}_\sigma(j\wedge k';\,y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})G_{k'}(f)(y_{\tau'}^{k',\nu'})\Bigg\}\\
&\times\Bigg\{\sum_{\tau''\in I_{k''}}\sum_{\nu''=1}^{N(k'',\tau'')}\mu(Q_{\tau''}^{k'',\nu''})\,\mathcal{K}_\sigma(j;\,y_\tau^{k,\nu},y_{\tau''}^{k'',\nu''})\mathcal{M}_\alpha(g)(y_{\tau''}^{k'',\nu''})\Bigg\},
\end{aligned}
$$

where $\alpha\in(0,\infty)$ is arbitrary and $\mathcal{M}_\alpha$ is as in (2.5) with S_k there replaced by P_k. Moreover, from Lemma 2.7(a) and Lemma 3.8, it follows that

$$
\begin{aligned}
\mathrm{J}_2\lesssim&\sum_{k'\in\mathbb{Z}}\sum_{k''=j+1}^{\infty}2^{-|j-k'|\sigma}2^{-|j-k''|\sigma}2^{[(j\wedge k')-k']n(1-1/\theta)}2^{(j-k'')n(1-1/\delta)}\\
&\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}|G_{k'}(f)(y_{\tau'}^{k',\nu'})|^\theta\chi_{Q_{\tau'}^{k',\nu'}}\Bigg)(y_\tau^{k,\nu})\Bigg\}^{1/\theta}\\
&\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau''\in I_{k''}}\sum_{\nu''=1}^{N(k'',\tau'')}|\mathcal{M}_\alpha(g)(y_{\tau''}^{k'',\nu''})|^\delta\chi_{Q_{\tau''}^{k'',\nu''}}\Bigg)(y_\tau^{k,\nu})\Bigg\}^{1/\delta}, \qquad (3.15)
\end{aligned}
$$

where $\theta,\ \delta\in(n/(n+\sigma),1]$. Due to the arbitrariness of $y_{\tau'}^{k',\nu'}\in Q_{\tau'}^{k',\nu'}$ and $y_{\tau''}^{k'',\nu''}\in Q_{\tau''}^{k'',\nu''}$, we obtain that (3.15) still holds if $|G_{k'}(f)(y_{\tau'}^{k',\nu'})|$ and $|\mathcal{M}_\alpha(g)(y_{\tau''}^{k'',\nu''})|$ are, respectively, replaced with

$$
\inf_{y_{\tau'}^{k',\nu'}\in Q_{\tau'}^{k',\nu'}}|G_{k'}(f)(y_{\tau'}^{k',\nu'})|\quad\text{and}\quad\inf_{y_{\tau''}^{k'',\nu''}\in Q_{\tau''}^{k'',\nu''}}|\mathcal{M}_\alpha(g)(y_{\tau''}^{k'',\nu''})|.
$$

For all $z\in\mathcal{X}$, by Lemma 3.3(i), we have

$$
\mathcal{M}\Bigg(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\inf_{y_{\tau'}^{k',\nu'}\in Q_{\tau'}^{k',\nu'}}|G_{k'}(f)(y_{\tau'}^{k',\nu'})|^\theta\chi_{Q_{\tau'}^{k',\nu'}}\Bigg)(z)\leqslant\mathcal{M}(|G_{k'}(f)|^\theta)(z). \qquad (3.16)
$$

Similarly, for all $z \in \mathcal{X}$,

$$\mathcal{M}\bigg(\sum_{\tau''\in I_{k''}}\sum_{\nu''=1}^{N(k'',\tau'')}\inf_{y_{\tau''}^{k'',\nu''}\in Q_{\tau''}^{k'',\nu''}}|\mathcal{M}_\alpha(g)(y_{\tau''}^{k'',\nu''})|^\delta\chi_{Q_{\tau''}^{k'',\nu''}}\bigg)(z) \leqslant \mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)(z).$$

Inserting this and (3.16) into (3.15), and invoking the fact

$$\sum_{k''=j+1}^{\infty} 2^{-|j-k''|\sigma+[j-k'']n(1-1/\delta)} \lesssim 1,$$

we finally obtain that

$$\mathrm{J}_2 \lesssim \sum_{k'\in\mathbb{Z}} 2^{-|j-k'|\sigma+[(j\wedge k')-k']n(1-1/\theta)}\{\mathcal{M}(|G_{k'}(f)|^\theta)(y_\tau^{k,\nu})\}^{1/\theta}\{\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)(y_\tau^{k,\nu})\}^{1/\delta}. \tag{3.17}$$

The estimate of J_1 is similar to that of J_2. In fact, an argument similar to (3.14) yields that for all $z \in Q_{\tau''}^{j,\nu''}$,

$$|G_k(D_j\widetilde{G}_{k'}(\cdot, y_{\tau'}^{k',\nu'})S_j\widetilde{P}_j(\cdot,z))(y_\tau^{k,\nu})| \lesssim 2^{-|j-k'|\sigma}\,\mathcal{K}_\sigma(j\wedge k';\, y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})\,\mathcal{K}_\sigma(j;\, y_\tau^{k,\nu}, z).$$

Moreover, by Lemma 2.7(a), it is not difficult to see that for all z, $y_{\tau''}^{j,\nu''} \in Q_{\tau''}^{j,\nu''}$,

$$\mathcal{K}_\sigma(j;\, y_\tau^{k,\nu}, z) \sim \mathcal{K}_\sigma(j;\, y_\tau^{k,\nu}, y_{\tau''}^{j,\nu''}) \tag{3.18}$$

with equivalence constants independent of $y_{\tau''}^{j,\nu''}$ and z. From Proposition 3.2, it follows that for any fixed $y_{\tau''}^{j,\nu''} \in Q_{\tau''}^{j,\nu''}$, we have

$$\begin{aligned}&\int_{Q_{\tau''}^{j,\nu''}} G_k(D_j\widetilde{G}_{k'}(\cdot, y_{\tau'}^{k',\nu'})S_j\widetilde{P}_j(\cdot,z))(y_\tau^{k,\nu})\,d\mu(z)\\ &\quad\lesssim 2^{-|j-k'|\sigma}\mu(Q_{\tau''}^{j,\nu''})\,\mathcal{K}_\sigma(j\wedge k';\, y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})\,\mathcal{K}_\sigma(j;\, y_\tau^{k,\nu}, y_{\tau''}^{j,\nu''}).\end{aligned} \tag{3.19}$$

From (3.8), the definition of $Q_{\tau''}^{j,\nu''}$ and Lemma 3.3(iv), we deduce that for any given $\alpha \in [1/3,\infty)$,

$$\left|\frac{1}{\mu(Q_{\tau''}^{j,\nu''})}\int_{Q_{\tau''}^{j,\nu''}} P_j(g)(z)\,d\mu(z)\right| \leqslant \mathcal{M}_\alpha(g)(y_{\tau''}^{j,\nu''}). \tag{3.20}$$

Applying (3.19) and (3.20), and following the same argument as that in the proof of (3.17), we also obtain that J_1 has the same upper bound estimate as in (3.17).

Combining the estimates of J_1 and J_2 yields that for all $j \in \mathbb{Z}$, $k \geqslant j+1$, $\tau \in I_k$ and $\nu \in \{1, 2, \ldots, N(k,\tau)\}$,

$$\begin{aligned}\big|G_k(D_j(f)S_j(g))(y_\tau^{k,\nu})\big| &\lesssim \sum_{k'\in\mathbb{Z}} 2^{-|j-k'|\sigma+[(j\wedge k')-k']n(1-1/\theta)}\\ &\quad\times\{\mathcal{M}(|G_{k'}(f)|^\theta)(y_\tau^{k,\nu})\}^{1/\theta}\{\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)(y_\tau^{k,\nu})\}^{1/\delta}.\end{aligned} \tag{3.21}$$

Now we estimate Z_2^N. Using (a) and (b) of Remark 2.9, we obtain that the kernel of $G_\ell E_j$ has the same size condition as those of $2^{-|j-\ell|\sigma}E_{j\wedge\ell}$. By this and $(j\wedge\ell) < k$, we further apply Remark 2.9(b) and obtain that for all $j, \ell \in \mathbb{Z}$, $k \geqslant j+1$, $\tau \in I_k$ and $\nu \in \{1, 2, \ldots, N(k,\tau)\}$,

$$\begin{aligned}|G_\ell E_j\widetilde{G}_k(x, y_\tau^{k,\nu})| &\lesssim 2^{-|j-\ell|\sigma}2^{-|(j\wedge\ell)-k|\sigma}\,\mathcal{K}_\sigma((j\wedge\ell)\wedge k;\, x, y_\tau^{k,\nu})\\ &\lesssim 2^{-|j-\ell|\sigma}2^{-|j-k|\sigma}\,\mathcal{K}_\sigma(j\wedge\ell;\, x, y_\tau^{k,\nu}).\end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22) yields that

$$|\mathrm{Z}_2^N| \lesssim \sum_{|j|\leqslant N}\sum_{k=j+1}^{\infty}\sum_{k'\in\mathbb{Z}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{-|j-\ell|\sigma}2^{-|j-k|\sigma}2^{-|j-k'|\sigma+[(j\wedge k')-k']n(1-1/\theta)}\mu(Q_\tau^{k,\nu})$$

$$\times \mathcal{K}_\sigma(j\wedge\ell;\, x, y_\tau^{k,\nu})\{\mathcal{M}(|G_{k'}(f)|^\theta)(y_\tau^{k,\nu})\}^{1/\theta}\{\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)(y_\tau^{k,\nu})\}^{1/\delta}.$$

Then using the arbitrariness of $y_\tau^{k,\nu}\in Q_\tau^{k,\nu}$, Lemma 3.8 and a similar argument as that used in (3.16), we obtain that for any given $\lambda\in(n/(n+\sigma),1]$,

$$\begin{aligned}
|\mathrm{Z}_2^N| &\lesssim \sum_{|j|\leqslant N}\sum_{k=j+1}^{\infty}\sum_{k'\in\mathbb{Z}} 2^{-|j-\ell|\sigma}2^{-|j-k|\sigma}2^{-|j-k'|\sigma+[(j\wedge k')-k']n(1-1/\theta)}2^{[(j\wedge\ell)-k]n(1-1/\lambda)}\\
&\quad\times\left\{\mathcal{M}\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\inf_{y_\tau^{k,\nu}\in Q_\tau^{k,\nu}}[\{\mathcal{M}(|G_{k'}(f)|^\theta)(y_\tau^{k,\nu})\}^{\lambda/\theta}\right.\right.\\
&\quad\left.\left.\times\{\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)(y_\tau^{k,\nu})\}^{\lambda/\delta}]\chi_{Q_\tau^{k,\nu}}\right)(x)\right\}^{1/\lambda}\\
&\lesssim \sum_{|j|\leqslant N}\sum_{k=j+1}^{\infty}\sum_{k'\in\mathbb{Z}} 2^{-|j-\ell|\sigma}2^{-|j-k|\sigma}2^{-|j-k'|\sigma+[(j\wedge k')-k']n(1-1/\theta)}2^{[(j\wedge\ell)-k]n(1-1/\lambda)}\\
&\quad\times\{\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)\}^{1/\lambda},
\end{aligned}$$

which combined with the fact that $\sum_{k=j+1}^{\infty}2^{-|j-k|\sigma}2^{[(j\wedge\ell)-k]n(1-1/\lambda)}\lesssim 2^{[(j\wedge\ell)-j]n(1-1/\lambda)}$ implies that

$$\begin{aligned}
|\mathrm{Z}_2^N| &\lesssim \sum_{|j|\leqslant N}\sum_{k'\in\mathbb{Z}} 2^{-|j-\ell|\sigma+[(j\wedge\ell)-j]n(1-1/\lambda)}2^{-|j-k'|\sigma+[(j\wedge k')-k']n(1-1/\theta)}\\
&\quad\times\{\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)\}^{1/\lambda}. \qquad (3.23)
\end{aligned}$$

From this and Hölder's inequality, we deduce that

$$\begin{aligned}
\sum_{\ell\in\mathbb{Z}}\left|\mathrm{Z}_2^N\right|^2 &\lesssim \sum_{\ell\in\mathbb{Z}}\left[\sum_{|j|\leqslant N}\sum_{k'\in\mathbb{Z}}2^{-|j-\ell|[\sigma-n(1/\lambda-1)]}2^{-|j-k'|[\sigma-n(1/\theta-1)]}\right]\\
&\quad\times\sum_{j\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}2^{-|j-\ell|[\sigma-n(1/\lambda-1)]}2^{-|j-k'|[\sigma-n(1/\theta-1)]}\\
&\quad\times\{\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)\}^{2/\lambda}. \qquad (3.24)
\end{aligned}$$

It is easy to verify that

$$\sum_{|j|\leqslant N}\sum_{k'\in\mathbb{Z}}2^{-|j-\ell|[\sigma-n(1/\lambda-1)]}2^{-|j-k'|[\sigma-n(1/\theta-1)]}\lesssim 1$$

and

$$\sum_{\ell\in\mathbb{Z}}\sum_{|j|\leqslant N}2^{-|j-\ell|[\sigma-n(1/\lambda-1)]}2^{-|j-k'|[\sigma-n(1/\theta-1)]}\lesssim 1.$$

Inserting these two estimates in (3.24) yields that

$$\sum_{\ell\in\mathbb{Z}}|\mathrm{Z}_2^N|^2\lesssim\sum_{k'\in\mathbb{Z}}[\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)]^{2/\lambda}. \qquad (3.25)$$

Since $\sigma\in(0,\epsilon_1\wedge\epsilon_2)$ and $p,\ q,\ r>n/(n+\epsilon)$, we choose $n/(n+\sigma)<\delta<(q\wedge 1)$, $n/(n+\sigma)<\theta<(p\wedge 1)$ and $n/(n+\sigma)<\lambda<(r\wedge 1)$. Then $r/\lambda>1$, $p/\theta>1$ and $q/\delta>1$. Using (3.25) and the Fefferman-Stein vector-valued maximal function inequality on spaces of homogeneous type (see [12, Theorem 1.2], [17, Lemma 3.14] or [21, (2.11)]) together with Hölder's inequality, we obtain

$$\begin{aligned}
\left\|\left\{\sum_{\ell\in\mathbb{Z}}|\mathrm{Z}_2^N|^2\right\}^{1/2}\right\|_{L^r(\mathcal{X})} &\lesssim \left\|\left\{\sum_{k'\in\mathbb{Z}}[\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})]^{2/\lambda}\right\}^{\lambda/2}\right\|_{L^{r/\lambda}(\mathcal{X})}^{1/\lambda}\\
&\lesssim \left\|\left\{\sum_{k'\in\mathbb{Z}}[\mathcal{M}(|G_{k'}(f)|^\theta)]^{2/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{2/\delta}\right\}^{\lambda/2}\right\|_{L^{r/\lambda}(\mathcal{X})}^{1/\lambda}
\end{aligned}$$

$$
\begin{aligned}
&\sim \left\| \left\{ \sum_{k'\in\mathbb{Z}} [\mathcal{M}(|G_{k'}(f)|^\theta)]^{2/\theta} \right\}^{1/2} [\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{1/\delta} \right\|_{L^r(\mathcal{X})} \\
&\lesssim \left\| \left\{ \sum_{k'\in\mathbb{Z}} [\mathcal{M}(|G_{k'}(f)|^\theta)]^{2/\theta} \right\}^{1/2} \right\|_{L^p(\mathcal{X})} \|[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{1/\delta}\|_{L^q(\mathcal{X})} \\
&\lesssim \left\| \left\{ \sum_{k'\in\mathbb{Z}} |G_{k'}(f)|^2 \right\}^{1/2} \right\|_{L^p(\mathcal{X})} \|\mathcal{M}_\alpha(g)\|_{L^q(\mathcal{X})} \\
&\lesssim \|f\|_{H^p(\mathcal{X})} \|g\|_{H^q(\mathcal{X})},
\end{aligned} \tag{3.26}
$$

where in the last step we used (2.4) and (2.6).

To obtain (3.10), we still need to show

$$
\left\| \left\{ \sum_{\ell\in\mathbb{Z}} |\mathrm{Z}_1^N|^2 \right\}^{1/2} \right\|_{L^r(\mathcal{X})} \lesssim \|f\|_{H^p(\mathcal{X})} \|g\|_{H^q(\mathcal{X})}. \tag{3.27}
$$

Observe that once we have obtained

$$
\begin{aligned}
|\mathrm{Z}_1^N| \lesssim \sum_{j\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} & 2^{-|j-\ell|\sigma + [(j\wedge\ell)-j]n(1-1/\lambda)} 2^{-|j-k'|\sigma + [(j\wedge k')-k']n(1-1/\theta)} \\
& \times \{\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta} [\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)\}^{1/\lambda},
\end{aligned} \tag{3.28}
$$

then (3.27) follows just by the arguments analogous to those used in the proof of (3.24) through (3.26). To prove (3.28), by an argument similar to (3.22), we use Remark 2.9 and obtain that for a certain $\sigma \in (0, \epsilon_1 \wedge \epsilon_2)$,

$$
|G_\ell E_j \widetilde{P}_j(x,z)| \lesssim 2^{-|j-\ell|\sigma} \mathcal{K}_\sigma(j\wedge\ell; x, z). \tag{3.29}
$$

Notice that for all $j\in\mathbb{Z}$, $\tau \in I_j$, $\nu \in \{1,2,\ldots,N(j,\tau)\}$, $x\in\mathcal{X}$ and all z, $y_\tau^{j,\nu} \in Q_\tau^{j,\nu}$,

$$
\mathcal{K}_\sigma(j\wedge\ell; x, z) \sim \mathcal{K}_\sigma(j\wedge\ell; x, y_\tau^{j,\nu}). \tag{3.30}
$$

Therefore, the combination of (3.29) and (3.30) yields that for any fixed $y_\tau^{j,\nu} \in Q_\tau^{j,\nu}$,

$$
\left| \int_{Q_\tau^{j,\nu}} G_\ell E_j \widetilde{P}_j(x,z)\, d\mu(z) \right| \lesssim 2^{-|j-\ell|\sigma} \mu(Q_\tau^{j,\nu})\, \mathcal{K}_\sigma(j\wedge\ell; x, y_\tau^{j,\nu}) \tag{3.31}
$$

with constant independent of $y_\tau^{j,\nu} \in Q_\tau^{j,\nu}$.

For any $z \in Q_\tau^{j,\nu}$, the estimate of $P_j(D_j(f)S_j(g))(z)$ is similar to that of (3.21). In fact, using (3.30) and proceeding as for estimates of J_1 and J_2 (see (3.13) through (3.21)), we also obtain that for any fixed $y_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ and all $z \in Q_\tau^{j,\nu}$,

$$
\begin{aligned}
|P_j(D_j(f)S_j(g))(z)| \lesssim \sum_{k'\in\mathbb{Z}} & 2^{-|j-k'|\sigma + [(j\wedge k')-k']n(1-1/\theta)} \\
& \times \{\mathcal{M}(|G_{k'}(f)|^\theta)(y_\tau^{j,\nu})\}^{1/\theta} \{\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)(y_\tau^{j,\nu})\}^{1/\delta},
\end{aligned} \tag{3.32}
$$

where the constant is independent of $z, y_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ and $j\in\mathbb{Z}$.

We insert (3.31) and (3.32) in the expression for Z_1^N. Using an argument similar to that used in the proof of estimate (3.23), we obtain that (3.26) also holds for Z_1^N. The details are omitted. Thus (3.27) holds for Z_1^N. Therefore we prove that (3.9) holds.

We still need to verify that $\Pi_N(f,g)$ is a Cauchy sequence in $H^r(\mathcal{X})$. Indeed, by (2.4) and Remark 2.6(ii), it suffices to show that for all $N, M \in \mathbb{N}$ and $N < M$, when $N\to\infty$,

$$
\|\Pi_M(f,g) - \Pi_N(f,g)\|_{H^r(\mathcal{X})} \sim \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \left| \sum_{N<|j|\leqslant M} G_\ell E_j\,(D_j(f)S_j(g)) \right|^2 \right\}^{1/2} \right\|_{L^r(\mathcal{X})} \to 0.
$$

To see this, observe that it suffices to show that when $N\to\infty$,

$$\bigg\|\bigg\{\sum_{\ell\in\mathbb{Z}}\big|\mathrm{Z}_1^N-\mathrm{Z}_1^M\big|^2\bigg\}^{1/2}\bigg\|_{L^r(\mathcal{X})}+\bigg\|\bigg\{\sum_{\ell\in\mathbb{Z}}\big|\mathrm{Z}_2^N-\mathrm{Z}_2^M\big|^2\bigg\}^{1/2}\bigg\|_{L^r(\mathcal{X})}\to 0. \tag{3.33}$$

We show (3.23) by proceeding as in the proof of (3.10) but with $\sum_{|j|\leqslant N}$ there replaced by $\sum_{N<|j|\leqslant M}$, and we obtain

$$\sum_{\ell\in\mathbb{Z}}\big|\mathrm{Z}_2^N-\mathrm{Z}_2^M\big|^2\lesssim\sum_{k'\in\mathbb{Z}}\sum_{N<|j|\leqslant M}2^{-|j-k'|[\sigma-n(1/\theta-1)]}[\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)]^{2/\lambda}.$$

Then we divide the summation over k' into two parts: $\sum_{|k'|>N/2}$ and $\sum_{|k'|\leqslant N/2}$. Observe that for all $k'\in\mathbb{Z}$, $\sum_{N<|j|\leqslant M}2^{-|j-k'|[\sigma-n(1/\theta-1)]}\lesssim 1$, and in particular, if $|k'|\leqslant N/2$ we have

$$\sum_{N<|j|\leqslant M}2^{-|j-k'|[\sigma-n(1/\theta-1)]}\lesssim 2^{-N[\sigma-n(1/\theta-1)]/2}.$$

From these, we further deduce that

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}}\big|\mathrm{Z}_2^N-\mathrm{Z}_2^M\big|^2\lesssim&\sum_{|k'|>N/2}[\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)]^{2/\lambda}\\&+2^{-N[\sigma-n(1/\theta-1)]/2}\sum_{k'\in\mathbb{Z}}[\mathcal{M}([\mathcal{M}(|G_{k'}(f)|^\theta)]^{\lambda/\theta}[\mathcal{M}([\mathcal{M}_\alpha(g)]^\delta)]^{\lambda/\delta})(x)]^{2/\lambda}.\end{aligned}$$

This combined with an argument similar to (3.26) yields that

$$\lim_{N<M,\,N\to\infty}\bigg\|\bigg\{\sum_{\ell\in\mathbb{Z}}|\mathrm{Z}_2^N-\mathrm{Z}_2^M|^2\bigg\}^{1/2}\bigg\|_{L^r(\mathcal{X})}=0.$$

Likewise, $\lim_{N<M,\,N\to\infty}\|\{\sum_{\ell\in\mathbb{Z}}|\mathrm{Z}_1^N-\mathrm{Z}_1^M|^2\}^{1/2}\|_{L^r(\mathcal{X})}=0$. Hence, (3.23) holds and we obtain that $\{\Pi_N(f,g)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $H^r(\mathcal{X})$ whenever $f\in H^p(\mathcal{X})$ and $g\in H^q(\mathcal{X})$. This finishes the proof of Theorem 1.2.

4 Proof of Theorem 1.3

We begin with some known results related to singular integrals on spaces of homogeneous type; see [6]. Let T be a linear operator bounded on $L^2(\mathcal{X})$ with kernel K, which is locally integrable on $(\mathcal{X}\times\mathcal{X})\setminus\{(x,x):\,x\in\mathcal{X}\}$. Assume that for any $f\in L^\infty(\mathcal{X})$ with bounded support and $x\notin\operatorname{supp}f$,

$$Tf(x)=\int_{\mathcal{X}}K(x,y)f(y)\,d\mu(y). \tag{4.1}$$

Moreover, there exist positive constants C and $\sigma\in(0,1]$ such that for all $x,\,y\in\mathcal{X}$,

$$|K(x,y)|\leqslant C\frac{1}{V(x,y)}; \tag{4.2}$$

and that when $d(x,x')\leqslant d(x,y)/2$,

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\leqslant C\frac{d(x,x')^\sigma}{d(x,y)^\sigma V(x,y)}. \tag{4.3}$$

Using the Calderón-Zygmund decomposition method, Coifman and Weiss [6, pp. 74–75] proved that if T satisfying (4.1) through (4.3) is bounded on $L^2(\mathcal{X})$, then T is bounded on $L^p(\mathcal{X})$ for all $p\in(1,\infty)$, and also bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$.

Recall that any locally integrable function f is said to be in $\mathrm{BMO}(\mathcal{X})$ if and only if

$$\|f\|_{\mathrm{BMO}(\mathcal{X})} := \sup_B \frac{1}{\mu(B)} \int_B |f(x) - f_B| \, d\mu(x) < \infty,$$

where the supremum is taken over all balls B of $\mathcal{X}$ and $f_B := \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y)$; see [7] for more details. For the operator T as above, by using the boundedness of T on $L^2(\mu)$ and the regular conditions of its kernel K, and proceeding as in [23, p. 156, Proposition 1], it is easy to see that T is bounded from $L^\infty(\mathcal{X})$ to $\mathrm{BMO}(\mathcal{X})$.

Recall that $T^*1 = 0$ means that for any $f \in L^2(\mathcal{X})$ with bounded support and

$$\int_{\mathcal{X}} f(x) \, d\mu(x) = 0,$$

we have that $\int_{\mathcal{X}} Tf(x) \, d\mu(x) = 0$. The following boundedness of T satisfying (4.1) through (4.3) on the Hardy spaces $H^p(\mathcal{X})$ was established in [28, Proposition 3.1].

Lemma 4.1. *Let T be a linear operator bounded on $L^2(\mathcal{X})$ with kernel K, which is locally integrable on $\mathcal{X} \times \mathcal{X} \setminus \{(x, x) : x \in \mathcal{X}\}$ and satisfies* (4.1) *through* (4.3)*. Moreover, assume that $T^*1 = 0$. Then T is bounded on $H^p(\mathcal{X})$ for all $p \in (n/(n + \sigma), 1]$, where σ is as in* (4.3)*.*

Proof of Theorem 1.3. To show (a) through (d), we temporarily freeze $b \in \mathrm{BMO}(\mathcal{X})$. By (1.4), the operator $\Pi(b, \cdot)$, namely,

$$\Pi(b, g)(x) := \sum_{j \in \mathbb{Z}} E_j(D_j(b) S_j(g))(x) \quad \text{for all } g \in (\mathcal{G}_0^\epsilon(\beta, \gamma))',$$

has kernel

$$K_b(x, y) = \sum_{j=-\infty}^{\infty} \int_{\mathcal{X}} E_j(x, z) D_j(b)(z) S_j(z, y) \, d\mu(z) \quad \text{for all } x,\, y \in \mathcal{X}.$$

It was proved in [17, Theorem 5.56] that $\Pi(b, \cdot)$ is bounded on $L^2(\mathcal{X})$ with operator norm a constant multiple of $\|b\|_{\mathrm{BMO}(\mathcal{X})}$, and moreover, $\Pi(b, \cdot)^*1 = 0$, the kernel K_b is a standard kernel satisfying that there exists a positive constant C such that for all $x,\, y \in \mathcal{X}$,

$$|K_b(x, y)| \leqslant C \|b\|_{\mathrm{BMO}(\mathcal{X})} \frac{1}{V(x, y)}; \tag{4.4}$$

and that when $d(x, x') \leqslant d(x, y)/2$,

$$|K_b(x, y) - K_b(x', y)| + |K_b(y, x) - K_b(y, x')| \leqslant C \|b\|_{\mathrm{BMO}(\mathcal{X})} \frac{d(x, x')^\epsilon}{d(x, y)^\epsilon V(x, y)}. \tag{4.5}$$

We remark that to obtain (4.5), we only need the regular condition of E_j with respect to the first variable and the regular condition of S_j with respect to the second variable. Using these facts and the previous discussion in this section, we know that (a) through (d) of Theorem 1.3 hold.

Now we show (e) of Theorem 1.3 for the case $p \in (1, \infty)$. For any $N \in \mathbb{N}$, $f \in L^p(\mathcal{X})$ and $g \in L^\infty(\mathcal{X})$, set $\Pi_N(f, g)(x) := \sum_{|j| \leqslant N} D_\ell E_j(D_j(f) S_j(g))(x)$. By Remark 2.6, Lemma 2.7(c) and (2.8) together with Hölder's inequality, we have that for $N,\, M \in \mathbb{N}$ and $N < M$,

$$\begin{aligned}
\|\Pi_N(f, g) - \Pi_M(f, g)\|_{L^p(\mathcal{X})} &\lesssim \left\| \left\{ \sum_{\ell \in \mathbb{Z}} \left| \sum_{N < |j| \leqslant M} D_\ell E_j(D_j(f) S_j(g)) \right|^2 \right\}^{1/2} \right\|_{L^p(\mathcal{X})} \\
&\lesssim \left\| \left\{ \sum_{\ell \in \mathbb{Z}} \left| \sum_{N < |j| \leqslant M} 2^{-|\ell - j|\epsilon'} \mathcal{M}\Big(D_j(f) S_j(g)\Big) \right|^2 \right\}^{1/2} \right\|_{L^p(\mathcal{X})} \\
&\lesssim \left\| \left\{ \sum_{N < |j| \leqslant M} |\mathcal{M}(D_j(f) S_j(g))|^2 \right\}^{1/2} \right\|_{L^p(\mathcal{X})}.
\end{aligned} \tag{4.6}$$

Furthermore, the Fefferman-Stein vector-valued maximal function inequality on $\mathcal{X}$ and

$$\sup_{j\in\mathbb{Z}}\|S_j(g)\|_{L^\infty(\mathcal{X})}\lesssim\|g\|_{L^\infty(\mathcal{X})}$$

imply that the last quantity in (4.6) above is bounded by a constant multiple of

$$\left\|\left\{\sum_{N<|j|\leqslant M}|D_j(f)S_j(g)|^2\right\}^{1/2}\right\|_{L^p(\mathcal{X})}\lesssim\|g\|_{L^\infty(\mathcal{X})}\left\|\left\{\sum_{N<|j|\leqslant M}|D_j(f)|^2\right\}^{1/2}\right\|_{L^p(\mathcal{X})},$$

which tends to 0 as $N\to\infty$. This shows that $\Pi_N(f,g)$ is a Cauchy sequence in $L^p(\mathcal{X})$ and, hence, $\Pi_N(f,g)$ converges to an element in $L^p(\mathcal{X})$ which we denote by $\Pi(f,g)$. The previous argument also proves that

$$\|\Pi(f,g)\|_{L^p(\mathcal{X})}\lesssim\|g\|_{L^\infty(\mathcal{X})}\left\|\left\{\sum_{j\in\mathbb{Z}}|D_j(f)|^2\right\}^{1/2}\right\|\lesssim\|f\|_{L^p(\mathcal{X})}\|g\|_{L^\infty(\mathcal{X})}.$$

To show (f) and the remaining part of (e) of Theorem 1.3, we freeze the second function $g\in L^\infty(\mathcal{X})$. Proceeding as in [17, Theorem 5.56], we obtain that the kernel of $\Pi(\cdot,g)$, say

$$K_g(x,y)=\sum_{j=-\infty}^{\infty}\int_{\mathcal{X}}E_j(x,z)D_j(z,y)S_j(g)(z)\,d\mu(z),$$

is also a standard kernel satisfying (4.4) and (4.5), with $\|b\|_{\mathrm{BMO}(\mathcal{X})}$ there replaced by $\|g\|_{L^\infty(\mathcal{X})}$. Then using the arguments in the beginning of this section and Lemma 4.1 together with the boundedness of $\Pi(\cdot,g)$ on $L^2(\mathcal{X})$, we obtain (e) and (f) of Theorem 1.3. This finishes the proof of Theorem 1.3.

Remark 4.2. From (2.5) and the existence of 1- AOTI with bounded support together with the independent of ϵ of $H^p(\mathcal{X})$ (see [27]), we deduce that the Hardy space $H^p(\mathcal{X})$ is well defined for all $p\in(n/(n+1),1]$. From this, we deduce that the results of Theorems 1.2 and 1.3 are valid for p, q, $r\in(n/(n+1),\infty)$.

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