CHARACTERIZATION OF $L^p(\mathbf{R}^n)$ USING THE GABOR FRAME

LOUKAS GRAFAKOS AND CHRIS LENNARD

Abstract. We characterize L^p norms of functions on $\mathbf{R}^n$ for $1 < p < \infty$ in terms of their Gabor coefficients. Moreover, we use the Carleson-Hunt theorem to show that the Gabor expansions of L^p functions converge to the functions almost everywhere and in L^p for $1 < p < \infty$. In L^1 we prove an analogous result: the Gabor expansions converge to the functions almost everywhere and in L^1 in a certain Cesàro sense. Consequently, we are able to establish that a large class of Gabor families generate Banach frames for $L^p(\mathbf{R}^n)$ when $1 \le p < \infty$.

1. Introduction

It is a well known fact that certain Gabor expansions form frames for the Hilbert space $L^2(\mathbf{R}^n)$. In this paper we show that that these expansions also generate Banach frames for $L^p(\mathbf{R}^n)$ when $1 \le p < \infty$. The case $p = 1$ must be handled separately using Cesàro-Fejér summability (instead of Dirichlet summability) of partial sums. We also establish that certain Gabor expansions of L^p functions converge to the functions almost everywhere and in $L^p(\mathbf{R}^n)$ for $1 \le p < \infty$, where the convergence is interpreted in the Cesàro sense when $p = 1$. Moreover, we obtain frame inequalities, that characterize the $L^p(\mathbf{R}^n)$ norm of f (up to equivalence) in terms of certain sequence-space norms evaluated at the sequence of Gabor coefficients of f.

Recall that a family $(f_i)_{i\in I}$ is a *frame* for a Hilbert space $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B||f||^2, \quad \text{for all } f \in \mathcal{H} \ .$$

In this paper the index set I will, as usual, be denumerable. Frames were introduced by Duffin and Schaeffer [6]. Also see Heil and Walnut [12] for an exposition.

Two analogues of Hilbert space frames for Banach spaces, *atomic decompositions* and *Banach frames* (which include atomic decompositions) were introduced by Gröchenig [10]. Also, see Christensen and Heil [4], Casazza and Christensen [3], and Walnut [22]. These papers contain examples of Banach frames, and results on the stability of Banach frames under perturbation.

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Our results show that Gabor expansions in $L^p(\mathbf{R}^n)$, $1 \le p < \infty$, once we slightly widen the definition (without altering its essential features), may be viewed as *Banach frames*. We restrict our attention to frames that are tight in the special case of $L^2(\mathbf{R}^n)$. We remark that Walnut [22] proves analogous results for certain Gabor expansions in weighted L^2 spaces i.e. that these Gabor expansions are atomic decompositions.

Our results for $1 < p < \infty$ may also be phrased in terms of a concept introduced in a paper of Kazarian, Soria and Zink [14]: Gabor systems $(\phi_{m,l})_{m,l\in\mathbf{Z}^n}$, of the type described below, form a *quasibasis* of $L^p(\mathbf{R}^n)$ (see [14] for the definition).

We note that simultaneously with this paper, K. Gröchenig and C. Heil [11] recently obtained frame characterizations of $L^p(\mathbf{R}^n)$ related to those of the authors. For $1 < p < \infty$, the frame characterizations of L^p in [11] hold for more general ϕ and ψ than those we consider. Moreover, the proof techniques used in [11], involving modulation spaces, are different from ours. Finally, we remark that our paper is more general than [11] in the sense that it includes a Gabor frame characterization of L^1.

Throughout this section I and J will denote countably infinite index sets. Moreover, $\mathcal{X}_d$ will denote a semi-normed vector space of scalar-valued sequences with domain I, containing $c_{00}(I)$, the space of all finitely non-zero scalar-valued sequences on I. Let $\mathcal{X}$ be a Banach space. We will denote the Banach dual of $\mathcal{X}$ by $\mathcal{X}^*$. The duality between any $x \in \mathcal{X}$ and $y \in \mathcal{X}^*$ will be given by $\langle x, y\rangle$.

Definition 1.1. Let $(f_i)_{i\in I}$ be a sequence in $\mathcal{X}$ and $(y_i)_{i\in I}$ be a sequence in $\mathcal{X}^*$. The pair $((y_i)_{i\in I}, (f_i)_{i\in I})$ is called an *atomic decomposition* for $\mathcal{X}$ with respect to $\mathcal{X}_d$, with bounds A_1 and B_1, if
(a) $(\langle x, y_i\rangle)_{i\in I} \in \mathcal{X}_d$ for each $x \in \mathcal{X}$;
(b) $A_1\|x\| \le \| (\langle x, y_i\rangle)_{i\in I} \|_{\mathcal{X}_d} \le B_1\|x\|$, for all $x \in \mathcal{X}$; and
(c) $x = \sum_{i\in I}\langle x, y_i\rangle f_i$, for all $x \in \mathcal{X}$, with convergence in the norm of $\mathcal{X}$.

Definition 1.2. Suppose we are given a family of linear operators $\Theta = (T_j)_{j\in J}$, where each $T_j : \mathcal{X}_d \to \mathcal{X}$, for some countable index set J, and a sequence $(y_i)_{i\in I}$ in $\mathcal{X}^*$. The pair $((y_i)_{i\in I}, \Theta)$ is called a *Banach frame* for $\mathcal{X}$ with respect to $\mathcal{X}_d$, with bounds A_1 and B_1, if we have
(a) $K(x) := (\langle x, y_i\rangle)_{i\in I} \in \mathcal{X}_d$ for each $x \in \mathcal{X}$;
(b) $A_1\|x\| \le \| (\langle x, y_i\rangle)_{i\in I} \|_{\mathcal{X}_d} \le B_1\|x\|$, for all $x \in \mathcal{X}$; and
(c) for all $x \in \mathcal{X}$, $\lim_{j\in J}\ T_j(K(x))$ exists in the norm of $\mathcal{X}$, and equals x.

We remark that in both Definitions 1.1 and 1.2 we have slightly broadened the definition of Gröchenig [10], where $\mathcal{X}_d$ is assumed to be a Banach space. We are assuming that $\mathcal{X}_d$ is merely a seminormed vector space.

Concerning Definition 1.2, in [10] it is assumed that there exists $T : \mathcal{X}_d \to \mathcal{X}$ that is linear and continuous on $\mathcal{X}_d$, and for which $x = T(K(x))$, for all $x \in \mathcal{X}$. Moreover, to establish frame perturbation theorems in Christensen and Heil [4], it is assumed that the usual delta sequence of vectors $(e_i)_{i\in I}$ in $c_{00}(I)$ is an unconditional Schauder basis for $\mathcal{X}_d$. Let us assume that $(e_i)_{i\in I}$ is a Schauder basis for $\mathcal{X}_d$ (perhaps not unconditional). For simplicity, and without loss of generality, let $I = \mathbf{N}$. Define

$J := \mathbf{N}$ also. Further, let $\Theta = (T_j)_{j\in J}$, where each $T_j : \mathcal{X}_d \to \mathcal{X}$, is defined by

$$T_j(\alpha) := \sum_{i=1}^{j} \alpha_i \, T(e_i) \quad , \quad \text{for all} \;\; \alpha = (\alpha_i)_{i\in\mathbf{N}} \quad , \quad \text{for all} \;\; j \in \mathbf{N} \quad .$$

Then, clearly,

$$\lim_{j\in J} \; T_j(\alpha)) = T\left(\lim_{j\to\infty} \; \sum_{i=1}^{j} \; \alpha_i \, e_i\right) = T(\alpha) \quad ,$$

in the norm of $\mathcal{X}$. Hence,

$$\lim_{j\in J} \; T_j(K(x))) = T(K(x)) = x \quad .$$

In summary, Definition 1.2 includes the original definition of a Banach frame.

We will be working on $\mathbf{R}^n$. We use the following definitions for the Fourier transform and the inverse Fourier transform of Schwartz functions f on $\mathbf{R}^n$:

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx, \qquad f^{\vee}(x) = \widehat{f}(-x) = \int_{\mathbf{R}^n} f(\xi)e^{2\pi i x\cdot\xi}d\xi, \tag{1}$$

where $x\cdot\xi = \sum_{j=1}^{n} x_j\xi_j$ if $x = (x_1,\ldots,x_n)$ and $\xi = (\xi_1,\ldots,\xi_n)$.

Throughout this article $Q = [0,1]^n$ will denote the unit cube in $\mathbf{R}^n$. Fix three functions ϕ, ψ and σ such that

(2) ϕ, ψ and σ are complex-valued Schwartz functions on $\mathbf{R}^n$; and

(3) $\widehat{\phi}$, $\widehat{\psi}$ and $\widehat{\sigma}$ are supported in Q.

We will assume the following structural relationship between ϕ and ψ:

$$\sum_{l\in\mathbf{Z}^n} \widehat{\phi}(\xi-\omega_0 l)\overline{\widehat{\psi}(\xi-\omega_0 l)} = A, \qquad \xi\in\mathbf{R}^n, \tag{4}$$

for some fixed ω_0 and A with $0<\omega_0<1$ and $0<A<\infty$. We remark that (2) (or (3)) implies that

$$B_\phi := \sup_{\xi\in\mathbf{R}^n} \sum_{l\in\mathbf{Z}^n} |\widehat{\phi}(\xi-\omega_0 l)|^2 < \infty \text{ and } B_\psi := \sup_{\xi\in\mathbf{R}^n} \sum_{l\in\mathbf{Z}^n} |\widehat{\psi}(\xi-\omega_0 l)|^2 < \infty. \tag{5}$$

Similarly, $B_\sigma < \infty$. We will also assume the following concentration condition

$$c = \int_Q \widehat{\psi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi - \sum_{\substack{(s_1,\ldots,s_n)\in\mathbf{Z}^n\setminus\{0\} \\ |s_j|<\omega_0^{-1}}} \left|\int_Q \widehat{\psi}(\xi)\overline{\widehat{\psi}(\xi+\omega_0 s)}\,d\xi\right| > 0, \tag{6}$$

basically saying the overlap of $\widehat{\psi}$ with all its translates on the grid $\mathbf{Z}^n\setminus\{0\}$ is relatively small compared with $\|\psi\|_{L^2}^2$. Since the number c in (6) tends to $\int_Q \widehat{\psi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi > 0$ as $\omega_0 \to 1$, it follows that (6) is not a serious restriction on ψ if ω_0 is near 1.

Finally, assume

$$c_\sigma = \int_Q \widehat{\sigma}(\xi)\overline{\widehat{\sigma}(\xi)}\,d\xi - \sum_{\substack{(s_1,\dots,s_n)\in\mathbf{Z}^n\setminus\{0\}\\ |s_j|<\omega_0^{-1}}} \left|\int_Q \widehat{\sigma}(\xi)\overline{\widehat{\sigma}(\xi+\omega_0 s)}\,d\xi\right| > 0\ . \tag{7}$$

We define a pair of Gabor families $(\phi_{m,l})_{m,l\in\mathbf{Z}^n}$ and $(\psi_{m,l})_{m,l\in\mathbf{Z}^n}$ as follows. For all m, l in $\mathbf{Z}^n$, let

$$\phi_{m,l}(x) = \phi(x-m)e^{2\pi i\omega_0 x\cdot l}, \qquad \psi_{m,l}(x) = \psi(x-m)e^{2\pi i\omega_0 x\cdot l}. \tag{8}$$

The Gabor family $(\sigma_{m,l})_{m,l\in\mathbf{Z}^n}$ is defined similarly. An easy calculation using (1) gives that

$$\widehat{\phi_{m,l}}(\xi) = \widehat{\phi}(\xi-\omega_0 l)e^{-2\pi i m\cdot(\xi-\omega_0 l)} \tag{9}$$

and similarly with ψ and σ. Let us denote by

$$\langle f,g\rangle = \int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx$$

the usual inner product on $\mathbf{R}^n$. We observe that the equality

$$\sum_{l\in\mathbf{Z}^n}\sum_{m\in\mathbf{Z}^n}\langle f,\phi_{m,l}\rangle\overline{\langle f,\psi_{m,l}\rangle} = A\|f\|_{L^2}^2 \tag{10}$$

is valid for all square integrable functions f on $\mathbf{R}^n$. Identity (10) is a consequence of Plancherel's theorem in $L^2(\mathbf{R}^n)$,

$$e^{2\pi i\omega_0 m\cdot l}\,\langle f,\phi_{m,l}\rangle = \left(\widehat{f}(\,\cdot\,)\overline{\widehat{\phi}(\,\cdot\,-\omega_0 l)}\right)^\vee(m)$$

and

$$e^{2\pi i\omega_0 m\cdot l}\,\langle f,\psi_{m,l}\rangle = \left(\widehat{f}(\,\cdot\,)\overline{\widehat{\psi}(\,\cdot\,-\omega_0 l)}\right)^\vee(m);$$

Parseval's formula in $L^2(Q)$,

$$\begin{aligned}&\sum_{m\in\mathbf{Z}^n}\left(\widehat{f}(\,\cdot\,)\overline{\widehat{\phi}(\,\cdot\,-\omega_0 l)}\right)^\vee(m)\,\overline{\left(\widehat{f}(\,\cdot\,)\overline{\widehat{\psi}(\,\cdot\,-\omega_0 l)}\right)^\vee(m)}\\ &=\int_Q\left(\sum_{r\in\mathbf{Z}^n}\widehat{f}(\xi-r)\overline{\widehat{\phi}(\xi-r-\omega_0 l)}\right)\left(\overline{\sum_{s\in\mathbf{Z}^n}\widehat{f}(\xi-s)\overline{\widehat{\psi}(\xi-s-\omega_0 l)}}\right)d\xi\,;\end{aligned}$$

and of the equalities below

$$\begin{aligned}&\sum_{l\in\mathbf{Z}^n}\sum_{m\in\mathbf{Z}^n}\langle f,\phi_{m,l}\rangle\overline{\langle f,\psi_{m,l}\rangle}\\ &=\sum_{r\in\mathbf{Z}^n}\sum_{s\in\mathbf{Z}^n}\int_Q\widehat{f}(\xi-r)\overline{\widehat{f}(\xi-s)}\sum_{l\in\mathbf{Z}^n}\overline{\widehat{\phi}(\xi-r-\omega_0 l)}\,\widehat{\psi}(\xi-s-\omega_0 l)\,d\xi\\ &=\sum_{r\in\mathbf{Z}^n}\int_Q|\widehat{f}(\xi-r)|^2A\,d\xi = A\|f\|_{L^2(\mathbf{R}^n)}^2,\end{aligned}$$

applying (4) and the fact that the last sum in l above vanishes when $r\neq s$, by (3).

Let us just observe that when $\phi = \psi$, (10) reduces to the tight frame identity

$$\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n} |\langle f, \phi_{m,l}\rangle|^2 = A\|f\|_{L^2}^2, \tag{11}$$

a well-known fact about Gabor frames under assumption (4) with $\phi = \psi$.

Also, note that condition (5) guarantees (and is equivalent to) the fact that for each $f \in L^2(\mathbf{R}^n)$, $(\langle f, \phi_{m,l}\rangle)_{m,l\in\mathbf{Z}^n}$ and $(\langle f, \psi_{m,l}\rangle)_{m,l\in\mathbf{Z}^n}$ are Bessel sequences. Indeed, calculations similar to those immediately above show that

$$\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n} |\langle f, \phi_{m,l}\rangle|^2 \le B_\phi \|f\|_{L^2}^2 \text{ and } \sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n} |\langle f, \psi_{m,l}\rangle|^2 \le B_\psi \|f\|_{L^2}^2 \,. \tag{12}$$

Moreover, when combined with (4), (5) tells us that both $(\phi_{m,l})_{m,l\in\mathbf{Z}^n}$ and $(\psi_{m,l})_{m,l\in\mathbf{Z}^n}$ are *frames* for $L^2(\mathbf{R}^n)$, with frame bounds $(A^2/B_\psi, B_\phi)$ and $(A^2/B_\phi, B_\psi)$, respectively.

In this paper we obtain results analogous to (10) for L^p spaces by characterizing the L^p norm of a function f on $\mathbf{R}^n$ in terms of its Gabor coefficients $\langle f, \phi_{m,l}\rangle$.

We shall now state our results in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, leaving the proofs and the L^1 case for later sections. We begin with some estimates valid for Schwartz functions.

Theorem 1.3. *Let ϕ and σ satisfy (2). Let $0 < p \le \infty$. Then there exists a constant $C_p > 0$ (depending only on p, n, and the functions ϕ and σ) such that for all f in the Schwartz class of $\mathbf{R}^n$ we have*

$$\left\| \sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right| \right\|_{L^p} \le C_p \|f\|_{L^p}. \tag{13}$$

In the opposite direction we have the following result.

Theorem 1.4. *Let ϕ and σ satisfy (2), (3), (4), and (7). Let $1 \le p \le \infty$. Then there exists a constant $D_p > 0$ (depending only on p, n, and on the functions ϕ, ψ, and σ) such that for all f in the Schwartz class of $\mathbf{R}^n$ we have*

$$\|f\|_{L^p} \le D_p \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^2 \right)^{1/2} \right\|_{L^p} . \tag{14}$$

Combining Theorems 1.3 and 1.4 we obtain a characterization of the L^p norm of a Schwartz function on $\mathbf{R}^n$ in terms of its Gabor coefficients $\langle f, \phi_{m,l}\rangle$ for $1 \le p \le \infty$.

Corollary 1.5. *Let ϕ and σ satisfy (2), (3), (4), and (7). Then for all f in the Schwartz class of $\mathbf{R}^n$ and all $1 \le p, q \le \infty$ we have the following equivalence of norms*

$$\|f\|_{L^p} \approx \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^q \right)^{1/q} \right\|_{L^p} , \tag{15}$$

with constants depending on p, n, and on the functions ϕ, ψ, and σ (but not on q).

Remark 1.6. Even when $p = q = 2$, the expression on the right of (15) is not a constant multiple of the expression on the left in (11). Nevertheless, (15) provides a generalization of (11) for $p \neq 2$. Moreover, the presence of the functions $\sigma_{m,l}$ indicates that the equivalent characterization of the L^p-norm of f given in (15) depends only on the sequence of Gabor coefficients $(\langle f, \phi_{m,l}\rangle)_{m,l}$, and not also on the functions $\phi_{m,l}$.

Next we show that Schwartz functions are limits of their Gabor expansions.

Theorem 1.7. *Let ϕ and ψ satisfy (2), (3) and (4). Every Schwartz function f can be represented as the pointwise limit of an absolutely convergent series of Gabor expansions as follows:*

$$f(x) = \frac{1}{A} \sum_{m\in\mathbf{Z}^n} \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \psi_{m,l}(x), \qquad x \in \mathbf{R}^n. \tag{16}$$

Having established some basic facts about Gabor expansions of Schwartz functions, we now pass to the main results of our paper. These results have to do with the behavior of the Gabor expansions of arbitrary L^p functions on $\mathbf{R}^n$ and the description of the L^p norm of a function in terms of its Gabor coefficients. In the theorem below we use square summation instead of spherical summation in l in view of the lack of L^p convergence (in general) of the spherical partials sums of a Fourier series of a function $f \in L^p(\mathbf{R}^n)$ when $n \geq 2$ and $p \neq 2$. See [8]. We will denote the coordinates of a lattice point $l \in \mathbf{Z}^n$ by $l_1, l_2, \dots, l_n$.

Theorem 1.8. *Let ϕ, ψ and σ satisfy (2), (3) (4), (6) and (7). Fix $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n)$. Then*

$$\frac{1}{A} \sum_{\substack{m\in\mathbf{Z}^n\\ |m|\leq M}} \sum_{|l_1|\leq L} \cdots \sum_{|l_n|\leq L} \langle f, \phi_{m,l}\rangle \psi_{m,l} \to f \tag{17}$$

in L^p and almost everywhere as $M, L \to \infty$. Moreover, for every $f \in L^p(\mathbf{R}^n)$ the series

$$\Delta_m^L(f) = \sum_{|l_1|\leq L} \cdots \sum_{|l_n|\leq L} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \tag{18}$$

converges in L^p and almost everywhere to some function which we denote by

$$\Delta_m(f) = \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}.$$

Further, for all $1 \leq q \leq \infty$ we have

$$\|f\|_{L^p} \approx \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^q \right)^{1/q} \right\|_{L^p} \tag{19}$$

with constants depending only on p, n, and on the functions ϕ, ψ and σ, but not on q.

Let us now see how to use Theorem 1.8 to view our Gabor family $\Phi = (\phi_{m,l})_{m,l\in\mathbf{Z}^n}$ as a Banach frame for $\mathcal{X} := L^p(\mathbf{R}^n)$.

Define $\mathcal{X}_d$ to be the set of all scalar-valued sequences $\alpha = (\alpha_{m,l})_{m,l\in\mathbf{Z}^n}$ for which, for all $m \in \mathbf{N}$, the sequence $(s_L^{(m)})_{L\in\mathbf{N}}$ of functions given by

$$s_L^{(m)}(\alpha) := \sum_{|l_1|\le L} \cdots \sum_{|l_n|\le L} \alpha_{m,l}\ \psi_{m,l}$$

converges in $L^p(\mathbf{R}^n)$ norm to some function $s^{(m)}(\alpha)$; and moreover, for which the sequence $(s^{(m)}(\alpha))_{m\in\mathbf{Z}^n}$ belongs to the Lebesgue-Bochner space $L^p(\mathbf{R}^n, \ell^2)$.

We remark that we could replace $L^p(\mathbf{R}^n, \ell^2)$ by $L^p(\mathbf{R}^n, \ell^q)$ in the definition of $\mathcal{X}_d$ above. See, for example, Diestel and Uhl [5] for more about Lebesgue-Bochner spaces.

Next, for all $\alpha \in \mathcal{X}_d$, define

$$\begin{aligned}\|\alpha\|_{\mathcal{X}_d} &:= \left\|(s^{(m)}(\alpha))_{m\in\mathbf{Z}^n}\right\|_{L^p(\mathbf{R}^n,\ell^2)} \\ &= \left\|\left(\sum_{m\in\mathbf{Z}^n} \left|s^{(m)}(\alpha)\right|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^n)} .\end{aligned}$$

Then $(\mathcal{X}_d, \|\cdot\|_{\mathcal{X}_d})$ is a semi-normed vector space. Let $J := \mathbf{N} \times \mathbf{N}$. Define $\Theta := (T_{M,L})_{(M,L)\in J}$ by setting, for each $(M, L) \in J$ and each $\alpha \in \mathcal{X}_d$,

$$T_{M,L}(\alpha) := \frac{1}{A} \sum_{\substack{m\in\mathbf{Z}^n\\ |m|\le M}} \sum_{|l_1|\le L} \cdots \sum_{|l_n|\le L} \alpha_{m,l}\ \psi_{m,l} \quad .$$

We have that Θ is a denumerable family of linear operators from $\mathcal{X}_d$ into $\mathcal{X}$ and $\Phi := (y_{m,l} := \phi_{m,l})_{m,l\in\mathbf{Z}^n}$ is a sequence in $\mathcal{X}^* = L^{p'}(\mathbf{R}^n)$. Theorem 1.8 now gives us properties (a), (b) and (c) of Definition 1.2; so that (Φ, Θ) is a *Banach frame* for $\mathcal{X} = L^p(\mathbf{R}^n)$ with respect to $\mathcal{X}_d$.

Finally, we have a version of Theorem 1.8 when $p = 1$ where the L^p-limit (18) is replaced by an L^1 limit of the square Fejér means of the function f. The details are in section 6: see Theorem 6.3. We remark that Theorem 6.3 may be viewed as a discrete analogue of Theorem 1.5 of Benedetto [1].

2. The proof of Theorem 1.3

We will use the Poisson summation formula to rewrite the expression

$$\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}$$

appearing in (13) in a simpler form. Recall that the Poisson summation formula says

$$\sum_{l\in\mathbf{Z}^n} \widehat{g}(l) e^{2\pi i x\cdot l} = \sum_{r\in\mathbf{Z}^n} g(x-r), \tag{20}$$

whenever g and $\widehat{g}$ satisfy $|g(x)| + |\widehat{g}(x)| \le C(1 + |x|)^{-n-\delta}$ for some $C, \delta > 0$. See for instance [21] p. 252.

Fix a Schwartz function f on $\mathbf{R}^n$. Observe that

$$\begin{aligned}
&\sum_{l\in\mathbf{Z}^n}\langle f,\phi_{m,l}\rangle\sigma_{m,l}(x)\\
=&\sigma(x-m)\sum_{l\in\mathbf{Z}^n}e^{2\pi i\omega_0 x\cdot l}\int_{\mathbf{R}^n}f(y)\overline{\phi(y-m)}e^{-2\pi i\omega_0 y\cdot l}dy\\
=&\frac{\sigma(x-m)}{\omega_0^n}\sum_{l\in\mathbf{Z}^n}e^{2\pi i\omega_0 x\cdot l}\int_{\mathbf{R}^n}f(\tfrac{y}{\omega_0})\overline{\phi(\tfrac{y}{\omega_0}-m)}e^{-2\pi i y\cdot l}dy\\
=&\frac{\sigma(x-m)}{\omega_0^n}\sum_{l\in\mathbf{Z}^n}e^{2\pi i\omega_0 x\cdot l}\Big(f(\tfrac{\cdot}{\omega_0})\overline{\phi(\tfrac{\cdot}{\omega_0}-m)}\Big)\widehat{\ }(l)\\
=&\frac{\sigma(x-m)}{\omega_0^n}\sum_{r\in\mathbf{Z}^n}f(x-\tfrac{1}{\omega_0}r)\overline{\phi(x-\tfrac{1}{\omega_0}r-m)}. \qquad (21)
\end{aligned}$$

Let us consider the case $p=\infty$ first. Since ϕ and σ are Schwartz functions, for all $\nu>0$ there exists a constant $C_\nu>0$ such that the estimate below holds:

$$|\phi(x)|+|\sigma(x)|\le C_\nu(1+|x|)^{-\nu}. \qquad (22)$$

Using (21) and (22) we obtain the bound

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^n}\Bigg(\sum_{m\in\mathbf{Z}^n}\Big|\sum_{l\in\mathbf{Z}^n}\langle f,\phi_{m,l}\rangle\sigma_{m,l}(x)\Big|\Bigg)\\
\le&\frac{C_\nu^2}{\omega_0^n}\|f\|_{L^\infty}\sup_{x\in\mathbf{R}^n}\Bigg(\sum_{m\in\mathbf{Z}^n}(1+|x-m|)^{-\nu}\Bigg(\sum_{r\in\mathbf{Z}^n}(1+|x-\tfrac{1}{\omega_0}r-m|)^{-\nu}\Bigg)\Bigg)\\
=&\frac{C_\nu^2(C_\nu')^2}{\omega_0^n}\|f\|_{L^\infty},
\end{aligned}$$

since for large enough ν,

$$\sup_{y\in\mathbf{R}^n}\sum_{m\in\mathbf{Z}^n}(1+|y-m|)^{-\nu}=C_\nu'<\infty.$$

We continue with the case $0<p<\infty$. Let

$$\Delta_m(f)=\sum_{l\in\mathbf{Z}^n}\langle f,\phi_{m,l}\rangle\sigma_{m,l}$$

We have that for all $x\in\mathbf{R}^n$,

$$\begin{aligned}
&\sum_{m\in\mathbf{Z}^n}|\Delta_m(f)(x)|\\
\le&\sum_{m\in\mathbf{Z}^n}\frac{|\sigma(x-m)|}{\omega_0^n}\sum_{r\in\mathbf{Z}^n}|f(x-\tfrac{1}{\omega_0}r)|\,|\phi(x-\tfrac{1}{\omega_0}r-m)| \qquad (23)\\
\le&\frac{C_\nu^2}{\omega_0^n}\sum_{r\in\mathbf{Z}^n}|f(x-\tfrac{1}{\omega_0}r)|\sum_{m\in\mathbf{Z}^n}(1+|x-m|)^{-\nu}(1+|x-\tfrac{1}{\omega_0}r-m|)^{-\nu}.
\end{aligned}$$

Now use that $(1+|x-m|)(1+|x-\frac{1}{\omega_0}r-m|) \geq (1+|\frac{1}{\omega_0}r|)$ to estimate the last expression in (23) by

$$\frac{C_\nu^2}{\omega_0^n}\sum_{r\in\mathbf{Z}^n}\frac{|f(x-\frac{r}{\omega_0})|}{(1+|\frac{r}{\omega_0}|)^{\nu/2}}\sum_{m\in\mathbf{Z}^n}\frac{1}{(1+|x-m|)^{\nu/2}} \leq \frac{C_\nu^2\,C'_{\nu/2}}{\omega_0^n}\sum_{r\in\mathbf{Z}^n}\frac{|f(x-\frac{r}{\omega_0})|}{(1+|\frac{r}{\omega_0}|)^{\nu/2}}.$$

Note that $C'_{\nu/2}<\infty$, for large enough ν. Clearly, the positive homogeneous mapping

$$f\mapsto \sum_{r\in\mathbf{Z}^n}\frac{|f(x-\frac{r}{\omega_0})|}{(1+|\frac{r}{\omega_0}|)^{\nu/2}}$$

is bounded on $L^p(\mathbf{R}^n)$ for $1\leq p<\infty$ by Minkowski's inequality. For $0<p<1$ we have

$$\Big\|\sum_{r\in\mathbf{Z}^n}\frac{|f(x-\frac{r}{\omega_0})|}{(1+|\frac{r}{\omega_0}|)^{\nu/2}}\Big\|_{L^p} \leq \left(\int_{\mathbf{R}^n}\sum_{r\in\mathbf{Z}^n}\frac{|f(x-\frac{r}{\omega_0})|^p}{(1+|\frac{r}{\omega_0}|)^{p\nu/2}}dx\right)^{1/p} \leq C''_\nu\|f\|_{L^p},$$

if $\nu>2n/p$. This gives the required conclusion.

3. The proof of Theorem 1.4 and of Corollary 1.5

A key observation we will use (proven below) is that for all $m,l,s\in\mathbf{Z}^n$

$$\langle\sigma_{m,l},\sigma_{m,s+l}\rangle=\int_{\mathbf{R}^n}\sigma_{m,l}(x)\overline{\sigma_{m,s+l}(x)}\,dx \tag{24}$$

is independent of l. To see this, use Plancherel's theorem and the simple formula in (9) to obtain

$$\begin{aligned}\langle\sigma_{m,l},\sigma_{m,s+l}\rangle &= \langle\widehat{\sigma_{m,l}},\widehat{\sigma_{m,s+l}}\rangle\\ &=\int_{\mathbf{R}^n}\widehat{\sigma_{m,l}}(\xi)\overline{\widehat{\sigma_{m,s+l}}(\xi)}\,d\xi\\ &=\int_{\mathbf{R}^n}\widehat{\sigma}(\xi-\omega_0 l)\,e^{-2\pi i m\cdot(\xi-\omega_0 l)}\,\overline{\widehat{\sigma}(\xi-\omega_0 s-\omega_0 l)}\,e^{2\pi i m\cdot(\xi-\omega_0 s-\omega_0 l)}\,d\xi\\ &=e^{-2\pi i\omega_0 m\cdot s}\int_{\mathbf{R}^n}\widehat{\sigma}(\xi-\omega_0 l)\,\overline{\widehat{\sigma}(\xi-\omega_0 s-\omega_0 l)}\,d\xi\\ &=e^{-2\pi i\omega_0 m\cdot s}\,\beta_s,\end{aligned}$$

where

$$\beta_s:=\int_{\mathbf{R}^n}\widehat{\sigma}(\eta)\,\overline{\widehat{\sigma}(\eta-\omega_0 s)}\,d\eta\ .$$

Note that our hypotheses (7) and (3) imply that for $k:=\langle\sigma,\sigma\rangle$, we have

$$|k|-\sum_{\substack{s\in\mathbf{Z}^n\setminus\{0\}\\ |s_j|<\omega_0^{-1}}}|\beta_s| = |k|-\sum_{s\in\mathbf{Z}^n\setminus\{0\}}|\beta_s| =: c>0\ .$$

(Here, s_j are the coordinates of the lattice point s.)

By a similar calculation to the derivation of (10) above, or by polarizing (10) using the identity

$$\langle f, g\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|f + i^k g\|_{L^2}^2 \ ,$$

we obtain the formula

$$\langle f, g\rangle = A^{-1}\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n}\langle f, \phi_{m,l}\rangle\overline{\langle g, \psi_{m,l}\rangle}, \tag{25}$$

for f and g Schwartz functions. Now fix $1 \le p \le \infty$ and let p' be the dual index. To complete the proof of Theorem 1.4, we will use (25) and duality; i.e.

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}}\le 1} |\langle f, g\rangle|,$$

noting that it is sufficient to only consider Schwartz functions g in the above supremum. Then

$$\begin{aligned}
\|f\|_{L^p} &= \sup_{\|g\|_{L^{p'}}\le 1} |\langle f, g\rangle| \\
&= A^{-1}\sup_{\|g\|_{L^{p'}}\le 1}\left|\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n}\langle f, \phi_{m,l}\rangle\overline{\langle g, \psi_{m,l}\rangle}\right| \\
&\le A^{-1}\sup_{\|g\|_{L^{p'}}\le 1}\sum_{m\in\mathbf{Z}^n}\left|\sum_{l\in\mathbf{Z}^n}\langle f, \phi_{m,l}\rangle\overline{\langle g, \psi_{m,l}\rangle}\right| \ .
\end{aligned}$$

Now for each Schwartz function g and $s \in \mathbf{Z}^n$ define g_s by

$$g_s(x) = g(x)\, e^{-2\pi i\omega_0 x\cdot s} \ , \text{ for all } x \in \mathbf{R}^n \ .$$

Define $\gamma(f, g)$ by

$$\gamma(f, g) := \sup_{s\in\mathbf{Z}^n}\sum_{m\in\mathbf{Z}^n}\left|\sum_{l\in\mathbf{Z}^n}\langle f, \phi_{m,l}\rangle\,\overline{\langle g_s, \psi_{m,l}\rangle}\right| \ .$$

By (12),

$$\begin{aligned}
\gamma(f, g) &:= \sup_{s\in\mathbf{Z}^n}\sum_{m\in\mathbf{Z}^n}\left|\sum_{l\in\mathbf{Z}^n}\langle f, \phi_{m,l}\rangle\,\overline{\langle g_s, \psi_{m,l}\rangle}\right| \\
&\le \sup_{s\in\mathbf{Z}^n}\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n}\left|\langle f, \phi_{m,l}\rangle\,\overline{\langle g_s, \psi_{m,l}\rangle}\right| \\
&\le \sup_{s\in\mathbf{Z}^n}\left(\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n}|\langle f, \phi_{m,l}\rangle|^2\right)^{1/2}\left(\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n}|\langle g_s, \psi_{m,l}\rangle|^2\right)^{1/2} \\
&\le \sup_{s\in\mathbf{Z}^n}\sqrt{B_\phi\, B_\psi}\,\|f\|_{L^2}\,\|g_s\|_{L^2} \\
&= \sqrt{B_\phi\, B_\psi}\,\|f\|_{L^2}\,\|g\|_{L^2} < \infty \ .
\end{aligned}$$

Fix Schwartz functions f and g. Then,

$$\begin{aligned}&|k| \sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,l}\rangle}\right| \\ &= \sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,l}\rangle} \langle \sigma_{m,l}, \sigma_{m,l}\rangle\right| \le U + \Gamma \text{ ; where}\end{aligned}$$

$$U := \sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \sum_{l'\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,l'}\rangle} \langle \sigma_{m,l}, \sigma_{m,l'}\rangle\right| \text{ , and}$$

$$\Gamma := \sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \sum_{l'\in\mathbf{Z}^n\setminus\{l\}} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,l'}\rangle} \langle \sigma_{m,l}, \sigma_{m,l'}\rangle\right| .$$

Now,

$$\begin{aligned}U &\le \int_{\mathbf{R}^n} \sum_{m\in\mathbf{Z}^n} \left| \left(\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}(x)\right) \left(\sum_{l'\in\mathbf{Z}^n} \overline{\langle g, \psi_{m,l'}\rangle \sigma_{m,l'}(x)}\right)\right| dx \\
&\le \int_{\mathbf{R}^n} \left(\sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}(x)\right|^2\right)^{1/2} \left(\sum_{m\in\mathbf{Z}^n} \left|\sum_{l'\in\mathbf{Z}^n} \langle g, \psi_{m,l'}\rangle \sigma_{m,l'}(x)\right|^2\right)^{1/2} dx \\
&\le \left\|\left(\sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}\right|^2\right)^{1/2}\right\|_{L^p} \left\|\left(\sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle g, \psi_{m,l}\rangle \sigma_{m,l}\right|^2\right)^{1/2}\right\|_{L^{p'}} \\
&\le \left\|\left(\sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}\right|^2\right)^{1/2}\right\|_{L^p} \left\|\sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle g, \psi_{m,l}\rangle \sigma_{m,l}\right|\right\|_{L^{p'}} \\
&\le \left\|\left(\sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}\right|^2\right)^{1/2}\right\|_{L^p} C_{p'} \|g\|_{L^{p'}} .\end{aligned}$$

On the other hand, from the definition of Γ and the verification of (24) above,

$$\begin{aligned}\Gamma &= \sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \sum_{s\in\mathbf{Z}^n\setminus\{0\}} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,s+l}\rangle} \langle \sigma_{m,l}, \sigma_{m,s+l}\rangle\right| \\
&\le \sum_{s\in\mathbf{Z}^n\setminus\{0\}} |\beta_s| \sum_{m\in\mathbf{Z}^n} \left|e^{-2\pi i\omega_0 m\cdot s} \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,s+l}\rangle}\right| \\
&= \sum_{s\in\mathbf{Z}^n\setminus\{0\}} |\beta_s| \sum_{m\in\mathbf{Z}^n} \left|\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g_s, \psi_{m,l}\rangle}\right| \\
&\le \left(\sum_{s\in\mathbf{Z}^n\setminus\{0\}} |\beta_s|\right) \gamma(f, g) .\end{aligned}$$

Again, fix Schwartz functions f and g. Then fix $t \in \mathbf{Z}^n$. Applying our argument immediately above to f and g_t instead of f and g yields

$$\begin{aligned}
&|k| \sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g_t, \psi_{m,l}\rangle} \right| \\
&\le \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^2 \right)^{1/2} \right\|_{L^p} C_{p'} \left\|g_t\right\|_{L^{p'}} + \left(\sum_{s\in\mathbf{Z}^n\setminus\{0\}} |\beta_s| \right) \gamma(f, g_t) \\
&= \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^2 \right)^{1/2} \right\|_{L^p} C_{p'} \left\|g\right\|_{L^{p'}} + \left(\sum_{s\in\mathbf{Z}^n\setminus\{0\}} |\beta_s| \right) \gamma(f, g) \ .
\end{aligned}$$

Since $t \in \mathbf{Z}^n$ is arbitrary, we may conclude that

$$|k|\, \gamma(f,g) \le \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^2 \right)^{1/2} \right\|_{L^p} C_{p'} \left\|g\right\|_{L^{p'}} + \left(\sum_{s\in\mathbf{Z}^n\setminus\{0\}} |\beta_s| \right) \gamma(f, g) \ ;$$

and therefore

$$c\, \gamma(f,g) \le \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^2 \right)^{1/2} \right\|_{L^p} C_{p'} \left\|g\right\|_{L^{p'}} \ .$$

Now f and g are arbitrary Schwartz functions on $\mathbf{R}^n$. So, recalling the duality introduced above, we see that

$$\begin{aligned}
\|f\|_{L^p} &= \sup_{\|g\|_{L^{p'}}\le 1} |\langle f, g\rangle| \\
&= A^{-1} \sup_{\|g\|_{L^{p'}}\le 1} \left| \sum_{m\in\mathbf{Z}^n} \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,l}\rangle} \right| \\
&\le A^{-1} \sup_{\|g\|_{L^{p'}}\le 1} \sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \overline{\langle g, \psi_{m,l}\rangle} \right| \\
&\le A^{-1} \sup_{\|g\|_{L^{p'}}\le 1} \gamma(f,g) \\
&\le \left(\frac{C_{p'}}{A\,c} \right) \sup_{\|g\|_{L^{p'}}\le 1} \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^2 \right)^{1/2} \right\|_{L^p} \|g\|_{L^{p'}} \\
&= \left(\frac{C_{p'}}{A\,c} \right) \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right|^2 \right)^{1/2} \right\|_{L^p} .
\end{aligned}$$

This proves the converse inequality with constant $D_p \le C_{p'}/(A\,c)$ and completes the proof of Theorem 1.4.

Let us now discuss Corollary 1.5. For a Schwartz function f recall our notation

$$\Delta_m(f) = \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle \sigma_{m,l}.$$

It follows from Theorems 1.3 and 1.4 that

$$\Big\| \sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big\|_{L^p} \le C_p D_p \Big\| \Big(\sum_{m\in\mathbf{Z}^n} |\Delta_m(f)|^2 \Big)^{1/2} \Big\|_{L^p} .$$

To introduce the ℓ^q norm, $1 \le q \le \infty$, inside the L^p norm above we argue as follows

$$\begin{aligned}
&\Big\| \Big(\sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big|^2 \Big)^{1/2} \Big\|_{L^p} \le \Big\| \Big(\sup_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big)^{1/2} \Big(\sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big)^{1/2} \Big\|_{L^p} \\
&\le \Big\| \sup_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big\|_{L^p}^{1/2} \Big\| \sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big\|_{L^p}^{1/2} \\
&\le \Big\| \sup_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big\|_{L^p}^{1/2} (C_p\, D_p)^{1/2} \Big\| \Big(\sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big|^2 \Big)^{1/2} \Big\|_{L^p}^{1/2} ,
\end{aligned}$$

which implies that

$$\Big\| \Big(\sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big|^2 \Big)^{1/2} \Big\|_{L^p} \le C_p\, D_p \Big\| \sup_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big\|_{L^p} .$$

Therefore

$$\|f\|_{L^p} \le D_p \Big\| \Big(\sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big|^2 \Big)^{1/2} \Big\|_{L^p} \le D_p\, C_p\, D_p \Big\| \Big(\sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big|^q \Big)^{1/q} \Big\|_{L^p} ,$$

while the converse inequality follows from Theorem 1.3.

4. The proof of Theorem 1.7

The absolute convergence of the series in (16) follows in a straightforward manner from hypotheses (2) and (3), and the fact that f is a Schwartz class function.

Let us now verify (16) for a given Schwartz class function f. We have

$$\begin{aligned}
&\sum_{m\in\mathbf{Z}^n} \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l} \rangle \psi_{m,l}(x) \\
&= \sum_{m\in\mathbf{Z}^n} \sum_{l\in\mathbf{Z}^n} \langle \widehat{f}, \widehat{\phi_{m,l}} \rangle \psi_{m,l}(x) \\
&= \int_{\mathbf{R}^n} \widehat{f}(\xi) \sum_{l\in\mathbf{Z}^n} \overline{\widehat{\phi}(\xi-\omega_0 l)}\, e^{2\pi i x\cdot\xi} \sum_{m\in\mathbf{Z}^n} \delta_{\xi,l}(x-m)\, d\xi \ ,
\end{aligned}$$

where $\delta_{\xi,l}(x) := \psi(x)\, e^{-2\pi i x\cdot(\xi-\omega_0 l)}$, for each $x \in \mathbf{R}^n$. Applying the Poisson summation formula, we see that

$$
\begin{aligned}
&\sum_{m\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n}\langle f,\phi_{m,l}\rangle\psi_{m,l}(x)\\
=&\int_{\mathbf{R}^n}\widehat{f}(\xi)\sum_{l\in\mathbf{Z}^n}\overline{\widehat{\phi}(\xi-\omega_0 l)}\,e^{2\pi i x\cdot\xi}\sum_{s\in\mathbf{Z}^n}\widehat{\delta_{\xi,l}}(s)\,e^{2\pi i x\cdot s}\,d\xi\\
=&\int_{\mathbf{R}^n}\widehat{f}(\xi)\sum_{l\in\mathbf{Z}^n}\overline{\widehat{\phi}(\xi-\omega_0 l)}\,e^{2\pi i x\cdot\xi}\sum_{s\in\mathbf{Z}^n}\widehat{\psi}(s+\xi-\omega_0 l)\,e^{2\pi i x\cdot s}\,d\xi\\
=&\int_{\mathbf{R}^n}\widehat{f}(\xi)\sum_{l\in\mathbf{Z}^n}\left(\sum_{s\in\mathbf{Z}^n}\overline{\widehat{\phi}(\xi-\omega_0 l)}\,\widehat{\psi}(\xi-\omega_0 l+s)\,e^{2\pi i x\cdot s}\right)e^{2\pi i x\cdot\xi}\,d\xi\\
=&\int_{\mathbf{R}^n}\widehat{f}(\xi)\sum_{l\in\mathbf{Z}^n}\overline{\widehat{\phi}(\xi-\omega_0 l)}\,\widehat{\psi}(\xi-\omega_0 l)\,e^{2\pi i x\cdot\xi}\,d\xi\\
=&\int_{\mathbf{R}^n}\widehat{f}(\xi)\,A\,e^{2\pi i x\cdot\xi}\,d\xi\\
=&A\,f(x)\ .
\end{aligned}
$$

The interchange of summation and integration above, and the use of the Poisson summation formula, are justified by the fact that the functions ϕ, ψ and f are in the Schwartz class.

5. The proof of Theorem 1.8

This theorem will be a consequence of the following Lemma.

Lemma 5.1. *Fix $1 < p < \infty$. Then for any $f \in L^p(\mathbf{R}^n)$ the expressions*

$$\Delta_m^L(f) = \sum_{|l_1|\le L}\cdots\sum_{|l_n|\le L}\langle f,\phi_{m,l}\rangle\sigma_{m,l}$$

converge in L^p and almost everywhere to some function $\Delta_m(f)$ as $L \to \infty$. Let

$$\mathcal{K}(f) = \sup_{L>0}\sum_{m\in\mathbf{Z}^n}\left|\Delta_m^L(f)\right|. \tag{26}$$

Then there exists a constant $C_{p,n} > 0$ such that for all $f \in L^p(\mathbf{R}^n)$ we have

$$\|\mathcal{K}(f)\|_{L^p} \le C_{p,n}\|f\|_{L^p}.$$

Let us prove Theorem 1.8 assuming Lemma 5.1.

Proof. Denote by

$$S_{M,L}(f) = \frac{1}{A}\sum_{\substack{m\in\mathbf{Z}^n\\|m|\le M}}\sum_{|l_1|\le L}\cdots\sum_{|l_n|\le L}\langle f,\phi_{m,l}\rangle\psi_{m,l}.$$

Given $\varepsilon > 0$, there exists a Schwartz function f_0 such that $\|f - f_0\|_{L^p(\mathbf{R}^n)} < \varepsilon$. Then

$$\begin{aligned} &\|S_{M,L}f - f\|_{L^p} \\ \le &\|\sup_{M,L} |S_{M,L}(f - f_0)|\,\|_{L^p} + \|S_{M,L}(f_0) - f_0\|_{L^p} + \|f - f_0\|_{L^p} \\ \le &\frac{C_{p,n}}{A}\,\varepsilon + \|S_{M,L}(f_0) - f_0\|_{L^p} + \varepsilon, \end{aligned}$$

which can be made arbitrarily small if $S_{M,L}(f_0) \to f_0$ in $L^p(\mathbf{R}^n)$ as $M, L \to \infty$. But Theorem 1.7 gives that $S_{M,L}(f_0) \to f_0$ a.e. (in fact everywhere). Since

$$|S_{M,L}(f_0)| \le \frac{1}{A}\,\mathcal{K}(f_0) \in L^p, \qquad \text{for all } M, L > 0,$$

by Lemma 5.1, we use the Lebesgue dominated convergence theorem to conclude that $\|S_{M,L}(f_0) - f_0\|_{L^p} < \varepsilon$ for M, L large enough.

To prove that $S_{M,L}f \to f$ almost everywhere as $M, L \to \infty$ we first define the oscillation O_f of an L^p function f as follows:

$$O_f(x) = \limsup_{M\to\infty}\limsup_{M'\to\infty}\limsup_{L\to\infty}\limsup_{L'\to\infty} |S_{M,L}(f)(x) - S_{M',L'}(f)(x)|$$

If we can show that for any $f \in L^p(\mathbf{R}^n)$ and $\delta > 0$ we have

$$|\{x \in \mathbf{R}^n :\ O_f(x) > \delta\}| = 0, \tag{27}$$

then the doubly indexed sequence $S_{M,L}(f)$ would be almost everywhere Cauchy and thus it would converge almost everywhere, to some measurable function h. But, since we also have L^p-norm convergence to f, it would follow that $h = f$. Given $\varepsilon > 0$, find f_0 a Schwartz function as before. Then

$$O_f(x) \le O_{f_0}(x) + O_{f-f_0}(x) = O_{f-f_0}(x)$$

since $O_{f_0}(x) \equiv 0$ identically. Then for $\delta > 0$ we have

$$\begin{aligned} &|\{x \in \mathbf{R}^n :\ O_f(x) > \delta\}| \\ \le &|\{x \in \mathbf{R}^n :\ O_{f-f_0}(x) > \delta\}| \\ \le &|\{x \in \mathbf{R}^n :\ 2\sup_{M,L} |S_{M,L}(f - f_0)(x)| > \delta\}| \\ \le &2^p\,\frac{\|\mathcal{K}(f - f_0)\|_{L^p}^p}{A^p\delta^p} \le 2^p C_{p,n}^p \frac{\|f - f_0\|_{L^p}^p}{A^p\delta^p} = 2^p C_{p,n}^p \frac{\varepsilon^p}{A^p\delta^p}, \end{aligned}$$

in view of the L^p boundedness of $\mathcal{K}$. Since the first term of this sequence of inequalities is independent of ε, letting $\varepsilon \to 0$ we obtain (27). This completes the first part of the proof of Theorem 1.8.

Let's now prove (19). From Lemma 5.1, for all $m \in \mathbf{Z}^n$, $\Delta_m^L(f) \to \Delta_m(f)$ a.e. as $L \to \infty$. Thus, $|\Delta_m^L(f)| \to |\Delta_m(f)|$ a.e. as $L \to \infty$. By Fatou's lemma in ℓ_1, the following inequality holds a.e.:

$$\sum_{m\in\mathbf{Z}^n} \left|\Delta_m(f)\right| \le \liminf_L \sum_{m\in\mathbf{Z}^n} \left|\Delta_m^L(f)\right| \le \mathcal{K}(f)\ .$$

Thus,

$$\Big\| \sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Big\|_{L^p} \le \|\mathcal{K}(f)\|_{L^p} \le C_{p,n}\|f\|_{L^p} \ ;$$

which proves one direction in (19) since

$$\Bigg\| \bigg(\sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big|^q \bigg)^{1/q} \Bigg\|_{L^p} \le \Bigg\| \sum_{m\in\mathbf{Z}^n} \big|\Delta_m(f)\big| \Bigg\|_{L^p} \le C_{p,n}\|f\|_{L^p}. \tag{28}$$

To prove the other direction in (19), we fix $\varepsilon > 0$ and pick a Schwartz function f_0 such that $\|f - f_0\|_{L^p} \le \varepsilon$. Then $\|f\|_{L^p} \le \varepsilon + \|f_0\|_{L^p}$ and Corollary 1.5 gives

$$\begin{aligned} \|f\|_{L^p} \le & \varepsilon + \|f_0\|_{L^p} \le \varepsilon + C'_{p,n}\big\|\big(\sum_{m\in\mathbf{Z}^n} |\Delta_m(f_0)|^q\big)^{1/q}\big\|_{L^p} \\ \le & \varepsilon + C'_{p,n}\big\|\big(\sum_{m\in\mathbf{Z}^n} |\Delta_m(f)|^q\big)^{1/q}\big\|_{L^p} + C'_{p,n}\big\|\big(\sum_{m\in\mathbf{Z}^n} |\Delta_m(f_0 - f)|^q\big)^{1/q}\big\|_{L^p} \\ \le & C'_{p,n}\big\|\big(\sum_{m\in\mathbf{Z}^n} |\Delta_m(f)|^q\big)^{1/q}\big\|_{L^p} + \varepsilon + C'_{p,n}C_{p,n}\varepsilon, \end{aligned}$$

where the last inequality follows from (28). Since $\varepsilon > 0$ was arbitrary, (19) is proved. □

We next prove Lemma 5.1.

Proof. It is here where the Carleson-Hunt theorem is needed. Let us first discuss some background material. For a positive integer L and $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, let

$$D_L(x_1, \dots, x_n) = \prod_{j=1}^{n} \left(\sum_{k=-L}^{L} e^{2\pi i k x_j} \right) = \prod_{j=1}^{n} \frac{\sin(2\pi(L+\frac{1}{2})x_j)}{\sin(\pi x_j)} \tag{29}$$

be the square Dirichlet kernel on the n-dimensional torus $\mathbf{T}^n$ which we are denoting by Q in this article. When $n = 1$, the Carleson-Hunt theorem [2], [13] says that for $1 < p < \infty$, there exists a constant $C_p > 0$ such that for all 1-periodic functions F on $\mathbf{R}$ the inequality below holds

$$\Big\| \sup_{N>0} |D_N * F| \Big\|_{L^p([0,1])} \le C_p \|F\|_{L^p([0,1])}.$$

The extension to higher dimensions for the summing of Fourier series over squares is a rather straightforward consequence of the one-dimensional result, and was obtained independently by Fefferman [7], Sjölin [16], and Tevzadze [20]. This result says that the inequality

$$\Big\| \sup_{L>0} |D_L * F| \Big\|_{L^p([0,1]^n)} \le C_p \|F\|_{L^p([0,1]^n)} \tag{30}$$

holds for all functions F on $\mathbf{R}^n$ which are 1-periodic in each variable. We will use this result to prove Lemma 5.1.

Let us denote by $\mathcal{C}$ the Carleson operator

$$\mathcal{C}(F) = \sup_{L>0} |D_L * F|,$$

acting on functions F on $\mathbf{R}^n$ that are 1-periodic in each variable, where D_L is the square Dirichlet kernel on Q defined in (29).

Now fix $f \in L^p(\mathbf{R}^n)$ for some $1 < p < \infty$. Our Gabor function $\phi \in L^\infty(\mathbf{R}^n) \cap L^{p'}(\mathbf{R}^n)$, and so $f(\cdot)\,\phi(\cdot - m) \in L^p(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$. So, we have

$$\begin{aligned}
\Delta_m^L(f)(x) =& \sigma(x-m) \sum_{|l_j|\le L} \int_{\mathbf{R}^n} f(y)\overline{\phi(y-m)}\, e^{2\pi i \omega_0 (x-y)\cdot l} dy \\
=& \sigma(x-m) \int_{\mathbf{R}^n} D_L(\omega_0(x-y)) f(y) \overline{\phi(y-m)}\, dy \\
=& \sigma(x-m) \int_{\mathbf{R}^n} D_L(\omega_0 x - y) f(\tfrac{y}{\omega_0}) \overline{\phi(\tfrac{y}{\omega_0} - m)}\, \frac{dy}{\omega_0^n} \\
=& \sigma(x-m) \int_{Q} D_L(\omega_0 x - y) \sum_{k\in\mathbf{Z}^n} f(\tfrac{y+k}{\omega_0}) \overline{\phi(\tfrac{y+k}{\omega_0} - m)}\, \frac{dy}{\omega_0^n} \\
=& \frac{1}{\omega_0^n} \sigma(x-m) \left(\Big(\sum_{k\in\mathbf{Z}^n} f(\tfrac{\cdot+k}{\omega_0}) \overline{\phi(\tfrac{\cdot+k}{\omega_0} - m)} \Big) * D_L \right) (\omega_0 x),
\end{aligned} \tag{31}$$

where we used the periodicity of D_L. Set

$$F_m(f)(x) = \sum_{k\in\mathbf{Z}^n} f(\tfrac{x+k}{\omega_0}) \overline{\phi(\tfrac{x+k}{\omega_0} - m)}.$$

Then $F_m(f)$ is 1-periodic and belongs to $L^p(Q)$. By the Carleson-Hunt theorem, $F_m(f) * D_L$ converges almost everywhere in $\mathbf{R}^n$ to the 1-periodic function $F_m(f)$ as $L \to \infty$. We conclude that

$$\Delta_m^L(f)(x) \to \frac{1}{\omega_0^n} \sigma(x-m) F_m(f)(\omega_0 x) \tag{32}$$

for almost all $x \in \mathbf{R}^n$ as $L \to \infty$. We now show that the convergence in (32) is also valid in the $L^p(\mathbf{R}^n)$-norm. Observe that the calculation in (31) implies that

$$\begin{aligned}
&\mathcal{K}(f)(x) = \sup_{L>0} \sum_{m\in\mathbf{Z}^n} |\Delta_m^L(f)(x)| \\
\le& \sum_{m\in\mathbf{Z}^n} |\sigma(x-m)| \sup_{L>0} \left| \sum_{|l_j|\le L} \int_{\mathbf{R}^n} f(y)\overline{\phi(y-m)} e^{2\pi i \omega_0 (x-y)\cdot l} dy \right| \\
=& \frac{1}{\omega_0^n} \sum_{m\in\mathbf{Z}^n} |\sigma(x-m)|\, \mathcal{C}\left(F_m(f)\right)(\omega_0 x) =: \widetilde{\mathcal{K}}(f)(x).
\end{aligned}$$

If we knew that the map $f \mapsto \widetilde{\mathcal{K}}(f)$ was a bounded operator on $L^p(\mathbf{R}^n)$, then we would be able to use the Lebesgue dominated convergence theorem to deduce that the convergence in (32) is also valid in $L^p(\mathbf{R}^n)$. But the L^p boundedness of $\widetilde{\mathcal{K}}$ would also imply that of $\mathcal{K}$, giving the second conclusion of Lemma 5.1. It therefore suffices to prove the boundedness of $\widetilde{\mathcal{K}}$ to finish the proof of Lemma 5.1.

Pick $\nu > 4n$ a large positive integer in the argument below. Fix $f \in L^p(\mathbf{R}^n)$. By the fact that

$$D_\nu := \sup_{x\in\mathbf{R}^n} \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|x-m|)^\nu} < \infty\ ,$$

and the convexity of the function $(\cdot)^p$ on $[0,\infty)$, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} \widetilde{\mathcal{K}}(f)(x)^p\,dx \leq & C_{\nu,n,\omega_0,p} \int_{\mathbf{R}^n} \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|x-m|)^\nu} \mathcal{C}(F_m(f))(\omega_0 x)^p\,dx \\
=& \frac{C_{\nu,n,\omega_0,p}}{\omega_0^n} \int_{\mathbf{R}^n} \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|\frac{x}{\omega_0}-m|)^\nu} \mathcal{C}(F_m(f))(x)^p\,dx \\
=& \frac{C_{\nu,n,\omega_0,p}}{\omega_0^n} \sum_{r\in\mathbf{Z}^n} \int_Q \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|\frac{x-r}{\omega_0}-m|)^\nu} \mathcal{C}(F_m(f))(x-r)^p\,dx \\
\leq & C'_{\nu,n,\omega_0,p} \sum_{r\in\mathbf{Z}^n} \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|\frac{r}{\omega_0}+m|)^\nu} \int_Q \mathcal{C}(F_m(f))(x-r)^p\,dx.
\end{aligned}$$

Now using the multidimensional Carleson-Hunt inequality (30) we obtain that the last expression above is controlled by

$$\begin{aligned}
& C''_{\nu,n,\omega_0,p} \sum_{r\in\mathbf{Z}^n} \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|\frac{r}{\omega_0}+m|)^\nu} \int_Q |F_m(f)(x-r)|^p\,dx \\
\leq & C'''_{\nu,n,\omega_0,p} \sum_{r\in\mathbf{Z}^n} \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|\frac{r}{\omega_0}+m|)^\nu} \int_Q \sum_{k\in\mathbf{Z}^n} \frac{1}{(1+|\frac{x-r+k}{\omega_0}-m|)^\nu} |f(\tfrac{x-r+k}{\omega_0})|^p\,dx \\
\leq & C''''_{\nu,n,\omega_0,p} \sum_{r\in\mathbf{Z}^n} \sum_{m\in\mathbf{Z}^n} \sum_{k\in\mathbf{Z}^n} \frac{1}{(1+|\frac{r}{\omega_0}+m|)^\nu} \frac{1}{(1+|\frac{r-k}{\omega_0}+m|)^\nu} \int_Q |f(\tfrac{x-r+k}{\omega_0})|^p\,dx\,.
\end{aligned}$$

Next we observe that the following inequality is valid

$$\begin{aligned}
& \frac{1}{(1+|\frac{r}{\omega_0}+m|)^\nu} \frac{1}{(1+|\frac{r-k}{\omega_0}+m|)^\nu} \\
\leq & \frac{1}{(1+|\frac{r}{\omega_0}+m|)^{\frac{\nu}{2}}} \frac{1}{(1+|\frac{r-k}{\omega_0}+m|)^{\frac{\nu}{2}}} \frac{1}{(1+|\frac{k}{\omega_0}|)^{\frac{\nu}{2}}}.
\end{aligned}$$

Using this fact and summing over $m \in \mathbf{Z}^n$ in the last triple sum above we obtain

$$\int_{\mathbf{R}^n} \widetilde{\mathcal{K}}(f)(x)^p\,dx \leq C'''''_{\nu,n,\omega_0,p} \sum_{r\in\mathbf{Z}^n} \sum_{k\in\mathbf{Z}^n} \frac{1}{(1+|\frac{k}{\omega_0}|)^{\frac{\nu}{2}}} \int_Q |f(\tfrac{x-r+k}{\omega_0})|^p\,dx\,.$$

Summing first over $r \in \mathbf{Z}^n$ we obtain

$$\int_{\mathbf{R}^n} \widetilde{\mathcal{K}}(f)(x)^p\,dx \leq C''''''_{\nu,n,\omega_0,p} \sum_{k\in\mathbf{Z}^n} \frac{1}{(1+|\frac{k}{\omega_0}|)^{\frac{\nu}{2}}} \int_{\mathbf{R}^n} |f(\tfrac{x+k}{\omega_0})|^p\,dx = C'''''''_{\nu,n,\omega_0,p}\|f\|_{L^p}^p.$$

This concludes the proof of the boundedness of $\widetilde{\mathcal{K}}$ and hence that of Lemma 5.1 $\square$

6. The L^1 case

For $g \in L^1(\mathbf{R}^n)$, it is easy to check that the *periodization* G of g,

$$G(t) := \sum_{r\in\mathbf{Z}^n} g(t-r) \quad , \quad t \in \mathbf{R}^n \quad .$$

belongs to $L^1(Q)$. For $H \in L^1(Q)$ we define the Fourier coefficient sequence $\overset{\triangle}{H}$ by

$$\overset{\triangle}{H}(l) := \int_Q H(t)\, e^{-2\pi i t\cdot l}\, dt \quad , \quad \text{for all} \quad l \in \mathbf{Z}^n \quad .$$

It is easy to verify that $\overset{\triangle}{G}(l) = \widehat{g}(l)$, for all $l \in \mathbf{Z}^n$.

Let Ω_N be the N-th Fejér kernel defined by

$$\Omega_N(t) := \frac{1}{(N+1)^n} \left[\frac{\sin\left((N+1)\pi t_1\right)}{\sin(\pi t_1)}\right]^2 \cdots \left[\frac{\sin\left((N+1)\pi t_n\right)}{\sin(\pi t_n)}\right]^2$$

for all $t = (t_1, \ldots, t_n) \in \mathbf{R}^n$, with the usual convention for evaluating the function $t \mapsto \sin(ct)/\sin(t)$ at $t = 0$. It is easy to see that for all $H \in L^1(Q)$ we have

$$(H * \Omega_N)(x) = \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \overset{\triangle}{H}(l)\, e^{2\pi i x\cdot l}\,.$$

It is well-known that Ω_N is an approximate identity as $N \to \infty$. This implies that for all $H \in L^1(Q)$ we have $H * \Omega_N \in L^1(Q)$ and $\|H - H * \Omega_N\|_{L^1(Q)} \to 0$ as $N \to \infty$.

Consider now a fixed member $f \in L^1(\mathbf{R}^n)$. As in the previous section, for all $m \in \mathbf{Z}^n$ let

$$F_m(f)(t) = \sum_{k\in\mathbf{Z}^n} f\left(\tfrac{t+k}{\omega_0}\right) \overline{\phi\left(\tfrac{t+k}{\omega_0} - m\right)} \quad , \quad t \in \mathbf{R}^n \quad .$$

$F_m(f)$ is the periodization of the function $f(\cdot/\omega_0)\, \overline{\phi((\cdot/\omega_0) - m)}$, and therefore $F_m(f) \in L^1(Q)$. In a similar manner to the previous section, we can show that for all $x \in \mathbf{R}^n$,

$$\frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l}\rangle\, \sigma_{m,l}(x) = \frac{\sigma(x-m)}{\omega_0^n} \left(F_m(f) * \Omega_N\right)(\omega_0 x).$$

Let's also consider the $L^1(\mathbf{R}^n)$ function

$$\Gamma_m(f)(x) = \frac{\sigma(x-m)}{\omega_0^n}\, F_m(f)(\omega_0 x) \quad . \tag{33}$$

This function coincides with $\sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle\, \sigma_{m,l}$ pointwise when f is in Schwartz class. For general $f \in L^1(\mathbf{R}^n)$, we are about to show that

$$\frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l}\rangle\, \sigma_{m,l}$$

converges in $L^1(\mathbf{R}^n)$ to the function $\Gamma_m(f)(x)$. We will indicate this fact by writing

$$\text{Cesàro-}L^1(\mathbf{R}^n) \sum_{l\in\mathbf{Z}^n} \langle f, \phi_{m,l}\rangle\, \sigma_{m,l} = \Gamma_m(f)$$

throughout the subsequent discussion.

Theorem 6.1. *For all $f \in L^1(\mathbf{R}^n)$ and for all $m \in \mathbf{Z}^n$, we have that*

$$\lim_{N\to\infty} \left\| \Gamma_m(f) - \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l}\rangle\, \sigma_{m,l} \right\|_{L^1(\mathbf{R}^n)} = 0 \quad .$$

Proof. Fix $f \in L^1(\mathbf{R}^n)$ and $m \in \mathbf{Z}^n$. Then

$$\begin{aligned}
E_N^{(m)} &:= \left\| \Gamma_m(f) - \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l}\rangle\, \sigma_{m,l} \right\|_{L^1(\mathbf{R}^n)} \\
&= \left\| \frac{\sigma(\cdot - m)}{\omega_0^n} \left[F_m(f)(\omega_0(\cdot)) - (F_m(f) * \Omega_N)(\omega_0(\cdot)) \right] \right\|_{L^1(\mathbf{R}^n)} \\
&\leq C_{\nu,\omega_0} \int_{\mathbf{R}^n} \frac{|F_m(f)(\omega_0 x) - (F_m(f) * \Omega_N)(\omega_0 x)|}{(1+|x-m|)^\nu}\, dx \\
&= C^{(1)}_{\nu,\omega_0} \int_{\mathbf{R}^n} \frac{|F_m(f)(y) - (F_m(f) * \Omega_N)(y)|}{\left(1+\left|\frac{y}{\omega_0} - m\right|\right)^\nu}\, dy \\
&= C^{(1)}_{\nu,\omega_0} \sum_{r\in\mathbf{Z}^n} \int_Q \frac{|F_m(f)(z) - (F_m(f) * \Omega_N)(z)|}{\left(1+\left|\frac{z}{\omega_0} + \frac{r}{\omega_0} - m\right|\right)^\nu}\, dz \\
&= \int_Q |F_m(f)(z) - (F_m(f) * \Omega_N)(z)|\; Q_m(z)\, dz \quad .
\end{aligned}$$

Here we set

$$0 \leq Q_m(z) := C^{(1)}_{\nu,\omega_0} \sum_{r\in\mathbf{Z}^n} \frac{1}{\left(1+\left|\frac{z}{\omega_0} + \frac{r}{\omega_0} - m\right|\right)^\nu} \leq B \quad ,$$

where B is finite and independent of z and m, for large enough $\nu \in \mathbf{N}$. Therefore,

$$E_N^{(m)} \leq B\, \|F_m(f) - F_m(f) * \Omega_N\|_{L^1(Q)} \to 0 \quad .$$

as $N \to \infty$. □

Fix $f \in L^1(\mathbf{R}^n)$. For all $M \in \mathbf{N}$ we define

$$P_M(f) := \sum_{|m|\leq M} \Gamma_m(f) \quad ,$$

where each $\Gamma_m(f)$ is defined as in (33). We will show that the series $(P_M)_{M\in\mathbf{N}}$ converges absolutely, for almost all $x \in \mathbf{R}$. We will do this, and prove that $(P_M)_M$ converges in $L^1(\mathbf{R}^n)$-norm, by proving that the corresponding series of absolute values is Cauchy in $L^1(\mathbf{R}^n)$.

Theorem 6.2. *For all $f \in L^1(\mathbf{R}^n)$, there exists a function $\Gamma(f) \in L^1(\mathbf{R}^n)$ such that*

$$\lim_{M\to\infty} \|P_M(f) - \Gamma(f)\|_{L^1(\mathbf{R}^n)} = 0 \quad .$$

Proof. Fix $f \in L^1(\mathbf{R}^n)$ and $M \in \mathbf{N}$. Define $F := \{m \in \mathbf{Z}^n : |m| \le M\}$. Then

$$\begin{aligned}
\Lambda_M(f) &:= \Big\| \sum_{m\in\mathbf{Z}^n\setminus F} |\Gamma_m(f)| \Big\|_{L^1(\mathbf{R}^n)} \\
&\le \Big\| \sum_{m\in\mathbf{Z}^n\setminus F} \frac{|\sigma(\cdot - m)|}{\omega_0^n} \sum_{r\in\mathbf{Z}^n} \left| f\left(\cdot - \tfrac{r}{\omega_0}\right)\right| \left|\phi\left(\cdot - \tfrac{r}{\omega_0} - m\right)\right| \Big\|_{L^1(\mathbf{R}^n)} \\
&\le C_\nu \sum_{m\in\mathbf{Z}^n\setminus F} \sum_{r\in\mathbf{Z}^n} \int_{\mathbf{R}^n} \frac{\left|f\left(x - \frac{r}{\omega_0}\right)\right|}{(1+|x-m|)^\nu \left(1 + \left|x - \frac{r}{\omega_0} - m\right|\right)^\nu}\, dx \\
&= C_\nu \sum_{m\in\mathbf{Z}^n\setminus F} \sum_{r\in\mathbf{Z}^n} \int_{\mathbf{R}^n} \frac{|f(y)|}{\left(1 + \left|y + \frac{r}{\omega_0} - m\right|\right)^\nu (1+|y-m|)^\nu}\, dy \\
&\le C_\nu \sum_{m\in\mathbf{Z}^n\setminus F} \sum_{r\in\mathbf{Z}^n} \frac{1}{\left(1 + \left|\frac{r}{\omega_0}\right|\right)^{\nu/2}} \int_{y\in\mathbf{R}^n} \frac{|f(y)|}{(1+|y-m|)^{\nu/2}}\, dy \\
&= \int_{y\in\mathbf{R}^n} |f(y)|\, \delta_M(y)\, dy \quad ;
\end{aligned}$$

where, for all $y \in \mathbf{R}^n$,

$$\delta_M(y) := \Big(\sum_{r\in\mathbf{Z}^n} \frac{1}{\left(1 + \left|\frac{r}{\omega_0}\right|\right)^{\nu/2}} \Big) \sum_{m\in\mathbf{Z}^n\setminus F} \frac{C_\nu}{(1+|y-m|)^{\nu/2}} \le C_\nu' \quad .$$

Now, for large enough $\nu \in \mathbf{N}$, $C_\nu' < \infty$, and so $0 \le |f(\cdot)|\, \delta_M(\cdot) \le C_\nu' |f(\cdot)| \in L^1(\mathbf{R}^n)$, for all $M \in \mathbf{N}$. Moreover, $|f(y)|\, \delta_M(y) \to 0$ pointwise almost everywhere as $M \to \infty$. Thus, via the dominated convergence theorem, we see that $\Lambda_M(f) \to 0$ as $M \to \infty$. ◻

When we let $\sigma := \psi$, it follows from Theorem 1.7 that $\Gamma(f) = A\, f$ pointwise on $\mathbf{R}^n$ for all Schwartz class functions f. The rest of this section is mainly devoted to establishing the following convergence result.

Theorem 6.3. *For every $f \in L^1(\mathbf{R}^n)$, we have that*

$$\frac{1}{A} \sum_{|m|\le M} \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l} \rangle \psi_{m,l} \to f$$

in $L^1(\mathbf{R}^n)$ and almost everywhere as N and M independently tend to ∞.

Proof. We will first show that for every $f \in L^1(\mathbf{R}^n)$, we have that

$$\left\| \sum_{|m|\le M} \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} - \Gamma(f) \right\|_{L^1(\mathbf{R}^n)} \tag{34}$$

tends to zero, as N and M independently tend to ∞.

Recall the operator P_M is defined, for all $f \in L^1(\mathbf{R}^n)$, by

$$P_M f := \sum_{|m|\le M} \Gamma_m(f) \quad .$$

Theorem 6.2 gives that

$$\lim_{M\to\infty} \|P_M(f) - \Gamma(f)\|_{L^1(\mathbf{R}^n)} = 0 \quad . \tag{35}$$

In view of (35), to establish (34) it is sufficient to show that for every $f \in L^1(\mathbf{R}^n)$, we have that

$$\left\| \sum_{|m|\le M} \left(\Gamma_m(f) - \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l}\rangle \sigma_{m,l} \right) \right\|_{L^1(\mathbf{R}^n)} \to 0$$

as $N \to \infty$, uniformly in M. Let us denote the expression above by $D_N^{(M)}(f)$.

Fix $f \in L^1(\mathbf{R}^n)$ and $N, M \in \mathbf{N}$. Then we have

$$\begin{aligned}
D_N^{(M)}(f) &= \left\| \sum_{|m|\le M} \frac{\sigma(\cdot - m)}{\omega_0} \left[F_m(f)(\omega_0(\cdot)) - (F_m(f) * \Omega_N)(\omega_0(\cdot)) \right] \right\|_{L^1(\mathbf{R}^n)} \\
&\le C_{\nu,\omega_0} \int_{\mathbf{R}^n} \sum_{m\in\mathbf{Z}^n} \frac{|F_m(f)(\omega_0 x) - (F_m(f) * \Omega_N)(\omega_0 x)|}{(1+|x-m|)^\nu}\, dx \\
&= C^{(1)}_{\nu,\omega_0} \int_{\mathbf{R}^n} \sum_{m\in\mathbf{Z}^n} \frac{|F_m(f)(y) - (F_m(f) * \Omega_N)(y)|}{\left(1 + \left|\frac{y}{\omega_0} - m\right|\right)^\nu}\, dy \\
&= C^{(1)}_{\nu,\omega_0} \sum_{r\in\mathbf{Z}^n} \int_Q \sum_{m\in\mathbf{Z}^n} \frac{|F_m(f)(z) - (F_m(f) * \Omega_N)(z)|}{\left(1 + \left|\frac{z}{\omega_0} + \frac{r}{\omega_0} - m\right|\right)^\nu}\, dz \\
&= \int_Q \sum_{m\in\mathbf{Z}^n} |F_m(f)(z) - (F_m(f) * \Omega_N)(z)|\; Q_m(z)\, dz \quad .
\end{aligned}$$

Here,

$$0 \le Q_m(z) := C^{(1)}_{\nu,\omega_0} \sum_{r\in\mathbf{Z}^n} \frac{1}{\left(1 + \left|\frac{z}{\omega_0} + \frac{r}{\omega_0} - m\right|\right)^\nu} \le B_\nu \quad ,$$

where B_ν is finite and independent of z and m, for large enough $\nu \in \mathbf{N}$. Therefore,

$$D_N^{(M)}(f) \leq B_\nu \int_Q \sum_{m\in\mathbf{Z}^n} \left| \int_{[-\frac{1}{2},\frac{1}{2}]^n} \left(F_m(f)(z) - F_m(f)(z-u)\right) \Omega_N(u)\, du \right| dz$$
$$\leq B_\nu \int_{[-\frac{1}{2},\frac{1}{2}]^n} \int_Q \left(\sum_{m\in\mathbf{Z}^n} |F_m(f)(z) - F_m(f)(z-u)| \right) dz\, \Omega_N(u)\, du \quad .$$

Fix z and u, and let us consider the integrand

$$W(z,u) := \sum_{m\in\mathbf{Z}^n} |F_m(f)(z) - F_m(f)(z-u)|$$
$$= \sum_{m\in\mathbf{Z}^n} \left| \sum_{r\in\mathbf{Z}^n} f\left(\tfrac{z-r}{\omega_0}\right) \overline{\phi\left(\tfrac{z-r}{\omega_0} - m\right)} - \sum_{r\in\mathbf{Z}^n} f\left(\tfrac{z-u-r}{\omega_0}\right) \overline{\phi\left(\tfrac{z-u-r}{\omega_0} - m\right)} \right|$$
$$\leq \sum_{r\in\mathbf{Z}^n} \left| f\left(\tfrac{z-r}{\omega_0}\right) - f\left(\tfrac{z-u-r}{\omega_0}\right) \right| \sum_{m\in\mathbf{Z}^n} \left| \phi\left(\tfrac{z-r}{\omega_0} - m\right) \right|$$
$$+ \sum_{r\in\mathbf{Z}^n} \left| f\left(\tfrac{z-u-r}{\omega_0}\right) \right| \sum_{m\in\mathbf{Z}^n} \left| \phi\left(\tfrac{z-r}{\omega_0} - m\right) - \phi\left(\tfrac{z-u-r}{\omega_0} - m\right) \right|$$
$$\leq C_\nu \left(\sum_{r\in\mathbf{Z}^n} \left| f\left(\tfrac{z-r}{\omega_0}\right) - f\left(\tfrac{z-u-r}{\omega_0}\right) \right| + \sum_{r\in\mathbf{Z}^n} \left| f\left(\tfrac{z-u-r}{\omega_0}\right) \right| \frac{|u|}{\omega_0} \right) \quad .$$

Here C_ν is a finite constant, for sufficiently large $\nu \in \mathbf{N}$. Note that $\tau_\alpha f := f(\cdot - \alpha)$, for all $\alpha \in \mathbf{R}^n$. It follows that

$$D_N^{(M)}(f) \leq B_\nu \int_{[-\frac{1}{2},\frac{1}{2}]^n} \int_Q W(z,u)\, dz\, \Omega_N(u)\, du$$
$$\leq B_\nu\, C'_\nu \int_{[-\frac{1}{2},\frac{1}{2}]^n} \left(\left\| f - \tau_{u/\omega_0} f \right\|_{L^1(\mathbf{R}^n)} + |u|\ \|f\|_{L^1(\mathbf{R}^n)} \right) \Omega_N(u)\, du \quad .$$

We see that the integrand in parentheses, $J(u)$ say, is a bounded function that is continuous at $u = 0$. A standard argument using Lebesgue's dominated convergence theorem (see, for example, Kicey [15], Appendix 2), now yields that $\int_{[-\frac{1}{2},\frac{1}{2}]^n} J(u)\Omega_N(u)\, du$ converges to 0; and so $D_N^{(M)}(f) \to 0$ uniformly in M as $N \to \infty$, as desired. Thus, (34) is true.

Finally, note that the bounded linear operators $P_{M,N}$ defined by

$$P_{M,N}(f) := \sum_{|m|\leq M} \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \langle f, \phi_{m,l} \rangle \sigma_{m,l}$$

all map $L^1(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$. Since for all $f \in L^1(\mathbf{R}^n)$,

$$\|P_{M,N}(f) - \Gamma(f)\|_{L^1(\mathbf{R}^n)} \to 0 \quad ,$$

as N and M independently tend to ∞, it follows from the Banach-Steinhaus theorem that

$$B := \sup_{M,N\in\mathbf{N}} \|P_{M,N}\|_{L^1,L^1} < \infty \quad .$$

Suppose now that $\sigma := \psi$. Because $\Gamma(f_0) = A\, f_0$ pointwise on $\mathbf{R}^n$ for all Schwartz class functions f_0, it follows from a standard $\frac{\varepsilon}{3}$-style argument, that

$$\|P_{M,N}(f) - A\, f\|_{L^1(\mathbf{R}^n)} \to 0 \quad ,$$

as N and M independently tend to ∞, for every $f \in L^1(\mathbf{R}^n)$.

We have now established convergence of the Cesàro means in $L^1(\mathbf{R}^n)$ and we turn our attention to their a.e. convergence, in the case where $\sigma := \psi$.

We will need to know that the operator

$$F \mapsto \sup_{N>0} |F * \Omega_N|$$

maps $L^1(Q)$ to $L^{1,\infty}(Q)$, where the latter space denotes weak L^1. This result is classical (at least in dimension one) but let us discuss its proof in general. We first observe that for $x \in [-\frac{1}{2}, \frac{1}{2}]^n$ we have

$$\Omega_N(x_1, \dots, x_n) \le C_n \frac{N+1}{1+((N+1)x_1)^2} \cdots \frac{N+1}{1+((N+1)x_n)^2}$$

for all $N > 0$. It is easy to show that the weak type $(1,1)$ boundedness of the operator $F \mapsto \sup_{N>0} |F * \Omega_N|$ on $[-\frac{1}{2}, \frac{1}{2}]^n$ is a simple consequence of the weak type $(1,1)$ boundedness of the operator

$$f \mapsto \sup_{\varepsilon>0} |f * \Phi_\varepsilon|$$

on $\mathbf{R}^n$, where $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(\varepsilon^{-1}x)$ and $\Phi(x) = \dfrac{1}{1+x_1^2} \cdots \dfrac{1}{1+x_n^2}$. For a proof of this last fact we refer to Stein [18], page 69. See also [19] page 82 and [17].

Let us now denote the operator $(\mathcal{H}F) := \sup_{N>0} |F * \Omega_N|$ acting on 1-periodic functions F. As in the proof of the almost everywhere result in Theorem 1.8 it will suffice to show that the operator

$$(\mathcal{G}f)(x) = \sum_{m\in\mathbf{Z}^n} \left|\sigma\left(\tfrac{x}{\omega_0} - m\right)\right| \sup_{N>0} |(F_m(f) * \Omega_N)(x)|,$$

maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$, where as before we set

$$F_m(f)(x) = \sum_{k\in\mathbf{Z}^n} f(\tfrac{x+k}{\omega_0})\overline{\phi(\tfrac{x+k}{\omega_0} - m)}.$$

Once this is established, the argument using the oscillation function O_f in the proof of Theorem 1.8, applied with $p = 1$, gives the required result: i.e. $P_{M,N}f \to \Gamma(f)$ a.e. as $M, N \to \infty$; and so $P_{M,N}f \to A\, f$ a.e. as $M, N \to \infty$ when $\sigma := \psi$.

Pick a number c_n such that

$$\frac{1}{c_n} \sum_{m\in\mathbf{Z}^n} \frac{1}{(1+|m|)^{n+1}} = 1.$$

Then

$$
\begin{aligned}
&|\{x\in\mathbf{R}^n:\ |(\mathcal{G}f)(x)|>\lambda\}| \le \left|\{x\in\mathbf{R}^n:\ \sum_{m\in\mathbf{Z}^n}\frac{C_\nu\,(\mathcal{H}(F_m(f)))(x)}{(1+|\frac{x}{\omega_0}-m|)^\nu}>\lambda\}\right|\\
&\le\sum_{m\in\mathbf{Z}^n}\left|\{x\in\mathbf{R}^n:\ \frac{C_\nu\,(\mathcal{H}(F_m(f)))(x)}{(1+|\frac{x}{\omega_0}-m|)^\nu}>\frac{\lambda}{c_n}\frac{1}{(1+|m|)^{n+1}}\}\right|\\
&\le\sum_{m\in\mathbf{Z}^n}\sum_{r\in\mathbf{Z}^n}\left|\{x\in Q-r:\ (\mathcal{H}(F_m(f)))(x)>\frac{\lambda}{c_{n,\nu}}\frac{(1+|\frac{r}{\omega_0}+m|)^\nu}{(1+|m|)^{n+1}}\}\right|\\
&\le\frac{c'_{n,\nu}}{\lambda}\sum_{m\in\mathbf{Z}^n}\sum_{r\in\mathbf{Z}^n}\frac{(1+|m|)^{n+1}}{(1+|\frac{r}{\omega_0}+m|)^\nu}\int_{Q-r}|F_m(f)(x)|\,dx\\
&\le\frac{c''_{n,\nu}}{\lambda}\sum_{m\in\mathbf{Z}^n}\sum_{r\in\mathbf{Z}^n}\frac{(1+|m|)^{n+1}}{(1+|\frac{r}{\omega_0}+m|)^\nu}\int_{Q}\sum_{k\in\mathbf{Z}^n}\frac{|f(\frac{x-r+k}{\omega_0})|}{(1+|\frac{x-r+k}{\omega_0}-m|)^\nu}\,dx\\
&\le\frac{c'''_{n,\nu}}{\lambda}\sum_{m\in\mathbf{Z}^n}\sum_{r\in\mathbf{Z}^n}\sum_{k\in\mathbf{Z}^n}\frac{(1+|m|)^{n+1}}{(1+|\frac{r}{\omega_0}+m|)^\nu}\frac{1}{(1+|\frac{r-k}{\omega_0}+m|)^\nu}\int_{Q}|f(\tfrac{x-r+k}{\omega_0})|\,dx.
\end{aligned}
$$

Now this expression is the same as the one that appears in the proof of Lemma 5.1 except for the harmless factor $(1+|m|)^{n+1}$. The same proof as in that argument gives that the last expression above is bounded by a constant multiple of $\|f\|_{L^1(\mathbf{R}^n)}$. □

The next corollary easily follows from the results and techniques developed so far. We leave the details of the proof to the reader.

Corollary 6.4. *Suppose that ϕ and σ satisfy (2), (3), (4), and (7). Then for all f in $L^1(\mathbf{R}^n)$ and all $1\le q\le\infty$ we have the following equivalence of norms*

$$
\|f\|_{L^1}\approx\left\|\left(\sum_{m\in\mathbf{Z}^n}\left|\text{Cesàro-}L^1(\mathbf{R}^n)\sum_{l\in\mathbf{Z}^n}\langle f,\phi_{m,l}\rangle\sigma_{m,l}\right|^q\right)^{1/q}\right\|_{L^1},
$$

with constants depending on n, and the functions ϕ, ψ and σ, but not on q.

Let us now see how to use Theorems 6.1 and 6.3, with Corollary 6.4, to view $\Phi:=(\phi_{m,l})_{m,l\in\mathbf{Z}^n}$ as a Banach frame for $\mathcal{X}:=L^1(\mathbf{R}^n)$. This time we define $\mathcal{X}_d$ to be the set of all scalar-valued sequences $\alpha=(\alpha_{m,l})_{m,l\in\mathbf{Z}^n}$ for which, for all $m\in\mathbf{N}$, the sequence $(c_N^{(m)})_{N\in\mathbf{N}}$ of functions given by

$$
c_N^{(m)}(\alpha):=\frac{1}{(N+1)^n}\sum_{k_1=0}^{N}\sum_{l_1=-k_1}^{k_1}\cdots\sum_{k_n=0}^{N}\sum_{l_n=-k_n}^{k_n}\alpha_{m,l}\ \psi_{m,l}
$$

converges in $L^1(\mathbf{R}^n)$ norm to some function $c^{(m)}$; and moreover, for which the sequence $(c^{(m)}(\alpha))_{m\in\mathbf{Z}^n}$ belongs to $L^1(\mathbf{R}^n,\ell^2)$.

Now, for all $\alpha \in \mathcal{X}_d$, define

$$\begin{aligned}\|\alpha\|_{\mathcal{X}_d} &:= \left\| (c^{(m)}(\alpha))_{m\in\mathbf{Z}^n} \right\|_{L^1(\mathbf{R}^n,\ell^2)} \\ &= \left\| \left(\sum_{m\in\mathbf{Z}^n} \left| \text{Cesàro-}L^1(\mathbf{R}^n) \sum_{l\in\mathbf{Z}^n} \alpha_{m,l}\ \psi_{m,l} \right|^2 \right)^{1/2} \right\|_{L^1(\mathbf{R}^n)} .\end{aligned}$$

Then $(\mathcal{X}_d, \|\cdot\|_{\mathcal{X}_d})$ is a semi-normed vector space. Let $J := \mathbf{N}\times\mathbf{N}$. Define $\Theta := (T_{M,N})_{(M,N)\in J}$ by setting, for each $(M,N)\in J$ and each $\alpha \in \mathcal{X}_d$,

$$T_{M,N}(\alpha) := \frac{1}{A} \sum_{\substack{m\in\mathbf{Z}^n \\ |m|\le M}} \frac{1}{(N+1)^n} \sum_{k_1=0}^{N} \sum_{l_1=-k_1}^{k_1} \cdots \sum_{k_n=0}^{N} \sum_{l_n=-k_n}^{k_n} \alpha_{m,l}\ \psi_{m,l} \ .$$

We have that Θ is a denumerable family of linear operators from $\mathcal{X}_d$ into $\mathcal{X}$ and $\Phi := (y_{m,l} := \phi_{m,l})_{m,l\in\mathbf{Z}^n}$ is a sequence in $\mathcal{X}^* = L^\infty(\mathbf{R}^n)$. Our results 6.1, 6.3 and 6.4 now tell us that (Φ,Θ) is a *Banach frame* for $L^1(\mathbf{R}^n)$ with respect to $\mathcal{X}_d$, in the sense of Definition 1.2.

7. Discussion of the continuous case

One motivation for considering the ℓ^q expression appearing in (15) comes from the wavelet (or ϕ-transform) characterizations of $L^p(\mathbf{R}^n)$ of Frazier and Jawerth [9]. Another motivation for considering this ℓ^q expression is the continuous case, where matters are much simpler.

Let us fix two nonzero Schwartz functions ζ and η on $\mathbf{R}^n$ and let us define the Gabor families associated with ζ and η

$$\begin{aligned}\zeta_{y,\xi}(x) &= e^{2\pi i \xi\cdot x}\zeta(x-y)\\ \eta_{y,\xi}(x) &= e^{2\pi i \xi\cdot x}\eta(x-y)\end{aligned}$$

by suitably translating and modulating (i.e. conjugating) the functions ζ and η, in a way similar to the discrete case.

The following identities can be easily checked

Proposition 7.1. *Let ζ and η as before. Then for all f Schwartz functions on $\mathbf{R}^n$ we have*

$$\int_{\mathbf{R}^n} \langle f, \zeta_{y,\xi}\rangle \eta_{y,\xi}(x) d\xi = f(x)\overline{\zeta(x-y)}\,\eta(x-y). \tag{36}$$

As a consequence we obtain

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \langle f, \zeta_{y,\xi}\rangle \eta_{y,\xi}(x) d\xi dy = \left\{ \int_{\mathbf{R}^n} \eta(z)\overline{\zeta(z)}dz \right\} f(x). \tag{37}$$

It follows that

$$\left\{ \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} \langle f, \zeta_{y,\xi}\rangle \eta_{y,\xi}(x) d\xi \right|^s dy \right\}^{1/s} = \left\{ \int_{\mathbf{R}^n} |\zeta(z)\eta(z)|^s dz \right\}^{1/s} \ |f(x)| \tag{38}$$

and that

$$\left\| \left\{ \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} \langle f, \zeta_{y,\xi} \rangle \eta_{y,\xi} d\xi \right|^s dy \right\}^{1/s} \right\|_{L^p} = C(\zeta, \eta) \|f\|_{L^p},$$

for all $0 < p, s \leq \infty$, an identity that provides motivation for the consideration of the expression in (15).

Let us now indicate why Proposition 7.1 is valid. Identity (36) is just a restatement of Fourier inversion for the function $x \mapsto f(x)\overline{\zeta(x-y)}$, while (37) and (38) are trivial consequences of (36).

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Loukas Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: loukas@math.missouri.edu

Chris Lennard, Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

E-mail address: lennard+@pitt.edu

and visiting:, Department of Mathematics and Statistics, Miami University, Oxford, OH 45056, USA

E-mail address: lennarcj@muohio.edu