Multiple Weighted Norm Inequalities for Maximal Multilinear Singular Integrals with Non-Smooth Kernels

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Abstract Weighted norm inequalities for maximal truncated operators of multilinear singular integrals with non-smooth kernels in the sense of Duong, Grafakos, and Yan are obtained; this class of operators extends the class of multilinear Calderón-Zygmund operators introduced by Coifman and Meyer and includes the higher order commutators of Calderón. The weighted norm inequalities obtained in this work are with respect to the new class of multiple weights of Lerner, Ombrosi, Pérez, Torres, and Trujillo-González. The key ingredient in the proof is the introduction of a new multi-sublinear maximal operator that plays the role of the Hardy-Littlewood maximal function in a version of Cotlar's inequality. As applications of these results, new weighted estimates for the m-th order Calderón commutators and their maximal counterparts are deduced.

1 Introduction

We consider multilinear operators T initially defined on the m-fold product of Schwartz spaces on $\mathbb{R}^n$ and taking values into the space of tempered distributions,

$$T : \overbrace{\mathcal{S} \times \cdots \times \mathcal{S}}^{m \text{ times}} \to \mathcal{S}'.$$

Every such operator is associated with a distribution kernel on $(\mathbb{R}^n)^{m+1}$. Throughout the paper, we assume that the distribution kernel coincides with a function K defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, and T is associated with the kernel K in the following way:

$$T(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m, \tag{1.1}$$

whenever $x \notin \cap_{j=1}^m \operatorname{supp} f_j$ and $f_1, \cdots, f_m$ are C^∞ functions with compact support.

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Recall that T is said to be an *m-linear Calderón-Zygmund operator* if it satisfies (1.1), moreover, there exist positive constants A and ϵ such that K satisfies the size condition

$$|K(y_0, y_1, \cdots, y_m)| \le \frac{A}{\left(\sum_{k,\ell=0}^m |y_k - y_\ell|\right)^{mn}} \tag{1.2}$$

for all $(y_0, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $y_0 \ne y_j$ for some $j \in \{1, 2, \cdots, m\}$; and the regularity condition

$$|K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y_j', \cdots, y_m)| \le \frac{A|y_j - y_j'|^\epsilon}{\left(\sum_{k,\ell=0}^m |y_k - y_\ell|\right)^{mn+\epsilon}}, \tag{1.3}$$

whenever $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2}\max_{0\le k\le m} |y_j - y_k|$. We denote by m-$CZK(A, \epsilon)$ the collection of all kernels K satisfying (1.2) and (1.3). A thorough study of such operators was undertaken in [8].

Recently, Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [10] developed a multiple weight theory that adapts to the multilinear Calderón-Zygmund operators. Precisely, for $\vec{P} = (p_1, \cdots, p_m)$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 \le p_1, \cdots, p_m < \infty$, we say that $\vec{w} = (w_1, \cdots, w_m)$ belongs to $A_{\vec{P}}$ if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_{\text{cubes } Q} \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right) \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{p/p_j'} < \infty, \tag{1.4}$$

where $\nu_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$; when $p_j = 1$, $(\frac{1}{|Q|}\int_Q w_j^{1-p_j'})^{p/p_j'}$ is understood as $(\inf_Q w_j)^{-p}$. In [10, Corollary 3.9] the following multiple-weight estimate concerning the classical multilinear Calderón-Zygmund operators was obtained:

Theorem A Let T be an m-linear operator with a kernel $K \in m$-$CZK(A, \epsilon)$. Suppose that for some $1 \le q_1, \cdots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, T maps $L^{q_1} \times \cdots \times L^{q_m}$ to L^q. Let $1 \le p_1, \cdots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{P} = (p_1, \cdots, p_m)$, $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P}}$ and $\nu_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$. Then

(i) T can be extended to a bounded operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ if all the exponents p_j are strictly greater than 1;

(ii) T can be extended to a bounded operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(\nu_{\vec{w}})$ if some of the exponents p_j are equal to 1.

In this paper, we replace the regularity condition (1.3) by weaker regularity conditions on the kernel K given by assumptions (H1), (H2) and (H3) described below. These assumptions were introduced in the work of Duong, Grafakos, and Yan; see [6, 5].

Let $\{A_t\}_{t>0}$ be a class of integral operators, which play the role of the approximation to the identity; see [4]. We always assume that the operators A_t are associated with kernels $a_t(x, y)$ in the sense that for all $f \in \cup_{p\in[1,\infty]} L^p$ and $x \in \mathbb{R}^n$,

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y)\, dy,$$

and that the kernels $a_t(x,y)$ satisfy the following size conditions

$$|a_t(x,y)| \le h_t(x,y) := t^{-n/s} h\left(\frac{|x-y|}{t^{1/s}}\right), \tag{1.5}$$

where s is a positive fixed constant and h is a positive, bounded, decreasing function satisfying that for some $\eta > 0$,

$$\lim_{r\to\infty} r^{n+\eta} h(r^s) = 0. \tag{1.6}$$

Assumption (H1) Assume that for each $j \in \{1, \cdots, m\}$, there exist operators $\{A_t^{(j)}\}_{t>0}$ with kernels $\{a_t\}_{t>0}$ that satisfy conditions (1.5) and (1.6) with constants s and η, and there exist kernels $\{K_t^{(j)}\}_{t>0}$ such that for all $t > 0$,

$$\begin{aligned} &T(f_1, \cdots, A_t^{(j)} f_j, \cdots, f_m)(x) \\ &= \int_{(\mathbb{R}^n)^m} K_t^{(j)}(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m \end{aligned}$$

for all $x \notin \cap_{j=1}^m \operatorname{supp} f_j$ and $f_1, \cdots, f_m$ are C^∞ functions with compact support. Assume also that there exist a non-negative function $\phi \in C(\mathbb{R})$ with $\operatorname{supp} \phi \subset [-1,1]$ and a positive constant ϵ such that for all $x, y_1, \cdots, y_m \in \mathbb{R}^n$ and $t > 0$ we have

$$\begin{aligned} &|K(x, y_1, \cdots, y_m) - K_t^{(j)}(x, y_1, \cdots, y_m)| \\ &\le \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{1\le k\le m, k\ne j} \phi\left(\frac{|y_j - y_k|}{t^{1/s}}\right) + \frac{A t^{\epsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\epsilon}} \end{aligned}$$

for some $A > 0$, whenever $2t^{1/s} \le |x - y_j|$.

Denote by m-$GCZK_0(A, s, \eta, \epsilon)$ the collection of all kernels K that satisfy the size estimate (1.2) and assumption (H1) with parameters m, A, s, η, ϵ.

The following weak type endpoint estimate, proved in [6, Proposition 2.1], is the analogous version of the m-linear Calderón-Zygmund theorem proved in [8, Theorem 1].

Theorem B Let T be an m-linear operator with a kernel $K \in m$-$GCZK_0(A, s, \eta, \epsilon)$ as in (1.1). Suppose that T maps $L^{q_1} \times \cdots \times L^{q_m}$ to L^q with norm $\|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^q}$, where $1 \le q_1, \cdots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Then T can be extended to a bounded operator from the m-fold product space $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$, with norm at most a positive constant multiple of $A + \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^q}$.

Recall that the j-th transpose T^{*j} of T is defined via

$$\langle T^{*j}(f_1, \cdots, f_m), h\rangle = \langle T(f_1, \cdots, f_{j-1}, h, f_{j+1}, \cdots, f_m), f_j\rangle$$

for all $f_1, \cdots, f_m, h$ in $\mathcal{S}(\mathbb{R}^n)$. Notice that the kernel K^{*j} of T^{*j} is related to the kernel K of T via the identity

$$K^{*j}(x, y_1, \cdots, y_{j-1}, y_j, y_{j+1}, \cdots, y_m) = K(y_j, y_1, \cdots, y_{j-1}, x, y_{j+1}, \cdots, y_m).$$

If a multilinear operator T maps a product of Banach spaces, $X_1 \times \cdots \times X_m$, to another Banach space X, then the transpose T^{*j} maps $X_1 \times \cdots \times X_{j-1} \times X \times X_{j+1} \times \cdots \times X_m$ to X_j. Moreover, the norms of T and T^{*j} are equal. To maintain uniform notation, we may occasionally denote T by T^{*0} and K by K^{*0}.

Assumption (H2) Assume that for each $i \in \{1, \cdots, m\}$, there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $\{a_t^{(i)}\}_{t>0}$ that satisfy conditions (1.5) and (1.6) with constants s and η, and that for every $j \in \{0, 1, \cdots, m\}$, there exist kernels $\{K_t^{*,j,(i)}\}_{t>0}$ such that for all $t > 0$ and all $f_1, \cdots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$ with $\cap_{k=1}^m \operatorname{supp} f_k \cap \operatorname{supp} g = \emptyset$,

$$\begin{aligned}&\langle T^{*j}(f_1, \cdots, A_t^{(i)} f_i, \cdots, f_m), g\rangle \\ &\quad = \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^m} K_t^{*,j,(i)}(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) g(x)\, dy_1 \cdots dy_m\, dx.\end{aligned}$$

Moreover, assume that there exist a non-negative function $\phi \in C(\mathbb{R})$ with $\operatorname{supp} \phi \subset [-1, 1]$ and a positive constant ϵ so that for every $j \in \{0, 1, \cdots, m\}$, every $i \in \{1, 2, \cdots, m\}$, all $t > 0$ and all $x, y_1, \cdots, y_m \in \mathbb{R}^n$, we have

$$\begin{aligned}&\left| K^{*j}(x, y_1, \cdots, y_m) - K_t^{*,j,(i)} x, y_1, \cdots, y_m) \right| \\ &\quad \le \frac{A}{(\sum_{k=1}^m |x - y_k|)^{mn}} \sum_{1 \le k \le m,\, k \ne i} \phi\left(\frac{|y_i - y_k|}{t^{1/s}}\right) + \frac{A t^{\epsilon/s}}{(\sum_{k=1}^m |x - y_k|)^{mn+\epsilon}}\end{aligned}$$

whenever $t^{1/s} \le |x - y_i|/2$.

Kernels K that satisfy the size estimate (1.2) and assumption (H2) with parameters m, A, s, η, ϵ are called *generalized Calderón-Zygmund kernels*, and their collection is denoted by m-$GCZK(A, s, \eta, \epsilon)$. We say that T is of class m-$GCZO(A, s, \eta, \epsilon)$ if T is associated with a kernel K in m-$GCZK(A, s, \eta, \epsilon)$. The following boundedness property was proved in [6, Theorems 3.1 and 3.2].

Theorem C Assume that T is a multilinear operator in m-$GCZO(A, s, \eta, \epsilon)$. Assume that for some $1 \le q_1, q_2, \cdots, q_{m-1} \le \infty$, $q_m \in (1, \infty)$ and $q \in (0, \infty)$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, T maps $L^{q_1} \times \cdots L^{q_m}$ to L^q. Let $1/m \le p < \infty$, $1 \le p_1, \cdots, p_m \le \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then the following hold:

(i) T can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p if all the exponents p_j are strictly greater than 1;

(ii) T can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{p,\infty}$ if some of the exponents p_j are equal to 1.

In either case, the norm of T is bounded by $C(A + \|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^q})$, where C is a positive constant depending on A, s, η, ϵ.

As in [9], we define the maximal truncated operator by

$$T^*(\vec{f})(x) := \sup_{\delta > 0} |T_\delta(f_1, \cdots, f_m)(x)|, \tag{1.7}$$

where, using the notation $\vec{y} := (y_1, \cdots, y_m)$ and $d\vec{y} := dy_1 \cdots dy_m$, we set

$$T_\delta(f_1, \cdots, f_m)(x) := \int_{\sum_{j=1}^m |x-y_j|^2 > \delta^2} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\vec{y}.$$

Such maximal truncated operators for multilinear integrals were first introduced in [9]. The size condition of K (see (1.2)) implies that $T^*(f_1, \cdots, f_m)$ is well-defined pointwise when every $f_j \in L^{q_j}$ with $q_j \in [1, \infty]$; see [9, p. 1263].

It was proved in [9] that if T is an m-linear operator associated with a kernel $K \in m$-$CZK(A, \epsilon)$, then boundedness of T on one point, say $T: L^{q_1} \times \cdots \times L^{q_m} \to L^q$ for some $1 \le q_1, \cdots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, implies that $T^*: L^{p_1}(w) \times \cdots \times L^{p_m}(w) \to L^p(w)$ provided that $w \in \cap_{1 \le j \le m} A_{p_j}$, where $1 < p_1, \cdots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. In [5], all results of [9] were generalized to multilinear operators of class m-$GCZO(A, s, \eta, \epsilon)$ with additional properties that their kernels satisfy the following assumption.

Assumption (H3) Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $\{b_t\}_{t>0}$ that satisfy conditions (1.5) and (1.6) with constants s and η, and also that there exist kernels $\{K_t^{(0)}\}_{t>0}$ such that for all $x, y_1, \cdots, y_m \in \mathbb{R}^n$,

$$K_t^{(0)}(x, y_1, \cdots, y_m) = \int_{\mathbb{R}^n} K(z, y_1, \cdots, y_m) b_t(x, z)\, dz$$

and that there exist a non-negative function $\phi \in C(\mathbb{R})$ with $\operatorname{supp} \phi \subset [-1, 1]$ and a positive constant ϵ such that for all $x, y_1, \cdots, y_m \in \mathbb{R}^n$ and $t > 0$ we have

$$\begin{aligned} &|K(x, y_1, \cdots, y_m) - K_t^{(0)}(x, y_1, \cdots, y_m)| \\ &\quad \le \frac{A}{\left(\sum_{k=1}^m |x - y_k|\right)^{mn}} \sum_{1 \le k \le m} \phi\left(\frac{|x - y_k|}{t^{1/s}}\right) + \frac{A t^{\epsilon/s}}{\left(\sum_{k=1}^m |x - y_k|\right)^{mn+\epsilon}} \end{aligned} \tag{1.8}$$

for some $A > 0$, whenever $2t^{1/s} \le \max_{1 \le j \le m} |x - y_j|$. Moreover, assume that for all $x, y_1, \cdots, y_m \in \mathbb{R}^n$,

$$|K_t^{(0)}(x, y_1, \cdots, y_m)| \le \frac{A}{\left(\sum_{k=1}^m |x - y_k|\right)^{mn}}, \tag{1.9}$$

whenever $2t^{1/s} \le \min_{1 \le j \le m} |x - y_j|$, and for all $x, x', y_1, \cdots, y_m \in \mathbb{R}^n$,

$$|K_t^{(0)}(x, y_1, \cdots, y_m) - K_t^{(0)}(x', y_1, \cdots, y_m)| \le \frac{A t^{\epsilon/s}}{\left(\sum_{k=1}^m |x - y_k|\right)^{mn+\epsilon}}, \tag{1.10}$$

whenever $2|x - x'| \le t^{1/s}$ and $2t^{1/s} \le \min_{1 \le j \le m} |x - y_j|$.

Using some ideas of [9, 5], we obtain the following multiple weighted variant of [5, Proposition 3.3].

Theorem 1.1 *Assume that T is a multilinear operator in m-GCZO(A,s,η,ϵ) and its kernel satisfies assumption (H3). Moreover, for some $1 \le q_1, q_2, \cdots, q_{m-1} \le \infty$, $q_m \in (1,\infty)$ and $q \in (0,\infty)$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, T maps $L^{q_1} \times \cdots L^{q_m}$ to L^q. Let $1 \le p_1, \cdots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P}}$ with $\vec{P} = (p_1, \cdots, p_m)$, and T^* be as in (1.7). Then,*

(i) T^* can be extended to a bounded operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ if all the exponents p_j are strictly greater than 1;

(ii) T^* can be extended to a bounded operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(\nu_{\vec{w}})$ if some exponent p_j is equal to 1.

In either case, the norm of T^* is bounded by $C(A + \|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^q})$, where C is a positive constant depending on A, s, η, ϵ and $[w]_{A_{\vec{P}}}$.

We remark that under the same assumptions as in Theorem 1.1 on T, Duong et al. in [5, Proposition 3.3] proved that if $w_1 = \cdots = w_m = \nu_{\vec{w}} \in \cap_{1\le j\le m} A_{p_j}$, then (i) of Theorem 1.1 holds.

To prove Theorem 1.1, we introduce in (2.2) modified versions of the multilinear maximal operator $\mathcal{M}$ used in [10, Definition 3.1] (see also (2.1) below), called $\mathcal{M}_\ell$, with $\ell \in \{0, 1, \ldots, m\}$. In Proposition 2.1 below, we show that these new operators $\mathcal{M}_\ell$ have the same weighted boundedness properties as those of $\mathcal{M}$. The proof of Theorem 1.1 is based on the boundedness properties of $\mathcal{M}_\ell$ and the following Cotlar-type inequality:

$$T^*(\vec{f})(x) \le C \left\{ \left[M(|T(\vec{f})|^\gamma)(x) \right]^{1/\gamma} + (A + W) \sum_{\ell=1}^{m} \mathcal{M}_\ell(\vec{f})(x) \right\},$$

where γ is some small positive number; see Proposition 3.1 below. Consequently, this inequality implies that Theorem 1.1 holds when all $w_j \equiv 1$, $1 \le j \le m$. The general result of Theorem 1.1 is proved in Section 4.

The weighted boundedness of T follows from Theorem 1.1 and a standard argument as that used in [9, Corollary 3.5].

Theorem 1.2 *With the hypothesis of Theorem 1.1, the following hold:*

(i) T extends to a bounded operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ if all the exponents p_j are strictly greater than 1;

(ii) T extends to a bounded operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(\nu_{\vec{w}})$ if some exponent p_j is equal to 1.

In either case, the norm of T is bounded by $C(A + \|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^q})$, where C is a positive constant depending on A, s, η, ϵ and $[w]_{A_{\vec{P}}}$.

Remark 1.1 Since classical Calderón-Zygmund kernels K in m-$CZK(A,\epsilon)$ satisfy assumptions (H1), (H2) and (H3), Theorems 1.1 and 1.2 are also valid for the maximal truncated operators of m-linear Calderón-Zygmund singular integrals.

Finally, we apply Theorems 1.1 and 1.2 to deduce the multiple weighted norm inequalities for the commutator of A. P. Calderón and the corresponding truncated maximal operators. Recall that the m-th Calderón commutator is given by

$$\mathcal{C}_{m+1}(a_1, \cdots, a_m, f)(x) := \int_{\mathbb{R}} \frac{\prod_{j=1}^{m}(A_j(x) - A_j(y))}{(x-y)^{m+1}} f(y)\, dy, \quad \forall\, x \in \mathbb{R}, \tag{1.11}$$

where $A_j' =: a_j$ for all $j \in \{1, 2, \cdots, m\}$. These operators first appeared in the study of Cauchy integrals along Lipschitz curves and, in fact, led to the first proof of the L^2-boundedness of the latter.

When $m = 1$, it is well known that $\mathcal{C}_2$ is bounded from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$ when $1 < p, q \le \infty$ and $0 < r < \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$; and moreover, it is bounded from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^{r,\infty}(\mathbb{R})$ if either $p = 1$ or $q = 1$ and in particular it is bounded from $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ to $L^{1/2,\infty}(\mathbb{R})$; see [1] and [2]. The corresponding result that $\mathcal{C}_3$ maps $L^1 \times L^1 \times L^1 \to L^{1/3,\infty}$ was proved by Coifman and Meyer; see [3], while the analogous result for the m-th commutator $\mathcal{C}_{m+1}$, $m \ge 3$, appeared in Duong, Grafakos, and Yan [6].

These estimates are consequences of the boundedness of the commutators $\mathcal{C}_{m+1}$ at a single $m + 1$ tuple of points, such as

$$\|\mathcal{C}_{m+1}(a_1, \cdots, a_m, f)\|_{L^2} \le C_m \|f\|_{L^2} \prod_{j=1}^{m} \|a_j\|_{L^\infty}\,,$$

a classical inequality proved by Calderón for $m = 1$, and Coifman and Meyer for $m \ge 2$. Define $e(x) := \chi_{(0,\infty)}(x)$ for all $x \in \mathbb{R} \setminus \{0\}$. Since $A_j' = a_j$, we write $\mathcal{C}_{m+1}$ in (1.11) as

$$\begin{aligned}&\mathcal{C}_{m+1}(a_1, \cdots, a_m, f)(x)\\ &\quad = \int_{\mathbb{R}^{m+1}} K(x, y_1, \cdots, y_{m+1}) a_1(y_1) \cdots a_m(y_m) f_{m+1}(y_{m+1})\, dy_1 \cdots dy_{m+1},\end{aligned}$$

where the kernel K is

$$K(x, y_1, \cdots, y_{m+1}) = \frac{(-1)^{me(y_{m+1}-x)}}{(x - y_{m+1})^{m+1}} \prod_{j=1}^{m} \chi_{(\min\{x,\, y_{m+1}\},\, \max\{x,\, y_{m+1}\})}(y_j). \tag{1.12}$$

It was proved in [6, Theorem 4.1] that such a kernel K is of class $(m+1)$-$GCZK(A, 1, 1, 1)$ for some constant $A > 0$. Consequently, for $1 \le p_1, \cdots, p_{m+1} \le \infty$ and $0 < p < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_{m+1}}$, the m-th commutator $\mathcal{C}_{m+1}$ maps $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_{m+1}}(\mathbb{R})$ to $L^{p,\infty}(\mathbb{R})$; and moreover, it maps $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_{m+1}}(\mathbb{R})$ to $L^p(\mathbb{R})$ if all $p_j \in (1, \infty)$; see [6, Corollary 4.2]. It was proved in [5, Proposition 4.1] that the kernel K as in (1.12) satisfies assumption (H3). As an application of Theorems 1.1 and 1.2 we have the following conclusion.

Corollary 1.1 *Let $\mathcal{C}_{m+1}$ be as in (1.11) and $\mathcal{C}^*_{m+1}$ the corresponding maximal truncated operator as defined in (1.7). Let $1 \le p_1, \cdots, p_{m+1} < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_{m+1}}$ and $\vec{w} = (w_1, \cdots, w_{m+1}) \in A_{\vec{P}}$ with $\vec{P} = (p_1, \cdots p_{m+1})$. Then,* *(i)* *both* $\mathcal{C}_{m+1}$ *and* $\mathcal{C}^*_{m+1}$ *extend to bounded operators from* $L^{p_1}(w_1)\times\cdots\times L^{p_{m+1}}(w_{m+1})$ *to* $L^p(\nu_{\vec{w}})$ *if all the exponents* p_j *are strictly greater than* 1*;*

(ii) *both* $\mathcal{C}_{m+1}$ *and* $\mathcal{C}^*_{m+1}$ *extend to bounded operators from* $L^{p_1}(w_1)\times\cdots\times L^{p_{m+1}}(w_{m+1})$ *to* $L^{p,\infty}(\nu_{\vec{w}})$ *if some exponent* p_j *is equal to* 1*.*

2 Multilinear maximal operators

Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [10] introduced the following multi(sub)linear operator $\mathcal{M}$ in order to control the multilinear Calderón-Zygmund operators. The operator $\mathcal{M}$ is defined by that for all locally integrable functions $\vec{f}=(f_1,\cdots,f_m)$ and $x\in\mathbb{R}^n$,

$$\mathcal{M}(\vec{f})(x):=\sup_{Q\ni x}\prod_{j=1}^m\frac{1}{|Q|}\int_Q|f_j(y_j)|\,dy_j, \tag{2.1}$$

where the supremum is taken over all cubes Q containing x; see [10, Definition 3.1]. Characterizations of the multiple weights in terms of $\mathcal{M}$ are proved in Theorem 3.3 and Theorem 3.7 of [10].

Lemma 2.1 *Let* $1\le p_1,\cdots,p_m<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$*, and* $\vec{P}=(p_1,\cdots,p_m)$*.*

(i) *If* $1<p_1,\cdots,p_m<\infty$*, then* $\mathcal{M}$ *is bounded from* $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ *to* $L^p(\nu_{\vec{w}})$ *if and only if* $\vec{w}=(w_1,\cdots,w_m)\in A_{\vec{P}}$*.*

(ii) *If* $1\le p_1,\cdots,p_m<\infty$*, then* $\mathcal{M}$ *is bounded from* $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ *to* $L^{p,\infty}(\nu_{\vec{w}})$ *if and only if* $\vec{w}=(w_1,\cdots,w_m)\in A_{\vec{P}}$*.*

For any $\tau>0$ and cube Q, denote by $\ell(Q)$ the side length of Q, and by τQ the cube with the same center as Q and of side length $\tau\ell(Q)$. Motivated by [10], for any $1\le\ell<m$, we define

$$\mathcal{M}_\ell(\vec{f})(x):=\sup_{Q\ni x}\sum_{k=0}^\infty 2^{-kn\ell}\left(\prod_{i=1}^\ell\frac{1}{|Q|}\int_Q|f_i|\right)\left(\prod_{j=\ell+1}^m\frac{1}{|2^kQ|}\int_{2^kQ}|f_j|\right), \tag{2.2}$$

where the supremum is taken over all cubes Q containing x. For the convenience of notation, we set $\mathcal{M}_m(\vec{f}):=\mathcal{M}(\vec{f})$.

Recall that the Hardy-Littlewood maximal function M is defined for all locally integrable functions f and all $x\in\mathbb{R}^n$ by

$$M(f)(x):=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|\,dy.$$

Obviously, for all $1\le\ell\le m$, $\vec{f}=(f_1,\cdots,f_m)$ and $x\in\mathbb{R}^n$,

$$\mathcal{M}(\vec{f})(x)\le\mathcal{M}_\ell(\vec{f})(x)\le 2\prod_{j=1}^m M(f_j)(x).$$

The modified maximal operator $\mathcal{M}_\ell$ has the same boundedness properties as those of $\mathcal{M}$.

Proposition 2.1 *Let $1 \le p_1, \cdots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{P} = (p_1, \cdots, p_m)$ and $\vec{w} \in A_{\vec{P}}$. Let $1 \le \ell < m$ and $\mathcal{M}_\ell$ be as in (2.2). Then,*

(i) $\mathcal{M}_\ell$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ if all the exponents p_j are strictly greater than 1;

(ii) $\mathcal{M}_\ell$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(\nu_{\vec{w}})$ if some of the exponents p_j are equal to 1.

Proof. We first prove (ii). It suffices to show that

$$\mathcal{M}_\ell(\vec{f})(x) \le C \prod_{j=1}^m \left\{M^c_{\nu_{\vec{w}}}(|f_j|^{p_j} w_j/\nu_{\vec{w}})(x)\right\}^{1/p_j}, \tag{2.3}$$

where $M^c_{\nu_{\vec{w}}}$ denotes the weighted centered Hardy-Littlewood maximal function, that is,

$$M^c_{\nu_{\vec{w}}}(f)(x) := \sup_{r>0} \frac{1}{\nu_{\vec{w}}(Q(x,r))} \int_{Q(x,r)} |f(y)| \nu_{\vec{w}}(y)\, dy,$$

where $Q(x,r)$ denotes the cube centered at x and of side length r. In fact, if (2.3) holds, then by an argument similar to that used in the proof of [10, Theorem 3.3] we obtain (ii).

Now we show (2.3). For any given $j \in \{1, \cdots, m\}$, by Hölder's inequality, we obtain that for all cubes Q,

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q |f(y_j)|\, dy_j \\
&\le \frac{1}{|Q|} \left[\int_Q |f(y_j)|^{p_j} w_j(y_j)\, dy_j\right]^{1/p_j} \left[\int_Q w_j(y_j)^{-p_j'/p_j}\, dy_j\right]^{1/p_j'} \\
&\le \left\{M^c_{\nu_{\vec{w}}}(|f_j|^{p_j} w_j/\nu_{\vec{w}})(x)\right\}^{1/p_j} \left[\frac{\nu_{\vec{w}}(Q)}{|Q|}\right]^{1/p_j} \left[\frac{1}{|Q|}\int_Q w_j(y_j)^{-p_j'/p_j}\, dy_j\right]^{1/p_j'},
\end{aligned}$$

where $(\frac{1}{|Q|}\int_Q w_j(y_j)^{-p_j'/p_j}\, dy_j)^{1/p_j'}$ is understood as $(\operatorname{essinf}_Q w_j)^{-1}$ if $p_j = 1$. Therefore,

$$\begin{aligned}
\mathcal{M}_\ell(\vec{f})(x) \le{}& \prod_{j=1}^m \left\{M^c_{\nu_{\vec{w}}}(|f_j|^{p_j} w_j/\nu_{\vec{w}})(x)\right\}^{1/p_j} \\
&\times \left\{\sum_{k=0}^\infty 2^{-kn\ell} \left(\prod_{j=1}^\ell \left[\frac{\nu_{\vec{w}}(Q)}{|Q|}\right]^{1/p_j} \left[\frac{1}{|Q|}\int_Q w_j(y_j)^{-p_j'/p_j}\, dy_j\right]^{1/p_j'}\right.\right. \\
&\left.\left.\times \prod_{j=\ell+1}^m \left[\frac{\nu_{\vec{w}}(2^kQ)}{|2^kQ|}\right]^{1/p_j} \left[\frac{1}{|2^kQ|}\int_{2^kQ} w_j(y_j)^{-p_j'/p_j}\, dy_j\right]^{1/p_j'}\right)\right\}.
\end{aligned} \tag{2.4}$$

For $k \ge 0$, denote by $I_{\ell,k}$ the quantity in the square bracket. Notice that for all $k \ge 0$, $(\operatorname{essinf}_Q w_j)^{-1} \le (\operatorname{essinf}_{2^kQ} w_j)^{-1}$. Then,

$$I_{\ell,k} \le [\vec{w}]^{1/p}_{A_{\vec{P}}} \prod_{j=1}^\ell \left[\frac{\nu_{\vec{w}}(Q)}{|Q|}\right]^{1/p_j} \left[\frac{|2^kQ|}{|Q|}\right]^{1/p_j'} \left[\frac{|2^kQ|}{\nu_{\vec{w}}(2^kQ)}\right]^{1/p_j}$$

$$\leq [\vec{w}]_{A_{\vec{P}}}^{1/p} 2^{kn\ell} \left[\frac{\nu_{\vec{w}}(Q)}{\nu_{\vec{w}}(2^k Q)}\right]^{\sum_{j=1}^{\ell} \frac{1}{p_j}}.$$

Since $\nu_{\vec{w}} \in A_{mp} \subset A_\infty$ (see [10, Theorem 3.6]), properties of A_∞ weights imply that there exists some $\theta \in (0,1)$ such that for all cubes Q and all sets $E \subset Q$,

$$\frac{\nu_{\vec{w}}(E)}{\nu_{\vec{w}}(Q)} \leq C\left(\frac{|E|}{|Q|}\right)^{\theta}. \tag{2.5}$$

Therefore,

$$\sum_{k=0}^{\infty} 2^{-kn\ell} I_{\ell,k} \leq C \sum_{k=0}^{\infty} \left[\frac{\nu_{\vec{w}}(Q)}{\nu_{\vec{w}}(2^k Q)}\right]^{\sum_{j=1}^{\ell} \frac{1}{p_j}} \leq C. \tag{2.6}$$

Inserting (2.6) into (2.4) we obtain (2.3). Hence, (ii) holds.

To prove (i), the assumption $\vec{w} \in A_{\vec{P}}$ and [10, Theorem 3.6] imply that $w_j^{-\frac{1}{p_j-1}}$ satisfies the reverse Hölder inequality, that is, there exist $r_j > 1$ and $\widetilde{C} > 0$ such that for all $r \in [1, r_j]$ and all cubes Q,

$$\left[\frac{1}{|Q|}\int_Q w_j(x)^{-\frac{r}{p_j-1}}\,dx\right]^{1/r} \leq \frac{\widetilde{C}}{|Q|}\int_Q w_j(x)^{-\frac{1}{p_j-1}}\,dx.$$

Let $\xi := \min_{1\leq j\leq m} r_j$ and

$$q := \max_{1\leq j\leq m} \frac{pm}{pm + (1-1/\xi)(p_j-1)}.$$

Observe that $q < 1$ and $qp_j > 1$ for all $1 \leq j \leq m$. We claim that for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_\ell(\vec{f})(x) \leq C \prod_{j=1}^{m} \left\{M^c_{\nu_{\vec{w}}}([|f_j|^{p_j} w_j/\nu_{\vec{w}}]^q)(x)\right\}^{\frac{1}{qp_j}}. \tag{2.7}$$

If (2.7) holds, then by Hölder's inequality and boundedness of $M^c_{\nu_{\vec{w}}}$ together with an argument as that used in the proof of [10, theorem 3.7] we obtain (i). Therefore, it suffices to show (2.7).

The estimates of (4.12) through (4.14) in [10] imply that for all cubes Q and $1 \leq j \leq m$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |f_j(y_j)|\,dy_j \\
&\quad\leq \frac{\widetilde{C}}{|Q|\nu_{\vec{w}}(Q)^{\frac{1-q}{qp_j}}} \left[\int_Q |f_j(y_j)|^{qp_j} w_j(y_j)^q \nu_{\vec{w}}(y_j)^{1-q}\,dy_j\right]^{\frac{1}{qp_j}} \left[\int_Q w_j(y_j)^{-\frac{1}{p_j-1}}\,dy_j\right]^{1-1/p_j} \\
&\quad\leq \widetilde{C}\left\{M^c_{\nu_{\vec{w}}}([|f_j|^{p_j} w_j/\nu_{\vec{w}}(y_j)]^q)(x)\right\}^{\frac{1}{qp_j}} \left[\frac{\nu_{\vec{w}}(Q)}{|Q|}\right]^{1/p_j} \left[\frac{1}{|Q|}\int_Q w_j(y_j)^{-p_j'/p_j}\,dy_j\right]^{1/p_j'}.
\end{aligned}$$

From this, the definition of $\mathcal{M}_\ell$ and (2.6), it follows that

$$\begin{aligned}\mathcal{M}_\ell(\vec{f})(x) &\le \widetilde{C}\left\{M^c_{\nu_{\vec{w}}}([|f_j|^{p_j}w_j/\nu_{\vec{w}}]^q)(x)\right\}^{\frac{1}{qp_j}}\left\{\sum_{k=0}^{\infty}2^{-kn\ell}I_{\ell,k}\right\}\\ &\le C\left\{M^c_{\nu_{\vec{w}}}([|f_j|^{p_j}w_j/\nu_{\vec{w}}]^q)(x)\right\}^{\frac{1}{qp_j}},\end{aligned}$$

which proves (2.7). This finishes the proof of (i) and hence Proposition 2.1. □

3 Cotlar's inequality

The main aim of this section is to prove the following Cotlar-type inequality.

Proposition 3.1 *Suppose that T is as in Theorem 1.1. Then, for all $\gamma > 0$, there exists a positive constant $C = C(\gamma, n, m, \eta, s, \epsilon, \|\phi\|_{L^\infty})$ such that for all $\vec{f}$ in the product space $L^{p_1} \times \cdots \times L^{p_m}$ with $1 \le p_1, \cdots, p_m < \infty$ and all $x \in \mathbb{R}^n$,*

$$T^*(\vec{f})(x) \le C\left\{\left[M(|T(\vec{f})|^\gamma)(x)\right]^{1/\gamma} + (A+W)\sum_{\ell=1}^{m}\mathcal{M}_\ell(\vec{f})(x)\right\}, \tag{3.1}$$

where $W := \|T\|_{L^1\times\cdots\times L^1\to L^{1/m,\infty}}$.

Proof. By Hölder's inequality, it suffices to show that (3.1) holds for $\gamma \in (0, \frac{1}{m})$. Fix $\gamma \in (0, \frac{1}{m})$ and $x \in \mathbb{R}^n$. Set

$$S_\delta(x) := \left\{\vec{y} = (y_1, \cdots, y_m) \in (\mathbb{R}^n)^m : \min_{1\le j\le m}|x - y_j| \le \delta/\sqrt{m}\right\}, \tag{3.2}$$

and

$$U_\delta(x) := \left\{\vec{y} \in S_\delta(x) : \sum_{j=1}^{m}|x-y_j|^2 > \delta^2\right\}.$$

From (1.2), it follows that

$$\begin{aligned}&\sup_{\delta>0}\left|\int_{U_\delta(x)}K(x, y_1, \cdots, y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y}\right|\\ &\quad\le \sup_{\delta>0}\int_{U_\delta(x)}\frac{A}{\left(\sum_{j=1}^m|x-y_j|\right)^{mn}}|f_1(y_1)\cdots f_m(y_m)|\,d\vec{y}.\end{aligned} \tag{3.3}$$

For any $y \in U_\delta(x)$, there exist $j_1, \cdots, j_\ell$ with $\ell < m$ such that $|x-y_j| \le \delta/\sqrt{m}$ if and only if $j \in \{j_1, \cdots, j_\ell\}$. Thus, the last integral in (3.3) can be written as a sum of integrals over sets $R_{j_1,\cdots,j_\ell}$ in $(\mathbb{R}^n)^m$ for some $\{j_1, \cdots, j_\ell\} \subset \{1, 2, \cdots, m\}$ and $1 \le \ell < m$ such that for $\vec{y} := (y_1, \cdots, y_m) \in R_{j_1,\cdots,j_\ell}$, we have that $|x - y_j| \le \delta/\sqrt{m}$ if and only if $j \in \{j_1, \cdots, j_\ell\}$. Set $\{k_1, \cdots, k_{m-\ell}\} := \{1, 2, \cdots, m\} \setminus \{j_1, \cdots, j_\ell\}$ and

$$\Omega_{k_1,\cdots,k_{m-\ell}} := \left\{(y_{k_1}, \cdots, y_{k_{m-\ell}}) \in (\mathbb{R}^n)^{m-\ell} : |x - y_{k_i}| > \delta/\sqrt{m}, \quad \forall\, i = 1, 2, \cdots, m-\ell\right\}.$$

Then,

$$\int_{R_{j_1,\cdots,j_\ell}} \frac{A}{\left(\sum_{j=1}^m |x-y_j|\right)^{mn}} |f_1(y_1)\cdots f_m(y_m)|\,d\vec{y}$$
$$\le \prod_{i=1}^{\ell} \int_{|x-y_{j_i}|\le\delta/\sqrt{m}} |f_{j_i}(y_{j_i})|\,dy_{j_i} \int_{\Omega_{k_1,\cdots,k_{m-\ell}}} \frac{A\prod_{i=1}^{m-\ell}|f_{k_i}(y_{k_i})|}{\left(\sum_{i=1}^{m-\ell}|x-y_{k_i}|\right)^{mn}}\,dy_{k_1}\cdots dy_{k_{m-\ell}}.$$

Since

$$\int_{\Omega_{k_1,\cdots,k_{m-\ell}}} \frac{A\prod_{i=1}^{m-\ell}|f_{k_i}(y_{k_i})|}{\left(\sum_{i=1}^{m-\ell}|x-y_{k_i}|\right)^{mn}}\,dy_{k_1}\cdots dy_{k_{m-\ell}}$$
$$\le C\sum_{\nu=0}^{\infty}\frac{A}{(2^\nu\delta)^{mn}}\int_{2^\nu\delta/\sqrt{m}<\sum_{i=1}^{m-\ell}|x-y_{k_i}|\le 2^{\nu+1}\delta/\sqrt{m}} \prod_{i=1}^{m-\ell}|f_{k_i}(y_{k_i})|\,dy_{k_1}\cdots dy_{k_{m-\ell}},$$

we have

$$\int_{R_{j_1,\cdots,j_\ell}} \frac{A}{\left(\sum_{j=1}^m |x-y_j|\right)^{mn}} |f_1(y_1)\cdots f_m(y_m)|\,d\vec{y}$$
$$\le CA\sum_{\nu=0}^{\infty}\frac{2^{\nu(m-\ell)}}{(2^\nu)^{mn}}\prod_{i=1}^{\ell}\frac{1}{|B(x,\delta/\sqrt{m})|}\int_{|x-y_{j_i}|\le\delta/\sqrt{m}}|f_{j_i}(y_{j_i})|\,dy_{j_i}$$
$$\times\prod_{i=1}^{m-\ell}\frac{1}{B(x,2^{\nu+1}\delta/\sqrt{m})}\int_{|x-y_{k_i}|\le 2^{\nu+1}\delta/\sqrt{m}}|f_{k_i}(y_{k_i})|\,dy_{k_1}\cdots dy_{k_{m-\ell}}$$
$$\le CA\mathcal{M}_\ell(\vec{f})(x),$$

which combined with (3.3) gives that

$$\sup_{\delta>0}\left|\int_{U_\delta(x)} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y}\right| \le CA\sum_{\ell=1}^{m-1}\mathcal{M}_\ell(\vec{f})(x). \tag{3.4}$$

Also, notice that

$$|T_\delta(\vec{f})(x)| \le \left|\int_{U_\delta(x)} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y}\right|$$
$$+\left|\int_{\{\vec{y}\notin S_\delta(x):\,\sum_{j=1}^m|x-y_j|^2>\delta^2\}} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y}\right|.$$

By this and (3.4), it suffices to prove that (3.1) holds with $T^*(\vec{f})$ replaced by

$$\widetilde{T}^*(\vec{f})(x) := \sup_{\delta>0}|\widetilde{T}_\delta(f_1,\cdots,f_m)(x)|, \tag{3.5}$$

where

$$\widetilde{T}_\delta(f_1,\cdots,f_m)(x) := \int_{\vec{y}\notin S_\delta(x)} K(x,y_1,\cdots,y_m) f_1(y_1)\cdots f_m(y_m)\,d\vec{y}.$$

Fix $\delta > 0$ and let $B := B(x, \frac{\delta}{9\sqrt{m}})$ be the ball centered at x with radius $\frac{\delta}{9\sqrt{m}}$. Since $\vec{f} \in \prod_{j=1}^m L^{p_j}$, it follows from Theorem C that $T(\vec{f}) \in L^{p,\infty}$, where $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and consequently $T(\vec{f})$ is finite almost everywhere. Set

$$G_\delta(\vec{f})(x,z) := \int_{\vec{y}\notin S_\delta(x)} K(z,y_1,\cdots,y_m) f_1(y_1)\cdots f_m(y_m)\,d\vec{y} \qquad \forall\, z \in \mathbb{R}^n,$$

and

$$\widetilde{U}_\delta(x) := \left\{ \vec{y} \in (\mathbb{R}^n)^m : \min_{1\le j\le m} |x-y_j| \le \frac{\delta}{\sqrt{m}} < \max_{1\le j\le m} |x-y_j| \right\}.$$

Observe that for all $z \in B$ we have

$$\begin{aligned} |\widetilde{T}_\delta(\vec{f})(x)| &\le |\widetilde{T}_\delta(\vec{f})(x) - G_\delta(\vec{f})(x,z)| + |T(\vec{f})(z) - T(\vec{f_0})(z)| \\ &\quad + \left| \int_{\widetilde{U}_\delta(x)} K(z,y_1,\cdots,y_m) f_1(y_1)\cdots f_m(y_m)\,d\vec{y} \right|, \end{aligned} \tag{3.6}$$

where $\vec{f_0} := (f_1\chi_{9B},\cdots,f_m\chi_{9B})$. By an argument similar to the proof of (3.4) we obtain that for all $z \in B(x, \frac{\delta}{9\sqrt{m}})$,

$$\left| \int_{\widetilde{U}_\delta(x)} K(x,y_1,\cdots,y_m) f_1(y_1)\cdots f_m(y_m)\,d\vec{y} \right| \le CA \sum_{\ell=1}^{m-1} \mathcal{M}_\ell(\vec{f})(x). \tag{3.7}$$

For all $z \in B$ and all $t > 0$, write

$$\begin{aligned} &|\widetilde{T}_\delta(\vec{f})(x) - G_\delta(\vec{f})(x,z)| \\ &\le \int_{\vec{y}\notin S_\delta(x)} |K(x,y_1,\cdots,y_m) - K_t^{(0)}(x,y_1,\cdots,y_m)| \prod_{j=1}^m |f_j(y_j)|\,d\vec{y} \\ &\quad + \int_{\vec{y}\notin S_\delta(x)} |K_t^{(0)}(x,y_1,\cdots,y_m) - K_t^{(0)}(z,y_1,\cdots,y_m)| \prod_{j=1}^m |f_j(y_j)|\,d\vec{y} \\ &\quad + \int_{\vec{y}\notin S_\delta(x)} |K_t^{(0)}(z,y_1,\cdots,y_m) - K(z,y_1,\cdots,y_m)| \prod_{j=1}^m |f_j(y_j)|\,d\vec{y} \\ &=: \mathrm{Z}_1 + \mathrm{Z}_2 + \mathrm{Z}_3. \end{aligned}$$

Let $t := (\frac{\delta}{4\sqrt{m}})^s$. Then, for all $z \in B$ and $\vec{y} \notin S_\delta(x)$,

$$4|z-x| < \frac{4\delta}{9\sqrt{m}} < 2t^{1/s} \le \min\left\{ \min_{1\le j\le m} |z-y_j|,\ \min_{1\le j\le m} |x-y_j| \right\}.$$

Thus, we can apply assumption (H3) to estimate $\mathrm{Z}_1, \mathrm{Z}_2$ and Z_3. In fact, by (1.8), we have

$$\begin{aligned}\mathrm{Z}_1 &\le \int_{\vec{y}\notin S_\delta(x)} \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{k=1}^m \phi\left(\frac{|x-y_k|}{t^{1/s}}\right) \prod_{j=1}^m |f_j(y_j)|\, d\vec{y} \\ &\quad + \int_{\vec{y}\notin S_\delta(x)} \frac{At^{\epsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\epsilon}} \prod_{j=1}^m |f_j(y_j)|\, d\vec{y} \\ &=: \mathrm{Z}_{1,1} + \mathrm{Z}_{1,2}.\end{aligned}$$

For all $1 \le k \le m$, since $\vec{y} \notin S_\delta(x)$, it follows that $|x-y_k| > \frac{\delta}{\sqrt{m}} > t^{1/s}$, which together with the support condition of ϕ implies that $\phi\left(\frac{|x-y_k|}{t^{1/s}}\right) = 0$. Hence, $\mathrm{Z}_{1,1} = 0$. From this and the definition of $S_\delta(x)$, we deduce that

$$\begin{aligned}\mathrm{Z}_1 = \mathrm{Z}_{1,2} &\le \sum_{\nu=0}^\infty \int_{2^\nu\delta/\sqrt{m} < \sum_{k=1}^m |x-y_k| \le 2^{\nu+1}\delta/\sqrt{m}} \frac{CA\delta^\epsilon}{(\sum_{k=1}^m |x-y_k|)^{mn+\epsilon}} \prod_{j=1}^m |f_j(y_j)|\, d\vec{y} \\ &\le \sum_{\nu=0}^\infty \frac{CA\delta^\epsilon}{(2^\nu\delta)^{mn+\epsilon}} \int_{\sum_{k=1}^m |x-y_k| \le 2^{\nu+1}\delta/\sqrt{m}} \prod_{j=1}^m |f_j(y_j)|\, d\vec{y} \\ &\le \sum_{\nu=0}^\infty 2^{-\nu\epsilon} \prod_{j=1}^m \frac{1}{|B(x, 2^{\nu+1}\delta/\sqrt{m})|} \int_{B(x,2^{\nu+1}\delta/\sqrt{m})} |f_j(y_j)|\, dy_j \\ &\le CA\mathcal{M}(\vec{f})(x).\end{aligned}$$

To estimate Z_2, notice that (1.10) implies that

$$\mathrm{Z}_2 \le \int_{\vec{y}\notin S_\delta(x)} \frac{At^{\epsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\epsilon}} \prod_{j=1}^m |f_j(y_j)|\, d\vec{y} = \mathrm{Z}_{1,2} \le CA\mathcal{M}(\vec{f})(x).$$

An argument similar to that of Z_1 gives that $\mathrm{Z}_3 \le CA\mathcal{M}(\vec{f})(x)$.

Combining the estimates of Z_1 through Z_3, (3.6) and (3.7) gives that for all $z \in B$,

$$|\widetilde{T}_\delta(\vec{f})(x)| \le CA \sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})(x) + |T(\vec{f})(z)| + |T(\vec{f_0})(z)|. \tag{3.8}$$

Raising (3.8) to the power γ, taking integral average over the ball B with respect to the variable z, we obtain

$$|\widetilde{T}_\delta(\vec{f})(x)|^\gamma \le \left[CA \sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})(x)\right]^\gamma + M(|T(\vec{f})|^\gamma)(x) + \frac{1}{|B|}\int_B |T(\vec{f_0})(z)|^\gamma\, dz. \tag{3.9}$$

Since T is bounded from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$, by Kolmogorov's inequality, we have

$$\int_B |T(\vec{f_0})(z)|^\gamma\, dz = m\gamma \int_0^\infty \lambda^{m\gamma-1} |\{z \in B : |T(\vec{f_0})(z)|^{1/m} > \lambda\}|\, d\lambda$$

$$\le m\gamma \int_0^\infty \lambda^{m\gamma-1} \min\left\{ |B|,\, \lambda^{-1} W^{1/m} \prod_{j=1}^m \|f_j \chi_{9B}\|_{L^1}^{1/m} \right\} d\lambda$$
$$\le C|B|^{1-m\gamma} W^\gamma \prod_{j=1}^m \|f_j \chi_{9B}\|_{L^1}^\gamma,$$

which further implies that

$$\left\{ \frac{1}{|B|} \int_B |T(\vec{f}_0)(z)|^\gamma \, dz \right\}^{1/\gamma} \le CW \prod_{j=1}^m \frac{\|f_j \chi_{9B}\|_{L^1}}{|B|} \le CW \mathcal{M}(\vec{f})(x).$$

Inserting this estimate into (3.9) we obtain (3.1), which concludes the proof of Proposition 3.1. □

Corollary 3.1 *Let T be as in Theorem 1.1 and T^* as in (1.7), $1 \le p_1, \cdots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then,*

(i) T^* has a bounded extension from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p if all the exponents p_j are strictly greater than 1;

(ii) T^* has a bounded extension from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{p,\infty}$ if some of the exponents p_j are equal to 1.

In either case, the norm of T^* is bounded by $C(A + \|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^q})$, where C is a positive constant depending on A, s, η, ϵ.

Proof. In fact, (i) and (ii) follow from Theorem C, Propositions 2.1 and 3.1, and an argument similar to that used in [9, Corollary 2.3]. We omit the details. □

4 Weighted norm inequalities

In what follows, we denote by L_c^∞ the collection of all functions in L^∞ with compact support.

Proposition 4.1 *Let T be as in Theorem 1.1 and $\widetilde{T}^*$ as in (3.5). Let $w \in A_\infty$ and θ be given as in (2.5). Then, there exists a positive constant C such that for all $\gamma > 0$ sufficiently small, and all $\alpha > 0$, and $\vec{f} = (f_1, \cdots, f_m)$ in the m-fold product space $L_c^\infty \times \cdots \times L_c^\infty$,*

$$w\left(\left\{ x \in \mathbb{R}^n : \widetilde{T}^*(\vec{f})(x) > 2^{m+1}\alpha,\, \sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})(x) \le \gamma\alpha \right\} \right)$$
$$\le C(A+W)^{\theta/m} \gamma^{\theta/m} w\left(\left\{ x \in \mathbb{R}^n : \widetilde{T}^*(\vec{f})(x) > \alpha \right\} \right), \qquad (4.1)$$

where $W := \|T\|_{L^{q_1} \times \cdots \times L^{q_m} \to L^q}$.

Proof. Set $\Omega := \{x \in \mathbb{R}^n : \widetilde{T}^*(\vec{f})(x) > \alpha\}$. Since Ω is an open proper subset of $\mathbb{R}^n$, we consider the Whitney decomposition of Ω. We write

(a) $\Omega = \bigcup_\nu Q_\nu$ and these Q_ν's have disjoint interiors;

(b) $\sqrt{n}\ell(Q_\nu) \le \operatorname{dist}(Q_\nu, \Omega^{\complement}) \le 4\sqrt{n}\ell(Q_\nu)$;

(c) If the boundaries of two cubes Q_ν and Q_μ touch, then $1/4 \le \ell(Q_\nu)/\ell(Q_\mu) \le 4$;

(d) For any given Q_ν there exist at most 12^n Q_μ's that touch it.

By this and (2.5), it suffices to show that for all Q_ν,

$$\left|\left\{x \in Q_\nu : \widetilde{T}^*(\vec{f})(x) > 2^{m+1}\alpha, \sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})(x) \le \gamma\alpha\right\}\right| \le C(A+W)^{1/m}\gamma^{1/m}|Q_\nu|. \tag{4.2}$$

Denote by E the set in the left hand side of (4.2). We may as well assume that there exists $\xi_\nu \in Q_\nu$ such that $\sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})(\xi_\nu) \le \gamma\alpha$; otherwise $E = \emptyset$ and (4.2) holds trivially.

Property (b) implies that $9\sqrt{n}Q_\nu$ intersects $\Omega^{\complement}$. For each Whitney cube Q_ν we fix $Q_\nu^* := 100nQ_\nu$ and $y_\nu \in \Omega^{\complement} \cap Q_\nu^*$ such that

$$\max_{z\in Q_\nu} |z - y_\nu| \le 9n\ell(Q_\nu) < 40n\ell(Q_\nu) < \operatorname{dist}(y_\nu, (Q_\nu^*)^{\complement}). \tag{4.3}$$

For each $j \in \{1, \cdots, m\}$, we set $f_j^0 := f_j\chi_{Q_\nu^*}$ and $f_j^\infty := f_j - f_j^0$. Then E is contained in the union of 2^m sets of the form

$$E_{\alpha_1,\cdots,\alpha_m} := \left\{x \in Q_\nu : \widetilde{T}^*(f_1^{\alpha_1}, \cdots, f_m^{\alpha_m})(x) > 2\alpha, \sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})(x) \le \gamma\alpha\right\},$$

where $\alpha_j \in \{0, \infty\}$ for all $1 \le j \le m$. Applying the boundedness of T^* from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$ (see Corollary 3.1) we obtain

$$\begin{aligned}|E_{0,\cdots,0}| &\le \frac{C(A+W)^{1/m}}{\alpha^{1/m}}\left[\int_{(\mathbb{R}^n)^m} \prod_{j=1}^m |f_j^0(y_j)|\, d\vec{y}\right]^{1/m} \\ &\le \frac{C(A+W)^{1/m}}{\alpha^{1/m}}\left[\mathcal{M}(\vec{f})(\xi_\nu)\right]^{1/m}|Q_\nu| \\ &\le C(A+W)^{1/m}\gamma^{1/m}|Q_\nu|.\end{aligned}$$

Thus, to obtain (4.1), it suffices to show that the remaining $2^m - 1$ sets $E_{\alpha_1,\cdots,\alpha_m}$ are empty if γ is chosen to be very small.

Suppose that there are exactly ℓ of $\{\alpha_1, \cdots, \alpha_m\}$ are 0, where $1 \le \ell < m$. By symmetry, we may as well assume that $\alpha_1 = \cdots = \alpha_\ell = 0$ and $\alpha_{\ell+1} = \cdots = \alpha_m = \infty$. Then, for all $\delta > 0$, $x \in Q_\nu$ and $S_\delta(x)$ as in (3.2),

$$\left|\int_{\vec{y}\notin S_\delta(x)} K(x, \vec{y}) \prod_{j=1}^\ell f_j^0(y_j) \prod_{i=\ell+1}^m f_i^\infty(y_i)\, d\vec{y}\right|$$

$$\begin{aligned}
&\le CA\prod_{j=1}^{\ell}\int_{Q_\nu^*}|f_j(y_j)|\,dy_j\int_{(\mathbb{R}^n\setminus Q_\nu^*)^{m-\ell}}\frac{\prod_{i=\ell+1}^m|f_i(y_i)|}{(\sum_{i=\ell+1}^m|x-y_i|)^{mn}}\,dy_{\ell+1}\cdots dy_m\\
&\le CA\sum_{k=0}^{\infty}\prod_{j=1}^{\ell}\int_{Q_\nu^*}|f_j(y_j)|\,dy_j\\
&\quad\times\int_{\{(y_{\ell+1},\cdots,y_m)\in(\mathbb{R}^n)^{m-\ell}:\,2^{k-1}\ell(Q_\nu^*)<\sum_{i=\ell+1}^m|x-y_i|\le 2^k\ell(Q_\nu^*)\}}\frac{\prod_{i=\ell+1}^m|f_i(y_i)|}{(2^k\ell(Q_\nu^*))^{mn}}\,dy_{\ell+1}\cdots dy_m\\
&\le CA\mathcal{M}_\ell(\vec{f})(\xi_\nu)\\
&\le CA\gamma\alpha,
\end{aligned}$$

which is bounded by 2α if we choose $\gamma>0$ small enough. In this way, we have that $E_{\alpha_1,\cdots,\alpha_m}=\emptyset$ when there exist some $\alpha_i=0$ and some $\alpha_j=\infty$.

Now it remains to prove that $E_{\infty,\cdots,\infty}=\emptyset$. Set $\vec{f}^\infty:=(f_1^\infty,\cdots,f_m^\infty)$. Observe that for all $\delta>0$ and all $x\in Q_\nu$,

$$|\widetilde{T}_\delta(\vec{f}^\infty)(x)|\le|\widetilde{T}_\delta(\vec{f}^\infty)(x)-G_\delta(\vec{f}^\infty)(x,y_\nu)|+|G_\delta(\vec{f}^\infty)(x,y_\nu)|, \tag{4.4}$$

where

$$G_\delta(\vec{f}^\infty)(x,y_\nu):=\int_{y\notin S_\delta(x)}K(y_\nu,\vec{y}\,)\prod_{j=1}^m f_j^\infty(y_j)\,d\vec{y}.$$

Let $t:=(18n\ell(Q_\nu))^s$. Then,

$$\begin{aligned}
|\widetilde{T}_\delta(\vec{f}^\infty)(x)-G_\delta(\vec{f}^\infty)(x,y_\nu)|\le&\left|\int_{y\notin S_\delta(x)}[K(x,\vec{y}\,)-K_t^{(0)}(x,\vec{y}\,)]\prod_{j=1}^m f_j^\infty(y_j)\,d\vec{y}\right|\\
&+\left|\int_{y\notin S_\delta(x)}[K_t^{(0)}(x,\vec{y}\,)-K_t^{(0)}(y_\nu,\vec{y}\,)]\prod_{j=1}^m f_j^\infty(y_j)\,d\vec{y}\right|\\
&+\left|\int_{y\notin S_\delta(x)}[K_t^{(0)}(y_\nu,\vec{y}\,)-K(y_\nu,\vec{y}\,)]\prod_{j=1}^m f_j^\infty(y_j)\,d\vec{y}\right|\\
=:&\ \mathrm{Z}_1+\mathrm{Z}_2+\mathrm{Z}_3.
\end{aligned}$$

By (4.3) and the support condition of $\vec{f}^\infty$, we have that for any $\vec{y}\in\operatorname{supp}\vec{f}^\infty$ and $x\in Q_\nu$,

$$2|x-y_\nu|\le t^{1/s},\quad 2t^{1/s}\le\min_{1\le j\le m}|y_\nu-y_j|,\quad 2t^{1/s}\le\min_{1\le j\le m}|x-y_j|.$$

Thus, we can apply assumption (H3) to estimate $\mathrm{Z}_1,\mathrm{Z}_2$ and Z_3. Indeed, by (1.8),

$$\mathrm{Z}_1\le\int_{(\mathbb{R}^n)^m}\frac{A}{(\sum_{k=1}^m|x-y_k|)^{mn}}\sum_{k=1}^m\phi\left(\frac{|x-y_k|}{t^{1/s}}\right)\prod_{j=1}^m|f_j^\infty(y_j)|\,d\vec{y}$$

$$+\int_{(\mathbb{R}^n)^m}\frac{At^{\epsilon/s}}{(\sum_{k=1}^m|x-y_k|)^{mn+\epsilon}}\prod_{j=1}^m|f_j^\infty(y_j)|\,d\vec{y}.$$

The support condition of ϕ and the fact $2t^{1/s}\le\min_{1\le j\le m}|x-y_j|$ imply that

$$\sum_{k=1}^m\phi\left(\frac{|x-y_k|}{t^{1/s}}\right)\prod_{j=1}^m|f_j^\infty(y_j)|=0,\quad\forall\,\vec{y}\in(\mathbb{R}^n)^m.$$

For $x\in Q_\nu$, $\xi_\nu\in Q_\nu$ and $y_k\notin Q_\nu^*$, we always have $|x-y_k|\approx|\xi_\nu-y_k|$. Therefore,

$$\begin{aligned}\mathrm{Z}_1&\le\int_{(\mathbb{R}^n\setminus Q_\nu^*)^m}\frac{At^{\epsilon/s}}{(\sum_{k=1}^m|x-y_k|)^{mn+\epsilon}}\prod_{j=1}^m|f_j(y_j)|\,d\vec{y}\\&\le C\int_{(\mathbb{R}^n\setminus Q_\nu^*)^m}\frac{At^{\epsilon/s}}{(\sum_{k=1}^m|\xi_\nu-y_k|)^{mn+\epsilon}}\prod_{j=1}^m|f_j(y_j)|\,d\vec{y}\\&\le C\sum_{i=0}^\infty\int_{\{\vec{y}\in(\mathbb{R}^n)^m:\,2^{i-1}\ell(Q_\nu^*)<\sum_{k=1}^m|\xi_\nu-y_k|\le2^i\ell(Q_\nu^*)\}}\frac{At^{\epsilon/s}\prod_{j=1}^m|f_j(y_j)|}{(\sum_{k=1}^m|x-y_k|)^{mn+\epsilon}}\,d\vec{y}\\&\le CA\mathcal{M}(\vec{f})(\xi_\nu).\end{aligned}$$

Likewise, we have

$$\mathrm{Z}_3\le CA\mathcal{M}(\vec{f})(\xi_\nu).$$

In view of (1.10), by using $|y_\nu-y_k|\approx|\xi_\nu-y_k|$, we estimate Z_2 by

$$\mathrm{Z}_2\le\int_{(\mathbb{R}^n\setminus Q_\nu^*)^m}\frac{At^{\epsilon/s}}{(\sum_{k=1}^m|y_\nu-y_k|)^{mn+\epsilon}}\prod_{j=1}^m|f_j(y_j)|\,d\vec{y}\le CA\mathcal{M}(\vec{f})(\xi_\nu).$$

Combining the estimates for Z_1 through Z_3 yields that

$$|\widetilde{T}_\delta(\vec{f^\infty})(x)-G_\delta(\vec{f^\infty})(x,y_\nu)|\le CA\mathcal{M}(\vec{f})(\xi_\nu).\tag{4.5}$$

Finally, we claim that $|G_\delta(\vec{f^\infty})(x,y_\nu)|\le(1+CA\gamma)\alpha$. If this claim holds, then by (4.5) and (4.4), there exists $\gamma>0$ small enough such that $\widetilde{T}^*(\vec{f^\infty})(x)\le2\alpha$. For such small positive γ, we have $E_{\infty,\cdots,\infty}=\emptyset$.

Now it remains to show the claim. Observe that

$$\left|G_\delta(\vec{f^\infty})(x,y_\nu)\right|=\left|\int_{(\mathbb{R}^n)^m}K(y_\nu,\vec{y})\prod_{j=1}^m f_j(y_j)\chi_{(B(x,\delta)\cup Q_\nu^*)^\complement}(y_j)\,d\vec{y}\right|.\tag{4.6}$$

We consider the following two cases.

Case 1: $\delta\ge18n\ell(Q_\nu)$. In this case, by $\xi_\nu\in Q_\nu$ and $x\in Q_\nu$, there exists a large constant $a_1>1$ such that

$$B(\xi_\nu,a_1^{-1}\delta)\subset(B(x,\delta)\cup Q_\nu^*)\subset B(\xi_\nu,a_1\delta),$$

and that for all $y_j \notin B(x,\delta)\cup Q_\nu^*$, by (4.3), we have

$$|y_j - y_\nu| \ge |y_j - x| - |x - y_\nu| > \delta/2,$$

which implies that $\min_{1\le j\le m}|y_\nu - y_j| > \delta/2$ for all $\vec{y}\in(\mathbb{R}^n\setminus(B(x,\delta)\cup Q_\nu^*))^m$. Thus,

$$\begin{aligned}\left|G_\delta(\vec{f}^\infty)(x,y_\nu)\right| &\le \left|\int_{\min\limits_{1\le j\le m}|y_\nu-y_j|>\delta/2} K(y_\nu,\vec{y}\,)\prod_{j=1}^m f_j(y_j)\,d\vec{y}\right| + \left|\int_{U_1} K(y_\nu,\vec{y}\,)\prod_{j=1}^m f_j(y_j)\,d\vec{y}\right|\\ &=: \mathrm{Z}_4 + \mathrm{Z}_5,\end{aligned}$$

where

$$U_1 := \left\{\vec{y}\in(\mathbb{R}^n)^m : \min_{1\le j\le m}|y_\nu - y_j| > \frac{\delta}{2},\ \exists\, j\in\{1,\cdots,m\}\ \text{s.\,t.}\ y_j\in B(x,\delta)\cup Q_\nu^*\right\}.$$

Obviously, $\mathrm{Z}_4 \le \widetilde{T}^*(y_\nu) \le \alpha$ since $y_\nu\notin\Omega$. To estimate Z_5, we may as well assume that, for some $1\le \ell\le m$, $y_j\in B(x,\delta)\cup Q_\nu^*$ when $1\le j\le \ell$ and $y_j\notin B(x,\delta)\cup Q_\nu^*$ for the remaining y_j's. Notice that when $\ell = m$, we have $y_j\in B(x,\delta)\cup Q_\nu^*$ for all $1\le j\le m$. Thus, by (1.2) and $|y_\nu - y_k|\approx|\xi_\nu - y_k|$,

$$\begin{aligned}\mathrm{Z}_5 &\le CA\prod_{j=1}^m\int_{B(\xi_\nu,a_1\delta)}\frac{\prod_{j=1}^m|f_j(y_j)|}{\delta^{mn}}\,dy_j\\ &\quad+\sum_{\ell=1}^{m-1}\prod_{j=1}^{\ell}\int_{B(\xi_\nu,a_1\delta)}|f_j(y_j)|dy_j\int_{\sum_{k=\ell+1}^m|y_\nu-y_k|>\frac{\delta}{2}}\frac{A\prod_{j=\ell+1}^m|f_j(y_j)|}{(\sum_{k=\ell+1}^m|y_\nu-y_k|)^{mn}}\,dy_{\ell+1}\cdots dy_m\\ &\le CA\sum_{\ell=1}^m\mathcal{M}_\ell(\vec{f}\,)(\xi_\nu)\\ &\le CA\gamma\alpha.\end{aligned}$$

Combining the estimates of Z_4 and Z_5 yields that $|G_\delta(\vec{f}^\infty)(x,y_\nu)|\le(1+CA\gamma)\alpha$.

Case 2: $\delta < 18n\ell(Q_\nu)$. In this case, there exists a large constant $a_2>1$, independent of $x\in Q_\nu$ and δ, such that

$$a_2^{-1}Q_\nu\subset(B(x,\delta)\cup Q_\nu^*)\subset a_2Q_\nu.$$

By this and (4.3), we have

$$\begin{aligned}\left|G_\delta(\vec{f}^\infty)(x,y_\nu)\right| &\le \left|\int_{\min\limits_{1\le j\le m}|y_\nu-y_j|>40n\ell(Q_\nu)} K(y_\nu,\vec{y}\,)\prod_{j=1}^m f_j(y_j)\,d\vec{y}\right|\\ &\quad+\left|\int_{U_2} K(y_\nu,\vec{y}\,)\prod_{j=1}^m f_j(y_j)\,d\vec{y}\right|\\ &=: \mathrm{Z}_6+\mathrm{Z}_7,\end{aligned}$$

where

$$U_2 := \left\{\vec{y} \in (\mathbb{R}^n)^m : \min_{1\le j\le m} |y_\nu - y_j| > 40n\ell(Q_\nu),\ \exists\, j \in \{1, \cdots, m\} \text{ s. t. } y_j \in B(x,\delta)\cup Q_\nu^*\right\}.$$

Obviously, $\mathrm{Z}_6 \le \widetilde{T}^*(y_\nu) \le \alpha$. Furthermore, similarly to the estimate of Z_5 and with the same assumptions, we have

$$\begin{aligned}
\mathrm{Z}_7 &\le CA\prod_{j=1}^m \int_{a_2Q_\nu} \frac{\prod_{j=1}^m |f_j(y_j)|}{\ell(Q_\nu)^{mn}}\,dy_j \\
&\quad + \sum_{\ell=1}^{m-1}\prod_{j=1}^{\ell}\int_{a_2Q_\nu} |f_j(y_j)|dy_j \int_{\sum_{k=\ell+1}^m |y_\nu-y_k|>\ell(Q_\nu)} \frac{A\prod_{j=\ell+1}^m |f_j(y_j)|}{(\sum_{k=\ell+1}^m |y_\nu - y_k|)^{mn}}\,dy_{\ell+1}\cdots dy_m \\
&\le CA\sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})(\xi_\nu) \\
&\le CA\gamma\alpha.
\end{aligned}$$

The estimates for Z_6 and Z_7 also give that $|G_\delta(\vec{f}^\infty)(x, y_\nu)| \le (1 + CA\gamma)\alpha$. This proves the claim, and concludes the proof of Proposition 4.1. □

Proof of Theorem 1.1. To show (i), for all $\vec{f} = (f_1, \cdots, f_m) \in L_c^\infty \times \cdots \times L_c^\infty$, applying (3.4), Proposition 4.1 and Proposition 2.1 yields that

$$\begin{aligned}
\|T^*(\vec{f})\|_{L^p(\nu_{\vec{w}})} &\le C\left(A\left\|\sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} + \left\|\widetilde{T}^*(\vec{f})\right\|_{L^p(\nu_{\vec{w}})}\right) \\
&\le C(A+W)\left\|\sum_{\ell=1}^m \mathcal{M}_\ell(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \le C(A+W)\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.
\end{aligned}$$

Then, the density of L_c^∞ in $L^{p_j}(w_j)$ for all $1 \le j \le m$ together with a standard argument implies (i). The weak-type estimate (ii) follows from an similar argument. We omit the details. □

Proof of Theorem 1.2. The proof for Theorem 1.2 follows from an argument similar to that used in [9, Corollary 3.5]. We omit the details. □

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