ON POINTWISE A.E. CONVERGENCE OF MULTILINEAR OPERATORS

LOUKAS GRAFAKOS, DANQING HE, PETR HONZÍK, AND BAE JUN PARK

ABSTRACT. In this work we obtain the pointwise almost everywhere convergence for two families of multilinear operators: (a) the doubly truncated homogeneous singular integral operators associated with L^q functions on the sphere; and (b) lacunary multiplier operators of limited smoothness. The a.e. convergence is deduced from the $L^2 \times \cdots \times L^2 \to L^{2/m}$ boundedness of the associated maximal multilinear operators.

1. INTRODUCTION AND PRELIMINARIES

The pointwise a.e. convergence of sequences of operators is of paramount importance and has been widely studied in several areas of analysis, such as harmonic analysis, PDE, and ergodic theory. This area boasts challenging problems, indicatively see [5, 6, 12, 24], and is intimately connected with the boundedness of the associated maximal operators; on this see [27]. Moreover, techniques and tools employed to study a.e. convergence have led to important developments in harmonic analysis.

Multilinear harmonic analysis has made significant advances in recent years. The founders of this area are Coifman and Meyer [8] who realized the applicability of multilinear operators and introduced their study in analysis in the mid 1970s. Focusing on operators that commute with translations, a fundamental difference between the multilinear and the linear theory is the existence of a straightforward characterization of boundedness at an initial point, usually $L^2 \to L^2$. The lack of an easy characterization of boundedness at an initial point in the multilinear theory creates difficulties in their study. Criteria that get very close to characterization of boundedness have recently been obtained by the first two authors and Slavíková [19] and also by Kato, Miyachi, and Tomita [25] in the bilinear case. These criteria were extended to the general m-linear case for $m \geq 2$ by the authors of this article in [18]. This reference

also contains initial $L^2 \times \cdots \times L^2 \to L^{2/m}$ estimates for rough homogeneous multilinear singular integrals associated with L^q functions on the sphere and multilinear multipliers of Hörmander type.

The purpose of this work is to obtain the pointwise a.e. convergence of doubly truncated multilinear homogeneous singular integrals and lacunary multilinear multipliers by establishing boundedness for their associated maximal operators.

We first introduce multilinear (singly) truncated singular integral operators. Let Ω be an integrable function, defined on the sphere $\mathbb{S}^{mn-1}$, satisfying the mean value zero property

$$(1.1) \qquad \int_{\mathbb{S}^{mn-1}} \Omega \, d\sigma_{mn-1} = 0.$$

Then we define

$$K(\vec{y}) := \frac{\Omega(\vec{y}\,')}{|\vec{y}|^{mn}}, \qquad \vec{y} \neq 0,$$

where $\vec{y}\,' := \vec{y}/|\vec{y}| \in \mathbb{S}^{mn-1}$, and the corresponding truncated multilinear operator $\mathcal{L}_\Omega^{(\epsilon)}$ by

$$\mathcal{L}_\Omega^{(\epsilon)}(f_1, \ldots, f_m)(x) := \int_{(\mathbb{R}^n)^m \setminus B(0,\epsilon)} K(\vec{y}) \prod_{j=1}^m f_j(x - y_j) \, d\vec{y}$$

acting on Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$, where $x \in \mathbb{R}^n$, $\vec{y} := (y_1, \ldots, y_m) \in (\mathbb{R}^n)^m$, and $B(0,\epsilon)$ is the ball centered at zero with radius $\epsilon > 0$ in $(\mathbb{R}^n)^m$. Moreover, by taking $\epsilon \searrow 0$, we obtain the multilinear homogeneous Calderón-Zygmund singular integral operator

$$(1.2) \qquad \mathcal{L}_\Omega(f_1, \ldots, f_m)(x) := \lim_{\epsilon \searrow 0} \mathcal{L}_\Omega^{(\epsilon)}(f_1, \ldots, f_m)(x)$$

$$= p.v. \int_{(\mathbb{R}^n)^m} K(\vec{y}) \prod_{j=1}^m f_j(x - y_j) \, d\vec{y}.$$

This is still well-defined for any Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$. In [18] we showed that if Ω lies in $L^q(\mathbb{S}^{mn-1})$ with $q > \frac{2m}{m+1}$, then the multilinear singular integral operator $\mathcal{L}_\Omega$ admits a bounded extension from $L^2(\mathbb{R}^n) \times \cdots \times L^2(\mathbb{R}^n)$ to $L^{2/m}(\mathbb{R}^n)$. In order words, given $f_j \in L^2(\mathbb{R}^n)$, $\mathcal{L}_\Omega(f_1, \ldots, f_m)$ is well-defined and is in $L^{2/m}(\mathbb{R}^n)$.

We now define the doubly truncated multilinear operator $\mathcal{L}_\Omega^{(\epsilon, \epsilon^{-1})}$ by

$$\mathcal{L}_\Omega^{(\epsilon, \epsilon^{-1})}(f_1, \ldots, f_m) := \mathcal{L}_\Omega^{(\epsilon)}(f_1, \ldots, f_m) - \mathcal{L}_\Omega^{(\epsilon^{-1})}(f_1, \ldots, f_m)$$

for Schwartz functions f_j, $j = 1, \ldots, m$. We observe that if $\Omega \in L^q(\mathbb{S}^{mn-1})$ for $\frac{2m}{m+1} < q \leq \infty$, then

$$\lim_{\epsilon \to 0} \mathcal{L}_\Omega^{(\epsilon, \epsilon^{-1})}(f_1, \ldots, f_m) = \lim_{\epsilon \to 0} \mathcal{L}_\Omega^{(\epsilon)}(f_1, \ldots, f_m)$$

for f_j in Schwartz class.

We define for fixed $0 < \epsilon_0 < 1$

$$\mathcal{L}_{\Omega}^{*,\epsilon_0}(\varphi_1,\ldots,\varphi_m) := \sup_{\epsilon \geq \epsilon_0} \left| \mathcal{L}_{\Omega}^{(\epsilon,\epsilon^{-1})}(\varphi_1,\ldots,\varphi_m) \right|$$

and

(1.3) $$\mathcal{L}_{\Omega}^{**}(\varphi_1,\ldots,\varphi_m) := \sup_{\epsilon > 0} \left| \mathcal{L}_{\Omega}^{(\epsilon,\epsilon^{-1})}(\varphi_1,\ldots,\varphi_m) \right| = \lim_{\epsilon_0 \to 0} \mathcal{L}_{\Omega}^{*,\epsilon_0}(\varphi_1,\ldots,\varphi_m)$$

for φ_j in the Schwartz class. One main difficulty to study the boundedness of $\mathcal{L}_{\Omega}^{**}$ is to show that the doubly truncated operator is well-defined pointwise a.e. for $f_j \in L^2(\mathbb{R}^n)$. To overcome this difficulty, we need to utilize the boundedness of $\mathcal{M}_{\Omega}$ introduced in Section 3; see Section 5 for the detailed proof.

Our first main result is as follows.

Theorem 1.1. *Let $m \geq 2$, $\frac{2m}{m+1} < q \leq \infty$ and $\Omega \in L^q(\mathbb{S}^{mn-1})$ satisfy (1.1). Then*

(1.4) $$\left\| \mathcal{L}_{\Omega}^{**}(f_1,\ldots,f_m) \right\|_{L^{2/m}(\mathbb{R}^n)} \leq C \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^{\infty} \|f_j\|_{L^2(\mathbb{R}^n)}$$

for $f_j \in L^2(\mathbb{R}^n)$. Moreover the doubly truncated singular integral $\mathcal{L}_{\Omega}^{(\epsilon,\epsilon^{-1})}(f_1,\ldots,f_m)$ converges to $\mathcal{L}_{\Omega}(f_1,\ldots,f_m)$ pointwise a.e. as $\epsilon \to 0$ when $f_j \in L^2(\mathbb{R}^n)$, $j = 1,\ldots,m$. That is, the multilinear singular integral $\mathcal{L}_{\Omega}(f_1,\ldots,f_m)$ is well-defined a.e. when $f_j \in L^2(\mathbb{R}^n)$, $j = 1,\ldots,m$.

In order to achieve this goal, we initially prove the following result, which provides the boundedness of the associated maximal singular integral operator

$$\mathcal{L}_{\Omega}^{*}(f_1,\ldots,f_m)(x) := \sup_{\epsilon > 0} \left| \mathcal{L}_{\Omega}^{(\epsilon)}(f_1,\ldots,f_m)(x) \right|$$

for Schwartz functions f_j, $j = 1,\ldots,m$.

Theorem 1.2. *Let $m \geq 2$, $\frac{2m}{m+1} < q \leq \infty$ and $\Omega \in L^q(\mathbb{S}^{mn-1})$ satisfy (1.1). Then there exists a constant $C > 0$ such that*

(1.5) $$\left\| \mathcal{L}_{\Omega}^{*}(f_1,\ldots,f_m) \right\|_{L^{2/m}(\mathbb{R}^n)} \leq C \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^{m} \|f_j\|_{L^2(\mathbb{R}^n)}$$

for Schwartz functions $f_1,\ldots,f_m$ on $\mathbb{R}^n$.

This extends and improves a result obtained in [3] which treated the case $m = 2$ and $q = \infty$. Theorem 1.2 follows from Propositions 4.1 and 4.2 below, which are counterparts of Propositions 5 and 4 in [3], respectively. We improve the two propositions in the m-linear settings. Remark that the assumption $\Omega \in L^2(\mathbb{S}^{2n-1})$ in Proposition 5 and Theorem 2 in [3] should be $\Omega \in L^\infty(\mathbb{S}^{2n-1})$. One of the main improvements is the $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ estimate for $\mathfrak{T}_K^{**}$ in (4.10) with a bound $\|\Omega\|_{L^1(\mathbb{S}^{mn-1})}$, while a simple m-linear extensions of the arguments in [3] requires the bound $\|\Omega\|_{L^\infty(\mathbb{S}^{mn-1})}$ for the estimate, which originated simply from the kernel estimate

(1.6) $$\left| K_{\tau}^{**}(\vec{y}) \right| \lesssim_N \|\Omega\|_{L^\infty(\mathbb{S}^{mn-1})} 2^{mn\tau} \left(1 + 2^{-\tau}|\vec{y}| \right)^{-N}$$

where $\mathfrak{T}_K^{**}$ is defined in (4.6) and its kernel K_τ^{**} is in (4.7). For the improvement, we incorporate a delicate decomposition, as we are unable to use the kernel estimate (1.6); see (4.8). To obtain the results in Proposition 4.2, we suitably combine Littlewood-Paley techniques and wavelet decompositions to reduce the boundedness of $\mathcal{L}_{\Omega,\mu}^\sharp$ to estimates for norms of maximal operators associated with lattice bumps with suitable decay. This is the essential contribution of this article in view of the fact that the bilinear argument in [3, Proposition 4] does not apply due to the complicated structure of general m-linear operators for $m \geq 3$; see (4.14) for the exact formulation. This result is actually proved in terms of Plancherel type inequalities, recently developed in [18] and stated in Proposition 2.1.

The tools used to establish Theorem 1.1 turn out to be useful in the study of pointwise convergence problems of several related operators. As an example let us take multilinear multipliers with limited decay to demonstrate our idea.

For a smooth function $\sigma \in \mathcal{C}^\infty((\mathbb{R}^n)^m)$ and $\nu \in \mathbb{Z}$ let

$$(1.7) \qquad S_\sigma^\nu(f_1,\ldots,f_m)(x) := \int_{(\mathbb{R}^n)^m} \sigma(2^\nu \vec{\xi})\Big(\prod_{j=1}^m \widehat{f_j}(\xi_j)\Big) e^{2\pi i \langle x, \sum_{j=1}^m \xi_j\rangle}\, d\vec{\xi}$$

for Schwartz functions $f_1,\ldots,f_m$ on $\mathbb{R}^n$, where $\vec{\xi} := (\xi_1,\ldots,\xi_m) \in (\mathbb{R}^n)^m$.

We are interested in the poinwise convergence of S_σ^ν when $\nu \to -\infty$. We pay particular attention to σ satisfying the limited decay property (for some fixed a)

$$\big|\partial^\beta \sigma(\vec{\xi})\big| \lesssim_\beta |\vec{\xi}|^{-a}$$

for sufficiently many β. Examples of multipliers of this type include $\widehat{\mu}$, the Fourier transform of the spherical measure μ; see [4, 9, 26] for the corresponding linear results.

The second contribution of this work is the following result.

Theorem 1.3. *Let $m \geq 2$ and $a > \frac{(m-1)n}{2}$. Let $\sigma \in \mathcal{C}^\infty((\mathbb{R}^n)^m)$ satisfy*

$$(1.8) \qquad \big|\partial^\beta \sigma(\vec{\xi})\big| \lesssim_\beta |\vec{\xi}|^{-a}$$

for all $|\beta| \leq \big[\frac{(m-1)n}{2}\big] + 1$, where $[r]$ denotes the integer part of r. Then for f_j in $L^2(\mathbb{R}^n)$, $j = 1,\ldots,m$, the functions $S_\sigma^\nu(f_1,\ldots,f_m)$ converge to $\sigma(0)f_1\cdots f_m$ pointwise a.e. as $\nu \to -\infty$ and to zero pointwise a.e. as $\nu \to \infty$.

The precise definition of the action of the multilinear operator S_σ^ν on L^2 functions will be discussed after Theorem 1.4.

The a.e. convergence claimed in Theorem 1.3 is related to the boundedness of the associated m-(sub)linear lacunary maximal multiplier operator defined by:

$$\mathcal{M}_\sigma(f_1,\ldots,f_m) := \sup_{\nu \in \mathbb{Z}}\big|S_\sigma^\nu(f_1,\ldots,f_m)\big|.$$

$\mathcal{M}_\sigma$ is the so-called multilinear spherical maximal function when $\sigma = \widehat{\mu}$, which was studied extensively recently by [1, 2, 10, 22, 23]. In particular a bilinear version of the following theorem was previously obtained in [17].

Theorem 1.4. *Let $m \geq 2$ and $a > \frac{(m-1)n}{2}$. Let $\sigma \in \mathcal{C}^\infty((\mathbb{R}^n)^m)$ be as in Theorem 1.3 Then there exists a constant $C > 0$ such that*

$$\big\|\mathcal{M}_\sigma(f_1, \ldots, f_m)\big\|_{L^{2/m}(\mathbb{R}^n)} \leq C \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)}$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

One of the main difficulties in dealing with general m-linear cases for $m \geq 3$ is that the target space $L^{2/m}$ is not a Banach space if $2/m < 1$. As a result, the condition $a > \frac{(m-1)n}{2}$ cannot be exploited by a simple adaptation of the bilinear argument in [17]. Additional combinatorial complexity arises from the multilinear extension, and in order to address these issues we apply a more refined decomposition, recently introduced in [18], so that l-linear Plancherel type estimates ($1 \leq l \leq m$) can be applied. These key estimates are stated in Proposition 2.1 below.

With the help of Theorem 1.4, we notice that the multilinear operator S_σ^ν is also well-defined for $f_j \in L^2(\mathbb{R}^n)$. Indeed, given f_j in $L^2(\mathbb{R}^n)$, we find a sequence of Schwartz functions f_j^k that converge to f_j in $L^2(\mathbb{R}^n)$ as $k \to \infty$. Then Theorem 1.4 implies that the sequence

$$\big\{ S_\sigma^\nu(f_1^k, \ldots, f_m^k) \big\}_k$$

is a Cauchy sequence in $L^{2/m}$ and thus it has a unique limit in $L^{2/m}$ which we call $S_\sigma^\nu(f_1, \ldots, f_m)$. It is easy to verify that this limit does not depend on the choice of f_j^k.

The paper is organized as follows. Section 2 is dedicated to preliminaries, introducing a wavelet decomposition that is one of the main ingredients to establish maximal inequalities in Theorems 1.2 and 1.4, and studying general properties of the decomposition. Another maximal inequality for rough singular integrals will be given in Section 3. We prove first Theorem 1.2 in Section 4 as it is necessary for the proof of Theorem 1.1 in Section 5. The proof of Theorems 1.4 and 1.3 will be given in turn in the last two sections.

2. Preliminary material

We adapt some notations and key estimates from [18]. For the sake of independent reading we review the main tools and notation. We begin with certain orthonormal bases of L^2 due to Triebel [30], that will be of great use in our work. The idea is as follows. For any fixed $L \in \mathbb{N}$ one can construct real-valued compactly supported functions ψ_F, ψ_M in $\mathcal{C}^L(\mathbb{R})$ satisfying the following properties: $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$, $\int_\mathbb{R} x^\alpha \psi_M(x) dx = 0$ for all $0 \leq \alpha \leq L$, and moreover, if $\Psi_{\vec{G}}$ is a function on $\mathbb{R}^{mn}$, defined by

$$\Psi_{\vec{G}}(\vec{x}) := \psi_{g_1}(x_1) \cdots \psi_{g_{mn}}(x_{mn})$$

for $\vec{x} := (x_1, \ldots, x_{mn}) \in \mathbb{R}^{mn}$ and $\vec{G} := (g_1, \ldots, g_{mn})$ in the set

$$\mathcal{I} := \big\{ \vec{G} := (g_1, \ldots, g_{mn}) : g_i \in \{F, M\} \big\},$$

then the family of functions

$$\bigcup_{\lambda\in\mathbb{N}_0}\bigcup_{\vec{k}\in\mathbb{Z}^{mn}}\left\{2^{\lambda mn/2}\Psi_{\vec{G}}(2^\lambda\vec{x}-\vec{k}):\vec{G}\in\mathcal{I}^\lambda\right\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^{mn})$, where $\mathcal{I}^0:=\mathcal{I}$ and for $\lambda\geq 1$, we set $\mathcal{I}^\lambda:=\mathcal{I}\setminus\{(F,\ldots,F)\}$.

We consistently use the notation $\vec{\xi}:=(\xi_1,\ldots,\xi_m)$ for elements of $(\mathbb{R}^n)^m$, $\vec{G}:=(G_1,\ldots,G_m)\in(\{F,M\}^n)^m$, and $\Psi_{\vec{G}}(\vec{\xi})=\Psi_{G_1}(\xi_1)\cdots\Psi_{G_m}(\xi_m)$. For each $\vec{k}:=(k_1,\ldots,k_m)\in(\mathbb{Z}^n)^m$ and $\lambda\in\mathbb{N}_0$, let

$$\Psi_{G_i,k_i}^\lambda(\xi_i):=2^{\lambda n/2}\Psi_{G_i}(2^\lambda\xi_i-k_i),\qquad 1\leq i\leq m$$

and

$$\Psi_{\vec{G},\vec{k}}^\lambda(\vec{\xi}):=\Psi_{G_1,k_1}^\lambda(\xi_1)\cdots\Psi_{G_m,k_m}^\lambda(\xi_m).$$

We also assume that the support of ψ_{g_i} is contained in $\{\xi\in\mathbb{R}:|\xi|\leq C_0\}$ for some $C_0>1$, which implies that

$$\text{Supp}(\Psi_{G_i,k_i}^\lambda)\subset\left\{\xi_i\in\mathbb{R}^n:|2^\lambda\xi_i-k_i|\leq C_0\sqrt{n}\right\}.$$

In other words, the support of Ψ_{G_i,k_i}^λ is contained in the ball centered at $2^{-\lambda}k_i$ and radius $C_0\sqrt{n}2^{-\lambda}$. Then we note that for a fixed $\lambda\in\mathbb{N}_0$, elements of $\{\Psi_{\vec{G},\vec{k}}^\lambda\}_{\vec{k}}$ have (almost) disjoint compact supports.

It is also known in [29] that if L is sufficiently large, then every tempered distribution H on $\mathbb{R}^{mn}$ can be represented as

$$\tag{2.1} H(\vec{x})=\sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\sum_{\vec{k}\in\mathbb{Z}^{mn}}b_{\vec{G},\vec{k}}^\lambda 2^{\lambda mn/2}\Psi_{\vec{G}}(2^\lambda\vec{x}-\vec{k})$$

and for $1<q<\infty$ and $s\geq 0$,

$$\left\|\left(\sum_{\vec{G},\,\vec{k}}|b_{\vec{G},\vec{k}}^\lambda\Psi_{\vec{G},\vec{k}}^\lambda|^2\right)^{1/2}\right\|_{L^q(\mathbb{R}^{mn})}\leq C2^{-s\lambda}\|H\|_{L_s^q(\mathbb{R}^{mn})}$$

where

$$b_{\vec{G},\vec{k}}^\lambda:=\int_{\mathbb{R}^{mn}}H(\vec{x})\Psi_{\vec{G},\vec{k}}^\lambda(\vec{x})\,d\vec{x}$$

and L_s^q is the Sobolev space of functions H such that $(I-\Delta)^{s/2}H\in L^q(\mathbb{R}^{mn})$. Moreover, it follows from the last estimate and from the (almost) disjoint support property of the $\Psi_{\vec{G},\vec{k}}^\lambda$'s that

$$\left\|\{b_{\vec{G},\vec{k}}^\lambda\}_{\vec{k}\in\mathbb{Z}^{mn}}\right\|_{\ell^q}\approx\left(2^{\lambda mn(1-q/2)}\int_{\mathbb{R}^{mn}}\left(\sum_{\vec{k}}|b_{\vec{G},\vec{k}}^\lambda\Psi_{\vec{G},\vec{k}}^\lambda(\vec{x})|^2\right)^{q/2}d\vec{x}\right)^{1/q}$$

$$\tag{2.2} \lesssim 2^{-\lambda(s-mn/q+mn/2)}\|H\|_{L_s^q(\mathbb{R}^{mn})}.$$

Now we study an essential estimate in [18] which will play a significant role in the proof of both Theorems 1.2 and 1.4. We define the operator $L_{G_i,k_i}^{\lambda,\gamma}$ by

$$\tag{2.3} L_{G_i,k_i}^{\lambda,\gamma}f:=\left(\Psi_{G_i,k_i}^\lambda(\cdot/2^\gamma)\widehat{f}\right)^\vee,\qquad\gamma\in\mathbb{Z}.$$

For $\mu \in \mathbb{Z}$ let

$$(2.4) \qquad \mathcal{U}^\mu := \left\{ \vec{k} \in (\mathbb{Z}^n)^m : 2^{\mu-2} \leq |\vec{k}| \leq 2^{\mu+2}, \; |k_1| \geq \cdots \geq |k_m| \right\}$$

and split the set into m disjoint subsets $\mathcal{U}_l^\mu$ as below:

$$\mathcal{U}_1^\mu := \left\{ \vec{k} \in \mathcal{U}^\mu : |k_1| \geq 2C_0\sqrt{n} > |k_2| \geq \cdots \geq |k_m| \right\}$$
$$\mathcal{U}_2^\mu := \left\{ \vec{k} \in \mathcal{U}^\mu : |k_1| \geq |k_2| \geq 2C_0\sqrt{n} > |k_3| \geq \cdots \geq |k_m| \right\}$$
$$\vdots$$
$$\mathcal{U}_m^\mu := \left\{ \vec{k} \in \mathcal{U}^\mu : |k_1| \geq \cdots \geq |k_m| \geq 2C_0\sqrt{n} \right\}.$$

Then we have the following two observations that appear in [18].

- For $\vec{k} \in \mathcal{U}_l^{\lambda+\mu}$,

$$(2.5) \qquad L_{G_j,k_j}^{\lambda,\gamma} f = L_{G_j,k_j}^{\lambda,\gamma} f^{\lambda,\gamma,\mu} \quad \text{for} \; 1 \leq j \leq l$$

  due to the support of Ψ_{G_j,k_j}^λ, where $\widehat{f^{\lambda,\gamma,\mu}}(\xi) := \widehat{f}(\xi)\chi_{C_0\sqrt{n}2^{\gamma-\lambda} \leq |\xi| \leq 2^{\gamma+\mu+3}}$.
- For $\mu \geq 1$ and $\lambda \in \mathbb{N}_0$,

$$(2.6) \qquad \left(\sum_{\gamma \in \mathbb{Z}} \|f^{\lambda,\gamma,\mu}\|_{L^2}^2 \right)^{1/2} \lesssim (\mu+\lambda)^{1/2}\|f\|_{L^2} \lesssim \mu^{1/2}(\lambda+1)^{1/2}\|f\|_{L^2}$$

  where Plancherel's identity is applied in the first inequality.

Proposition 2.1 ([18, Proposition 2.4]). *Let m be a positive integer with $m \geq 2$ and $0 < q < \frac{2m}{m-1}$. Fix $\lambda \in \mathbb{N}_0$ and $\vec{G} \in \mathcal{I}^\lambda$. Suppose that $\{b_{\vec{G},\vec{k}}^{\lambda,\gamma,\mu}\}_{\vec{G} \in \mathcal{I}^\lambda, \gamma, \mu \in \mathbb{Z}, \vec{k} \in (\mathbb{Z}^n)^m}$ is a sequence of complex numbers satisfying*

$$\sup_{\gamma \in \mathbb{Z}} \left\| \{b_{\vec{G},\vec{k}}^{\lambda,\gamma,\mu}\}_{\vec{k} \in (\mathbb{Z}^n)^m} \right\|_{\ell^\infty} \leq A_{\vec{G},\lambda,\mu}$$

and

$$\sup_{\gamma \in \mathbb{Z}} \left\| \{b_{\vec{G},\vec{k}}^{\lambda,\gamma,\mu}\}_{\vec{k} \in (\mathbb{Z}^n)^m} \right\|_{\ell^q} \leq B_{\vec{G},\lambda,\mu,q}.$$

Then the following statements hold:

(1) For $1 \leq r \leq 2$, there exists a constant $C > 0$, independent of $, \vec{G}, \lambda, \mu$, such that

$$\left\| \left(\sum_{\gamma \in \mathbb{Z}} \left| \sum_{\vec{k} \in \mathcal{U}_1^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\gamma,\mu} L_{G_1,k_1}^{\lambda,\gamma} f_1^{\lambda,\gamma,\mu} \prod_{j=2}^m L_{G_j,k_j}^{\lambda,\gamma} f_j \right|^r \right)^{1/r} \right\|_{L^{2/m}}$$

$$\leq C A_{\vec{G},\lambda,\mu} 2^{\lambda mn/2} \left(\sum_{\gamma \in \mathbb{Z}} \|f_1^{\lambda,\gamma,\mu}\|_{L^2}^r \right)^{1/r} \prod_{j=2}^m \|f_j\|_{L^2}$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

(2) *For $2 \le l \le m$ there exists a constant $C > 0$, independent of $\vec{G}, \lambda, \mu$, such that*

$$\Big\| \sum_{\gamma \in \mathbb{Z}} \Big| \sum_{\vec{k} \in \mathcal{U}_l^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\gamma,\mu} \Big(\prod_{j=1}^{l} L_{G_j,k_j}^{\lambda,\gamma} f_j^{\lambda,\gamma,\mu} \Big) \Big(\prod_{j=l+1}^{m} L_{G_j,k_j}^{\lambda,\gamma} f_j \Big) \Big| \Big\|_{L^{2/m}}$$

$$\le C A_{\vec{G},\lambda,\mu}^{1-\frac{(l-1)q}{2l}} B_{\vec{G},\lambda,\mu,q}^{\frac{(l-1)q}{2l}} 2^{\lambda mn/2} \Big[\prod_{j=1}^{l} \Big(\sum_{\gamma \in \mathbb{Z}} \|f_j^{\lambda,\gamma,\mu}\|_{L^2}^2 \Big)^{1/2} \Big] \Big[\prod_{j=l+1}^{m} \|f_j\|_{L^2} \Big]$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$, where $\prod_{m+1}^{m}$ is understood as empty.

In view of (2.5), (2.6) and Proposition 2.1, we actually obtain

$$\Big\| \Big(\sum_{\gamma \in \mathbb{Z}} \Big| \sum_{\vec{k} \in \mathcal{U}_1^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\gamma,\mu} \prod_{j=1}^{m} L_{G_j,k_j}^{\lambda,\gamma} f_j \Big|^2 \Big)^{1/2} \Big\|_{L^{2/m}}$$

$$(2.7) \qquad \lesssim A_{\vec{G},\lambda,\mu} \mu^{1/2} 2^{\lambda mn/2} (\lambda+1)^{1/2} \prod_{j=1}^{m} \|f_j\|_{L^2}$$

and for $2 \le l \le m$

$$\Big\| \sum_{\gamma \in \mathbb{Z}} \Big| \sum_{\vec{k} \in \mathcal{U}_l^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\gamma,\mu} \prod_{j=1}^{m} L_{G_j,k_j}^{\lambda,\gamma} f_j \Big| \Big\|_{L^{2/m}}$$

$$(2.8) \qquad \lesssim A_{\vec{G},\lambda,\mu}^{1-\frac{(l-1)q}{2l}} B_{\vec{G},\lambda,\mu,q}^{\frac{(l-1)q}{2l}} \mu^{l/2} 2^{\lambda mn/2} (\lambda+1)^{l/2} \prod_{j=1}^{m} \|f_j\|_{L^2}.$$

3. An auxiliary lemma

We have the following extension of Lemma 5 in [3].

Lemma 3.1. *Let $1 < q \le \infty$ and $\Omega \in L^q(\mathbb{S}^{mn-1})$. Suppose $1 < p_1, \ldots, p_m < \infty$ and $1/m < p < \infty$ satisfies $1/p = 1/p_1 + \cdots + 1/p_m$ and*

$$(3.1) \qquad \frac{1}{p} < \frac{1}{q} + \frac{m}{q'}.$$

Given $f_j \in L^{p_j}(\mathbb{R}^n)$, there is a set of measure zero E such that for $x \in \mathbb{R}^n \setminus E$

$$(3.2) \qquad \int \cdots \int_{|\vec{y}| \le R} |\Omega(\vec{y}')| \prod_{j=1}^{m} |f_j(x - y_j)| \, d\vec{y} < \infty$$

for all $R > 0$. Then for $x \in \mathbb{R}^n \setminus E$ the maximal operator

$$(3.3) \qquad \mathcal{M}_\Omega(f_1, \ldots, f_m)(x) = \sup_{R>0} \frac{1}{R^{mn}} \int \cdots \int_{|\vec{y}| \le R} |\Omega(\vec{y}')| \prod_{j=1}^{m} |f_j(x - y_j)| \, d\vec{y}$$

is well defined and maps $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$ *with norm bounded by a constant multiple of* $\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}$. *Precisely, there is a constant* $C > 0$ *such that*

$$(3.4) \qquad \|\mathcal{M}_\Omega(f_1, \ldots, f_m)\|_{L^p} \leq C\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\|f_1\|_{L^{p_1}} \cdots \|f_m\|_{L^{p_m}}$$

for functions $f_j \in L^{p_j}(\mathbb{R}^n)$, $1 \leq j \leq m$.

Proof. Since $\|\Omega\|_{L^r(\mathbb{S}^{mn-1})} \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{mn-1})}$ for all $1 < r < \infty$ and there exists $1 < q < \infty$ such that $1/p < 1/q + m/q' < m \ (= 1/\infty + m/1)$, we may assume $1 < q < \infty$. Without loss of generality, we may also assume that $\|\Omega\|_{L^q(\mathbb{S}^{mn-1})} = 1$.

We split

$$\Omega = \Omega_0 + \sum_{l=1}^{\infty} \Omega_l,$$

where $\Omega_0 = \Omega\chi_{|\Omega|\leq 2}$ and $\Omega_l = \Omega\chi_{2^l < |\Omega| \leq 2^{l+1}}$ for $l \geq 1$. Then Hölder's inequality and Chebyshev's inequality give

$$\|\Omega_l\|_{L^1} \leq |\mathrm{Supp}\,\Omega_l|^{\frac{1}{q'}} \leq \|\Omega\|_{L^q}^{\frac{q}{q'}} 2^{-l\frac{q}{q'}} = 2^{-l\frac{q}{q'}}$$

and obviously

$$(3.5) \qquad \|\Omega_l\|_{L^\infty} \leq 2^{l+1}.$$

We first claim that for $1 < r, r_1, \ldots, r_m < \infty$ with $1/r = 1/r_1 + \cdots + 1/r_m$ we have

$$(3.6) \qquad \left\|\mathcal{M}_{\Omega_l}(S_1, \ldots, S_m)\right\|_{L^{r_1} \times \cdots \times L^{r_m} \to L^r} \lesssim 2^{-l\frac{q}{q'}} \prod_{j=1}^{m} \|S_j\|_{L^{r_j}(\mathbb{R}^n)}$$

for simple functions S_j. To verify this estimate, we choose indices $\mu_1, \ldots, \mu_m$ satisfying

$$1/\mu_1 + \cdots + 1/\mu_m = 1$$

and

$$1 < \mu_j < r_j \qquad \text{for each } \ 1 \leq j \leq m.$$

Then a direct computation using Hölder's inequality yields

$$\mathcal{M}_{\Omega_l}(S_1, \ldots, S_m)(x) \leq \int_{\mathbb{S}^{mn-1}} |\Omega_l(\vec{\theta})| \prod_{j=1}^{m} \mathcal{M}_{\mu_j}^{\theta_j} S_j(x) \, d\vec{\theta},$$

where the directional maximal operator $\mathcal{M}_{\mu_j}^{\theta_j}$ is defined by

$$\mathcal{M}_{\mu_j}^{\theta_j} g(x) := \sup_{R>0} \left(\frac{1}{R} \int_0^R |g(x - t\theta_j)|^{\mu_j} \, dt\right)^{1/\mu_j}.$$

It follows from this that

$$\left\|\mathcal{M}_{\Omega_l}(S_1, \ldots, S_m)\right\|_{L^r} \leq \int_{\mathbb{S}^{mn-1}} |\Omega_l(\vec{\theta})| \prod_{j=1}^{m} \left\|\mathcal{M}_{\mu_j}^{\theta_j} S_j\right\|_{L^{r_j}} \, d\vec{\theta},$$

where Minkowski's inequality and Hölder's inequality are applied. Using the L^{r_j} boundedness of $\mathcal{M}_{\mu_j}^{\theta_j}$ for $0 < \mu_j < r_j$ with constants independent of θ_j (by the method of rotations), we obtain (3.6).

Then the case $p > 1$ (for which $q > 1$ implies the assumption (3.1)) in the assertion follows from summing the estimates (3.6) over $l \geq 0$.

The other case $1/m < p \leq 1$ can be proved by interpolation with the $L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$ estimate. Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator. Then, using (3.5), it is easy to verify the pointwise estimate

$$\mathcal{M}_{\Omega_l}(f_1, \ldots, f_m)(x) \leq 2^{l+1} \prod_{j=1}^{m} \mathcal{M}f_j(x),$$

for $f_1, \ldots, f_m$ in $L^1(\mathbb{R}^n)$, and this yields that

(3.7) $$\|\mathcal{M}_{\Omega_l}\|_{L^1 \times \cdots \times L^1 \to L^{1/m,\infty}} \lesssim 2^l,$$

using Hölder's inequality for weak type spaces ([14, p 16]) and the weak $(1,1)$ boundedness of $\mathcal{M}$. Now we fix $0 < p_1, \ldots, p_m < \infty$ and $1/m < p \leq 1$, and choose $r > 1$ such that

$$\frac{1}{p} < \frac{1}{rq} + \frac{m}{q'} \quad \left(< \frac{1}{q} + \frac{m}{q'} \right),$$

or, equivalently,

$$\frac{q(m - 1/p)}{q'(m - 1/r)} - \frac{1/p - 1/r}{m - 1/r} > 0.$$

Interpolating between (3.7) and (3.6) with appropriate $(r_1, \ldots, r_m)$ satisfying $1/r = 1/r_1 + \cdots + 1/r_m$ (using [15, Theorem 7.2.2]) yields

$$\|\mathcal{M}_{\Omega_l}(S_1, \ldots, S_m)\|_{L^p} \leq C 2^{-l\left(\frac{q(m-1/p)}{q'(m-1/r)} - \frac{1/p-1/r}{m-1/r}\right)} \|S_1\|_{L^{p_1}} \cdots \|S_m\|_{L^{p_m}}$$

for simple functions S_j. The exponential decay in l obtained above together with the fact that $\| \cdot \|_{L^p}^p$ is a subadditive quantity for $0 < p \leq 1$ implies, for p and q satisfying (3.1),

(3.8) $$\|\mathcal{M}_{\Omega}(S_1, \ldots, S_m)\|_{L^p} \leq C \|S_1\|_{L^{p_1}} \cdots \|S_m\|_{L^{p_m}}.$$

for simple functions $S_1, \ldots, S_m$.

Next we extend $\mathcal{M}_{\Omega}(f_1, \ldots, f_m)$ to functions $f_j \in L^{p_j}(\mathbb{R}^n)$. To achieve this goal, we choose nonnegative simple functions S_j^k that increase pointwise to $|f_j|$ as $k \to \infty$.

It follows from (3.8) that for any $R > 0$

$$\left\{ \int_{\mathbb{R}^n} \left[\frac{1}{R^{mn}} \int \cdots \int_{|\vec{y}| \leq R} |\Omega(\vec{y}')| \prod_{j=1}^{m} S_j^k(x - y_j) \, d\vec{y} \right]^p dx \right\}^{\frac{1}{p}} \leq C \|f_1\|_{L^{p_1}} \cdots \|f_m\|_{L^{p_m}}$$

and from this we obtain

$$\left\{ \int_{\mathbb{R}^n} \left[\frac{1}{R^{mn}} \int \cdots \int_{|\vec{y}| \leq R} |\Omega(\vec{y}')| \prod_{j=1}^{m} |f_j^k(x - y_j)| \, d\vec{y} \right]^p dx \right\}^{\frac{1}{p}} \leq C \|f_1\|_{L^{p_1}} \cdots \|f_m\|_{L^{p_m}}$$

via Lebesgue's monotone convergence theorem. We conclude that for any $R \in \mathbb{Z}^+$ there is a set of measure zero E^R such that

$$(3.9) \qquad \frac{1}{R^{mn}} \int \cdots \int_{|\vec{y}| \leq R} |\Omega(\vec{y}')| \prod_{j=1}^{m} |f_j^k(x - y_j)| \, d\vec{y} < \infty$$

for all $x \in \mathbb{R}^n \setminus E^R$. Setting $E = \cup_{R=1}^{\infty} E^R$ we obtain (3.2) for $x \in \mathbb{R}^n \setminus E$.

This allows us to define $\mathcal{M}_\Omega(f_1, \ldots, f_m)(x)$ for $f_j \in L^{p_j}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n \setminus E$ as the supremum of the expressions in (3.9). Now for $x \in \mathbb{R}^n \setminus E$,

$$\mathcal{M}_\Omega(f_1, \ldots, f_m)(x) = \sup_{R>0} \frac{1}{R^{mn}} \int \cdots \int_{|\vec{y}| \leq R} |\Omega(\vec{y}')| \prod_{j=1}^{m} |f_j(x - y_j)| \, d\vec{y}$$

$$= \sup_{R>0} \lim_{k\to\infty} \frac{1}{R^{mn}} \int \cdots \int_{|\vec{y}| \leq R} |\Omega(\vec{y}')| \prod_{j=1}^{m} S_j^k(x - y_j) \, d\vec{y}$$

$$\leq \liminf_{k\to\infty} \mathcal{M}_\Omega(S_1^k, \ldots, S_m^k)(x).$$

As $\mathcal{M}_\Omega(S_1^k, \ldots, S_m^k)(x)$ is increasing in k, we obtain from (3.8) by Fatou's lemma that

$$\|\mathcal{M}_\Omega(f_1, \ldots, f_m)\|_{L^p(\mathbb{R}^n)} \lesssim \liminf_{k\to\infty} \|\mathcal{M}_\Omega(S_1^k, \ldots, S_m^k)\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

In particular, this shows that for $f_j \in L^{p_j}(\mathbb{R}^n)$ there is a set E' of measure zero such that

$$\sup_{R>0} \frac{1}{R^{mn}} \int \cdots \int_{|\vec{y}| \leq R} |\Omega(\vec{y}')| \prod_{j=1}^{m} |f_j(x - y_j)| \, d\vec{y} < \infty$$

for all $x \in \mathbb{R}^n \setminus E'$, and moreover, on $\mathbb{R}^n \setminus (E \cup E')$, the preceding supremum pointwise controls all expressions in (3.9).

$\square$

4. Proof of Theorem 1.2

Let $\frac{2m}{m+1} < q < 2$ and Ω in $L^q(\mathbb{S}^{mn-1})$. We use a dyadic decomposition introduced by Duoandikoetxea and Rubio de Francia [13]. We choose a Schwartz function $\Phi^{(m)}$ on $(\mathbb{R}^n)^m$ such that its Fourier transform $\widehat{\Phi^{(m)}}$ is supported in the annulus $\{\vec{\xi} \in (\mathbb{R}^n)^m : 1/2 \leq |\vec{\xi}| \leq 2\}$ and satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Phi_j^{(m)}}(\vec{\xi}) = 1$ for $\vec{\xi} \neq \vec{0}$ where $\widehat{\Phi_j^{(m)}}(\vec{\xi}) := \widehat{\Phi^{(m)}}(\vec{\xi}/2^j)$. For $\gamma \in \mathbb{Z}$ let

$$K^\gamma(\vec{y}) := \widehat{\Phi^{(m)}}(2^\gamma \vec{y}) K(\vec{y}), \quad \vec{y} \in (\mathbb{R}^n)^m$$

and then we observe that $K^\gamma(\vec{y}) = 2^{\gamma mn} K^0(2^\gamma \vec{y})$. For $\mu \in \mathbb{Z}$ we define

$$(4.1) \qquad K_\mu^\gamma(\vec{y}) := \Phi_{\mu+\gamma}^{(m)} * K^\gamma(\vec{y}) = 2^{\gamma mn} [\Phi_\mu^{(m)} * K^0](2^\gamma \vec{y}).$$

It follows from this definition that

$$\widehat{K_\mu^\gamma}(\vec{\xi}) = \widehat{\Phi^{(m)}}(2^{-(\mu+\gamma)}\vec{\xi})\widehat{K^0}(2^{-\gamma}\vec{\xi}) = \widehat{K_\mu^0}(2^{-\gamma}\vec{\xi}),$$

which implies that $\widehat{K_\mu^\gamma}$ is bounded uniformly in γ while they have almost disjoint supports, so it is natural to add them together as follows:

$$K_\mu(\vec{y}) := \sum_{\gamma \in \mathbb{Z}} K_\mu^\gamma(\vec{y}).$$

4.1. **Reduction.** We introduce the maximal operator

$$\mathcal{L}_\Omega^\sharp(f_1, \dots, f_m)(x) := \sup_{\tau \in \mathbb{Z}} \left| \sum_{\gamma < \tau} \int_{(\mathbb{R}^n)^m} K^\gamma(\vec{y}) \prod_{j=1}^m f_j(x - y_j)\, d\vec{y} \right|$$

for $x \in \mathbb{R}^n$. Then we claim that

(4.2) $$\mathcal{L}_\Omega^*(f_1, \dots, f_m) \leq \mathcal{M}_\Omega(f_1, \dots, f_m)(x) + \mathcal{L}_\Omega^\sharp(f_1, \dots, f_m)$$

To prove (4.2) we introduce the notation

$$K^{(\epsilon)}(\vec{y}) := K(\vec{y})\chi_{|\vec{y}| \geq \epsilon}, \qquad \widetilde{K}^{(\epsilon)}(\vec{y}) := K(\vec{y})\big(1 - \widehat{\Theta^{(m)}}(\vec{y}/\epsilon)\big),$$

setting $\widehat{\Theta^{(m)}}(\vec{y}) := 1 - \sum_{\gamma \in \mathbb{N}} \widehat{\Phi^{(m)}}(\vec{y}/2^\gamma)$ so that

$$\mathrm{Supp}(\widehat{\Theta^{(m)}}) \subset \{\vec{y} \in (\mathbb{R}^n)^m : |\vec{y}| \leq 2\}$$

and $\widehat{\Theta^{(m)}}(\vec{y}) = 1$ for $|\vec{y}| \leq 1$.

Given $\epsilon > 0$ choose $\rho \in \mathbb{Z}$ such that $2^\rho \leq \epsilon < 2^{\rho+1}$. Then we write

$$\left| \int_{(\mathbb{R}^n)^m \setminus B(0,\epsilon)} K(\vec{y}) \prod_{j=1}^m f_j(x - y_j)\, d\vec{y} \right|$$

(4.3) $$\leq \left| \int_{(\mathbb{R}^n)^m} \big(K^{(\epsilon)}(\vec{y}) - \widetilde{K}^{(2^\rho)}(\vec{y})\big) \prod_{j=1}^m f_j(x - y_j)\, d\vec{y} \right|$$

(4.4) $$+ \left| \int_{(\mathbb{R}^n)^m} \widetilde{K}^{(2^\rho)}(\vec{y}) \prod_{j=1}^m f_j(x - y_j)\, d\vec{y} \right|.$$

Term (4.4) is clearly less than

$$\left| \sum_{\gamma \in \mathbb{Z}: \gamma < -\rho} \int_{(\mathbb{R}^n)^m} K^\gamma(\vec{y}) \prod_{j=1}^m f_j(x - y_j)\, d\vec{y} \right| \leq \mathcal{L}_\Omega^\sharp(f_1, \dots, f_m)(x),$$

while (4.3) is controlled by $\mathcal{M}_\Omega(f_1, \dots, f_m)(x)$ as

$$\left| K^{(\epsilon)}(\vec{y}) - \widetilde{K}^{(2^\rho)}(\vec{y}) \right| \lesssim |K(\vec{y})| \chi_{|\vec{y}| \approx 2^\rho} \lesssim \frac{|\Omega(\vec{y}')|}{2^{\rho mn}} \chi_{|\vec{y}| \lesssim 2^\rho}.$$

Thus (4.2) follows after taking the supremum over all $\epsilon > 0$.

Since the boundedness of $\mathcal{M}_\Omega$ follows from Lemma 3.1 with the fact that $q > \frac{2m}{m+1}$ implies $\frac{m}{2} < \frac{1}{q} + \frac{m}{q'}$, matters reduce to the boundedness of $\mathcal{L}_\Omega^\sharp$.

For each $\gamma \in \mathbb{Z}$ let

$$K_\mu := \sum_{\gamma \in \mathbb{Z}} K_\mu^\gamma.$$

In the study of multilinear rough singular integral operators $\mathcal{L}_\Omega$ in [18] whose kernel is $\sum_{\gamma \in \mathbb{Z}} K^\gamma = \sum_{\mu \in \mathbb{Z}} \sum_{\gamma \in \mathbb{Z}} K_\mu^\gamma = \sum_{\mu \in \mathbb{Z}} K_\mu$, the part where μ is less than a constant is relatively simple because the Fourier transform of K_μ satisfies the estimate

$$(4.5) \qquad \left| \partial^\alpha \widehat{K_\mu}(\vec{\xi}) \right| \lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} |\vec{\xi}|^{-|\alpha|} Q(\mu), \qquad 1 < q \le \infty$$

for all multiindices α and $\vec{\xi} \in \mathbb{R}^{mn} \setminus \{0\}$, where $Q(\mu) = 2^{(mn-\delta')\mu}$ if $\mu \ge 0$ and $Q(\mu) = 2^{\mu(1-\delta')}$ if $\mu < 0$ for some $0 < \delta' < 1/q'$, which is the condition of the Coifman-Meyer multiplier theorem [7], [15, Theorem 7.5.3] with constant $\|\Omega\|_{L^q(\mathbb{S}^{mn-1})} Q(\mu)$. The remaining case when μ is large enough was handled by using product-type wavelet decompositions. We expect a similar strategy would work in handling $\mathcal{L}_\Omega^\sharp$.

To argue strictly, we write

$$\mathcal{L}_\Omega^\sharp(f_1, \ldots, f_m) \le \widetilde{\mathcal{L}}_\Omega^\sharp(f_1, \ldots, f_m) + \sum_{\mu \in \mathbb{Z}:\, 2^{\mu-10} > C_0\sqrt{mn}} \mathcal{L}_{\Omega,\mu}^\sharp(f_1, \ldots, f_m),$$

where we set

$$\widetilde{\mathcal{L}}_\Omega^\sharp(f_1, \ldots, f_m)(x) := \sup_{\tau \in \mathbb{Z}} \left| \int_{(\mathbb{R}^n)^m} \sum_{\gamma < \tau} \sum_{\mu \in \mathbb{Z}:\, 2^{\mu-10} \le C_0\sqrt{mn}} K_\mu^\gamma(\vec{y}) \prod_{j=1}^m f_j(x-y_j)\, d\vec{y} \right|$$

and

$$\mathcal{L}_{\Omega,\mu}^\sharp(f_1, \ldots, f_m)(x) := \sup_{\tau \in \mathbb{Z}} \left| \sum_{\gamma < \tau} \int_{(\mathbb{R}^n)^m} K_\mu^\gamma(\vec{y}) \prod_{j=1}^m f_j(x-y_j)\, d\vec{y} \right|.$$

Then Theorem 1.2 follows from the following two propositions:

Proposition 4.1. *Let* $1 < p_1, \ldots, p_m \le \infty$ *and* $1/p = 1/p_1 + \cdots + 1/p_m$. *Suppose that* $1 < q < \infty$ *and* $\Omega \in L^q(\mathbb{S}^{mn-1})$ *with* $\int_{\mathbb{S}^{mn-1}} \Omega d\sigma = 0$. *Then there exists a constant* $C > 0$ *such that*

$$\left\| \widetilde{\mathcal{L}}_\Omega^\sharp(f_1, \ldots, f_m) \right\|_{L^p} \le C \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^m \|f_j\|_{L^{p_j}}$$

for Schwartz functions $f_1, \ldots, f_m$ *on* $\mathbb{R}^n$.

Proposition 4.2. *Let* $\frac{2m}{m+1} < q \le \infty$ *and* $\Omega \in L^q(\mathbb{S}^{mn-1})$ *with* $\int_{\mathbb{S}^{mn-1}} \Omega d\sigma = 0$. *Suppose that* $\mu \in \mathbb{Z}$ *satisfies* $2^{\mu-10} > C_0\sqrt{mn}$. *Then there exist* $C, \epsilon_0 > 0$ *such that*

$$\left\| \mathcal{L}_{\Omega,\mu}^\sharp(f_1, \ldots, f_m) \right\|_{L^{2/m}} \lesssim 2^{-\epsilon_0\mu} \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^m \|f_j\|_{L^2}$$

for Schwartz functions $f_1, \ldots, f_m$ *on* $\mathbb{R}^n$.

4.2. **Proof of Proposition 4.1.** We decompose $\widetilde{\mathcal{L}}_\Omega^\sharp$ further so that the Coifman-Meyer multiplier theorem is involved: Setting

$$\widetilde{K}(\vec{y}) := \sum_{\mu \in \mathbb{Z}: 2^{\mu-10} \leq C_0\sqrt{mn}} K_\mu(\vec{y}) = \sum_{\mu \in \mathbb{Z}: 2^{\mu-10} \leq C_0\sqrt{mn}} \sum_{\gamma \in \mathbb{Z}} K_\mu^\gamma(\vec{y}),$$

$\widetilde{\mathcal{L}}_\Omega^\sharp(f_1, \ldots, f_m)(x)$ is controlled by the sum of

$$T_{\widetilde{K}}^*(f_1, \ldots, f_m)(x) := \sup_{\tau \in \mathbb{Z}} \left| \int_{|y| > 2^{-\tau}} \widetilde{K}(\vec{y}) \prod_{j=1}^m f_j(x - y_j) \, d\vec{y} \right|$$

and

$$(4.6) \qquad \mathfrak{T}_K^{**}(f_1, \ldots, f_m)(x) := \sup_{\tau \in \mathbb{Z}} \left| \int_{(\mathbb{R}^n)^m} K_\tau^{**}(\vec{y}) \prod_{j=1}^m f_j(x - y_j) \, d\vec{y} \right|,$$

where

$$(4.7) \qquad K_\tau^{**}(\vec{y}) := \left(\sum_{\mu \in \mathbb{Z}: 2^{\mu-10} \leq C_0\sqrt{mn}} \sum_{\gamma < \tau} K_\mu^\gamma(\vec{y}) \right) - \widetilde{K}(\vec{y}) \chi_{|\vec{y}| > 2^{-\tau}}.$$

To obtain the boundedness of $T_{\widetilde{K}}^*$, we claim that $\widetilde{K}$ is an m-linear Calderón-Zygmund kernel with constant $C\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}$ for $1 < q < \infty$. Indeed, it follows from (4.5) that

$$\left| \partial^\alpha \widehat{\widetilde{K}}(\vec{\xi}) \right| \leq \sum_{\mu \in \mathbb{Z}: 2^{\mu-10} \leq C_0\sqrt{mn}} \left| \partial^\alpha \widehat{K_\mu}(\vec{\xi}) \right| \lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} |\vec{\xi}|^{-|\alpha|}$$

as the sum of $Q(\mu)$ over μ satisfying $2^{\mu-10} \leq C_0\sqrt{mn}$ converges. Then $\widetilde{K}$ satisfies the size and smoothness conditions for m-linear Calderón-Zygmund kernel with constant $C\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}$, as mentioned in the proof of [21, Proposition 6]. Since $\widetilde{K}$ is a Calderón-Zygmund kernel, Cotlar's inequality in [20, Theorem 1] yields that $T_{\widetilde{K}}^*$ is bounded on the full range of exponents with constant $C\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}$.

To handle the boundedness of the operator $\mathfrak{T}_K^{**}$, we observe that the kernel K_τ^{**} can be written as

$$(4.8) \qquad K_\tau^{**}(\vec{y}) = \sum_{\mu \in \mathbb{Z}: 2^{\mu-10} \leq C_0\sqrt{mn}} \left(\sum_{\gamma < \tau} K_\mu^\gamma(\vec{y}) \chi_{|\vec{y}| \leq 2^{-\tau}} - \sum_{\gamma \geq \tau} K_\mu^\gamma(\vec{y}) \chi_{|\vec{y}| > 2^{-\tau}} \right)$$

and thus

$$\mathfrak{T}_K^{**}(f_1, \ldots, f_m)(x) \leq \sup_{\tau \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}: 2^{\mu-10} \leq C_0\sqrt{mn}} \mathcal{I}_{\mu,\tau}(x) + \mathcal{J}_{\mu,\tau}(x)$$

where

$$\mathcal{I}_{\mu,\tau}(x) := \sum_{\gamma < \tau} \left| \int_{|\vec{y}| < 2^{-\tau}} K_\mu^\gamma(\vec{y}) \prod_{j=1}^m f_j(x - y_j) \, d\vec{y} \right|,$$

$$\mathcal{J}_{\mu,\tau}(x) := \sum_{\gamma \geq \tau} \left| \int_{|\vec{y}| \geq 2^{-\tau}} K_\mu^\gamma(\vec{y}) \prod_{j=1}^m f_j(x - y_j) \, d\vec{y} \right|.$$

We claim that there exists $\epsilon > 0$ such that

$$(4.9) \qquad \mathcal{I}_{\mu,\tau} + \mathcal{J}_{\mu,\tau} \lesssim_{C_0,m,n} 2^{\epsilon\mu} \|\Omega\|_{L^1(\mathbb{S}^{mn-1})} \prod_{j=1}^{m} \mathcal{M} f_j \quad \text{uniformly in } \tau \in \mathbb{Z}$$

for μ satisfying $2^{\mu-10} \leq C_0 \sqrt{mn}$, where we recall $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Then, using Hölder's inequality and the boundedness of $\mathcal{M}$, we obtain

$$(4.10) \quad \left\| \mathfrak{T}_K^{**}(f_1, \ldots, f_m) \right\|_{L^p} \lesssim \|\Omega\|_{L^1(\mathbb{S}^{mn-1})} \left\| \prod_{j=1}^{m} \mathcal{M} f_j \right\|_{L^p} \lesssim \|\Omega\|_{L^1(\mathbb{S}^{mn-1})} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}}$$

for $1 < p_1, \ldots, p_m \leq \infty$ and $0 < p \leq \infty$ satisfying $1/p = 1/p_1 + \cdots + 1/p_m$ as $\sum_{\mu:2^{\mu-10}\leq C_0\sqrt{mn}} 2^{\epsilon\mu}$ converges. Therefore, let us prove (4.9).

Using (4.1), we have

$$\mathcal{I}_{\mu,\tau}(x) \lesssim \sum_{\gamma<\tau} \int_{|\vec{y}|<2^{-\tau}} \int_{|\vec{z}|\approx 1} 2^{\gamma mn} 2^{\mu mn} |\Omega(\vec{z}')| d\vec{z} \prod_{j=1}^{m} |f_j(x-y_j)| \, d\vec{y}$$

$$\lesssim 2^{\mu mn} \|\Omega\|_{L^1(\mathbb{S}^{mn-1})} \frac{1}{2^{-\tau mn}} \int_{|\vec{y}|<2^{-\tau}} \prod_{j=1}^{m} |f_j(x-y_j)| \, d\vec{y}$$

$$\lesssim 2^{\mu mn} \|\Omega\|_{L^1(\mathbb{S}^{mn-1})} \prod_{j=1}^{m} \mathcal{M} f_j(x)$$

as desired.

In addition,

$$\mathcal{J}_{\mu,\tau}(x) \leq \sum_{\gamma\geq\tau} \int_{|\vec{y}|\geq 2^{-\tau}} 2^{\gamma mn} \left| \int_{|\vec{z}|\approx 1} \Phi_\mu(2^\gamma \vec{y} - \vec{z}) \Omega(\vec{z}') d\vec{z} \right| \prod_{j=1}^{m} |f_j(x-y_j)| \, d\vec{y}.$$

Since Ω has vanishing mean, we have

$$\left| \int_{|\vec{z}|\approx 1} \Phi_\mu(2^\gamma \vec{y} - \vec{z}) \, \Omega(\vec{z}') \, d\vec{z} \right|$$

$$\lesssim 2^{\mu(mn+1)} \int_{|\vec{z}|\approx 1} \int_0^1 \left| \nabla\Phi(2^{\mu+\gamma}\vec{y} - 2^\mu t\vec{z}) \right| dt \, |\Omega(\vec{z}')| \, d\vec{z}.$$

Now we choose a constant M such that $mn < M < mn + 1$ and see that

$$\left| \nabla\Phi(2^{\mu+\gamma}\vec{y} - 2^\mu t\vec{z}) \right| \lesssim_M \frac{1}{(1 + |2^{\mu+\gamma}\vec{y} - 2^\mu t\vec{z}|)^M}$$

$$\lesssim_{C_0,m,n,M} \frac{1}{(1 + 2^{\mu+\gamma}|\vec{y}|)^M} \leq \frac{1}{2^{M(\mu+\gamma)}} \frac{1}{|\vec{y}|^M}$$

as $|\vec{z}| \approx 1$, $0 < t < 1$, and $2^{\mu-10} \leq C_0\sqrt{mn}$. This yields that

$$\mathcal{J}_{\mu,\tau}(x) \lesssim 2^{\mu(mn+1-M)} \|\Omega\|_{L^1(\mathbb{S}^{mn-1})} \left(\sum_{\gamma\geq\tau} 2^{-\gamma(M-mn)} \right)$$

$$\times \int_{|\vec{y}| \geq 2^{-\tau}} \frac{1}{|\vec{y}|^M} \prod_{j=1}^m |f_j(x - y_j)| \, d\vec{y}.$$

Since $M > mn$, the sum over $\gamma \geq \tau$ converges to $2^{-\tau(M-mn)}$ and the integral over $|\vec{y}| \geq 2^{-\tau}$ is estimated by

$$\sum_{l=0}^\infty \int_{2^{-\tau+l} \leq |\vec{y}| < 2^{-\tau+l+1}} \frac{1}{|\vec{y}|^M} \prod_{j=1}^m |f_j(x - y_j)| \, d\vec{y}$$

$$\lesssim 2^{\tau(M-mn)} \sum_{l=0}^\infty 2^{-l(M-mn)} \Big(\frac{1}{2^{(-\tau+l+1)mn}} \int_{|\vec{y}| \leq 2^{-\tau+l+1}} \prod_{j=1}^m |f_j(x - y_j)| \, d\vec{y} \Big)$$

$$\lesssim 2^{\tau(M-mn)} \prod_{j=1}^m \mathcal{M} f_j(x).$$

Finally, we have

$$\mathcal{J}_{\mu,\tau} \lesssim 2^{\mu(mn+1-M)} \|\Omega\|_{L^1(\mathbb{S}^{mn-1})} \prod_{j=1}^m \mathcal{M} f_j,$$

which completes the proof of (4.9).

4.3. **Proof of Proposition 4.2.** The proof is based on the wavelet decomposition and the recent developments in [18]. Recalling that $\widehat{K_\mu^0} \in L^{q'}$, we apply the wavelet decomposition (2.1) to write

$$\widehat{K_\mu^0}(\vec{\xi}) = \sum_{\lambda \in \mathbb{N}_0} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\vec{k} \in (\mathbb{Z}^n)^m} b_{\vec{G},\vec{k}}^{\lambda,\mu} \Psi_{G_1,k_1}^\lambda(\xi_1) \cdots \Psi_{G_m,k_m}^\lambda(\xi_m)$$

where

$$b_{\vec{G},\vec{k}}^{\lambda,\mu} := \int_{(\mathbb{R}^n)^m} \widehat{K_\mu^0}(\vec{\xi}) \Psi_{\vec{G},\vec{k}}^\lambda(\vec{\xi}) \, d\vec{\xi}.$$

It is known in [18] that for any $0 < \delta < 1/q'$,

$$(4.11) \qquad \big\| \{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}} \big\|_{\ell^\infty} \lesssim 2^{-\delta\mu} 2^{-\lambda(M+1+mn)} \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}$$

where M is the number of vanishing moments of $\Psi_{\vec{G}}$. Moreover, it follows from the inequality (2.2), the Hausdorff-Young inequality, and Young's inequality that

$$(4.12) \qquad \big\| \{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}} \big\|_{\ell^{q'}} \lesssim 2^{-\lambda mn(1/2-1/q')} \|\widehat{K_\mu^0}\|_{L^{q'}} \lesssim 2^{-\lambda mn(1/q-1/2)} \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}.$$

Now we may assume that $2^{\lambda+\mu-2} \leq |\vec{k}| \leq 2^{\lambda+\mu+2}$ due to the compact supports of $\widehat{K_\mu^0}$ and $\Psi_{\vec{G},\vec{k}}^\lambda$. In addition, by symmetry, it suffices to focus on the case $|k_1| \geq \cdots \geq |k_m|$. Since $\widehat{K_\mu^\gamma}(\vec{\xi}) = \widehat{K_\mu^0}(\vec{\xi}/2^\gamma)$, the boundedness of $\mathcal{L}_{\Omega,\mu}^\sharp$ is reduced to the inequality

$$\Big\| \sup_{\tau \in \mathbb{Z}} \Big| \sum_{\lambda \in \mathbb{N}_0} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\gamma \in \mathbb{Z} : \gamma < \tau} \sum_{\vec{k} \in \mathcal{U}^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\mu} \prod_{j=1}^m L_{G_j,k_j}^{\lambda,\gamma} f_j \Big| \Big\|_{L^{2/m}}$$

$$(4.13) \qquad \lesssim 2^{-\epsilon_0\mu}\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\prod_{j=1}^m \|f_j\|_{L^2}$$

where the operators $L_{G_j,k_j}^{\lambda,\gamma}$ and the set $\mathcal{U}^{\lambda+\mu}$ are defined as in (2.5) and (2.4). We split $\mathcal{U}^{\lambda+\mu}$ into m disjoint subsets $\mathcal{U}_l^{\lambda+\mu}$ ($1 \le l \le m$) as before such that for $k \in \mathcal{U}_l^{\lambda+\mu}$ we have

$$|k_1| \ge \cdots \ge |k_l| \ge 2C_0\sqrt{n} \ge |k_{l+1}| \ge \cdots \ge |k_m|.$$

Then the left-hand side of (4.13) is estimated by

$$\left(\sum_{l=1}^m \sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\left\|\sup_{\tau\in\mathbb{Z}}\Big|\sum_{\gamma\in\mathbb{Z}:\gamma<\tau}\mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m)\Big|\right\|_{L^{2/m}}^{2/m}\right)^{m/2}$$

where $\mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}$ is defined by

$$\mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m) := \sum_{\vec{k}\in\mathcal{U}_l^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\mu}\Big(\prod_{j=1}^m L_{G_j,k_j}^{\lambda,\gamma}f_j\Big).$$

We claim that for each $1 \le l \le m$ there exists $\epsilon_0, M_0 > 0$ such that

$$(4.14)$$
$$\left\|\sup_{\tau\in\mathbb{Z}}\Big|\sum_{\gamma\in\mathbb{Z}:\gamma<\tau}\mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m)\Big|\right\|_{L^{2/m}}$$
$$\lesssim 2^{-\epsilon_0\mu_0}2^{-\lambda M_0}\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\prod_{j=1}^m \|f_j\|_{L^2},$$

which concludes (4.13). Therefore it remains to prove (4.14).

Proof of (4.14). When $2 \le l \le m$, we apply (2.8) with $2 < q' < \frac{2m}{m-1}$, along with (4.11), and (4.12) to obtain

$$\left\|\sup_{\tau\in\mathbb{Z}}\Big|\sum_{\gamma\in\mathbb{Z}:\gamma<\tau}\mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m)\Big|\right\|_{L^{2/m}} \le \left\|\sum_{\gamma\in\mathbb{Z}}\big|\mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m)\big|\right\|_{L^{2/m}}$$

$$\lesssim \big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}}\big\|_{\ell^\infty}^{1-\frac{(m-1)q'}{2m}}\big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}}\big\|_{\ell^{q'}}^{\frac{(m-1)q'}{2m}}2^{\lambda mn/2}(\lambda+1)^{l/2}\mu^{l/2}\prod_{j=1}^m \|f_j\|_{L^2}$$

$$\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}2^{-\delta\mu(1-\frac{(m-1)q'}{2m})}\mu^{m/2}2^{-\lambda C_{M,m,n,q}}(\lambda+1)^{m/2}\prod_{j=1}^m \|f_j\|_{L^2},$$

where

$$C_{M,m,n,q} := (M+1+mn)(1-\frac{(m-1)q'}{2m}) + mn(1/q-1/2)\frac{(m-1)q'}{2m} - \frac{mn}{2}.$$

Here we used the fact that $\frac{l-1}{2l} \le \frac{m-1}{2m}$ for $l \le m$. Then (4.14) follows from choosing M sufficiently large so that $C_{M,m,n,q} > 0$ since $1 - \frac{(m-1)q'}{2m} > 0$.

Now let us prove (4.14) for $l = 1$. In this case, we first see the estimate

$$(4.15) \quad \left\| \Big(\sum_{\gamma \in \mathbb{Z}} \big| \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m) \big|^2 \Big)^{1/2} \right\|_{L^{2/m}} \lesssim 2^{-\epsilon_0 \mu} 2^{-M_0 \lambda} \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^m \|f_j\|_{L^2}$$

for some $\epsilon_0, M_0 > 0$, which can be proved, as in [18, Section 6], by using (2.7) and (4.11).

Choose a Schwartz function Γ on $\mathbb{R}^n$ whose Fourier transform is supported in the ball $\{\xi \in \mathbb{R}^n : |\xi| \le 2\}$ and is equal to 1 for $|\xi| \le 1$, and define $\Gamma_k := 2^{kn}\Gamma(2^k \cdot)$ so that $\mathrm{Supp}(\widehat{\Gamma_k}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2^{k+1}\}$ and $\widehat{\Gamma_k}(\xi) = 1$ for $|\xi| \le 2^k$.

Since the Fourier transform of $\mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m)$ is supported in the set $\{\xi \in \mathbb{R}^n : 2^{\gamma+\mu-5} \le |\xi| \le 2^{\gamma+\mu+4}\}$, we can write

$$\sum_{\gamma \in \mathbb{Z}: \gamma < \tau} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m) = \Gamma_{\tau+\mu+3} * \Big(\sum_{\gamma \in \mathbb{Z}: \gamma < \tau} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m) \Big)$$

and then split the right-hand side into

$$\Gamma_{\tau+\mu+3} * \Big(\sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m) \Big) - \Gamma_{\tau+\mu+3} * \Big(\sum_{\gamma \in \mathbb{Z}: \gamma \ge \tau} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m) \Big).$$

Due to the Fourier support conditions of $\Gamma_{\tau+\mu+3}$ and $\mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m)$, the sum in the second term can be actually taken over $\tau \le \gamma \le \tau + 9$. Therefore, the left-hand side of (4.14) is controlled by the sum of

$$(4.16) \qquad I := \left\| \sup_{\nu \in \mathbb{Z}} \Big| \Gamma_\nu * \Big(\sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \dots, f_m) \Big) \Big| \right\|_{L^{2/m}}$$

and

$$(4.17) \qquad II := \sum_{\gamma=0}^9 \left\| \sup_{\tau \in \mathbb{Z}} \big| \Gamma_{\tau+\mu+3} * T_{\vec{G},1}^{\lambda,\tau+\gamma,\mu}(f_1, \dots, f_m) \big| \right\|_{L^{2/m}}.$$

First of all, when $0 \le \gamma \le 9$, the Fourier supports of both $\Gamma_{\tau+\mu+3}$ and $T_{\vec{G},1}^{\lambda,\tau+\gamma,\mu}(f_!, \dots, f_m)$ are $\{\xi \in \mathbb{R}^n : |\xi| \sim 2^{\tau+\mu}\}$. This implies that for any $0 < r < 1$,

$$\big| \Gamma_{\tau+\mu+3} * T_{\vec{G},1}^{\lambda,\tau+\gamma,\mu}(f_1, \dots, f_m)(x) \big|$$

$$\lesssim 2^{(\tau+\mu)(n/r-n)} \Big(\int_{\mathbb{R}^n} \big| \Gamma_{\tau+\mu+3}(x-y) \big|^r \big| T_{\vec{G},1}^{\lambda,\tau+\gamma,\mu}(f_1, \dots, f_m)(y) \big|^r \, dy \Big)^{1/r}$$

$$\lesssim \Big(\mathcal{M}\big(|T_{\vec{G},1}^{\lambda,\tau+\gamma,\mu}(f_1, \dots, f_m)|^r \big)(x) \Big)^{1/r}$$

where the Nikolskii inequality (see [28, Proposition 1.3.2]) is applied in the first inequality. Setting $0 < r < 2/m$, and using the maximal inequality for $\mathcal{M}$ and the embedding $\ell^2 \hookrightarrow \ell^\infty$ we obtain

$$(4.18) \qquad II \lesssim \big\| \sup_{\tau \in \mathbb{Z}} \big| T_{\vec{G},1}^{\lambda,\tau,\mu}(f_1, \dots, f_m) \big| \big\|_{L^{2/m}}$$

$$\leq \Big\| \Big(\sum_{\gamma \in \mathbb{Z}} |\mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m)|^2 \Big)^{1/2} \Big\|_{L^{2/m}}.$$

Then the $L^{2/m}$ norm is bounded by the right-hand side of (4.14), thanks to (4.15). This completes the estimate for II defined in (4.17) and we turn our attention to I defined in (4.16).

In the sequel we will make use of the following inequality: if $\widehat{g_\gamma}$ is supported on $\{\xi \in \mathbb{R}^n : C^{-1} 2^{\gamma+\mu} \leq |\xi| \leq C 2^{\gamma+\mu}\}$ for some $C > 1$ and $\mu \in \mathbb{Z}$, then

$$(4.19) \qquad \Big\| \Big\{ \Phi_j^{(1)} * \Big(\sum_{\gamma \in \mathbb{Z}} g_\gamma \Big) \Big\}_{j \in \mathbb{Z}} \Big\|_{L^p(\ell^q)} \lesssim_C \big\| \{g_j\}_{j \in \mathbb{Z}} \big\|_{L^p(\ell^q)} \quad \text{uniformly in } \mu$$

for $0 < p < \infty$. The proof of (4.19) is elementary and standard, so it is omitted here; see [16, (13)] and [31, Theorem 3.6] for related arguments.

To obtain the bound of I, we note that

$$I \approx \Big\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m) \Big\|_{H^{2/m}}$$

where $H^{2/m}$ is the Hardy space. We refer to [15, Corollary 2.1.8] for the above estimate. Then, using the Littlewood-Paley theory for Hardy space (see for instance [15, Theorem 2.2.9]) and (4.19), there exists a unique polynomial $Q^{\lambda,\mu,\vec{G}}(x)$ such that

$$\Big\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m) - Q^{\lambda,\mu,\vec{G}} \Big\|_{H^{2/m}} \lesssim \Big\| \Big(\sum_{\gamma \in \mathbb{Z}} |\mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m)|^2 \Big)^{1/2} \Big\|_{L^{2/m}}$$

$$(4.20) \qquad\qquad\qquad\qquad\qquad \lesssim 2^{-\epsilon_0 \mu} 2^{-M_0 \lambda} \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^{m} \|f_j\|_{L^2}$$

where (4.15) is applied. Furthermore,

$$\Big\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m) \Big\|_{H^{2/m}}$$

$$\approx \Big\| \sup_{\nu \in \mathbb{Z}} \Big| \Gamma_\nu * \Big(\sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m) \Big) \Big| \Big\|_{L^{2/m}}$$

$$= \Big\| \sup_{\nu \in \mathbb{Z}} \Big| \Gamma_\nu * \Big(\sum_{\gamma \in \mathbb{Z} : \gamma \leq \nu - \mu + 5} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m) \Big) \Big| \Big\|_{L^{2/m}}$$

$$\lesssim \Big\| \sup_{\nu \in \mathbb{Z}} \Big| \sum_{\gamma \in \mathbb{Z} : \gamma \leq \nu - \mu + 5} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m) \Big| \Big\|_{L^{2/m}}$$

$$\leq \Big\| \sum_{\gamma \in \mathbb{Z}} |\mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1,\ldots,f_m)| \Big\|_{L^{2/m}}$$

where the argument that led to (4.18) is applied in the first inequality. As we have discussed in [18, Section 6.1], this quantity is finite for all Schwartz functions $f_1,\ldots,f_m$.

Accordingly, we have

$$\sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \ldots, f_m) - Q^{\lambda,\mu,\vec{G}} \in H^{2/m}$$

and

$$\sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}(f_1, \ldots, f_m) \in H^{2/m},$$

and thus $Q^{\lambda,\mu,\vec{G}} = 0$. Now it follows from (4.20) that

$$I \lesssim 2^{-\epsilon_0 \mu} 2^{-M_0 \lambda} \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^{m} \|f_j\|_{L^2},$$

as expected. This completes the proof of (4.14).

5. Proof of Theorem 1.1

We now prove Theorem 1.1 by making use of Theorem 1.2. Recall that in Theorem 1.2 we established the boundedness only for Schwartz functions. An important obstacle we address is the pointwise definition of the maximal operator $\mathcal{L}^{**}$ (see (1.3)) for general L^2 functions. This definition can be given via an abstract extension (see [11]), this is not as useful for our purposes. We provide below a concrete approach that preserves the pointwise bounds provided by the supremum.

5.1. **A variant of Theorem 1.2 for general L^2 functions.** We note that when $f_j \in L^2(\mathbb{R}^n)$, by Lemma 3.1, there exists a set $E_{f_1,\ldots,f_m}^{\Omega}$ of measure 0 such that

$$\mathcal{M}_{\Omega}(f_1, \ldots, f_m)(x) < \infty$$

when $x \notin E_{f_1,\ldots,f_m}^{\Omega}$. Therefore for $x \notin E_{f_1,\ldots,f_m}^{\Omega}$ we have

$$(5.1) \qquad \int_{\epsilon_0 \leq |\vec{y}| \leq \epsilon_0^{-1}} \frac{|\Omega(\vec{y}')|}{|\vec{y}|^{mn}} \prod_{j=1}^{m} |f_j(x - y_j)| \, d\vec{y} \leq C \frac{1}{\epsilon_0^{2mn}} \mathcal{M}_{\Omega}(f_1, \ldots, f_m)(x),$$

and thus $\mathcal{L}_{\Omega}^{(\epsilon,\epsilon^{-1})}(f_1, \ldots, f_m)(x)$ and $\mathcal{L}_{\Omega}^{*,\epsilon_0}(f_1, \ldots, f_m)(x)$ are well-defined, for f_j in $L^2(\mathbb{R}^n)$ and Ω in $L^q(\mathbb{S}^{mn-1})$. Moreover,

$$\mathcal{L}_{\Omega}^{(\epsilon,\epsilon^{-1})}(f_1, \ldots, f_m)(x) \leq \mathcal{L}_{\Omega}^{*,\epsilon_0}(f_1, \ldots, f_m)(x)$$

pointwise for $x \in \mathbb{R}^n \setminus E_{f_1,\ldots,f_m}^{\Omega}$.

For given $f_j \in L^2(\mathbb{R}^n)$, we pick sequences of Schwartz functions f_j^k converging to f_j in L^2 as $k \to \infty$, by density. Using the identity

$$(5.2) \qquad a_1 a_2 \cdots a_m - b_1 b_2 \cdots b_m = \sum_{j=1}^{m} b_1 \cdots b_{j-1}(a_j - b_j) a_{j+1} \cdots a_m$$

(with the obvious modification when $j = 1$ or $j = m$), the inequality

$$\mathcal{L}_{\Omega}^{*,\epsilon_0}(f_1, \ldots, f_m) \leq 2\mathcal{L}_{\Omega}^{*}(f_1^k, \ldots, f_m^k) + \sum_{j=1}^{m} \mathcal{L}_{\Omega}^{*,\epsilon_0}(f_1^k, \ldots f_{j-1}^k, f_j - f_j^k, f_{j+1} \ldots, f_m)$$

is valid poinwisely on the complement of the set

$$(5.3) \qquad E^{\Omega} := E^{\Omega}_{f_1,\ldots,f_m} \cup \Big(\bigcup_{k=1}^{\infty} E^{\Omega}_{f_1^k,\ldots,f_m^k} \Big) \cup \Big(\bigcup_{j=1}^{m} \bigcup_{k=1}^{\infty} E^{\Omega}_{f_1^k,\ldots f_{j-1}^k, f_j - f_j^k, f_{j+1}\ldots,f_m} \Big)$$

which is a set of measure zero. Now we take the $L^{2/m}$ quasi-norm on both sides and apply Theorem 1.3 for Schwartz functions and the estimate (5.1), combined with Lemma 3.1, for the terms in the sum. Taking $k \to \infty$, we obtain

$$\big\| \mathcal{L}^{*,\epsilon_0}_{\Omega}(f_1,\ldots,f_m) \big\|_{L^{2/m}(\mathbb{R}^n)} \le C \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^{\infty} \|f_j\|_{L^2(\mathbb{R}^n)}.$$

Finally, letting $\epsilon_0 \to 0$ and using the monotone convergence theorem, we conclude

$$(5.4) \qquad \big\| \mathcal{L}^{**}_{\Omega}(f_1,\ldots,f_m) \big\|_{L^{2/m}(\mathbb{R}^n)} \le C \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^{\infty} \|f_j\|_{L^2(\mathbb{R}^n)}$$

for $f_j \in L^2(\mathbb{R}^n)$.

5.2. **Proof of Theorem 1.1.** Let $f_1,\ldots,f_m$ be given L^2 functions and pick sequences $\{f_j^k\}$ of Schwartz functions such that f_j^k converges to f_j in $L^2(\mathbb{R}^n)$ as $k \to \infty$. Recall that $\mathcal{L}_{\Omega}(f_1,\ldots,f_m)$ is defined as the $L^{2/m}$ limit of $\mathcal{L}_{\Omega}(f_1^k,\ldots,f_m^k)$ as $k \to \infty$. Then there exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\mathcal{L}_{\Omega}(f_1^{k_l},\ldots,f_m^{k_l}) \to \mathcal{L}_{\Omega}(f_1,\ldots,f_m)$ pointwise on $\mathbb{R}^n \setminus E$, for some set E of measure zero. Let us denote the subsequence $\{k_l\}$ still by $\{k\}$ for notational convenience. Then on $\mathbb{R}^n \setminus (E \cup E^{\Omega})$, where E^{Ω} is as in (5.3), we have

$$\big| \mathcal{L}^{(\epsilon,\epsilon^{-1})}_{\Omega}(f_1,\ldots,f_m) - \mathcal{L}_{\Omega}(f_1,\ldots,f_m) \big|$$
$$\le \big| \mathcal{L}^{(\epsilon,\epsilon^{-1})}_{\Omega}(f_1,\ldots,f_m) - \mathcal{L}^{(\epsilon,\epsilon^{-1})}_{\Omega}(f_1^k,\ldots,f_m^k) \big|$$
$$+ \big| \mathcal{L}^{(\epsilon,\epsilon^{-1})}_{\Omega}(f_1^k,\ldots,f_m^k) - \mathcal{L}_{\Omega}(f_1^k,\ldots,f_m^k) \big|$$
$$+ \big| \mathcal{L}_{\Omega}(f_1^k,\ldots,f_m^k) - \mathcal{L}_{\Omega}(f_1,\ldots,f_m) \big|.$$

We first take the $\limsup_{\epsilon \to 0}$ on both sides and then the middle term on the right vanishes. Then we apply $\liminf_{k \to \infty}$ so that the last term vanishes. Consequently, we have

$$\limsup_{\epsilon \to 0} |\mathcal{L}^{(\epsilon,\epsilon^{-1})}_{\Omega}(f_1,\ldots,f_m) - \mathcal{L}_{\Omega}(f_1,\ldots,f_m)|$$
$$\le \liminf_{k \to \infty} \sum_{j=1}^{m} \mathcal{L}^{**}_{\Omega}(f_1,\ldots,f_{j-1},f_j - f_j^k, f_{j+1}^k,\ldots,f_m^k)$$

on $\mathbb{R}^n \setminus \big(E \cup E^{\Omega} \big)$, where the identity (5.2) is applied. It follows that

$$\Big| \big\{ x : \ \limsup_{\epsilon \to 0} |\mathcal{L}^{(\epsilon,\epsilon^{-1})}_{\Omega}(f_1,\ldots,f_m) - \mathcal{L}_{\Omega}(f_1,\ldots,f_m)| > \lambda \big\} \Big|$$
$$\le \Big| \big\{ x : \ \liminf_{k \to \infty} \sum_{j=1}^{m} \mathcal{L}^{**}_{\Omega}(f_1,\ldots,f_{j-1},f_j - f_j^k, f_{j+1}^k,\ldots,f_m^k) > \lambda \big\} \Big|$$

$$\leq \lambda^{-\frac{2}{m}} \sum_{j=1}^{m} \Big\| \liminf_{k\to\infty} \sum_{j=1}^{m} \mathcal{L}_\Omega^{**}(f_1,\ldots,f_{j-1},f_j-f_j^k,f_{j+1}^k,\ldots,f_m^k) \Big\|_{L^{\frac{2}{m}}}^{\frac{2}{m}}$$

by Chebyshev's inequality. But this last expression tends to zero as $k \to \infty$ in view of Fatou's lemma and (5.4). We conclude that

$$\limsup_{\epsilon\to 0} |\mathcal{L}_\Omega^{(\epsilon,\epsilon^{-1})}(f_1,\ldots,f_m) - \mathcal{L}_\Omega(f_1,\ldots,f_m)|$$

equals zero a.e. and this finishes the proof.

6. Proof of Theorem 1.4

Let μ_0 be the smallest integer satisfying $2^{\mu_0-3} > C_0\sqrt{mn}$ and

$$\widehat{\Theta_{\mu_0-1}^{(m)}}(\vec{\xi}) := 1 - \sum_{\mu=\mu_0}^{\infty} \widehat{\Phi_\mu^{(m)}}(\vec{\xi}).$$

Clearly,

$$\widehat{\Theta_{\mu_0-1}^{(m)}}(\vec{\xi}) + \sum_{\mu=\mu_0}^{\infty} \widehat{\Phi_\mu^{(m)}}(\vec{\xi}) = 1$$

and thus we can write

$$\sigma(\vec{\xi}) = \widehat{\Theta_{\mu_0-1}^{(m)}}(\vec{\xi})\sigma(\vec{\xi}) + \sum_{\mu=\mu_0}^{\infty} \widehat{\Phi_\mu^{(m)}}(\vec{\xi})\sigma(\vec{\xi}) =: \sigma_{\mu_0-1}(\vec{\xi}) + \sum_{\mu=\mu_0}^{\infty} \sigma_\mu(\vec{\xi}).$$

Note that σ_{μ_0-1} is a compactly supported smooth function and thus the corresponding maximal multiplier operator $\mathcal{M}_{\sigma_{\mu_0-1}}$, defined by

$$\mathcal{M}_{\sigma_{\mu_0-1}}(f_1,\ldots,f_m)(x)$$
$$:= \sup_{\nu\in\mathbb{Z}} \Big| \int_{(\mathbb{R}^n)^m} \sigma_{\mu_0-1}(2^\nu\vec{\xi})\Big(\prod_{j=1}^{m} \widehat{f_j}(\xi_j)\Big)e^{2\pi i\langle x,\sum_{j=1}^{m}\xi_j\rangle}d\vec{\xi}\Big|,$$

is bounded by a constant multiple of $\mathcal{M}f_1(x)\cdots\mathcal{M}f_m(x)$ where $\mathcal{M}$ is the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ as before. Using Hölder's inequality and the L^2-boundedness of $\mathcal{M}$, we can prove

$$\big\|\mathcal{M}_{\sigma_{\mu_0-1}}(f_1,\ldots,f_m)\big\|_{L^{2/m}} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^2}.$$

It remains to show that

(6.1)
$$\Big\| \sum_{\mu=\mu_0}^{\infty} \mathcal{M}_{\sigma_\mu}(f_1,\ldots,f_m) \Big\|_{L^{2/m}} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^2}.$$

Using the decomposition (2.1), write

(6.2)
$$\sigma_\mu(\vec{\xi}) = \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \sum_{\vec{k}\in(\mathbb{Z}^n)^m} b_{\vec{G},\vec{k}}^{\lambda,\mu} \Psi_{G_1,k_1}^\lambda(\xi_1)\cdots\Psi_{G_m,k_m}^\lambda(\xi_m)$$

where

$$b_{\vec{G},\vec{k}}^{\lambda,\mu} := \int_{(\mathbb{R}^n)^m} \sigma_\mu(\vec{\xi}) \Psi_{\vec{G},\vec{k}}^\lambda(\vec{\xi}) d\vec{\xi}.$$

Let $M := \left[\frac{(m-1)n}{2}\right] + 1$ and choose $1 < q < \frac{2m}{m-1}$ such that

(6.3) $$\frac{(m-1)n}{2} < \frac{mn}{q} < \min(a, M).$$

In view of (2.2), we have

$$\left\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\right\|_{\ell^q} \lesssim 2^{-\lambda(M-mn/q+mn/2)}\|\sigma_\mu\|_{L_M^q((\mathbb{R}^n)^m)}$$

(6.4) $$\lesssim 2^{-\lambda(M-mn/q+mn/2)}2^{-\mu(a-mn/q)}$$

where the assumption (1.8) is applied in the last inequality.

We observe that if $\mu \geq \mu_0$, then $b_{\vec{G},\vec{k}}^{\lambda,\mu}$ vanishes unless $2^{\lambda+\mu-2} \leq |\vec{k}| \leq 2^{\lambda+\mu+2}$ due to the compact supports of σ_μ and $\Psi_{\vec{G},\vec{k}}^\lambda$, which allows us to replace the sum over $\vec{k} \in (\mathbb{Z}^n)^m$ in (6.2) by the sum over $2^{\lambda+\mu-1} \leq |\vec{k}| \leq 2^{\lambda+\mu+1}$. Moreover, we may consider only the case $|k_1| \geq \cdots \geq |k_m|$ as in the previous section. Therefore, in the rest of the section, we assume

$$\sigma_\mu(\vec{\xi}) = \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \sum_{\vec{k}\in\mathcal{U}^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\mu} \Psi_{G_1,k_1}^\lambda(\xi_1) \cdots \Psi_{G_m,k_m}^\lambda(\xi_m)$$

$$= \sum_{l=1}^{m} \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \sum_{\vec{k}\in\mathcal{U}_l^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\mu} \Psi_{G_1,k_1}^\lambda(\xi_1) \cdots \Psi_{G_m,k_m}^\lambda(\xi_m)$$

$$=: \sum_{l=1}^{m} \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \sigma_{\mu,l}^{\lambda,\vec{G}}(\vec{\xi})$$

where the sets $\mathcal{U}^{\lambda+\mu}, \mathcal{U}_l^{\lambda+\mu}$ are defined as before. Then the left-hand side of (6.1) can be controlled by

(6.5) $$\left(\sum_{l=1}^{m} \sum_{\mu=\mu_0}^{\infty} \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \left\|\mathcal{M}_{\sigma_{\mu,l}^{\lambda,\vec{G}}}(f_1,\ldots,f_m)\right\|_{L^{2/m}}^{2/m}\right)^{m/2}.$$

Now we claim that

$$\left\|\mathcal{M}_{\sigma_{\mu,l}^{\lambda,\vec{G}}}(f_1,\ldots,f_m)\right\|_{L^{2/m}}$$

(6.6) $$\lesssim 2^{-\lambda(M-mn/q)}(\lambda+1)^{l/2}2^{-\mu(a-mn/q)}\mu^{l/2}\prod_{j=1}^{m}\|f_j\|_{L^2}.$$

Then (6.5) is less than a constant multiple of $\prod_{j=1}^{m}\|f_j\|_{L^2}$ as desired, due to the choice of q in (6.3).

In order to prove (6.6), we use the estimates (2.7) and (2.8). We first rewrite

$$\mathcal{M}_{\sigma^{\lambda,\vec{G}}_{\mu,l}}(f_1,\dots,f_m)(x) = \sup_{\gamma\in\mathbb{Z}} \Big| \sum_{\vec{k}\in\mathcal{U}_l^{\lambda+\mu}} b^{\lambda,\mu}_{\vec{G},\vec{k}} \Big(\prod_{j=1}^m L^{\lambda;\gamma}_{G_j,k_j} f_i(x) \Big) \Big|$$

where $L^{\lambda;\gamma}_{G,k}$ is defined as in (2.3).

When $l=1$, applying the embeddings $\ell^2 \hookrightarrow \ell^\infty$, $\ell^q \hookrightarrow \ell^\infty$, and (2.7), the left-hand side of (6.6) is less than

$$\Big\| \Big(\sum_{\gamma\in\mathbb{Z}} \Big| \sum_{\vec{k}\in\mathcal{U}_1^{\lambda+\mu}} b^{\lambda,\mu}_{\vec{G},\vec{k}} \prod_{j=1}^m L^{\lambda;\gamma}_{G_j,k_j} f_j \Big|^2 \Big)^{1/2} \Big\|_{L^{2/m}}$$

$$\lesssim \big\| \{b^{\lambda,\mu}_{\vec{G},\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m} \big\|_{\ell^q} \mu^{1/2} 2^{\lambda mn/2} (\lambda+1)^{1/2} \prod_{j=1}^m \|f_j\|_{L^2}$$

$$\lesssim 2^{-\lambda(M-mn/q)} (\lambda+1)^{1/2} 2^{-\mu(a-mn/q)} \mu^{1/2} \prod_{j=1}^m \|f_j\|_{L^2}$$

where (6.4) is applied in the last inequality.

For the case $2 \le l \le m$, we can bound the left-hand side of (6.6) by

$$\Big\| \sum_{\gamma\in\mathbb{Z}} \Big| \sum_{\vec{k}\in\mathcal{U}_l^{\lambda+\mu}} b^{\lambda,\mu}_{\vec{G},\vec{k}} \prod_{j=11}^m L^{\lambda;\gamma}_{G_j,k_j} f_j \Big| \Big\|_{L^{2/m}}$$

$$\lesssim \big\| \{b^{\lambda,\mu}_{\vec{G},\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m} \big\|_{\ell^q} \mu^{l/2} 2^{\lambda mn/2} (\lambda+1)^{l/2} \prod_{j=1}^m \|f_j\|_{L^2}.$$

Here, we used the inequality (2.8) and the embedding $\ell^q \hookrightarrow \ell^\infty$. Then the preceding expression is estimated by

$$2^{-\lambda(M-mn/q)} (\lambda+1)^{l/2} 2^{-\mu(a-mn/q)} \mu^{l/2} \prod_{j=1}^m \|f_j\|_{L^2}$$

in view of (6.4). This completes the proof of (6.6).

7. Proof of Theorem 1.3

We now prove Theorem 1.3 taking Theorem 1.4 for granted.

First of all, it is easy to see that if f_j are Schwartz functions on $\mathbb{R}^n$, then

$$(7.1) \qquad \lim_{\nu\to-\infty} S^\nu_\sigma(f_1,\dots,f_m)(x) = \sigma(0) f_1(x) \cdots f_m(x)$$

and

$$\lim_{\nu\to\infty} S^\nu_\sigma(f_1,\dots,f_m)(x) = 0,$$

using the Lebesgue dominated convergence theorem and the property that

$$\lim_{\nu\to\infty} \sigma(2^\nu \vec{\xi}) = 0.$$

7.1. **Extension of Theorem 1.4 to general** $f_j \in L^2(\mathbb{R}^n)$. Let $f_1, \ldots, f_m$ be given L^2 functions on $\mathbb{R}^n$. As $S_\sigma^\nu(f_1, \ldots, f_m)$ is finite a.e. for each $\nu \in \mathbb{Z}$, there exists a set E_ν of measure zero so that $\left|S_\sigma^\nu(f_1, \ldots, f_m)\right| < \infty$ on the complement of E_ν. Since ν ranges over a countable set $\mathbb{Z}$, the measure of $E := \bigcup_{\nu \in \mathbb{Z}} E_\nu$ is clearly zero and thus we can define

$$\mathcal{M}_\sigma(f_1, \ldots, f_m) = \sup_{\nu \in \mathbb{Z}} |S_\sigma^\nu(f_1, \ldots, f_m)|$$

on E^c. That is, $\mathcal{M}_\sigma(f_1, \ldots, f_m)$ is well-defined pointwise a.e. Morever, it controls every $\left|S_\sigma^\nu(f_1, \ldots, f_m)\right|$ pointwise, whenever the latter is finite.

We now extend Theorem 1.5 to $f_j \in L^2(\mathbb{R}^n)$ using the above definition. Without loss of generality we only consider the case when $\nu \to -\infty$ as the case $\nu \to \infty$ follows similarly. As every sequence that converges in $L^{2/m}$ has a subsequence that converges a.e., there are a set of measure zero $E_{f_1,\ldots,f_m}^1$ and a subsequence

$$k_1^1 < k_2^1 < k_3^1 < \cdots < k_\ell^1 < \cdots$$

of the sequence of k's such that

$$S_\sigma^{-1}(f_1^{k_\ell^1}, \ldots, f_m^{k_\ell^1})(x) \to S_\sigma^{-1}(f_1, \ldots, f_m)(x)$$

for all $x \in \mathbb{R}^n \setminus E_{f_1,\ldots,f_m}^1$. Next, there are a set of measure zero $E_{f_1,\ldots,f_m}^2$ and a subsequence

$$k_1^2 < k_2^2 < k_3^2 < \cdots < k_\ell^2 < \cdots$$

of

$$k_1^1 < k_2^1 < k_3^1 < \cdots < k_\ell^1 < \cdots$$

such that

$$S_\sigma^{-2}(f_1^{k_\ell^2}, \ldots, f_m^{k_\ell^2})(x) \to S_\sigma^{-2}(f_1, \ldots, f_m)(x)$$

for all $x \in \mathbb{R}^n \setminus (E_{f_1,\ldots,f_m}^2 \cup E_{f_1,\ldots,f_m}^2)$. There are a set of measure zero $E_{f_1,\ldots,f_m}^3$ and a subsequence

$$k_1^3 < k_2^3 < k_3^3 < \cdots < k_\ell^3 < \cdots$$

of

$$k_1^2 < k_2^2 < k_3^2 < \cdots < k_\ell^2 < \cdots$$

such that

$$S_\sigma^{-3}(f_1^{k_\ell^3}, \ldots, f_m^{k_\ell^3})(x) \to S_\sigma^{-3}(f_1, \ldots, f_m)(x)$$

for all $x \in \mathbb{R}^n \setminus (E_{f_1,\ldots,f_m}^1 \cup E_{f_1,\ldots,f_m}^2 \cup E_{f_1,\ldots,f_m}^3)$. Iterating this process, we can take a diagonal sequence

$$k_1^1 < k_2^2 < k_3^3 < \cdots < k_\ell^\ell < \cdots,$$

which is a subsequence of all subsequences, for which $f_j^{k_\ell^\ell}(x) \to f_j(x)$ for all $1 \leq j \leq m$ and

$$S_\sigma^\nu(f_1^{k_\ell^\ell}, \ldots, f_m^{k_\ell^\ell})(x) \to S_\sigma^\nu(f_1, \ldots, f_m)(x)$$

as $\ell \to \infty$ for $x \in \mathbb{R}^n \setminus \bigcup_{\rho=1}^\infty E_{f_1,\ldots,f_m}^\rho$ and all $\nu = -1, -2, \ldots$.

Now on the set $\mathbb{R}^n \setminus \bigcup_{\rho=1}^\infty E_{f_1,\ldots,f_m}^\rho$, we have

$$\left|S_\sigma^\nu(f_1, \ldots, f_m)\right| = \lim_{\ell \to \infty} \left|S_\sigma^\nu(f_1^{k_\ell^\ell}, \ldots, f_m^{k_\ell^\ell})\right| = \liminf_{\ell \to \infty} \left|S_\sigma^\nu(f_1^{k_\ell^\ell}, \ldots, f_m^{k_\ell^\ell})\right|$$

for every $\nu = -1, -2, \ldots$, and thus

$$\left| S_\sigma^\nu(f_1, \ldots, f_m) \right| \le \liminf_{\ell \to \infty} \mathcal{M}_\sigma(f_1^{k_\ell}, \ldots, f_m^{k_\ell}).$$

This deduces

$$\mathcal{M}_\sigma(f_1, \ldots, f_m)| \le \liminf_{\ell \to \infty} \mathcal{M}_\sigma(f_1^{k_\ell}, \ldots, f_m^{k_\ell})$$

on $\mathbb{R}^n \setminus \bigcup_{\rho=1}^\infty E_{f_1, \ldots, f_m}^\rho$. Taking the $L^{2/m}$ quasi-norm on the both sides and using Fatou's lemma and Theorem 1.4 for Schwartz functions, we finally obtain

$$(7.2) \qquad \left\| \mathcal{M}_\sigma(f_1, \ldots, f_m) \right\|_{L^{2/m}(\mathbb{R}^n)} \lesssim \prod_{j=1}^m \| f_j \|_{L^2(\mathbb{R}^n)}$$

for $f_j \in L^2(\mathbb{R}^n)$.

7.2. **Proof of Theorem 1.3.** Let f_j, $j = 1, \ldots, m$, be functions in $L^2(\mathbb{R}^n)$ and $\{ f_j^{k_\ell} \}_\ell$ be sequences that appeared above so that on $(\bigcup_{\rho=1}^\infty E_{f_1, \ldots, f_m}^\rho)^c$,

$$(7.3) \qquad \lim_{\ell \to \infty} S_\sigma^\nu(f_1^{k_\ell}, \ldots, f_m^{k_\ell}) = S_\sigma^\nu(f_1, \ldots, f_m)$$

for each $\nu = -1, -2, \ldots$.

On $(\bigcup_{\rho=1}^\infty E_{f_1, \ldots, f_m}^\rho)^c$, we write

$$\left| S_\sigma^\nu(f_1, \ldots, f_m)(x) - \sigma(0) f_1(x) \cdots f_m(x) \right|$$
$$\le \left| S_\sigma^\nu(f_1, \ldots, f_m) - S_\sigma^\nu(f_1^{k_\ell}, \ldots, f_m^{k_\ell}) \right|$$
$$+ \left| S_\sigma^\nu(f_1^{k_\ell}, \ldots, f_m^{k_\ell}) - \sigma(0) f_1^{k_\ell} \cdots f_m^{k_\ell} \right|$$
$$+ \left| \sigma(0) f_1^{k_\ell} \cdots f_m^{k_\ell} - \sigma(0) f_1 \cdots f_m \right|.$$

We first take $\limsup_{\nu \to -\infty}$ and use (7.1) to make the middle term on the right vanish, and then apply $\liminf_{\ell \to \infty}$ to handle the last term on the right which will vanish as well. As a result, we obtain

$$\limsup_{\nu \to -\infty} \left| S_\sigma^\nu(f_1, \ldots, f_m) - \sigma(0) f_1 \cdots f_m \right|$$
$$\le \liminf_{\ell \to \infty} \sup_{\nu < 0} \left| S_\sigma^\nu(f_1, \ldots, f_m) - S_\sigma^\nu(f_1^{k_\ell}, \ldots, f_m^{k_\ell}) \right|.$$

Using the identity (5.2), we control the preceding expression pointwise by

$$\liminf_{\ell \to \infty} \sum_{i=1}^m \mathcal{M}_\sigma(f_1^{k_\ell}, \ldots, f_{i-1}^{k_\ell}, f_i^{k_\ell} - f_i, f_{i+1}, \ldots, f_m)$$

on the complement of the set

$$\bigcup_{\rho=1}^\infty \left[E_{f_1, \ldots, f_m}^\rho \cup \left(\bigcup_{i=1}^m \bigcup_{\ell=1}^\infty E_{f_1^{k_\ell}, \ldots, f_{i-1}^{k_\ell}, f_i^{k_\ell} - f_i, f_{i+1}, \ldots, f_m}^\rho \right) \right]$$

which has full measure. Since

$$\left\| \liminf_{\ell \to \infty} \sum_{i=1}^{m} \mathcal{M}_\sigma(f_1^{k_\ell^\ell}, \ldots, f_{i-1}^{k_\ell^\ell}, f_i^{k_\ell^\ell} - f_i, f_{i+1}, \ldots, f_m) \right\|_{L^{2/m}(\mathbb{R}^n)} = 0$$

in view of Fatou's lemma and (7.2), we finally obtain

$$\limsup_{\nu \to -\infty} \left| S_\sigma^\nu(f_1, \ldots, f_m) - \sigma(0) f_1 \cdots f_m \right| = 0$$

for almost all points in $\mathbb{R}^n$, which proves one part of the claimed a.e. convergences.

8. Concluding remarks

As of this writing, we are uncertain how to extend Theorem 1.4 in the non-lacunary case. A new ingredient may be necessary to accomplish this.

We have addressed the boundedness of several multilinear and maximal multilinear operators at the initial point $L^2 \times \cdots \times L^2 \to L^{2/m}$. Our future investigation related to this project has two main directions: (a) to extend this initial estimate to many other operators, such as the general maximal multipliers considered in [17, 26], and (b) to obtain $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ bounds for all of these operators in the largest possible range of exponents possible. Additionally, one could consider the study of related endpoint estimates. We hope to achieve this goal in future publications.

References

[1] T. Anderson, E. Palsson. *Bounds for discrete multilinear spherical maximal functions.* Collect. Math. **73** (2022), no. 1, 75–87.

[2] J. Barrionuevo, L. Grafakos, D. He, P. Honzík, and L. Oliveira. *Bilinear spherical maximal function.* Math. Res. Lett. **25** (2018), no. 5, 1369–1388.

[3] E. Buriánková and P. Honzík, *Rough maximal bilinear singular integrals*, Collect. Math. **70** (2019), 431–446.

[4] C. Calderón, *Lacunary spherical means*, Illinois J. Math. **23** (1979), no. 3, 476–484.

[5] A. Carbery, J. Rubio de Francia, L. Vega. *Almost everywhere summability of Fourier integrals.* J. London Math. Soc. **38** (1988), no. 3, 513–524.

[6] L. Carleson. *On convergence and growth of partial sums of Fourier series.* Acta Math. **116** (1966), 135–157.

[7] R. R. Coifman, Y. Meyer, *Commutateurs d' intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[8] R. R. Coifman, Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[9] R. Coifman, G. Weiss, *Book Review: Littlewood-Paley and multiplier theory*, Bull. Amer. Math. Soc. **84** (1978), no. 2, 242–250.

[10] G. Dosidis. *Multilinear spherical maximal function.* Proc. Amer. Math. Soc. **149** (2021), no. 4, 1471–1480.

[11] G. Dosidis and L. Grafakos, *On families between the Hardy-Littlewood and spherical maximal functions*, Ark. Mat. **59** (2021), no. 2, 323–343.

[12] X. Du, L. Guth, X. Li. em A sharp Schrödinger maximal estimate in $\mathbb{R}^2$. Ann. of Math. **186** (2017), no. 2, 607–640.

[13] J. Duoandikoetxea and J.-L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986) 541–561.

[14] L. Grafakos, *Classical Fourier Analysis*, 3rd edition, Graduate Texts in Mathematics 249, Springer, New York, 2014.

[15] L. Grafakos, *Modern Fourier Analysis*, 3rd edition, Graduate Texts in Mathematics 250, Springer, New York, 2014.

[16] L. Grafakos, D. He, and P. Honzík, *Rough bilinear singular integrals*, Adv. Math. **326** (2018) 54–78.

[17] L. Grafakos, D. He, and P. Honzík, *Maximal operators associated with bilinear multipliers of limited decay*, J. Anal. Math. **143** (2021) 231-251.

[18] L. Grafakos, D. He, P. Honzík, and B. Park, *Initial $L^2 \times \cdots \times L^2$ bounds for multilinear operators*, Trans. Amer. Math. Soc. **376** (2023) 3445-3472.

[19] L. Grafakos, D. He, and L. Slavíková, $L^2 \times L^2 \to L^1$ *boundedness criteria*, Math. Ann. **376** (2020) 431–455.

[20] L. Grafakos and R. H. Torres, *Maximal operator and weighted norm inequalities for multilinear singular integrals*, Indiana Univ. Math. J. **51** (2002), 1261–1276.

[21] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund Theory*, Adv. Math. **165** (2002), 124–164.

[22] Y.Heo, S. Hong, C. Yang. *Improved bounds for the bilinear spherical maximal operators.* Math. Res. Lett. **27** (2020), no. 2, 397–434.

[23] E. Jeong, S. Lee. *Maximal estimates for the bilinear spherical averages and the bilinear Bochner-Riesz operators.* J. Funct. Anal. **279** (2020), no. 7, 108629, 29 pp.

[24] B. Jessen, J. Marcinkiewicz, and A. Zygmund. *Note on the differentiability of multiple integrals.* Fundamenta Mathematicae **25** (1935) no.1, 217–234.

[25] T. Kato, A. Miyachi, N. Tomita, *Boundedness of bilinear pseudo-differential operators of $S_{0,0}$-type on $L^2 \times L^2$*, J. Pseudo-Differ. Oper. Appl. **12** (2021), 38pp.

[26] J. L. Rubio de Francia, *Maximal functions and Fourier transforms*, Duke Math. J. **53** (1986) 395-404.

[27] E. Stein. *On limits of sequences of operators.* Ann. of Math. **74** (1961), no.2 140–170.

[28] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel-Boston-Stuttgart, 1983.

[29] H. Triebel, *Theory of function spaces. III*, Birkhäuser, Basel-Boston-Stuttgart, 2006.

[30] H. Triebel, *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*, EMS Tracts in Mathematics, vol. 11, European Mathematical Society (EMS), Zürich, 2010.

[31] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., **33** (1986) 131-174.

L. Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
Email address: `grafakosl@missouri.edu`

D. He, School of Mathematical Sciences, Fudan University, People's Republic of China
Email address: `hedanqing@fudan.edu.cn`

P. Honzík, Department of Mathematics, Charles University, 116 36 Praha 1, Czech Republic
Email address: `honzik@gmail.com`

B. Park, Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea
Email address: `bpark43@skku.edu`