EXTRAPOLATION OF OPERATORS OF MANY VARIABLES AND APPLICATIONS

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Abstract. Two versions of Rubio de Francia's extrapolation theorem for multivariable operators of functions are obtained. One version assumes an initial estimate with different weights in each space and implies boundedness on all products of Lebesgue spaces. Another version assumes an initial estimate with the same weight but yields boundedness on a product of Lebesgue spaces whose indices lie on a line. Applications are given in the context of multilinear Calderón-Zygmund operators for which vector-valued inequalities are obtained. A multilinear extension of the Marcinkiewicz and Zygmund theorem on ℓ^2-valued extensions of bounded linear operators is also obtained.

1. Introduction

The Rubio de Francia extrapolation theorem [16] provides a powerful tool that enables one to deduce the boundedness of a given operator on all spaces $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided this operator is bounded on $L^{p_0}(w)$ for a single p_0 and all weights $w \in A_{p_0}$. Our goal in this article is to extend this theorem to operators of many functions (which may not necessarily be linear in each variable) .

Some differences appear in this context compared to the case of operators of one variable. Multilinear operators may map into L^r for $r < 1$ (c.f. [11], [15]) and as there is no appropriate definition for A_r when $r < 1$ one needs to consider weights that "match" the spaces in some other natural way. To achieve this, we consider powers of weights that match each of the domain spaces and products of the r^{th} power of these weights in the target space L^r, even when $r < 1$. We also prove a version of multivariable extrapolation in which only one weight appears in all the spaces in question. These formulations provide an appropriate setting to study extrapolation abstractly as they appear in many "natural" examples, such as, for instance, that of multilinear Calderón-Zygmund operators [11].

We note that in the case of many weights one obtains boundedness in the full simplex of exponents possible but in the case of one weight boundedness only follows for indices lying on a line contained in this simplex. We discuss this issue in section 5. We also consider the situation in which the initial estimates are of weak type. See section 6.

We begin by recalling that, for $1 < p < \infty$, an A_p weight w is a locally integrable function on $\mathbb{R}^n$ which satisfies

$$[w]_{A_p} = \sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty\,.$$

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The quantity $[w]_{A_p}$ is called the A_p constant of the weight w. For $p=1$, we say that $w \in A_1$ if it satisfies

$$[w]_{A_1} = \sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \big\| w^{-1} \big\|_{L^\infty(Q)} < \infty .$$

We present the general setup. In the sequel, T will be defined on $\prod_{j=1}^m L^{p_j}(w_j^{p_j})$ for all m tuples of indices $(p_1, \dots, p_m)$ with $1 \le p_j < \infty$ and all tuples of weights $(w_1^{p_1}, \dots, w_m^{p_m})$ in $(A_{p_1}, \dots, A_{p_m})$. It could happen that these weights are all equal. If T happens to be an m-linear operator, then it only needs to be initially defined on a dense subspace of all these spaces, such as $(C_0^\infty(\mathbb{R}^n))^m$. The sort of initial assumption we will impose is that for some fixed indices $1 \le q_1, \dots, q_m < \infty$ and $\frac{1}{m} \le q < \infty$ that satisfy

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q} \tag{1.1}$$

and for all tuples of weights $(w_1^{q_1}, \dots, w_m^{q_m}) \in (A_{q_1}, \dots, A_{q_m})$ and all functions $f_j \in L^{q_j}(w_j^{q_j})$ we have

$$\big\| T(f_1, \dots, f_m) \big\|_{L^q(w_1^q \cdots w_m^q)} \le C_0 \prod_{j=1}^m \big\| f_j \big\|_{L^{q_j}(w_j^{q_j})} \tag{1.2}$$

for some constant C_0. In this article we show that a single estimate (1.2) allows one to "extrapolate" other estimates similar to (1.2); in particular, for all indices $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{m} < p < \infty$ that satisfy

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p} \tag{1.3}$$

and for all weights $(w_1^{p_1}, \dots, w_m^{p_m}) \in (A_{p_1}, \dots, A_{p_m})$ there is a constant C such that

$$\big\| T(f_1, \dots, f_m) \big\|_{L^p(w_1^p \cdots w_m^p)} \le C \prod_{j=1}^m \big\| f_j \big\|_{L^{p_j}(w_j^{p_j})} \tag{1.4}$$

for all $f_j \in L^{p_j}(w_j^{p_j})$. We now precisely state our main theorems.

Throughout this article m will be a fixed integer greater than or equal to 2, although the results obtained equally apply to the known case $m = 1$. In both theorems below T is defined on $L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$ for all m tuples of indices $(p_1, \dots, p_m)$ with $1 \le p_j < \infty$ and all tuples of weights $(w_1^{p_1}, \dots, w_m^{p_m})$ in $(A_{p_1}, \dots, A_{p_m})$.

Theorem 1. *Let $1 \le q_1, \dots, q_m < \infty$ and $\frac{1}{m} \le q < \infty$ be fixed indices that satisfy (1.1). We suppose that for all $B > 1$, there is a constant $C_0(B) > 0$ such that for all tuples of weights $(w_1^{q_1}, \dots, w_m^{q_m}) \in (A_{q_1}, \dots, A_{q_m})$ with $[w_j^{q_j}]_{A_{q_j}} \le B$ and all functions $f_j \in L^{q_j}(w_j^{q_j})$ estimate (1.2) holds with $C_0 = C_0(B)$. Then for all indices $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{m} < p < \infty$ that satisfy (1.3), all $B > 1$, and all weights $(w_1^{p_1}, \dots, w_m^{p_m})$ in $(A_{p_1}, \dots, A_{p_m})$ with $[w_j^{p_j}]_{A_{p_j}} \le B$, there is a constant $C = C(B)$ such that estimate (1.4) is valid.*

We also have a version of this theorem in which there is only one weight.

Theorem 2. *Let $1 \le q_1, \dots, q_m < \infty$ and $\frac{1}{m} \le q < \infty$ be fixed indices that satisfy (1.1) and suppose that for every $B > 1$, there is a constant $C_0(B) > 0$ such that for all weights w in $A_{q_1} \cap \cdots \cap A_{q_m}$ with $[w]_{A_{q_j}} \le B$ and all functions $f_j \in L^{q_j}(w)$ we have the estimate*

$$\big\| T(f_1, \dots, f_m) \ \big\|_{L^q(w)} \le C_0(B) \prod_{j=1}^m \big\| f_j \big\|_{L^{q_j}(w)} . \tag{1.5}$$

Then for all indices $1 < p_1, \dots, p_m < \infty$ *and* $\frac{1}{m} < p < \infty$ *that satisfy* $p_j = q_j/\theta$ *and* $p = q/\theta$ *for some* $0 < \theta < \infty$, *all* $B > 1$, *and all weights* $w \in A_{p_1} \cap \cdots \cap A_{p_m}$ *with* $[w]_{A_{p_j}} \le B$ *for all* j, *there is a constant* $C = C(B)$ *such that the estimate below holds*

$$\big\|T(f_1, \dots, f_m)\,\big\|_{L^p(w)} \le C \prod_{j=1}^{m} \big\|f_j\big\|_{L^{p_j}(w)} \tag{1.6}$$

for all $f_j \in L^{p_j}(w)$.

Before we pass to the proofs of these theorems and our applications we make some comments. The assumption that the initial estimates hold for weights w with $[w]_{A_{q_j}} \le B$ is not new, but as a careful examination shows, is implicitly contained in all proofs of the Rubio de Francia extrapolation theorem, although not often stated as a hypothesis. Moreover, this assumption is naturally satisfied in all the applications. The reason for this is that any dependence on the A_q constant of a weight is usually in a continuous (or bounded) way and any continuous function $\varphi([w]_{A_q})$ is bounded in the set $[1, B]$ for all $B > 1$; (recall $[w]_{A_q} \ge 1$). On the other hand the conclusions of both theorems above are stronger as stated than they would be, if they had been stated with a constant simply dependent on the weights in some unspecified way. The proofs of Theorems 1 and 2 given below are inspired by that of García-Cuerva [6] in the one-variable case.

We also note that in neither theorem there is an assumption restricting T to be a multilinear or multi-sublinear operator (i.e. sublinear in each variable.) The only assumption needed is that T is well-defined on all products of weighted L^q spaces. If T happens to be a multilinear operator, then one may relax the hypotheses of Theorems 1 and 2 by initially assuming that T is well-defined on a dense subspace of all these spaces, such as $(C_0^\infty(\mathbb{R}^n))^m$. This is natural in many applications involving singular integral operators which are not a priori defined on all products of weighted L^q spaces but only on a dense subspace of them.

The article is organized as follows. We first discuss the proof of Theorem 1 which we present in the following three sections. The proof of Theorem 2 is given in section 5. In section 6 we discuss some extrapolation results in which the initial estimates are of weak type. As an application, in section 7 we obtain weighted vector-valued estimates. The key idea here is that one can apply the extrapolation results to a vector-valued extension of a given operator. In section 8, we apply all these results to multilinear Calderón-Zygmund operators, in particular developing a vector-valued theory for them. Finally, we use a different technique to obtain another kind of ℓ^2-valued estimates for general multilinear operators. This is discussed in section 9, in which we prove a multilinear version of the classical Marcinkiewicz and Zygmund theorem [14] on ℓ^2-valued extensions of linear operators.

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2. The case $p_1 > q_1, \dots, p_m > q_m$

In this section we prove Theorem 1 when $p_j > q_j$ for all j. Let us fix $1 \le q_j < p_j < \infty$, for $j = 1, \dots, m$, and $0 < p < \infty$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}\,.$$

Note in particular that $\frac{1}{m} \le q < p < \infty$. We also fix a tuple of weights $(w_1^{p_1}, \dots, w_m^{p_m}) \in (A_{p_1}, \dots, A_{p_m})$ and $f_j \in L^{p_j}(w_j^{p_j})$ for $1 \le j \le m$. We set

$$s = \frac{p}{q} > 1\,, \qquad \theta_j = \frac{s'}{\frac{q_j}{q}\left(\frac{p_j}{q_j}\right)'}\,, \qquad 1 \le j \le m.$$

Then $0 < \theta < 1$ and $\sum_{j=1}^m \theta_j = 1$. We have

$$\begin{aligned} \big\|T(f_1, \dots, f_m)\big\|_{L^p(w_1^p \dots w_m^p)} &= \big\||T(f_1, \dots, f_m)\, w_1 \dots w_m|^q\big\|_{L^s}^{\frac{1}{q}} \\ &= \sup_h \Big(\int_{\mathbb{R}^n} |T(f_1, \dots, f_m)|^q\, w_1^q \dots w_m^q\, h\, dx\Big)^{\frac{1}{q}} \end{aligned} \tag{2.1}$$

where the supremum is taken over all functions $0 \le h \in L^{s'}$ with $\|h\|_{L^{s'}} = 1$. Given a function h_0 for which the supremum above is attained, we define functions

$$h_j = h_0^{\theta_j}\, w_j^{-\frac{p}{s'}}\,.$$

Then we have

$$\|h_j\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})} = 1 \qquad \text{and} \qquad w_1^q \dots w_m^q\, h = (h_1\, w_1^p) \dots (h_m\, w_m^p)\,.$$

At this point we are going to introduce a suitable Rubio de Francia algorithm R_j to be precisely defined (and shown to exist) later.

Lemma 1. *With the notation above, for any nonnegative function h in $L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})$ there exists a function $R_j(h)$ such that:*

(*a*) *$h(x) \le R_j(h)(x)$ for almost every $x \in \mathbb{R}^n$.*

(*b*) *$\|R_j(h)\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})} \le 2\, \|h\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})}$.*

(*c*) *$w_j^{\frac{p}{q} q_j} R_j(h)^{\frac{q_j}{q}} \in A_{q_j}$, in particular $\big[w_j^{\frac{p}{q} q_j} R_j(h)^{\frac{q_j}{q}}\big]_{A_{q_j}} \le C\big([w_j^{p_j}]_{A_{p_j}}\big) < \infty$, where C is a constant that grows as its argument grows.*

We will prove this result at the end of this section. Now write $W_j = w_j^{\frac{p}{q}} R_j(h_j)^{\frac{1}{q}}$. Then,

$$\begin{aligned} \Big(\int_{\mathbb{R}^n} &|T(f_1, \dots, f_m)|^q\, w_1^q \dots w_m^q\, h\, dx\Big)^{\frac{1}{q}} \\ &= \Big(\int_{\mathbb{R}^n} |T(f_1, \dots, f_m)|^q\, h_1\, w_1^p \dots h_m\, w_m^p\, dx\Big)^{\frac{1}{q}} \\ &\le \Big(\int_{\mathbb{R}^n} |T(f_1, \dots, f_m)|^q\, R_1(h_1)\, w_1^p \dots R_m(h_m)\, w_m^p\, dx\Big)^{\frac{1}{q}} \\ &= \big\|T(f_1, \dots, f_m)\big\|_{L^q(W_1^q \dots W_m^q)} \\ &\le C \prod_{j=1}^m \|f_j\|_{L^{q_j}(W_j^{q_j})}, \end{aligned} \tag{2.2}$$

when the first estimate follows from (a) in Lemma 1 and in the last one we used (1.2) in view of condition (c) of Lemma 1. We note here that if $[w_j^{p_j}]_{A_{p_j}} \le B$, then $[W_j^{q_j}]_{A_{q_j}} \le C(B)$

and the hypothesis (1.2) applies. Now we analyze each norm above. First we use Hölder's inequality with $\frac{p_j}{q_j} > 1$ to get

$$\begin{aligned} \|f_j\|_{L^{q_j}(W_j^{q_j})} &= \Big(\int_{\mathbb{R}^n} |f_j|^{q_j}\, W_j^{q_j}\, w_j^{q_j}\, w_j^{-q_j}\, dx\Big)^{\frac{1}{q_j}} \\ &\le \Big(\int_{\mathbb{R}^n} |f_j|^{p_j} w_j^{p_j}\, dx\Big)^{\frac{1}{p_j}} \Big(\int_{\mathbb{R}^n} W_j^{q_j\left(\frac{p_j}{q_j}\right)'}\, w_j^{-q_j\left(\frac{p_j}{q_j}\right)'}\, dx\Big)^{\frac{1}{q_j\,(p_j/q_j)'}} \\ &= \|f_j\|_{L^{p_j}(w_j^{p_j})}\, I_j, \end{aligned} \tag{2.3}$$

where I_j stands for the second factor. But this term satisfies:

$$I_j^{q_j\left(\frac{p_j}{q_j}\right)'} = \int_{\mathbb{R}^n} W_j^{q_j\left(\frac{p_j}{q_j}\right)'}\, w_j^{-q_j\left(\frac{p_j}{q_j}\right)'}\, dx = \|R_j(h_j)\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})}^{\frac{\theta_j}{s'}} \le 2^{\frac{\theta_j}{s'}}\, \|h_j\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})}^{\frac{\theta_j}{s'}} = 2^{\frac{\theta_j}{s'}},$$

where we used (b) of Lemma 1. Combining this estimate with (2.1), (2.2), and (2.3) we obtain the desired inequality

$$\|T(f_1,\dots,f_m)\|_{L^p(w_1^p\dots w_m^p)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j^{p_j})}.$$

Proof of Lemma 1. Set

$$t_j = \frac{p_j - q_j}{p_j - 1} \le 1 \qquad \text{and} \qquad \alpha_j = p_j + \Big(\frac{p}{q}\, q_j - p_j\Big)\frac{1}{t_j}.$$

For every $0 \le h \in L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})$ we consider the operator

$$S_j(h) = \Big(M\big(h^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j^{\alpha_j}\big)\, w_j^{-\alpha_j}\Big)^{\frac{q}{q_j}t_j},$$

where M is the Hardy-Littlewood maximal operator. We observe that S_j is bounded on $L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})$. Indeed,

$$\|S_j(h)\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})} = \Big\|M\big(h^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j^{\alpha_j}\big)\Big\|_{L^{p_j'}(w_j^{-p_j'})}^{\frac{\theta_j}{s'}p_j} \le C\, \Big\|h^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j^{\alpha_j}\Big\|_{L^{p_j'}(w_j^{-p_j'})}^{\frac{\theta_j}{s'}p_j} = C\, \|h\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})},$$

where we have used the fact that $w_j^{p_j} \in A_{p_j}$ or, equivalently that $w_j^{-p_j'} \in A_{p_j'}$, which yields the boundedness of M in $L^{p_j'}(w_j^{-p_j'})$. We denote the norm of S_j as a bounded operator in the previous space as $\|S_j\|_*$. Now we define the Rubio de Francia algorithm as follows

$$R_j(h)(x) = \sum_{k=0}^{\infty} \frac{S_j^k(h)(x)}{2^k\, \|S_j\|_*^k},$$

where S_j^k is the operator S_j iterated k times for $k \ge 1$ and for $k = 0$ is just the identity operator. The fact that S_j^0 is the identity gives part (a). For part (b), note that $\frac{s'}{\theta_j} > s' > 1$, thus

$$\|R_j(h)\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})} \le \sum_{k=0}^{\infty} \frac{\|S_j^k(h)\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})}}{2^k\, \|S_j\|_*^k} \le \sum_{k=0}^{\infty} 2^{-k}\, \|h\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})} = 2\, \|h\|_{L^{\frac{s'}{\theta_j}}(w_j^{\frac{p}{\theta_j}})}.$$

Finally let us check that (c) holds. We observe that S_j is sublinear because $\frac{q_j}{q}\frac{1}{t_j} > 1$ and then,

$$S_j(R_j(h)) = S_j\bigg(\sum_{k=0}^{\infty} \frac{S_j^k(h)}{2^k\, \|S_j\|_*^k}\bigg) \le \sum_{k=0}^{\infty} \frac{S_j^{k+1}(h)}{2^k\, \|S_j\|_*^k} \le 2\, \|S_j\|_* \sum_{k=0}^{\infty} \frac{S_j^k(h)}{2^k\, \|S_j\|_*^k} = 2\, \|S_j\|_*\, R_j(h).$$

This estimate yields

$$\begin{aligned}\frac{1}{|Q|}\int_Q R_j(h)^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j^{\alpha_j}\, dx &\le M\big(R_j(h)^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j^{\alpha_j}\big)(x)\\ &= S_j(R_j(h))(x)^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j(x)^{\alpha_j}\\ &\le C\, R_j(h)(x)^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j(x)^{\alpha_j}\end{aligned}$$

for almost every $x \in Q$. If $t_j = 1$, or equivalently $q_j = 1$, this inequality turns out to be

$$\frac{1}{|Q|}\int_Q R_j(h)^{\frac{q_j}{q}}\, w_j^{\frac{p}{q}\,p_j}\, dx \le C\, R_j(h)(x)^{\frac{q_j}{q}}\, w_j(x)^{\frac{p}{q}\,p_j}$$

for almost every $x \in Q$. This proves that $R_j(h)^{\frac{q_j}{q}}\, w_j^{\frac{p}{q}\,q_j} \in A_1 = A_{q_j}$ with constant smaller than C. In the other case, $t_j < 1$, we use Hölder's inequality with exponents $\frac{1}{t_j}$ and $\frac{1}{1-t_j}$ to get

$$\begin{aligned}\frac{1}{|Q|}\int_Q R_j(h)^{\frac{q_j}{q}}\, w_j^{\frac{p}{q}\,q_j}\, dx &= \frac{1}{|Q|}\int_Q R_j(h)^{\frac{q_j}{q}}\, w_j^{\frac{p}{q}\,q_j - p_j\,(1-t_j)}\, w_j^{p_j\,(1-t_j)} dx\\ &\le \bigg(\frac{1}{|Q|}\int_Q R_j(h)^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j^{\alpha_j}\, dx\bigg)^{t_j} \bigg(\frac{1}{|Q|}\int_Q w_j^{p_j}\, dx\bigg)^{1-t_j}\\ &\le C\, \operatorname*{ess\,inf}_Q \Big(R_j(h)^{\frac{q_j}{q}\frac{1}{t_j}}\, w_j^{\alpha_j}\Big)^{t_j} \bigg(\frac{1}{|Q|}\int_Q w_j^{p_j}\, dx\bigg)^{1-t_j}.\end{aligned}$$

This last inequality allows us to estimate the A_{q_j} constant of $R_j(h)^{\frac{q_j}{q}}\, w_j^{\frac{p}{q}\,q_j}$. Indeed, for any cube Q we have

$$\begin{aligned}&\bigg(\frac{1}{|Q|}\int_Q R_j(h)^{\frac{q_j}{q}}\, w_j^{\frac{p}{q}\,q_j}\, dx\bigg) \bigg(\frac{1}{|Q|}\int_Q \Big(R_j(h)^{\frac{q_j}{q}}\, w_j^{\frac{p}{q}\,q_j}\Big)^{-\frac{q_j'}{q_j}}\, dx\bigg)^{q_j-1}\\ &\le C\bigg(\frac{1}{|Q|}\int_Q w_j^{p_j}\, dx\bigg)^{1-t_j} \bigg(\frac{1}{|Q|}\int_Q R_j(h)^{-\frac{q_j'}{q}}\, w_j^{-\frac{p}{q}\,q_j'}\, \operatorname*{ess\,inf}_Q \Big(R_j(h)^{\frac{q_j}{q}\frac{1}{t_j}} w_j^{\alpha_j}\Big)^{\frac{t_j}{q_j-1}} dx\bigg)^{q_j-1}\\ &= C\bigg(\frac{1}{|Q|}\int_Q w_j^{p_j}\, dx\bigg)^{1-t_j} \bigg(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\, dx\bigg)^{(p_j-1)\,(1-t_j)}\\ &\le C\, [w_j^{p_j}]_{A_{p_j}}^{1-t_j},\end{aligned}$$

which proves (c). The proof of Lemma 1 is now complete. □

3. THE CASE $p_1 < q_1, \ldots, p_m < q_m$

First of all, we observe that this case does not appear if $q_j = 1$ for some j. Otherwise, let us fix $1 < p_j < q_j$, for $j = 1, \ldots, m$, and $0 < p < \infty$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} .$$

Note that in particular $\frac{1}{m} < p < q < \infty$. We also fix a tuple of weights $(w_1^{p_1}, \ldots, w_m^{p_m})$ in $(A_{p_1}, \ldots, A_{p_m})$ and $f_j \in L^{p_j}(w_j^{p_j})$ for $1 \le j \le m$. We define

$$h_j = |f_j|^{\frac{q}{q_j}(q_j - p_j)} w_j^{\frac{q}{q_j}(\frac{p}{q}q_j - p_j)}, \qquad \lambda_j = p_j - \tfrac{p_j}{q_j - p_j}(\tfrac{p}{q}q_j - p_j), \tag{3.1}$$

and we observe that

$$\|h_j\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})} = \|f_j\|_{L^{p_j}(w_j^{p_j})}^{\frac{q_j-p_j}{q_j}q} .$$

We will need a lemma analogous to Lemma 1 used in the previous section.

Lemma 2. *With the notation above, for any nonnegative function h in $L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})$ there exists a function $G_j(h)$ such that*

- (a) *$h(x) \le G_j(h)(x)$ for almost every $x \in \mathbb{R}^n$.*
- (b) $\|G_j(h)\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})} \le 2^{q_j-1} \|h\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})}$.
- (c) *$w_j^{\frac{p}{q}q_j} G_j(h)^{-\frac{q_j}{q}} \in A_{q_j}$, in particular $[w_j^{\frac{p}{q}q_j} G_j(h)^{-\frac{q_j}{q}}]_{A_{q_j}} \le C([w_j^{p_j}]_{A_{p_j}})$, where C is a constant that grows as its argument grows.*

Using Lemma 2 we can now write

$$\begin{aligned}
&\|T(f_1, \ldots, f_m)\|_{L^p(w_1^p \ldots w_m^p)} \\
&\quad = \||T(f_1, \ldots, f_m)|^q\|_{L^{\frac{p}{q}}(w_1^p \ldots w_m^p)}^{\frac{1}{q}} \\
&\quad = \||T(f_1, \ldots, f_m)|^q G_1(h_1) \ldots G_m(h_m) G_1(h_1)^{-1} \ldots G_m(h_m)^{-1}\|_{L^{\frac{p}{q}}(w_1^p \ldots w_m^p)}^{\frac{1}{q}} \\
&\quad \le \Big\||T(f_1, \ldots, f_m)|^q \prod_{j=1}^m G_j(h_j)^{-1}\Big\|_{L^1(w_1^p \ldots w_m^p)}^{\frac{1}{q}} \Big\|\prod_{j=1}^m G_j(h_j)\Big\|_{L^{\frac{p}{q-p}}(w_1^p \ldots w_m^p)}^{\frac{1}{q}}
\end{aligned} \tag{3.2}$$

where in the last step we used Hölder's inequality. We set

$$W_j = w_j^{\frac{p}{q}} G_j(h_j)^{-\frac{1}{q}}$$

for $j = 1, \ldots, m$. Then it follows that

$$\begin{aligned}
\Big\||T(f_1, \ldots, f_m)|^q \prod_{j=1}^m G_j(h_j)^{-1}\Big\|_{L^1(w_1^p \ldots w_m^p)}^{\frac{1}{q}} &= \||T(f_1, \ldots, f_m)|^q W_1^q \ldots W_m^q\|_{L^1}^{\frac{1}{q}} \\
&= \|T(f_1, \ldots, f_m)\|_{L^q(W_1^q \ldots W_m^q)}.
\end{aligned}$$

Conclusion (c) of Lemma 2 gives that $[W_j^{q_j}]_{A_{q_j}} \le C\big([w_j^{p_j}]_{A_{p_j}}\big) \le C(B)$ for all $1 \le j \le m$. By (1.2) we conclude that

$$\Big\| |T(f_1,\dots,f_m)|^q \prod_{j=1}^m G_j(h_j)^{-1} \Big\|_{L^1(w_1^p\dots w_m^p)}^{\frac{1}{q}} \le C \prod_{j=1}^m \|f_j\|_{L^{q_j}(W_j^{q_j})}. \tag{3.3}$$

Using conclusion (a) of Lemma 2 we obtain

$$\|f_j\|_{L^{q_j}(W_j^{q_j})} = \Big(\int |f_j|^{q_j} w_j^{\frac{pq_j}{q}} G_j(h_j)^{-\frac{q_j}{q}}\,dx\Big)^{\frac{1}{q_j}} \le \Big(\int |f_j|^{p_j} w_j^{p_j}\,dx\Big)^{\frac{1}{q_j}} = \|f_j\|_{L^{p_j}(w_j^{p_j})}^{\frac{p_j}{q_j}}.$$

Combining this with (3.3) we get

$$\big\||T(f_1,\dots,f_m)|^q G_1(h_1)^{-1}\dots G_m(h_m)^{-1}\big\|_{L^1(w_1^p\dots w_m^p)}^{\frac{1}{q}} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j^{p_j})}^{\frac{p_j}{q_j}}. \tag{3.4}$$

Since

$$\sum_{j=1}^m \Big(\frac{p_j}{q_j-p_j}\frac{q_j}{q}\Big)^{-1} = \Big(\frac{p}{q-p}\Big)^{-1},$$

Hölder's inequality yields

$$\begin{aligned} \Big\| \prod_{j=1}^m G_j(h_j)\Big\|_{L^{\frac{p}{q-p}}(w_1^p\dots w_m^p)}^{\frac{1}{q}} &= \Big\| \prod_{j=1}^m G_j(h_j) w_j^{q-p}\Big\|_{L^{\frac{p}{q-p}}}^{\frac{1}{q}} \\ &\le \prod_{j=1}^m \big\|G_j(h_j) w_j^{q-p}\big\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}}^{\frac{1}{q}} \\ &= \prod_{j=1}^m \big\|G_j(h_j)\big\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})}^{\frac{1}{q}} \end{aligned} \tag{3.5}$$

where we used (3.1) in the last step above. Conclusion (b) of Lemma 2 gives that the last expression above is bounded by

$$C\prod_{j=1}^m \|h_j\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})}^{\frac{1}{q}} = C\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j^{p_j})}^{\frac{1}{q}\frac{q_j-p_j}{q_j}q} = C\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j^{p_j})}^{1-\frac{p_j}{q_j}} \tag{3.6}$$

Combining (3.5) and (3.6) with (3.4) and (3.2) we obtain the required conclusion.
It remains to prove Lemma 2.

Proof of Lemma 2. We set

$$\tau_j = \frac{q_j-p_j}{q_j-1}, \qquad \beta_j = \Big(p_j' - \frac{p}{q}\,q_j'\Big)\frac{1}{\tau_j} - p_j'.$$

For every $0 \le h \in L^{\frac{p_j}{\tau_j}\frac{q_j}{q}}(w_j^{\lambda_j})$ we define the operator

$$S_j(h) = \Big(M\big(h^{\frac{q_j}{q}\frac{1}{\tau_j}} w_j^{\beta_j}\big)\, w_j^{-\beta_j}\Big)^{\frac{q}{q_j}\tau_j}.$$

Let us see that S_j is not only well defined on $L^{\frac{p_j}{\tau_j}\frac{q_j}{q}}(w_j^{\lambda_j})$ but also bounded on this space:

$$\|S_j(h)\|_{L^{\frac{p_j}{\tau_j}\frac{q_j}{q}}(w_j^{\lambda_j})} = \Big\|M\big(h^{\frac{q_j}{q}\frac{1}{\tau_j}} w^{\beta_j}\big)\Big\|_{L^{p_j}(w_j^{p_j})}^{\tau_j\frac{q}{q_j}} \le C\,\Big\|h^{\frac{q_j}{q}\frac{1}{\tau_j}} w^{\beta_j}\Big\|_{L^{p_j}(w_j^{p_j})}^{\tau_j\frac{q}{q_j}} = C\,\|h\|_{L^{\frac{p_j}{\tau_j}\frac{q_j}{q}}(w_j^{\lambda_j})},$$

where we have used that $w_j^{p_j} \in A_{p_j}$ which gives that M is bounded on $L^{p_j}(w_j^{p_j})$. We write $\|S_j\|_*$ for the norm of S_j as a bounded operator on that space. For every $0 \le h \in L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})$, we define the Rubio de Francia algorithm via

$$G_j(h)(x) = \left(\sum_{k=0}^{\infty} \frac{S_j^k\left(h^{\frac{1}{q_j-1}}\right)(x)}{2^k \|S_j\|_*^k} \right)^{q_j-1},$$

where S_j^k stands for the operator S_j iterated k times for $k \ge 1$ and for $k = 0$ is just the identity operator. Let us see that G_j is well defined and bounded on $L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})$. Indeed,

$$\begin{aligned}
\|G_j(h)\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})} &= \left\| \sum_{k=0}^{\infty} \frac{S_j^k\left(h^{\frac{1}{q_j-1}}\right)}{2^k \|S_j\|_*^k} \right\|_{L^{\frac{p_j}{\tau_j}\frac{q_j}{q}}(w_j^{\lambda_j})}^{q_j-1} \\
&\le \left(\sum_{k=0}^{\infty} \frac{\left\| S_j^k\left(h^{\frac{1}{q_j-1}}\right)\right\|_{L^{\frac{p_j}{\tau_j}\frac{q_j}{q}}(w_j^{\lambda_j})}}{2^k \|S_j\|_*^k} \right)^{q_j-1} \\
&\le \Big(\sum_{k=0}^{\infty} 2^{-k} \Big)^{q_j-1} \left\| h^{\frac{1}{q_j-1}} \right\|_{L^{\frac{p_j}{\tau_j}\frac{q_j}{q}}(w_j^{\lambda_j})}^{q_j-1} \\
&= 2^{q_j-1} \|h\|_{L^{\frac{p_j}{q_j-p_j}\frac{q_j}{q}}(w_j^{\lambda_j})},
\end{aligned}$$

where we were allowed to use the triangle inequality because $\frac{p_j}{\tau_j}\frac{q_j}{q} > p_j > 1$. This proves conclusion (b) of Lemma 2. On the other hand, in order to get (a) it is enough to realize that everything is positive and the sum is bigger than the term with $k = 0$. Finally we have to show conclusion (c) of Lemma 2, that is, $w_j^{\frac{p}{q}q_j} G_j(h)^{-\frac{q_j}{q}} \in A_{q_j}$. We are going to find a constant $C > 0$ such that for every cube Q we have,

$$(3.7) \quad \left(\frac{1}{|Q|} \int_Q w_j^{\frac{p}{q}q_j} G_j(h)^{-\frac{q_j}{q}}\, dx \right) \left(\frac{1}{|Q|} \int_Q \left(w_j^{\frac{p}{q}q_j} G_j(h)^{-\frac{q_j}{q}} \right)^{-\frac{q_j'}{q_j}} dx \right)^{q_j-1} \le C[w_j^{p_j}]_{A_{p_j}}.$$

First of all, we observe that S_j is a sublinear operator since $\frac{q_j}{q}\frac{1}{\tau_j} > 1$. Since $G_j(h)$ is defined as a sum of iterations of S_j we obtain:

$$\begin{aligned}
S_j\left(G_j(h)^{\frac{1}{q_j-1}}\right) &= S_j \left(\sum_{k=0}^{\infty} \frac{S_j^k\left(h^{\frac{1}{q_j-1}}\right)}{2^k \|S_j\|_*^k} \right) \\
&\le \sum_{k=0}^{\infty} \frac{S_j^{k+1}\left(h^{\frac{1}{q_j-1}}\right)}{2^k \|S_j\|_*^k} \\
&\le 2\|S_j\|_* \sum_{k=0}^{\infty} \frac{S_j^k\left(h^{\frac{1}{q_j-1}}\right)}{2^k \|S_j\|_*^k}
\end{aligned}$$

$$= 2\,\|S_j\|_*\, G_j(h)^{\frac{1}{q_j-1}}.$$

This fact, which means that $G_j(h)$ is an A_1 weight for the operator S_j (see [7]), can be rewritten in the following way: for almost every $x \in Q$,

$$\begin{aligned}
\frac{1}{|Q|}\int_Q G_j(h)^{\frac{q_j'}{q}\frac{1}{\tau_j}}\, w_j^{\beta_j}\, dx &\le M\Big(G_j(h)^{\frac{1}{q_j-1}\frac{q_j}{q}\frac{1}{\tau_j}}\, w_j^{\beta_j}\Big)(x)\\
&= S_j\Big(G_j(h)^{\frac{1}{q_j-1}}\Big)(x)^{\frac{q_j}{q}\frac{1}{\tau_j}}\, w_j(x)^{\beta_j}\\
&\le C\, G_j(h)(x)^{\frac{q_j'}{q}\frac{1}{\tau_j}}\, w_j(x)^{\beta_j}.
\end{aligned}$$

We can use this estimate to bound the second factor on the left hand side in (3.7):

$$\begin{aligned}
\frac{1}{|Q|}\int_Q G_j(h)^{\frac{q_j'}{q}}\, w_j^{-\frac{p}{q}q_j'}\, dx &= \frac{1}{|Q|}\int_Q G_j(h)^{\frac{q_j'}{q}}\, w_j^{-\frac{p}{q}q_j'+p_j'(1-\tau_j)}\, w_j^{-p_j'(1-\tau_j)}\, dx\\
&\le \left(\frac{1}{|Q|}\int_Q G_j(h)^{\frac{q_j'}{q}\frac{1}{\tau_j}}\, w_j^{(-\frac{p}{q}q_j'+p_j'(1-\tau_j))\frac{1}{\tau_j}}\, dx\right)^{\tau_j}\left(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\, dx\right)^{1-\tau_j}\\
&= \left(\frac{1}{|Q|}\int_Q G_j(h)^{\frac{q_j'}{q}\frac{1}{\tau_j}}\, w_j^{\beta_j}\, dx\right)^{\tau_j}\left(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\, dx\right)^{1-\tau_j}\\
&\le C \operatorname*{ess\,inf}_{x\in Q}\Big(G_j(h)(x)^{\frac{q_j'}{q}\frac{1}{\tau_j}}\, w_j(x)^{\beta_j}\Big)^{\tau_j}\left(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\, dx\right)^{1-\tau_j},
\end{aligned}$$

where we used Hölder's inequality with exponents $\frac{1}{\tau_j}$ and $\frac{1}{1-\tau_j}$. Thus,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q w_j^{\frac{p}{q}q_j}\, G_j(h)^{-\frac{q_j}{q}}\, dx\right)\left(\frac{1}{|Q|}\int_Q G_j(h)^{\frac{q_j'}{q}}\, w_j^{-\frac{p}{q}q_j'}\, dx\right)^{q_j-1}\\
&\le C\left(\frac{1}{|Q|}\int_Q w_j^{\frac{p}{q}q_j}\, G_j(h)^{-\frac{q_j}{q}} \operatorname*{ess\,inf}_{Q}\Big(G_j(h)^{\frac{q_j'}{q}\frac{1}{\tau_j}}\, w_j^{\beta_j}\Big)^{\tau_j(q_j-1)} dx\right)\left(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\, dx\right)^{(1-\tau_j)(q_j-1)}\\
&\le C\left(\frac{1}{|Q|}\int_Q w_j^{p_j}\, dx\right)\left(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\, dx\right)^{p_j-1} \le C\,[w_j^{p_j}]_{A_{p_j}}.
\end{aligned}$$

This proves (3.7) and the proof of Lemma 2 is completed. □

4. The general case

To obtain the general case we are going to use a bootstrapping argument. First we define the set of admissible exponents:

$$\mathcal{U} = \Big\{\Big(\frac{1}{p_1},\dots,\frac{1}{p_m},\frac{1}{p}\Big) : 1 < p_1,\dots,p_m < \infty,\ \frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}\Big\}.$$

We also set $\overline{\mathcal{U}}$ defined in the same way but with $1 \le p_1,\dots,p_m < \infty$. In Figure 1 below we have represented the two-variable case, that is, the case where $m = 2$. The set $\mathcal{U}$ corresponds to the points inside the rhombus. In the set $\overline{\mathcal{U}}$ we additionally include the points that are in the two upper edges. For a fixed $\left(\frac{1}{q_1}, \ldots, \frac{1}{q_m}, \frac{1}{q}\right) \in \overline{\mathcal{U}}$ we also define:

$$\mathcal{U}^-(q_1, \ldots, q_m) = \left\{\left(\frac{1}{p_1}, \ldots, \frac{1}{p_m}, \frac{1}{p}\right) \in \mathcal{U} : p_1 > q_1, \ldots, p_m > q_m\right\}.$$

and

$$\mathcal{U}^+(q_1, \ldots, q_m) = \left\{\left(\frac{1}{p_1}, \ldots, \frac{1}{p_m}, \frac{1}{p}\right) \in \mathcal{U} : p_1 < q_1, \ldots, p_m < q_m\right\}.$$

As we can see in Figure 2, $\mathcal{U}^+$ and $\mathcal{U}^-$ represents respectively the shaded areas above and below the given point. We observe that $\mathcal{U}^+ = \emptyset$ if some $q_j = 1$ and in this case this triple of exponents lies in one of the upper edges.

A $(m+1)$-tuple $\left(\frac{1}{p_1}, \ldots, \frac{1}{p_m}, \frac{1}{p}\right) \in \overline{\mathcal{U}}$ is said to be in $\mathcal{W}(T)$ if and only if T satisfies:

$$\left\|T(f_1, \ldots, f_m)\right\|_{L^p(w_1^p \ldots w_m^p)} \leq C \prod_{j=1}^{m} \left\|f_j\right\|_{L^{p_j}(w_j^{p_j})}, \tag{4.8}$$

for all weights $(w_1^{p_1}, \ldots, w_m^{p_m}) \in (A_{p_1}, \ldots, A_{p_m})$. Remember that our only hypothesis is that $\left(\frac{1}{q_1}, \ldots, \frac{1}{q_m}, \frac{1}{q}\right) \in \mathcal{W}(T)$. In sections 2 and 3 we have respectively shown that

$$\mathcal{U}^-(q_1, \ldots, q_m) \subset \mathcal{W}(T) \qquad \text{and} \qquad \mathcal{U}^+(q_1, \ldots, q_m) \subset \mathcal{W}(T).$$

Our goal now is to obtain that $\mathcal{W}(T) = \mathcal{U}$. In Figures 1 and 2, we have started with some point and we have proved that the shaded regions $\mathcal{U}^+$ and $\mathcal{U}^-$ are contained in $\mathcal{W}(T)$ or, what it is the same, that there are weighted norm estimates for exponents in these two sets. The aim now is to show that we can shade the whole rhombus.

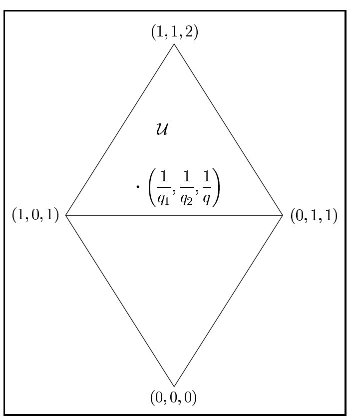

Figure 1

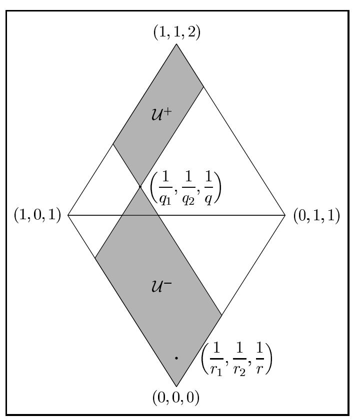

Figure 2

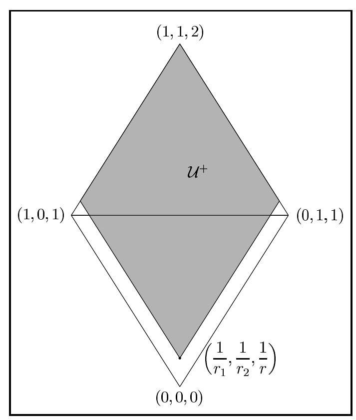

Figure 3

Let us take $\left(\frac{1}{p_1}, \ldots, \frac{1}{p_m}, \frac{1}{p}\right) \in \mathcal{U}$. Then, there exists N big enough such that

$$1 < \max\{p_1, \ldots, p_m, q_1 \ldots, q_m\} < N < \infty.$$

We take $r_1 = r_2 = \cdots = r_m = N$ and $r = \frac{N}{m}$. It is clear that the corresponding tuple of exponents is in $\mathcal{U}$. Furthermore, the way we have chosen N guarantees the following

$$\left(\frac{1}{r_1}, \ldots, \frac{1}{r_m}, \frac{1}{r}\right) \in \mathcal{U}^-(q_1, \ldots, q_m) \qquad \text{and} \qquad \left(\frac{1}{p_1}, \ldots, \frac{1}{p_m}, \frac{1}{p}\right) \in \mathcal{U}^+(r_1, \ldots, r_m).$$

The fact that $\mathcal{U}^-(q_1,\dots,q_m)\subset\mathcal{W}(T)$ allows us to use Theorem 1 with starting point $\left(\frac{1}{r_1},\dots,\frac{1}{r_m},\frac{1}{r}\right)$, which lies in $\mathcal{W}(T)$. In particular, $\mathcal{U}^+(r_1,\dots,r_m)\subset\mathcal{W}(T)$ which yields $\left(\frac{1}{p_1},\dots,\frac{1}{p_m},\frac{1}{p}\right)\in\mathcal{W}(T)$. This proves that $\mathcal{U}=\mathcal{W}(T)$. The idea of this last part of the proof is given in Figure 3. Namely, we take a point close to $(0,0,0)$ that lies in the shaded area $\mathcal{U}^-$ (see Figure 2). Then we apply the result proved in section 3 starting with this point and consequently the shaded region $\mathcal{U}^+$ is contained in $\mathcal{W}(T)$ (see Figure 3). Finally, if we let this point approach to $(0,0,0)$ we shade the whole rhombus.

We now discuss how this result is affected in the case when some $q_j=1$. First, the case where all $q_j=1$ corresponds with the upper vertex and then everything follows just from section 2. Otherwise, the tuple of exponents lies in one of the upper edges (or faces when $m\geq 3$). Then $\mathcal{U}^+=\emptyset$ but we still get $\mathcal{U}^-$ that allows us to use the previous argument to shade the whole rhombus. □

5. The proof of Theorem 2

Without loss of generality we may assume that

$$1\leq q_1\leq q_2\leq\cdots\leq q_m<\infty\,.$$

Then we also have $1<p_1\leq p_2\leq\cdots\leq p_m<\infty$. Fix a weight $w\in A_{p_1}\cap\cdots\cap A_{p_m}=A_{p_1}$. We will consider the following two cases: $q<p$ and $q>p$.

Case 1: $q<p$. In this case we set $s=\frac{1}{\theta}=\frac{p}{q}=\frac{p_j}{q_j}>1$. Fix $f_j\in L^{p_j}(w)$ and let h be a function in $L^{s'}(w)$ with norm at most 1 such that

$$\begin{aligned}\|T(f_1,\dots,f_m)\|_{L^p(w)}&=\Big\||T(f_1,\dots,f_m)|^q\Big\|_{L^s(w)}^{\frac{1}{q}}\\&=\left(\int_{\mathbb{R}^n}|T(f_1,\dots,f_m)|^q\,h\,w\,dx\right)^{\frac{1}{q}}.\end{aligned}\tag{5.9}$$

We have the following lemma.

Lemma 3. *For any nonnegative function h in $L^{s'}(w)$, there exists a function $R(h)$ such that*

(1) $h(x)\leq R(h)(x)$ *for almost every* $x\in\mathbb{R}^n$.

(2) $\|R(h)\|_{L^{s'}(w)}\leq 2\|h\|_{L^{s'}(w)}$

(3) $w\,R(h)\in A_{q_1}$, *in particular* $[w\,R(h)]_{A_{q_1}}\leq C\big([w]_{A_{p_1}}\big)<\infty$, *where C is a constant that grows as its argument grows.*

Assuming Lemma 3 we complete Case 1. Using conclusion (1) in Lemma 3 we can estimate the last expression in (5.9) by

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^n} |T(f_1,\dots,f_m)|^q\, R(h)\, w\, dx\Big)^{\frac{1}{q}} &\le C\prod_{j=1}^m \|f_j\|_{L^{q_j}(R(h)w)}\\
&\le C\prod_{j=1}^m \|R(h)\|_{L^{s'}(w)}^{\frac{1}{q_j}}\, \|f_j\|_{L^{p_j}(w)}\\
&\le C\|h\|_{L^{s'}(w)}^{\frac{1}{q}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w)}\\
&\le C\prod_{j=1}^m \|f_j\|_{L^{p_j}(w)}
\end{aligned} \tag{5.10}
$$

where we used the hypothesis of the Theorem 2, Hölder's inequality and the fact that $\|h\|_{L^{s'}(w)} \le 1$.

Case 2: $q > p$. If $q_1 = 1$ there is nothing to prove. Otherwise, we set $s = \theta = \frac{q}{p} = \frac{q_j}{p_j}$. Fix $f_j \in L^{p_j}(w)$. We have the following lemma.

Lemma 4. *For any nonnegative function g in $L^{\frac{p}{q-p}}(w)$, there exists a function $G(g)$ such that*

(4) *$g(x) \le G(g)(x)$ for almost every $x \in \mathbb{R}^n$.*

(5) $\|G(g)\|_{L^{\frac{p}{q-p}}(w)} \le 2^{q_1-1}\|g\|_{L^{\frac{p}{q-p}}(w)}$

(6) *$w\,G(g)^{-1} \in A_{q_1}$, in particular $[w\,G(g)^{-1}]_{A_{q_1}} \le C\big([w]_{A_{p_1}}\big)$, where C is a constant that grows as its argument grows.*

Assuming Lemma 4 we complete Case 2. We take

$$
h = \left(\sum_{j=1}^m \left(\frac{|f_j|}{\|f_j\|_{L^{p_j}(w)}}\right)^{p_j}\right)^{\frac{q-p}{p}}
$$

and we observe that $\|h\|_{L^{\frac{p}{q-p}}(w)} \le m^{\frac{q-p}{p}} < \infty$. Using conclusions (4), (5), (6) in Lemma 4 and the hypothesis of the theorem, we obtain the sequence of inequalities below:

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^n} |T(f_1,\dots,f_m)|^p w\, dx\Big)^{\frac{1}{p}} &= \Big\||T(f_1,\dots,f_m)|^q\Big\|_{L^{\frac{p}{q}}(w)}^{\frac{1}{q}}\\
&= \Big\||T(f_1,\dots,f_m)|^q G(h)^{-1} G(h)\Big\|_{L^{\frac{p}{q}}(w)}^{\frac{1}{q}}\\
&\le \Big\||T(f_1,\dots,f_m)|^q G(h)^{-1}\Big\|_{L^1(w)}^{\frac{1}{q}} \Big\|G(h)\Big\|_{L^{\frac{p}{q-p}}(w)}^{\frac{1}{q}}\\
&\le C\Big(\int_{\mathbb{R}^n} |T(f_1,\dots,f_m)|^q G(h)^{-1} w\, dx\Big)^{\frac{1}{q}} \Big\|h\Big\|_{L^{\frac{p}{q-p}}(w)}^{\frac{1}{q}}
\end{aligned}
$$

$$\le C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} |f_j|^{q_j} G(h)^{-1} w \, dx \right)^{\frac{1}{q_j}}$$

$$\le C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} |f_j|^{q_j} h^{-1} w \, dx \right)^{\frac{1}{q_j}} .$$

We use that for all j,

$$\left(\frac{|f_j(x)|}{\|f_j\|_{L^{p_j}(w)}} \right)^{p_j \, \frac{q-p}{p}} \le h(x) ,$$

and that $p/q = p_j/q_j$ to get that the last expression in the sequence of inequalities above is bounded by

$$C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} |f_j|^{p_j} w \, dx \right)^{\frac{1}{p_j}}$$

which proves the required conclusion.

It remains to prove Lemmata 3 and 4. We are going to sketch here the proof of these results for the sake of completeness. For the original proofs, which are different from the ones here, the reader is referred to [6] or [7].

Proof of Lemma 3. Set $t = \frac{p_1'}{s'} \le 1$ and define the sublinear operator

$$S(h)(x) = \Big(M\big(h^{\frac{1}{t}} \, w\big)(x) \, w(x)^{-1} \Big)^{t}$$

which is bounded on $L^{s'}(w)$ since $w \in A_{p_1}$. We denote the norm of this operator by $\|S\|_*$. Next, we consider the Rubio de Francia algorithm given by

$$R(h)(x) = \sum_{k=0}^{\infty} \frac{S^k(h)(x)}{2^k \, \|S\|_*^k} .$$

Conclusions (1) and (2) are left to the reader. On the other hand, by the sublinearity of S we get $S(R(h))(x) \le 2 \, \|S\|_* \, R(h)(x)$ which gives

$$\frac{1}{|Q|} \int_Q R(h)^{\frac{1}{t}} \, w \, dx \le C \, R(h)(x)^{\frac{1}{t}} \, w(x), \qquad \text{for a.e. } x \in Q.$$

If $t = 1$ this inequality says that $w \, R(h) \in A_1 = A_{q_1}$. Otherwise, the fact that $w \, R(h) \in A_{q_1}$ follows from Hölder's inequality with $\frac{1}{t} > 1$ and the previous estimate. □

Proof of Lemma 4. We set $\tau = \frac{q_1 - p_1}{q_1 - 1} < 1$ and define $S(h)(x) = M\big(h^{\frac{1}{\tau}}\big)(x)^{\tau}$ that is clearly sublinear. This operator is bounded on $L^{\frac{p_1}{\tau}}(w)$, since $w \in A_{p_1}$, and denote its norm as $\|S\|_*$. The suitable Rubio de Francia algorithm for this case is

$$G(h)(x) = \left(\sum_{k=0}^{\infty} \frac{S^k\Big(h^{\frac{1}{q_1-1}}\Big)(x)}{2^k \, \|S\|_*} \right)^{q_1 - 1} .$$

Condition (4) is automatic because S^0 is the identity operator. The boundedness of S yields (5). To obtain (6) observe that since S is sublinear it follows

$$S\Big(G(h)^{\frac{1}{q_1-1}}\Big)(x) \le 2 \, \|S\|_* \, G(h)(x)^{\frac{1}{q_1-1}}$$

which implies

$$\Big(\frac{1}{|Q|}\int_Q G(h)^{\frac{1}{q_1-p_1}}\,dx\Big)^{q_1-p_1} \le C\,G(h)(x), \qquad \text{for a.e. } x\in\mathbb{R}^n.$$

Use Hölder's inequality with $\frac{1}{\tau}>1$, this estimate and the fact that $w\in A_{p_1}$ to conclude that $w\,G(h)^{-1}\in A_{q_1}$. □

The proof of Theorem 2 is shown pictorially in Figure 4 in the two-variable case. One easily sees that starting at a point yields estimates for all the exponents that lie on the line passing through this point and the origin. Note that if $q_j=1$ the initial point lies in some of the upper edges and we obtain boundedness for points on the same line.

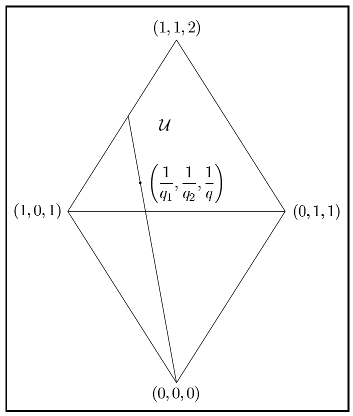

Figure 4

6. Extrapolation from weak type estimates

The main goal of this section is to obtain an extrapolation theorem from an initial weak type estimate. We precisely state our theorems and we discuss their proofs in the remaining part of this section.

Theorem 3. *Suppose that in Theorem 1 condition (1.2) is replaced by*

$$\big\|T(f_1,\dots,f_m)\big\|_{L^{q,\infty}(w_1^q\dots w_m^q)} \le C_0(B)\prod_{j=1}^m \big\|f_j\big\|_{L^{q_j}(w_j^{q_j})}. \tag{6.11}$$

Then, we obtain that T maps

$$L^{p_1}(w_1^{p_1})\times\cdots\times L^{p_m}(w_m^{p_m}) \longrightarrow L^{p,\infty}(w_1^p\dots w_m^p) \tag{6.12}$$

for any $1<p_1,\dots,p_m<\infty$ and $\frac{1}{m}<p<\infty$ such that $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, and for all weights $(w_1^{p_1},\dots,w_m^{p_m})\in(A_{p_1},\dots,A_{p_m})$.

Theorem 4. *Suppose that in Theorem 2 condition (1.5) is replaced by*

$$\big\|T(f_1,\dots,f_m)\big\|_{L^{q,\infty}(w)} \le C_0(B)\prod_{j=1}^m \big\|f_j\big\|_{L^{q_j}(w)}.$$

Then, we have that T maps $L^{p_1}(w)\times\cdots\times L^{p_m}(w)$ into $L^{p,\infty}(w)$ for any $1<p_1,\dots,p_m<\infty$ and $\frac{1}{m}<p<\infty$ that satisfy $p_j=q_j/\theta$ and $p=q/\theta$ for some $0<\theta<\infty$, and for all weights $w\in A_{p_1}\cap\cdots\cap A_{p_m}$.

Proof or Theorem 3. For this proof one may follow the ideas of [6]. But we have a new very short proof. For any $\lambda > 0$ we define a new operator $T_\lambda(f_1, \dots, f_m) = \lambda\, \chi_{E_\lambda}$ where $E_\lambda = \{y \in \mathbb{R}^n : |T(f_1, \dots, f_m)(y)| > \lambda\}$. We first see that T_λ satisfies (1.4) uniformly on λ. This follows from (6.11):

$$\begin{aligned} \big\|T_\lambda(f_1, \dots, f_m)\big\|_{L^q(w_1^q \dots w_m^q)} &= \lambda \left(w_1^q \dots w_m^q\right)(E_\lambda)^{\frac{1}{q}} \le \big\|T(f_1, \dots, f_m)\big\|_{L^{q,\infty}(w_1^q \dots w_m^q)} \\ &\le C_0 \prod_{j=1}^m \big\|f_j\big\|_{L^{q_j}(w_j^{q_j})} \end{aligned}$$

where we have used the standard notation $\left(w_1^q \dots w_m^q\right)\left(E_\lambda\right) = \int_{E_\lambda} w_1^q \dots w_m^q \, dx$. Then we apply Theorem 1 to T_λ and since its the norm is uniformly bounded on λ we get

$$\lambda \left(w_1^p \dots w_m^p\right)(E_\lambda)^{\frac{1}{p}} = \big\|T_\lambda(f_1, \dots, f_m)\big\|_{L^p(w_1^p \dots w_m^p)} \le C \prod_{j=1}^m \big\|f_j\big\|_{L^{p_j}(w_j^{p_j})}.$$

This estimate gives that T maps $L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$ into $L^{p,\infty}(w_1^p \dots w_m^p)$ because the constant C is independent of λ. □

The proof of Theorem 4 is left to the reader, since the main idea is contained in the proof of the argument for Theorem 3.

We now discuss two cases in which Theorems 3 and 4 can be applied.

Corollary 1. *In Theorem 3 also assume that T is sublinear in each variable. Then for all $1 < p_1, \dots, p_m < \infty$ and $1/m < p < \infty$ which satisfy $1/p = 1/p_1 + \cdots + 1/p_m$, T maps $L^{p_1}(w) \times \cdots \times L^{p_m}(w)$ into $L^p(w)$ for all weights $w \in A_{p_1} \cap \cdots \cap A_{p_m}$. In particular this estimate holds in the unweighted case.*

Proof. The proof is an easy consequence of the multilinear Marcinkiewicz interpolation theorem between $m+1$ weak type estimates of the type (6.12). To do that, we fix the exponents $p_1, \dots, p_m, p$ as before and a weight $w \in A_{p_1} \cap \cdots \cap A_{p_m}$. By the reverse Hölder inequality, it is well known that the Muckenhoupt classes are open from the left. Then we can apply Theorem 3 to prove weighted weak type estimates for exponents that lie in a ball whose center is the given point. In all these estimates the weight remains fixed, or equivalently, the underlying measure in every space is $w(x)\,dx$. Applying the multilinear Marcinkiewicz interpolation theorem (see for instance [8] Theorem 4.6) we obtain the strong type estimate at the desired exponents. We leave the details for the reader. □

We remark that it is still an open question whether one can extend the conclusion of Theorem 3 to the case in which the space $L^{p,\infty}$ is replaced by L^p in (6.12). To achieve this one needs a multilinear version of the Marcinkiewicz interpolation theorem with a change of measure, (see Stein and Weiss [19] for the linear case). At present it is not known to us whether such a theorem holds.

Finally, Theorem 4 is applicable to cases in which the operator T is m-linear and its adjoints are of the same nature, in the sense that if an estimate holds for the operator, then it also holds for all of its m adjoints. Then one can apply multilinear interpolation to obtain boundedness of T on products of unweighted Lebesgue spaces for a wide range of exponents. The multilinear interpolation is straightforward when $q > 1$. When $q \le 1$ one needs to apply multilinear interpolation between adjoint operators as in [10]. Theorem 4, for instance could be used to obtain unweighted estimates for the bilinear Hilbert transform in the range of exponents $1 < p_1, p_2 \leq \infty$, $1/2 < p < \infty$, from the family of weighted estimates $L^2(w) \times L^2(w) \longrightarrow L^{1,\infty}(w)$ for all $w \in A_2$. This line of investigation will be pursued elsewhere.

7. Vector-valued inequalities

One of the main applications of extrapolation of operators is in the area of vector-valued inequalities. The use of extrapolation on this context was pioneered by Rubio de Francia [16] who was the first to observe the intimate connection between weighted norm inequalities and vector-valued estimates. An analogous connection is valid for multivariable operators and is investigated in this section. Our results are very general as the only hypotheses essentially needed for the multivariable operators in question is that they are well-defined on products of weighted Lebesgue spaces. This application will require only a very slight modification of the proofs of our results discussed in the previous sections.

Section 8 deals with multilinear Calderón-Zygmund operators for which we prove vector-valued inequalities as a consequence of extrapolation. There is no vector-valued theory developed for multilinear operators in the literature (only recently a vector-valued estimate for the bilinear Hilbert transforms was obtained by Grafakos and Li [9]). Our approach is based on theory of the weights but another way to obtain such estimates would be via a multilinear extension of the results in [1], [17] for linear Calderón-Zygmund operators.

Before stating a precise result about vector-valued inequalities we investigate what kind of estimates one might expect. In the one-variable case the inequalities that yield from extrapolation are:

$$\Big\| \Big(\sum_k |Tf_k|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w)} \leq C \Big\| \Big(\sum_k |f_k|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w)}$$

for $1 < p, s < \infty$ and for every $w \in A_p$. We point out that by convexity one can prove this estimates with two different powers, namely, if $s \leq r$ then

$$\Big\| \Big(\sum_k |Tf_k|^r \Big)^{\frac{1}{r}} \Big\|_{L^p(w)} \leq \Big\| \Big(\sum_k |Tf_k|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w)} \leq C \Big\| \Big(\sum_j |f_k|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w)}.$$

This condition $s \leq r$ is also necessary. To see this, one only needs to take $f_j = f$ for $1 \leq j \leq N$ and $f_j = 0$ otherwise. Applying the inequality above to this sequence we get

$$N^{\frac{1}{r}} \, \|Tf\|_{L^p(w)} \leq C \, N^{\frac{1}{s}} \, \|f\|_{L^p(w)}.$$

Since C does not depends on N, which can be taken arbitrarily big, it follows that $s \leq r$. In any case, the result with the same power is the optimal and is the one that arises as a consequence of the classical extrapolation result (see [7]).

We apply the same idea to multivariable operators. The estimates that we would like to handle are the following:

$$\Big\| \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^r \Big)^{\frac{1}{r}} \Big\|_{L^p} \leq C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}} \Big\|_{L^{p_j}}. \tag{7.1}$$

In the Lebesgue spaces involved in this estimate we have intentionally omitted the underlying measure or weight since the argument below is independent of these. We consider exponents $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{m} < p < \infty$ that satisfy (1.3). Comparing with the one-variable case it seems natural to assume that $1 < s_1, \dots, s_m < \infty$ and $0 < r < \infty$. In the previous inequality we take $f_j^k = f_j$ for $1 \le k \le N$, $1 \le j \le m$, and $f_j^k = 0$ otherwise to obtain

$$N^{\frac{1}{r}} \big\| T(f_1, \dots, f_m) \big\|_{L^p} \le C \prod_{j=1}^m N^{\frac{1}{s_j}} \big\| f_j \big\|_{L^{p_j}}.$$

where C is independent of N. Then we are forced to have

$$\frac{1}{r} \le \frac{1}{s_1} + \dots + \frac{1}{s_m}.$$

Thus estimate (7.1) will be optimal if r is replaced by s where

$$\frac{1}{s} = \frac{1}{s_1} + \dots + \frac{1}{s_m}, \tag{7.2}$$

and therefore $\frac{1}{m} < s < \infty$. We have the following theorem.

Theorem 5. *Let $1 \le q_1, \dots, q_m < \infty$ and $\frac{1}{m} \le q < \infty$ be fixed indices that satisfy (1.1). We suppose that for all $B > 1$, there is a constant $C_0(B) > 0$ such that for all tuples of weights $(w_1^{q_1}, \dots, w_m^{q_m}) \in (A_{q_1}, \dots, A_{q_m})$ with $[w_j^{q_j}]_{A_{q_j}} \le B$ and all functions $f_j \in L^{q_j}(w_j^{q_j})$ estimate (1.2) holds with $C_0 = C_0(B)$. Then for all indices $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{m} < p < \infty$ that satisfy (1.3), $1 < s_1, \dots, s_m < \infty$ and $\frac{1}{m} < s < \infty$ that satisfy (7.2), all $B > 1$, and all weights $(w_1^{p_1}, \dots, w_m^{p_m})$ in $(A_{p_1}, \dots, A_{p_m})$ with $[w_j^{p_j}]_{A_{p_j}} \le B$, there is a constant $C = C(B)$ such that*

$$\Big\| \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w_1^p \dots w_m^p)} \le C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}} \Big\|_{L^{p_j}(w_j^{p_j})}.$$

Remark 1. *Taking $w_1^{p_1} = \dots = w_m^{p_m} = w$ for some $w \in A_{p_1} \cap \dots \cap A_{p_m}$ yields the following one-weight norm inequality:*

$$\Big\| \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w)} \le C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}} \Big\|_{L^{p_j}(w)}.$$

In particular, if $w = 1$ one obtains the corresponding unweighted vector-valued estimate.

Proof. We introduce some notation: for $1 \le j \le m$,

$$F_j = \{f_j^k\}_k, \qquad \|F_j\|_{\ell^{s_j}} = \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}}$$

and we define a new multivariable operator

$$\widetilde{T}(F_1, \dots, F_m) = \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^s \Big)^{\frac{1}{s}}.$$

Note that the estimate we want to prove can be written as

$$\big\| \widetilde{T}(F_1, \dots, F_m) \big\|_{L^p(w_1^p \dots w_m^p)} \le C \prod_{j=1}^m \Big\| \|F_j\|_{\ell^{s_j}} \Big\|_{L^{p_j}(w_j^{p_j})}. \tag{7.3}$$

for all $(w_1^{p_1}, \dots, w_m^{p_m})$ in $(A_{p_1}, \dots A_{p_m})$. On the other hand, we can apply Theorem 1 and for all weights $(w_1^{s_1}, \dots, w_m^{s_m}) \in (A_{s_1}, \dots A_{s_m})$ we have

$$\begin{aligned}
\big\|\widetilde{T}(F_1, \dots, F_m)\big\|_{L^s(w_1^s \dots w_m^s)} &= \Big(\sum_k \big\|T(f_1^k, \dots, f_m^k)\big\|^s_{L^s(w_1^s \dots w_m^s)}\Big)^{\frac{1}{s}} \\
&\leq C\Big(\sum_k \prod_{j=1}^m \|f_j^k\|^s_{L^{s_j}(w_j^{s_j})}\Big)^{\frac{1}{s}} \\
&\leq C\prod_{j=1}^m \Big(\sum_k \|f_j^k\|^{s_j}_{L^{s_j}(w_j^{s_j})}\Big)^{\frac{1}{s_j}} \\
&= C\prod_{j=1}^m \Big\|\Big(\sum_k |f_j^k|^{s_j}\Big)^{\frac{1}{s_j}}\Big\|_{L^{s_j}(w_j^{s_j})} \\
&= C\prod_{j=1}^m \Big\|\|F_j\|_{\ell^{s_j}}\Big\|_{L^{s_j}(w_j^{s_j})}.
\end{aligned}$$

Note that it was in the first inequality where we used the extrapolation result since we know that T satisfies weighted estimates for all the admissible exponents and in particular for $s_1, \dots, s_m$ and s. We also point out that the second estimate is just a consequence of Hölder's inequality. We have proved that

$$\big\|\widetilde{T}(F_1, \dots, F_m)\big\|_{L^s(w_1^s \dots w_m^s)} \leq C\prod_{j=1}^m \Big\|\|F_j\|_{\ell^{s_j}}\Big\|_{L^{s_j}(w_j^{s_j})} \tag{7.4}$$

holds for all weights $(w_1^{s_1}, \dots, w_m^{s_m}) \in (A_{s_1}, \dots, A_{s_m})$. One now needs to extrapolate from this estimate to obtain (7.3). However, we cannot use Theorem 1 in a straightforward way since on the right hand side of (7.4) we have ℓ^{s_j} norms of the sequences F_j instead of a function f_j. But a careful examination of the proof of Theorem 1 yields that at no step it was crucial that we were dealing with scalar-valued functions. Using this observation and thinking of F_j as a Banach-valued (ℓ^{s_j}-valued) function instead of a scalar-valued function, one obtains that the proof of Theorem 1 equally applies in this setting. Consequently we can extrapolate from (7.4) to obtain (7.3). □

The argument that we have just shown works actually for more general Banach spaces. In fact, one can prove extrapolation results for Banach-valued operators. We have the following:

Proposition 1. *Let $\mathbb{A}_1, \dots, \mathbb{A}_m$ and $\mathbb{B}$ be Banach spaces. Consider T a multivariable operator such that for any $(\mathbb{A}_1, \dots, \mathbb{A}_m)$-valued m-tuple of "good" functions $(f_1, \dots, f_m)$ we have that $T(f_1, \dots, f_m)$ is an element of $\mathbb{B}$. Suppose that for some fixed exponents $1 \leq q_1, \dots, q_m < \infty$ and $\frac{1}{m} \leq q < \infty$ such that (1.1) holds, the operator T satisfies*

$$\Big\|\|T(f_1, \dots, f_m)\|_{\mathbb{B}}\Big\|_{L^q(w_1^q \dots w_m^q)} \leq C\prod_{j=1}^m \Big\|\|f_j\|_{\mathbb{A}_j}\Big\|_{L^{q_j}(w_j^{q_j})}$$

for any tuple of weights $(w_1^{q_1}, \dots, w_m^{q_m}) \in (A_{q_1}, \dots A_{q_m})$. Then we can extrapolate and the previous estimate holds for all the admissible exponents. As before, some sequence-valued inequalities can be obtained for T.

Remark 2. *One can also obtain vector-valued estimates as a consequence of Theorem 2. We leave precise formulation and the simple details of the verification to the interested reader.*

8. Multilinear Calderón-Zygmund operators

Multilinear operators arise in the study of expressions that involve product-like operations. The study of this subject has recently enjoyed a resurgence of renewed interest and activity. In analogy with the linear theory, the class of multilinear singular integrals with standard Calderón-Zygmund kernels provides a fundamental topic of investigation within the framework of the general theory. Multilinear Calderón-Zygmund operators were introduced and first studied by Coifman and Meyer [2], [3], [4], and later by Grafakos and Torres [11], [12].

We recall the relevant background from the general theory. We start with a function $K(y_0, y_1, \dots, y_m)$ defined *away from the diagonal* $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ which satisfies the following estimates

$$|\partial_{y_0}^{\alpha_0} \dots \partial_{y_m}^{\alpha_m} K(y_0, y_1, \dots, y_m)| \le \frac{A_\alpha}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l|\Big)^{mn+|\alpha|}}, \qquad \text{for all } |\alpha| \le 1, \tag{8.5}$$

where $\alpha = (\alpha_0, \dots, \alpha_m)$ is an ordered set of n-tuples of nonnegative integers, $|\alpha| = |\alpha_0| + \cdots + |\alpha_m|$, and $|\alpha_j|$ is the order of each multiindex α_j. Such functions K are called multilinear standard kernels. We assume below that T is a weakly continuous m-linear operator from $\mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ such that for some multilinear standard kernel K, the integral representation below is valid

$$T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \dots, y_m) \prod_{j=1}^{m} f_j(y_j)\, dy_1 \dots dy_m, \tag{8.6}$$

whenever f_j are smooth functions with compact support and $x \notin \cap_{j=1}^m \mathrm{supp} f_j$. In the case $m = 1$ conditions (8.5) are called standard estimates and operators given by (8.6) are called Calderón-Zygmund if they are bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. In the multilinear case we call T a multilinear Calderón-Zygmund operator if it is associated to a multilinear standard kernel as in (8.6) and has a bounded extension from a product of some L^{q_j} spaces into another L^q space for some choice of $1 < q_j < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$. If this is the case, it was shown in [11], [13] that these operators map any other product of Lebesgue spaces $\prod_{j=1}^m L^{p_j}(\mathbb{R}^n)$ with $p_j > 1$ into the corresponding $L^p(\mathbb{R}^n)$ space and they also map $\prod_{j=1}^m L^1(\mathbb{R}^n)$ into $L^{1/m,\infty}(\mathbb{R}^n)$.

The kind of weighted estimates we want to study for multilinear Calderón-Zygmund operators have been considered in [12]. By means of a good-λ inequality it was shown in [12] that for any $w \in A_\infty$, one has

$$\big\|T(f_1, \dots, f_m)\big\|_{L^p(w)} \le C \Big\| \prod_{j=1}^{m} M f_j \Big\|_{L^p(w)} \tag{8.7}$$

whenever the left-hand side is finite, where M is the Hardy-Littlewood maximal operator. We point out that this inequality is a consequence of the one proved for the multilinear maximal singular integral T_* defined as

$$T_*(f_1,\dots,f_m)(x)=\sup_{\delta>0}\left|\int\cdots\int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} K(x,y_1,\dots,y_m)f_1(y_1)\dots f_m(y_m)\,dy_1\dots dy_m\right|.$$

This inequality is not stated there in this way but (8.7) can be easily derived from the good-λ estimate in [12]. As a consequence of (8.7), in that paper it was shown that T and T_* map $L^{p_1}(w)\times\cdots\times L^{p_m}(w)$ into $L^p(w)$ for $1<p_1,\dots,p_m<\infty$ and p satisfying (1.3) and for any $w\in A_{p_1}\cap\cdots\cap A_{p_m}$. But these inequalities allow us also to derive some other weighted estimates using Theorem 1. We need the following lemma.

Lemma 5. *For $(w_1,\dots,w_m)\in(A_{p_1},\dots,A_{p_m})$ with $1\le p_1,\dots,p_m<\infty$ and for $0<\theta_1,\dots,\theta_m<1$ such that $\theta_1+\cdots+\theta_m=1$, we have $w_1^{\theta_1}\dots w_m^{\theta_m}\in A_{\max\{p_1,\dots,p_m\}}$.*

Proof. We prove the lemma by induction. For $m=2$, we clearly have $w_1,w_2\in A_{\max\{p_1,p_2\}}$ and by Hölder's inequality it follows that $w_1^{\theta}w_2^{1-\theta}\in A_{\max\{p_1,p_2\}}$. Now assume that the case m is proved and we want to obtain the case $m+1$. Let us observe that

$$w_1^{\theta_1}\dots w_{m+1}^{\theta_{m+1}}=\left(w_1^{\frac{\theta_1}{1-\theta_{m+1}}}\dots w_m^{\frac{\theta_m}{1-\theta_{m+1}}}\right)^{1-\theta_{m+1}}w_{m+1}^{\theta_{m+1}}=W_m^{1-\theta_{m+1}}w_{m+1}^{\theta_{m+1}}.$$

We use the induction hypothesis to get that $W_m\in A_{\max\{p_1,\dots,p_m\}}$ and then we use the case $m=2$ to eventually deduce that $w_1^{\theta_1}\dots w_{m+1}^{\theta_{m+1}}\in A_{\max\{p_1,\dots,p_{m+1}\}}$. □

This result allows us to prove more general weighted estimates for multilinear Calderón-Zygmund operators.

Corollary 2. *Let T be an m-linear Calderón-Zygmund operator as above. Consider an m-tuple $(w_1^{p_1},\dots,w_m^{p_m})\in(A_{p_1},\dots,A_{p_m})$ where $1<p_1,\dots,p_m<\infty$ and $\frac{1}{m}<p<\infty$ satisfy $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. Then there exists a constant C that only depends on the p_j's, on the weights, and on size estimate constants for the kernel K of T such that*

$$\big\|T(f_1,\dots,f_m)\big\|_{L^p(w_1^p\dots w_m^p)}\le C\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j^{p_j})}$$

and

$$\big\|T_*(f_1,\dots,f_m)\big\|_{L^p(w_1^p\dots w_m^p)}\le C\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j^{p_j})}.$$

Proof. We use Lemma 5 to obtain

$$w_1^p\dots w_m^p=\big(w_1^{p_1}\big)^{\frac{p}{p_1}}\dots\big(w_m^{p_m}\big)^{\frac{p}{p_m}}\in A_{\max\{p_1,\dots,p_m\}}\subset A_\infty.$$

Then we can apply (8.7) with this A_∞ weight to get

$$\begin{aligned}\big\|T(f_1,\dots,f_m)\big\|_{L^p(w_1^p\dots w_m^p)}&\le C\Big\|\prod_{j=1}^m Mf_j\Big\|_{L^p(w_1^p\dots w_m^p)}\\&\le C\prod_{j=1}^m\|Mf_j\|_{L^{p_j}(w_j^{p_j})}\\&\le C\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j^{p_j})},\end{aligned}$$

where we have used Hölder's inequality and that $w_j^{p_j} \in A_{p_j}$ for $1 \le j \le m$. For T_* the proof is similar. □

The previous result is just a consequence of the good-λ estimate obtained in [12]. Our extrapolation Theorem 1 may be applied to show that if one had a weighted estimate for some fixed exponents then weighted estimates hold for all exponents. Nevertheless, we are going to use the results in section 7 to obtain a stronger vector-valued weighted estimate for multilinear Calderón-Zygmund operators. The following corollary arises as a straightforward application of Theorem 5. Note that Corollary 2 yields the "starting" estimate for the result in Corollary 3 below.

Corollary 3. *Let T be a multilinear Calderón-Zygmund operator as before. Consider an m-tuple $(w_1^{p_1}, \dots, w_m^{p_m}) \in (A_{p_1}, \dots, A_{p_m})$ where $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{m} < p < \infty$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. Take $1 < s_1, \dots, s_m < \infty$ and $\frac{1}{m} < s < \infty$ such that $\frac{1}{s_1} + \cdots + \frac{1}{s_m} = \frac{1}{s}$. Then there exists a constant C that depends only on the allowable parameters such that*

$$\Big\| \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w_1^p \cdots w_m^p)} \le C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}} \Big\|_{L^{p_j}(w_j^{p_j})} \tag{8.8}$$

and

$$\Big\| \Big(\sum_k |T_*(f_1^k, \dots, f_m^k)|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w_1^p \cdots w_m^p)} \le C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}} \Big\|_{L^{p_j}(w_j^{p_j})}.$$

Remark 3. *As a consequence of this result one can prove estimates for just one weight. Namely, suppose that the exponents satisfy the previous hypotheses. Then for any weight $w \in A_{p_1} \cap \cdots \cap A_{p_m}$ we have*

$$\Big\| \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w)} \le C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}} \Big\|_{L^{p_j}(w)} \tag{8.9}$$

and

$$\Big\| \Big(\sum_k |T_*(f_1^k, \dots, f_m^k)|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(w)} \le C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^{s_j} \Big)^{\frac{1}{s_j}} \Big\|_{L^{p_j}(w)}$$

In particular, one can take $w = 1$ and the corresponding unweighted vector-valued estimates for T and T_* hold.

Remark 4. *These vector-valued inequalities can be independently proved using some extrapolation results obtained in [5]. This approach employs estimates of another nature, more in the spirit of (8.7), where the weights involved are in A_∞. The reader is referred to this article for details.*

9. A multilinear extension of a theorem of Marcinkiewicz and Zygmund

In addition to the vector-valued inequalities proved as a consequence of the mutlivariable extrapolation theory developed in the previous sections, there are ℓ^2-valued estimates that arise from a multilinear version of the classical theorem of Marcinkiewicz and Zygmund [14] on ℓ^2-valued extensions of linear operators. Let us note that from the extrapolation results proved here, one can not derive the expected vector-valued weighted weak type norm estimates when some of the exponents p_j are equal to 1. This can be done, however, with this technique that we discuss next.

The classical result of Marcinkiewicz and Zygmund [14] says that every linear operator that maps L^p into L^q for some $0 < p, q < \infty$ admits an ℓ^2 bounded extension. In this section we extend this theorem to the multilinear setting. The following result holds for general measure spaces (X_j, μ_j) and (Y, ν).

Theorem 6. *(a) Let T be an m-linear operator that maps $L^{p_1}(X_1, \mu_1) \times \cdots \times L^{p_m}(X_m, \mu_m)$ into $L^q(Y, \nu)$ for some $0 < p_1, p_2, \ldots, p_m, q < \infty$ with norm $\|T\|$. Then there is a constant C such that for all sequences of functions $\{f_j^k\}^{k\in\mathbb{Z}}$ in $L^{p_j}(X_j)$, $1 \le j \le m$, we have*

$$\Big\| \Big(\sum_{k_1} \cdots \sum_{k_m} |T(f_1^{k_1}, \ldots, f_m^{k_m})|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q} \le C \, \|T\| \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p_j}} \tag{9.10}$$

and in particular one has the estimate

$$\Big\| \Big(\sum_k |T(f_1^k, \ldots, f_m^k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q} \le C \, \|T\| \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p_j}} . \tag{9.11}$$

(b) Suppose that T be an m-linear operator that maps $L^{p_1}(X_1, \mu_1) \times \cdots \times L^{p_m}(X_m, \mu_m)$ into $L^{q,\infty}(Y, \nu)$ for some $0 < p_1, p_2, \ldots, p_m, q < \infty$ with norm $\|T\|_{weak}$. Then T has an ℓ^2-valued extension, i.e.

$$\Big\| \Big(\sum_{k_1} \cdots \sum_{k_m} |T(f_1^{k_1}, \ldots, f_m^{k_m})|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q,\infty}} \le C \, \|T\|_{weak} \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p_j}} \tag{9.12}$$

for some constant C that depends only on p_j and q. In particular one has the estimate

$$\Big\| \Big(\sum_k |T(f_1^k, \ldots, f_m^k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q,\infty}} \le C \, \|T\|_{weak} \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p_j}} . \tag{9.13}$$

Proof. The result in part (a) is an easy consequence of the estimate below valid for the Rademacher functions r_j:

$$B_q^m \bigg(\sum_{k_1} \cdots \sum_{k_m} |c_{k_1,\ldots,k_m}|^2 \bigg)^{\frac{1}{2}} \le \big\| F_m \big\|_{L^q([0,1]^m)} \le D_q^m \bigg(\sum_{k_1} \cdots \sum_{k_m} |c_{k_1,\ldots,k_m}|^2 \bigg)^{\frac{1}{2}} \tag{9.14}$$

where

$$F_m(t_1, \ldots, t_m) = \sum_{k_1} \cdots \sum_{k_m} c_{k_1,\ldots,k_m} r_{k_1}(t_1) \ldots r_{k_m}(t_m),$$

$0 < q < \infty$, $0 < D_q, B_q < \infty$, $t_j \in [0,1]$, and $c_{k_1,\ldots,k_n}$ is a sequence of complex numbers. We refer the reader to the Appendix in [18] for a proof of this estimate. The proof of (9.10) follows by a linearization of the square function and an application of (9.14). Indeed, we let $1/p = 1/p_1 + \cdots + 1/p_m$ and we consider the following two cases:

Case 1: $q \le p$. In this case we fix a positive integer n. Using both estimates in (9.14) and the multilinearity of T we obtain

$$\begin{aligned} &\Big\| \Big(\sum_{|k_1| \le n} \cdots \sum_{|k_m| \le n} |T(f_1^{k_1}, \ldots, f_m^{k_m})|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q}^q \\ &\quad \le B_q^{-qm} \int_Y \int_{[0,1]^m} \Big| \sum_{|k_1| \le n} \cdots \sum_{|k_m| \le n} T(f_1^{k_1}, \ldots, f_m^{k_m}) r_{k_1}(t_1) \ldots r_{k_m}(t_m) \Big|^q dt_1 \ldots dt_m \, d\nu \end{aligned}$$

$$
\begin{aligned}
&\leq B_q^{-qm} \int_{[0,1]^m} \int_Y \Big| T\Big(\sum_{|k_1|\leq n} r_{k_1}(t_1) f_1^{k_1}, \dots, \sum_{|k_m|\leq n} r_{k_m}(t_m) f_m^{k_m} \Big)\Big|^q d\nu\, dt_1 \dots dt_m \\
&\leq B_q^{-qm} \|T\|^q \int_{[0,1]^m} \prod_{j=1}^m \Big\| \sum_{|k_j|\leq n} r_{k_j}(t_j) f_j^{k_j} \Big\|_{L^{p_j}(X_j)}^q dt_1 \dots dt_m \\
&\leq B_q^{-qm} \|T\|^q \prod_{j=1}^m \Big(\int_0^1 \Big\| \sum_{|k_j|\leq n} r_{k_j}(t_j) f_j^{k_j} \Big\|_{L^{p_j}(X_j)}^{p_j} dt_j \Big)^{\frac{q}{p_j}} \\
&\leq B_q^{-qm} \|T\|^q \prod_{j=1}^m \Big(D_{p_j}^{p_j} \Big\| \Big(\sum_{|k_j|\leq n} |f_j^{k_j}|^2 \Big)^{\frac12} \Big\|_{L^{p_j}(X_j)}^{p_j} \Big)^{\frac{q}{p_j}} \\
&\leq B_q^{-qm} D_{p_1}^q \dots D_{p_m}^q \, \|T\|^q \prod_{j=1}^m \Big\| \Big(\sum_{k_j\in\mathbb{Z}} |f_j^{k_j}|^2 \Big)^{\frac12} \Big\|_{L^{p_j}(X_j)}^q
\end{aligned}
$$

where we used the fact that each $p_j \geq q$ in Hölder's inequality in the fourth inequality above. Letting $n \to \infty$ yields the required conclusion in case 1.

Case 2: $p < q$. Using duality we can write

$$
\begin{aligned}
&\Big\| \Big(\sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_m\in\mathbb{Z}} |T(f_1^{k_1}, \dots, f_m^{k_m})|^2 \Big)^{\frac12} \Big\|_{L^q} \\
&\qquad = \sup_{\|g\|_{L^{(q/p)'}}\leq 1} \Big(\int_Y \Big(\sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_m\in\mathbb{Z}} |T(f_1^{k_1}, \dots, f_m^{k_m})|^2 \Big)^{\frac{p}{2}} |g|\, d\nu \Big)^{\frac1p}
\end{aligned}
\tag{9.15}
$$

and motivated by this we define an m-linear operator T_g by setting

$$
T_g(f_1, \dots, f_m) = |g|^{\frac1p} T(f_1, \dots, f_m)
$$

for some fixed function g in $L^{(q/p)'}$ with norm at most 1. We can easily verify that T_g is bounded from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p with norm at most $\|T\|$. Indeed, for all $\|f_j\|_{L^{p_j}(X_j)} \leq 1$, we have

$$
\big\| T_g(f_1, \dots, f_m) \big\|_{L^p} = \Big(\int_Y |g|\, |T(f_1, \dots, f_m)|^p d\nu \Big)^{\frac1p} \leq \|g\|_{L^{(\frac{q}{p})'}} \big\| |T(f_1, \dots, f_m)|^p \big\|_{L^{\frac{q}{p}}}^{\frac1p} \leq \|T\|
$$

since $\|g\|_{L^{(q/p)'}} \leq 1$. Applying case 1 to T_g yields

$$
\Big(\int_Y \Big(\sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_m\in\mathbb{Z}} |T(f_1^{k_1}, \dots, f_m^{k_m})|^2 \Big)^{\frac{p}{2}} |g|\, d\nu \Big)^{\frac1p} \leq C_{p_j,q,m} \|T\| \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^2 \Big)^{\frac12} \Big\|_{L^{p_j}}
$$

and this estimate combined with (9.15) gives (9.10) in case 2.

We now turn our attention to part (b) of the theorem. We recall the following well-known characterization of weak L^q:

$$
\|f\|_{L^{q,\infty}} \leq \sup_{0<\nu(E)<\infty} \nu(E)^{\frac1q - \frac1r} \Big(\int_E |f|^r \, d\nu \Big)^{\frac1r} \leq \Big(\frac{q}{q-r} \Big)^{\frac1r} \|f\|_{L^{q,\infty}},
\tag{9.16}
$$

where $0 < r < q$ and the supremum is taken over all E subsets of Y of positive and finite ν measure. Using (9.16) we obtain

$$
\begin{aligned}
&\Big\| \Big(\sum_{k_1} \cdots \sum_{k_m} |T(f_1^{k_1}, \dots, f_m^{k_m})|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q,\infty}(\nu)} \\
&\leq \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{q}-\frac{1}{r}} \Big(\int_E \Big(\sum_{k_1} \cdots \sum_{k_m} |T(f_1^{k_1}, \dots, f_m^{k_m})|^2 \Big)^{\frac{r}{2}} d\nu \Big)^{\frac{1}{r}} \\
&= \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{q}-\frac{1}{r}} \Big(\int_Y \Big(\sum_{k_1} \cdots \sum_{k_m} |\chi_E \, T(f_1^{k_1}, \dots, f_m^{k_m})|^2 \Big)^{\frac{r}{2}} d\nu \Big)^{\frac{1}{r}} \\
&\leq \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{q}-\frac{1}{r}} \|T_E\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^r} \prod_{j=1}^m \Big(\int_{X_j} \Big(\sum_k |f_j^k|^2 \Big)^{\frac{p_j}{2}} d\mu_j \Big)^{\frac{1}{p_j}} \qquad (9.17)
\end{aligned}
$$

where we defined $T_E(f_1, \dots, f_m) = \chi_E \, T(f_1, \dots, f_m)$ and we used the result in part (a). (We denote by $\| \cdot \|_{L^{p_1}\times\cdots\times L^{p_m}\to L^r}$ the norm of an m-linear operator from $L^{p_1} \times \cdots \times L^{p_m}$ into L^r.) But since for all m-tuples of functions $(f_1, \dots, f_m) \in L^{p_1} \times \cdots \times L^{p_m}$ we have

$$
\begin{aligned}
\nu(E)^{\frac{1}{q}-\frac{1}{r}} \|T_E(f_1, \dots, f_m)\|_{L^r} &\leq \Big(\frac{q}{q-r} \Big)^{\frac{1}{r}} \|T(f_1, \dots, f_m)\|_{L^{q,\infty}} \\
&\leq \Big(\frac{q}{q-r} \Big)^{\frac{1}{r}} \|T\|_{weak} \prod_{j=1}^m \|f_j\|_{L^{p_j}} ,
\end{aligned}
$$

it follows that for any measurable set E of finite measure the estimate

$$
\nu(E)^{\frac{1}{q}-\frac{1}{r}} \|T_E\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^r} \leq \Big(\frac{q}{q-r} \Big)^{\frac{1}{r}} \|T\|_{weak} \qquad (9.18)
$$

is valid. Now returning to (9.17) and using (9.18) we obtain the required conclusion. We note that our proof in the spirit of that given in [7] for linear operators. □

The following corollary easily follows from part (b) in Theorem 6 and the weak type weighted inequalities proved in [12].

Corollary 4. *Let $1 \leq p_1, \dots, p_m < \infty$ and $\frac{1}{m} < p < \infty$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$ and let T be as in Corollary 3. Suppose that at least one $p_j = 1$. Then for every $w \in A_1$ we have*

$$
\Big\| \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p,\infty}(w)} \leq C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p_j}(w)},
$$

and, in particular,

$$
\Big\| \Big(\sum_k |T(f_1^k, \dots, f_m^k)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{1/m,\infty}(w)} \leq C \prod_{j=1}^m \Big\| \Big(\sum_k |f_j^k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^1(w)}.
$$

We note that these ℓ^2-valued estimates do not follow from the extrapolation results that we have obtained.

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