ENDPOINT BOUNDS FOR AN ANALYTIC FAMILY OF HILBERT TRANSFORMS

Loukas Grafakos

University of California, Los Angeles

Abstract. In $\mathbb{R}^2$, we consider an analytic family of operators H_z, $z \in \mathbb{C}$, whose convolution kernel is obtained by taking $-z-1$ derivatives of arclength measure on the parabola (t, t^2) in a homogeneous way, defined in such a way so that H_{-1} be the standard parabolic Hilbert transform. For a fixed z, we study the set of p for which H_z is bounded on $L^p(\mathbb{R}^2)$ and for the critical z that captures the degree of singularity of this operator on $L^p(\mathbb{R}^2)$, we prove a positive endpoint result.

1. Introduction. The role of curvature in Harmonic Analysis has received increasing attention in recent years. The point of departure for work in this area has been the connection between submanifolds of $\mathbb{R}^n$ and decay of the Fourier transform of compactly supported surface distributions. Such decay estimates fail for submanifolds contained completely in some hyperplane and in general the "amount" of curvature of the submanifold is related to the rate of decay of the Fourier transform of the distribution.

Well known operators whose L^p boundness is affected by curvature are singular integrals along submanifolds of $\mathbb{R}^n$. Consider for example the case of an operator given by convolution with a distribution which is singular along a submanifold of codimension 1. Certain distributions give rise to convolution operators which are bounded on some but not all L^p. If a distribution depends analytically on a parameter z, for a given z, what is the set of all p's for which the associated operator is bounded on L^p?

We study the case where the analytic family of distributions is obtained by taking $-z-1$ transverse derivatives of arclength measure on the parabola and doing so in a homogeneous way. For $1 < p \leq 2$, the operators H_z are easily seen to be unbounded on L^p when $\mathrm{Re} z < 1/p-2$ and one can show using Calderón-Zygmund theory and interpolation that H_z are bounded on L^p when the above inequality is reversed. For the critical $z = 1/p-2+i\theta$, the kernel of H_z lacks the amount of smoothness required by the usual singular integral theory to establish L^p boundedness. Nevertheless, the curvature of the parabola makes

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

up for this lack of smoothness and enables us to prove positive results when the usual methods don't apply. For $p = 1$ we prove that these operators map H^1 to weak L^1 and for $1 < p < 2$ that they map L^p to weak L^p. We also prove that the first result is sharp in the sense that for $p = 1$ all these operators, except one, don't map L^1 to weak L^1. Precise statements of results are given in section 2.

2. Preliminaries and statements of results. We denote by C_0^∞ the set of smooth functions with compact support. Fix ψ an even function in $C_0^\infty(\mathbb{R})$ such that $\psi \geq 0$, $\psi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\psi \equiv 0$ off $[-1, 1]$. For $\operatorname{Re} z > -1$, define an analytic family of distributions D_z acting on test functions of the real variable u as follows:

$$\langle D_z, f\rangle = 2\Gamma\left(\tfrac{z+1}{2}\right)^{-1} \int |u-1|^z\, \psi(u-1) f(u)\; du\,.$$

By analytic continuation, see [GS], D_z may be extended to a distribution–valued entire function of z. For example, use (2.1) to define D_z for $\operatorname{Re} z > -2$

$$\langle D_z, f\rangle = (z+1)\Gamma\left(\tfrac{z+3}{2}\right)^{-1} \int |u-1|^z \psi(u-1)\left(f(u) - f(1)\right)\, du + a_z f(1) \tag{2.1}$$

for a suitable constant a_z . Because of the Γ function normalization we have

$$\langle D_{-1}, f\rangle = f(1)\,. \tag{2.2}$$

We now define an analytic family of distributions K_z, acting on the Schwartz class, $\mathcal{S}(\mathbb{R}^2)$, as follows:

$$\langle K_z, h\rangle = pv \int \langle D_z(u), h(t, ut^2)\rangle\; \frac{dt}{t}\,, \tag{2.3}$$

where the integrand in (2.3) denotes the result of the action of D_z on the function $u \to h(t, ut^2)$. Our analytic family H_z is given by convolution with K_z, that is,

$$(H_z f)(x) = pv \int \langle D_z(u), f(x_1 - t, x_2 - ut^2\rangle\; \frac{dt}{t}\,.$$

In view of (2.2), H_{-1} is the Hilbert transform along the parabola (t, t^2) studied in [SWA].

Fourier transform calculations and the method of stationary phase give the following:

Theorem 1. *For* $\operatorname{Re} z > -2$*,* $\hat{K}_z(\xi)$ *is a* C^∞ *function on* $\mathbb{R}^2 \setminus \{\xi_2 = 0\}$ *and for fixed* $\xi_1 \neq 0$ *equals*

$$(\operatorname{sgn}\, \xi_1) C_{0,z} + C_{1,z} \left|\frac{\sqrt{|\xi_2|}}{\xi_1}\right|^{2z+3} e^{i\,\frac{\pi}{2}\;\frac{\xi_1^2}{\xi_2}} + O\left(\left|\frac{\sqrt{|\xi_2|}}{\xi_1}\right|^{2z+4}\right)$$

as $\xi_2 \to 0$*.* ($C_{0,z}, C_{1,z}$ *are nonzero constants.*)

As a corollary we get that H_z maps L^2 to L^2 if and only if $\operatorname{Re} z \geq -3/2$. Our next result is the following:

Theorem 2. *For* $\operatorname{Re} z = -1$, H_z *maps* H^1 *to* $L^{1,\infty}$.

Here H^1 denotes the usual parabolic real Hardy space homogeneous under the family of dilations $(x_1, x_2) \to (rx_1, r^2x_2)$ as defined in [CT1]. This result is an extension of Theorem 3 in [C1]. Surprisingly, this theorem is sharp in the sense that H_z are not of weak type $(1,1)$ when $\operatorname{Re} z = -1$ and z is not -1. Therefore we have explicit examples of operators with the same homogeneity as the parabolic Hilbert transform H_{-1} which are not of weak type $(1,1)$. However, we don't know whether H_{-1} is of weak type $(1,1)$.

In section 7 we discuss an interpolation theorem (Theorem 3), that enables us to replace L^1 by H^1 in the usual analytic interpolation when the target spaces are arbitrary Lorentz spaces $L^{p,q}$. As a corollary we obtain:

Theorem 4. *For* $\operatorname{Re} z = 1/p - 2$, $1 < p < 2$, H_z *maps* L^p *to* $L^{p,p'}$.

Our result is the best possible in the sense that H_z doesn't map L^p to $L^{p,\infty}$ when $\operatorname{Re} z < 1/p - 2$. However, we don't know whether H_z maps $L^p \to L^p$ when $\operatorname{Re} z = 1/p - 2$.

Finally we would like to make the following notational convention. Throughout this paper, C_z, c_z will denote constants positive or complex that depend only on the fixed parameters of the problem and on z and are allowed to grow at most exponentially in $\operatorname{Im} z$ as $|\operatorname{Im} z| \to \infty$.

3. Fourier transform asymptotics and L^2 estimate. In this section we will compute the Fourier transforms $\hat{K}_z$ of our distributions K_z. It will turn out that

$$\hat{K}_z(\xi_1, \xi_2) = \lim_{\substack{\delta \to 0 \\ N \to \infty}} \int_{\delta \le |t| \le N} \hat{D}_z(t^2\xi_2) e^{-2\pi i t \xi_1} \, \frac{dt}{t}\,, \tag{3.1}$$

when $\operatorname{Re} z > -2$. Before we prove (3.1) we will study the functions

$$G_{z,\delta,N}(\xi_1, \xi_2) = \int_{\delta \le |t| \le N} \hat{D}_z(t^2\xi_2) e^{-2\pi i t \xi_1} \, \frac{dt}{t}$$

and

$$G_z(\xi) = \lim_{\substack{\delta \to 0 \\ N \to \infty}} G_{z,\delta,N}(\xi)\,.$$

We start with the following

Lemma 3.1. *For all* z *with* $\operatorname{Re} z > -2$ *and all* $\xi \in \mathbb{R}^2$,

$$\lim_{\substack{\delta \to 0 \\ N \to \infty}} G_{z,\delta,N}(\xi) = G_z(\xi) \quad \textit{exists}\,.$$

Proof. Fix z with $\operatorname{Re} z > -2$. First note that

$$\hat{D}_z(v) = 2 \left(\frac{|u|^z \psi(u)}{\Gamma\left(\frac{z+1}{2}\right)} \right)^{\wedge} (v) e^{-2\pi i v} . \tag{3.2}$$

By a formula on page 359 in [GS], (3.2) is equal to

$$c \, \frac{2^z}{\Gamma\left(-\frac{z}{2}\right)} \, (|\cdot|^{-z-1} * \hat{\psi})(v) e^{-2\pi i v}, \quad c \neq 0 . \tag{3.3}$$

Let's call

$$L_z(v) = c \, \frac{2^z}{\Gamma\left(-\frac{z}{2}\right)} \, (|\cdot|^{-z-1} * \hat{\psi})(v) .$$

L_z is a C^∞ even function on the real line because ψ was chosen to be even. We will need the following lemma whose proof we postpone until the end of this section.

Lemma 3.2. *There exists a nonzero constant C_z such that $L_z(v) = C_z |v|^{-z-1} + R_z(v)$ where $R_z(v)$ as well as all of its derivatives are*

$$O(|v|^{-M}) \quad \forall\, M > 0 \quad as \quad |v| \to \infty$$

with bounds that grow at most exponentially in $|\operatorname{Im} z|$ as $|\operatorname{Im} z| \to \infty$.

We now continue the proof of Lemma 3.1. Fix $(\xi_1, \xi_2) = \xi \in \mathbb{R}^2$. If $\xi_2 = 0$ the assertion of Lemma 3.1 is trivial. We may therefore assume that $\xi_2 \neq 0$. Set $\lambda = \xi_1/\sqrt{|\xi_2|}$. Also set $\varepsilon_1 = \operatorname{sgn}\, \xi_1$, $\varepsilon_2 = \operatorname{sgn}\, \xi_2$, $\delta' = \delta|\xi_2|^{-1/2}$, $N' = N|\xi_2|^{-1/2}$. ($\operatorname{sgn} x$ is by definition 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.) We then have

$$\begin{aligned} G_{z,\delta,N}(\xi) &= \int_{\delta \leq |t| \leq N} L_z(t^2 \xi_2) e^{-2\pi i (t\xi_1 + t^2 \xi_2)} \, \frac{dt}{t} \\ &= \int_{\delta' \leq |t| \leq N'} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \, \frac{dt}{t} \end{aligned} \tag{3.4}$$

where we used the evenness of L_z in the change of variables in (3.4). Since $\delta \to 0$, $N \to \infty$ and ξ is fixed we can assume that $\delta' \leq 1 \leq N'$. Write (3.4) as (3.5) + (3.6) where

$$\int_{\delta' \leq |t| \leq 1} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \, \frac{dt}{t} \tag{3.5}$$

and

$$\int_{1 \leq |t| \leq N'} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \, \frac{dt}{t} . \tag{3.6}$$

(3.5) is equal to

$$\begin{aligned}(3.7)\qquad &\int_{\delta'\le|t|\le 1}(L_z(t^2)e^{-2\pi i\varepsilon_2 t^2}-L_z(0))e^{-2\pi i\lambda t}\ \frac{dt}{t}\\ &\qquad + L_z(0)\int_{\delta'\le|t|\le 1}\ \frac{e^{-2\pi i\lambda t}-1}{t}\ dt\,.\end{aligned}$$

Because of the smoothness of L_z, (3.7) has a limit as $\delta'\to 0$ (equivalently $\delta\to 0$) equal to

$$(3.8)\qquad \int_{|t|\le 1}(L_z(t^2)e^{-2\pi i\varepsilon_2 t^2}-L_z(0))e^{-2\pi i\lambda t}\ \frac{dt}{t}+L_z(0)\int_{|t|\le 1}\frac{e^{-2\pi i\lambda t}-1}{t}\ dt\,.$$

We now treat (3.6). We make use of Lemma 3.2 to rewrite (3.6) as

$$(3.9)\qquad c_z\int_{1\le|t|\le N'}|t|^{-2z-2}e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}\ \frac{dt}{t}+R_z^1(\lambda,N')$$

where

$$R_z^1(\lambda,N')=\int_{1\le|t|\le N'}R_z(t^2)e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}\ \frac{dt}{t}$$

(R_z as in Lemma 3.2). The estimates for R_z show that the integrand above decays like $|t|^{-M}$ $\forall\ M>0$ as $|t|\to 0$ as therefore $R_z^1(\lambda,N')$ has a limit as $N'\to\infty$ (or $N\to\infty$). We now prove a similar result for the main term in (3.9). We write it as

$$(3.10)\qquad c_z\left[A_\lambda(t)e^{-2\pi i\varepsilon_2 t^2}\right]_{|t|=1}^{N'}-c_z\int_{|t|=1}^{N'}\frac{dA_\lambda(t)}{dt}\ e^{-2\pi i\varepsilon_2 t^2}\ dt$$

where $A_\lambda(t)=(-4\pi i\varepsilon_2)^{-1}|t|^{-2z-4}e^{-2\pi it\lambda}$. We integrate by parts again to write (3.10) as

$$\begin{aligned}(3.11)\qquad & c_{z,1}\varepsilon_2^{-1}\left[|t|^{-2z-4}e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}\right]_{|t|=1}^{N'}\\ &+c_{z,2}\varepsilon_2^{-2}\left[t^{-1}|t|^{-2z-5}e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}\right]_{|t|=1}^{N'}\\ &+c_{z,3}\varepsilon_2^{-2}\lambda\left[t^{-1}|t|^{-2z-4}e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}\right]_{|t|=1}^{N'}\\ &+c_{z,4}\varepsilon_2^{-2}\int_{|t|=1}^{N'}e^{-2\pi i\varepsilon_2 t^2}\ \{\frac{d}{dt}\left(\frac{1}{t}\frac{dA_\lambda}{dt}\right)\}\ dt\,.\end{aligned}$$

It is easy to see that the expression inside the curly brackets in (3.11) decays at least like $|t|^{-2\mathrm{Re}z-5}$ as $|t| \to \infty$. Since $\mathrm{Re}\, z > -2$, (3.11) has a limit as $N' \to \infty$. We have now proved that (3.6) has a limit as $N' \to \infty$ (equivalently $N \to \infty$) which is equal to

$$\begin{aligned} & c'_{z,1}\varepsilon_2^{-1}e^{-2\pi i\lambda} + c'_{z,2}e^{-2\pi i\lambda} + c'_{z,3}\lambda e^{-2\pi i\lambda} \\ & + c_{z,4}\int_{|t|=1}^{\infty} e^{-2\pi i\varepsilon_2 t^2}\left\{\frac{d}{dt}\left(\frac{1}{t}\frac{dA_\lambda}{dt}\right)\right\} dt \\ & + \int_{|t|=1}^{\infty} R_z(t^2)e^{-2\pi i(\varepsilon_2 t^2 + t\lambda)}\ \frac{dt}{t} \quad . \end{aligned} \tag{3.12}$$

Lemma 3.1 is now proved. Notice that $G_z(\xi) = (3.8) + (3.12)$.

Next, we study the functions G_z, $\mathrm{Re}\, z > -2$. We prove that they are C^∞ off the ξ_1-axis and we find their asymptotic behavior as ξ_2 approaches zero. Later we prove that $G_z = \hat{K}_z$, $\mathrm{Re}\, z > -2$ and therefore Theorem 1 will describe the behavior of the Fourier transforms of K_z. Until the end of this section, z will denote a complex number with real part greater than -2.

Theorem 1. *$G_z(\xi)$ is a C^∞ function except at $\xi_2 = 0$ and behaves asymptotically like*

$$\begin{aligned} & (\mathrm{sgn}\ \xi_1)\cdot C_{0,z} + C_{1,z}\left|\frac{\sqrt{|\xi_2|}}{\xi_1}\right|^{2z+3} e^{i\,\frac{\pi}{2}\,\frac{\xi_1^2}{\xi_2}} \\ & + O\left(\left|\frac{\sqrt{|\xi_2|}}{\xi_1}\right|^{2z+4}\right) \quad as\ \xi_2 \to 0\,. \end{aligned}$$

$C_{0,z}$ is a fixed nonzero constant and $C_{1,z} = C_{1,z}(\mathrm{sgn}\ \xi_2)$ is a nonzero constant depending on sgn ξ_2.

Proof. We start by proving the smoothness of $G_z(\xi)$ when $\xi_2 \neq 0$. It suffices to show that (3.8) and (3.12) are smooth functions of $\lambda = \xi_1 / \sqrt{|\xi_2|}$.

Near ξ, when $\xi_2 \neq 0$, ε_2 is a constant. Then differentiation under the integral sign shows that (3.8) is a C^∞ function of λ. We will now prove the same for (3.12). Clearly (3.12) is a continuous function of λ. To prove that it is C^∞ we need to be able to differentiate under the integral signs. Each time we differentiate with respect to λ we pick up a factor of t which worsens the convergence of the integrals in (3.12). Suppose we want to show that (3.12) is C^k. After $k-2$ partial integrations we write (3.12) as

$$\sum_{j=0}^{k-1} c'_{z,j}\lambda^j e^{-2\pi i\lambda} + \sum_{j=0}^{k} c_{z,j}\lambda^j \int_{|t|=1}^{\infty} e^{-2\pi i(\varepsilon_2 t^2 + \lambda t)}A_{z,j}(t)\ dt \tag{3.13}$$

where each $A_{z,j}(t)$ decays at least like $|t|^{-2\operatorname{Re} z-3-2k+j}$ as $|t| \to \infty$ and the constants $c'_{z,j}$, $c_{z,j}$ depend on ε_2. Near ξ, ε_2 is a constant and differentiation under the integral sign shows that (3.13) is C^k for all k. Since k was arbitrary, (3.12)=(3.13) is C^∞.

To study the asymptotics of G_z as $\xi_2 \to 0$ introduce two even C^∞ functions ζ, $\phi \geq 0$ with compact support on the real line such that

(i) ϕ is supported in $|t| \in [\frac{1}{4}, 1]$ and is equal to 1 for $|t| \in [\frac{3}{8}, \frac{6}{8}]$.

(ii) $\zeta(t)$ is supported in $|t| \leq 100$ and is equal to 1 for $|t| \leq 50$.

We may assume that $|\lambda| > 1000$. Because of (3.4), $G_z(\xi)$ is equal to

$$\lim_{\substack{\delta' \to 0 \\ N' \to \infty}} \int_{\delta' \leq |t| \leq N'} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \, \frac{dt}{t} = (3.14) + (3.15) + (3.16) + (3.17)\,,$$

where

$$pv \int_{|t| \leq 100} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \zeta(t) \, \frac{dt}{t} \tag{3.14}$$

$$\int_{50 \leq |t| \leq \frac{3|\lambda|}{8}} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} (1 - \zeta(t)) \left(1 - \phi(|\lambda|^{-1} t)\right) \frac{dt}{t} \tag{3.15}$$

$$\int_{\frac{|\lambda|}{4} \leq |t| \leq |\lambda|} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \phi(|\lambda|^{-1} t) \, \frac{dt}{t} \tag{3.16}$$

$$\lim_{N' \to \infty} \int_{\frac{6|\lambda|}{8} \leq |t| \leq N'} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \left(1 - \phi(|\lambda|^{-1} t)\right) \, \frac{dt}{t}\,. \tag{3.17}$$

Notice that $1 - \zeta$ doesn't appear in (3.16) or (3.17) because $\zeta(t) = 0$ when $|t| \geq 1$. To treat (3.14) we need the following lemma whose proof we postpone until the end of this section.

Lemma 3.3. *Let $a(u)$ be a $C_0^\infty(\mathbb{R})$ function. Then*

$$pv \int e^{i\lambda u} a(u) \, \frac{du}{u} = a(0) i\pi \operatorname{sgn} \lambda + O(|\lambda|^{-M})$$
$$\textit{for all } M > 0 \textit{ as} |\lambda| \to \infty\,.$$

To apply the lemma, set $a(t) = L_z(t^2) e^{-2\pi i \varepsilon_2 t^2} \zeta(t)$. A simple change of variables $t \to -2\pi t$ shows that (3.14) $= -i\pi L_z(0) \operatorname{sgn} \xi_1 + O(|\lambda|^{-M})$ for all M as $|\lambda| \to \infty$.

Using the fact that $L_z(0) \neq 0$ and by choosing $M > 2\operatorname{Re} z + 4$ we conclude that (3.14) $= (\operatorname{sgn} \xi_1) \cdot C_{0,z} + O(|\lambda|^{-2z-4})$ as $|\lambda| \to +\infty$, $C_{0,z} \neq 0$.

We now turn to (3.15). Change variables $t \to |\lambda|^{-1}t$ to rewrite (3.15) as

$$\int_{\frac{50}{\lambda}\le|t|\le\frac{3}{8}} L_z(|\lambda|^2t^2)e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)}(1-\phi(t))(1-\zeta(|\lambda|t))\ \frac{dt}{t}\,. \tag{3.18}$$

Set $\phi_1 = 1-\phi$, $\zeta_1 = 1-\zeta$ for simplicity.

By Lemma 3.2, (3.18) is equal to

$$C_z|\lambda|^{-2z-2}\int_{\frac{50}{|\lambda|}\le|t|\le\frac{3}{8}} |t|^{-2z-2}e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)}\phi_1(t)\zeta_1(|\lambda|t)\ \frac{dt}{t}+R_z^2(\lambda) \tag{3.19}$$

where

$$R_z^2(\lambda) = \int_{\frac{50}{|\lambda|}\le|t|\le\frac{3}{8}} R_z(|\lambda|^2t^2)e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)}\phi_1(t)\zeta_1(|\lambda|t)\ \frac{dt}{t}\,.$$

We treat the main term in (3.19) by a sequence of partial integrations. The phase function $-2\pi(\varepsilon_1 t+\varepsilon_2 t^2)$ has no critical points in $\{t : |t| \le \frac{3}{8}\}$ and all the boundary terms vanish. If we set $B_0(t) = |t|^{-2z-2}t^{-1}\phi_1(t)\zeta_1(|\lambda|t)$ and for $n \ge 0$

$$B_{n+1}(t) = \left(\frac{B_n(t)}{2\varepsilon_2 t+\varepsilon_1}\right)'$$

we can write the main term in (3.19) as

$$C_z|\lambda|^{-2z-2-2M}\int_{\frac{50}{|\lambda|}\le|t|\le\frac{3}{8}} e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)}B_M(t)\ dt\,. \tag{3.20}$$

It remains to control B_M in terms of λ. An easy inductive argument shows that

$$|B_M(t)| \le C_M\sum_{k=0}^{M}|B_0^{(k)}(t)|\,.$$

An application of Leibniz's rule gives

$$\begin{aligned}|B_0^{(k)}(t)| &\le C_{z,k}\sum_{j=0}^{k}\binom{k}{j}|(|t|^{-2z-2}t^{-1}\phi_1(t))^{(j)}\zeta_1(|\lambda|t)^{(k-j)}| \\ &\le C_{z,k}\sum_{j=0}^{k}|t|^{-2\mathrm{Re}z-3-j}|\lambda|^{k-j}\end{aligned}$$

$$\leq C_{z,k}\sum_{j=0}^{k}|\lambda|^{2\mathrm{Re}z+3+j}|\lambda|^{k-j}\leq C_{z,k}|\lambda|^{\mathrm{Re}z+3+k}$$

on the support of $\phi_1(t)\zeta_1(|\lambda|t)$.

It follows that $\|B_M\|_{L^\infty}\leq C_{z,M}|\lambda|^{\mathrm{Re}\,z+3+M}$.

We now have that for all $M>0$

$$\begin{aligned}|(3.20)|&\leq C_{z,M}|\lambda|^{-2\mathrm{Re}\,z-2-2M}|\lambda|^{\mathrm{Re}\,z+3+M}\left(\frac{3}{8}-\frac{50}{|\lambda|}\right)\\&\leq C_{z,M}|\lambda|^{\mathrm{Re}\,z+1-M}\,.\end{aligned}$$

The same argument, together with the estimates for the derivatives of R_z (Lemma 3.2) prove that $R_z^2(\lambda)$ is $O(|\lambda|^{-M})\ \forall\ M$ as $|\lambda|\to\infty$.

By choosing M large enough we get that (3.15) = (3.18) = (3.19) is $O(|\lambda|^{-2z-4})$ as $|\lambda|\to\infty$.

We now treat (3.16). First change variables $t\to|\lambda|^{-1}t$ and then use Lemma 3.2 to write (3.16) as

$$C_z|\lambda|^{-2z-2}\int_{\frac{1}{4}\leq|t|\leq 1}|t|^{-2z-2}e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)}\phi(t)\ \frac{dt}{t}+R_z^3(\lambda) \tag{3.21}$$

where

$$R_z^3(\lambda)=\int_{\frac{1}{4}\leq|t|\leq 1}R_z(|\lambda|^2t^2)e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon 2t^2)}\phi(t)\ \frac{dt}{t}\,.$$

The behavior of R_z at infinity shows that $R_z^3(\lambda)$ is $O(|\lambda|^{-M})\ \forall\ M>0$ as $|\lambda|\to\infty$. The derivative of the phase function $-2\pi(\varepsilon_1 t+\varepsilon_2 t^2)$ of the main term in (3.21) has only one zero $t_0=-\varepsilon_1/2\varepsilon_2$ on the support of ϕ and the second derivative of the phase function never vanishes. By the method of stationary phase ([HO] Theorem 7.7.5 Vol. I) the main term in (3.21) behaves asymptotically as $|\lambda|\to\infty$ like

$$\begin{aligned}&C_z|\lambda|^{-2z-2}\left[e^{-2\pi i|\lambda|^2\left(\frac{\varepsilon_2}{4}+\varepsilon_1\frac{-\varepsilon_1}{2\varepsilon_2}\right)}\left(\frac{|\lambda|^2(-4\pi\varepsilon_2)}{2\pi i}\right)^{-1/2}+O(|\lambda|^{-2})\right]=\\&C_{1,z}(\varepsilon_2)|\lambda|^{-2z-3}e^{i\frac{\pi}{2}\,\varepsilon_2|\lambda|^2}+O(|\lambda|^{-2z-4})\end{aligned} \tag{3.22}$$

for some nonzero constant $C_{1,z}$ depending on ε_2. We have now proved that (3.22) describes the asymptotic behavior of (3.16) as $|\lambda|\to\infty$. Finally we treat (3.17). Change variables $t\to|\lambda|^{-1}t$ and use Lemma 3.2 to write (3.17) as

$$\lim_{N'\to\infty}C_z\int_{|t|=\frac{6}{8}}^{\frac{N'}{|\lambda|}}|\lambda t|^{-2z-2}e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)}\phi_1(t)\ \frac{dt}{t}+\lim_{N'\to\infty}R_z^4(\lambda,N') \tag{3.23}$$

where

$$R_z^4(\lambda) = \int_{|t|=\frac{6}{8}}^{\frac{N'}{|\lambda|}} R_z(|\lambda|^2 t^2) e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)} \phi_1(t)\ \frac{dt}{t}\,.$$

Using that $|R_z(v)| \le C_{z,M}|v|^{-M}\ \forall\, M > 0$ we immediately deduce that $\lim_{N'\to\infty} R_z^4(\lambda, N')$ is $O(|\lambda|^{-M})\ \forall\ M > 0$ as $|\lambda| \to \infty$. Since the phase function of the main term in (3.23) has no critical points on the range of integration, a partial integration gives that the main term of (3.23) is equal to

$$C_z|\lambda|^{-2z-4} \lim_{N'\to\infty} \left\{ A(|\lambda|^{-1}N') - \int_{\frac{6}{8}\le|t|\le\frac{N'}{|\lambda|}} A'(t) e^{-2\pi i|\lambda|^2(\varepsilon_1 t+\varepsilon_2 t^2)}\ dt \right\} \tag{3.24}$$

where

$$A(t) = \frac{|t|^{-2z-2}\phi_1(t)}{-2\pi i t(\varepsilon_1 + 2\varepsilon_2 t)} \quad .$$

$A(t)$ decays like $|t|^{-2z-4}$ as $|t| \to \infty$ and its derivative decays like $|t|^{-2z-5}$ as $|t| \to \infty$.

It follows that the integral inside the curly brackets in (3.24) converges absolutely, uniformly in λ and that the (3.24) is $O(|\lambda|^{-2z-4})$ as $|\lambda| \to \infty$. This estimate concludes the proof of Theorem 1.

An immediate corollary is the following:

Proposition 1. (i) $\hat{K}_z = G_z$,

(ii) *H_z maps L^2 boundedly onto itself if and only if $\operatorname{Re} z \ge -3/2$. If the latter happens the bound grows at most exponentially in $|\operatorname{Im} z|$ as $|\operatorname{Im} z| \to \infty$.*

Proof. (i) We will first prove an estimate of the form

$$|G_{z,\delta,N}(\xi)| \le C_z p(\xi) \tag{3.25}$$

uniformly in $\delta \le 1 \le N$, where $p(\xi)$ is a function that is bounded in any compact set and has at most polynomial growth in $|\xi|$. Set as before

$$\lambda = \frac{\xi_1}{\sqrt{|\xi_2|}}\,, \qquad N' = \frac{N}{\sqrt{|\xi_2|}}\,, \qquad \delta' \frac{\delta}{\sqrt{|\xi_2|}}\,.$$

Consider first the case when λ is small. It suffices to show that (3.5) and (3.6) satisfy (3.25). We have (3.5) = (3.7) which is clearly bounded uniformly in δ. Also (3.6) = (3.9) = (3.10) + $R_z^1(\lambda, N')$ = (3.11) + $R_z^1(\lambda, N')$ which is clearly bounded by $C_z + C_z'|N'|^{-2\mathrm{Re} z-4} \le$

$C_z + C'_z|\xi_2|^{\mathrm{Re}\, z+2}$ uniformly in $N \geq 1$. Consider now the case when λ is large. Write $G_{z,\delta,N}(\xi) = (3.14)' + (3.15) + (3.16) + (3.17)'$ where

$$(3.14)' = \int_{\delta' \leq |t| \leq 100} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \zeta(t) \, \frac{dt}{t}$$

and

$$(3.17)' = \int_{\frac{6|\lambda|}{8} \leq |t| \leq N'} L_z(t^2) e^{-2\pi i (t\lambda + \varepsilon_2 t^2)} \phi_1(|\lambda|^{-1} t) \, \frac{dt}{t} \, .$$

We have

$$\begin{aligned} |(3.14)'| \leq & \left| \int_{\delta' \leq |t| \leq 100} (L_z(t^2)\zeta(t) e^{2\pi i \varepsilon_2 t^2} - L_z(0)) \, \frac{dt}{t} \right| \\ & + \left| L_z(0) \int_{\delta' \leq |t| \leq 100} e^{-2\pi i t\lambda} \, \frac{dt}{t} \right| \leq C_z \end{aligned}$$

where we made use of the simple fact that for all $0 < a < b < \infty$

$$\left| \int_{a \leq |t| \leq b} \frac{e^{it}}{t} \, dt \right| \leq 10 \, .$$

Also, an easy examination of (3.23) and (3.24) shows that

$$\begin{aligned} |(3.17)'| \leq C_z |\lambda|^{-2\mathrm{Re}\, z - 4} & \left\{ \left| A(|\lambda|^{-1} N') \right| + \int_{\frac{6}{8} \leq |t| \leq \frac{N'}{|\lambda|}} |A'(t)| \, dt \right\} \\ & + C_z \int_{\frac{6}{8} \leq |t| \leq \frac{N'}{|\lambda|}} |R_z(|\lambda|^2 t^2)| \, \frac{dt}{t} \end{aligned}$$

where $A(t)$ is as in (3.24).

Clearly the expression above grows at most polynomially in ξ and (3.25) is now proved. The value of (3.25) lies in the fact that for any $f \in \mathcal{S}(\mathbb{R}^2)$, $\int |f(\xi) G_{z,\delta,N}(\xi)| \, d\xi \leq C_z$ uniformly in δ, N. We now prove that $\hat{K}_z = G_z$, $\mathrm{Re}\, z > -2$. $\hat{K}_z$ is originally defined as a tempered distribution acting on functions $f \in \mathcal{S}(\mathbb{R}^2)$ as follows:

$$\begin{aligned} & \langle \hat{K}_z, f \rangle = \langle K_z, \hat{f} \rangle \\ = & \, pv \int \left\langle D_z(u), \iint f(\xi_1, \xi_2) e^{-2\pi i (t\xi_1 + t^2 \xi_2 u)} \, d\xi_1 \, d\xi_2 \right\rangle \frac{dt}{t} \end{aligned}$$

$$= pv \iint h(\xi) \left\langle D_z(u), e^{-2\pi i (t\xi_1 + ut^2 \xi_2)} \right\rangle \, d\xi \, \frac{dt}{t}$$

$$= \lim_{\substack{\delta \to 0 \\ N \to \infty}} \int_{\delta \le |t| \le N} \left[\iint h(\xi) e^{-2\pi i t \xi_1} \hat{D}_z(t\xi_2) \, d\xi \right] \frac{dt}{t}$$

$$= \lim_{\substack{\delta \to 0 \\ N \to \infty}} \iint h(\xi) G_{z,\delta,N}(\xi) \, d\xi$$

$$= \iint h(\xi) G_z(\xi) \, d\xi$$

where we made use of (3.25) when we applied the Lebesgue dominated theorem in the last equality.

(ii) The smoothness of $G_z = \hat{K}_z$, clearly implies that $\hat{K}_z(\xi)$ is always bounded for $|\lambda| \le C$. For λ large in view of the asymptotics of Theorem 1, $\hat{K}_z$ is bounded if and only if $-2 \operatorname{Re} z - 3 \le 0$.

We end this section by proving Lemmas 3.2 and 3.3 .

Proof of Lemma 3.2. Since ψ was chosen to be equal to 1 in some neighborhood of the origin, it follows that $\hat{\psi}$ has integral equal to 1 and vanishing moments of all orders. Fix $v \in \mathbb{R}$ so that $|v|$ is large. If $|w| \le \frac{1}{2}|v|$, the function $w \to |v - w|^{-z-1}$ is smooth and has a Taylor expansion about $w = 0$. Assume first that $\operatorname{Re} z < 0$. Then $|v|^{-z-1} \in L^1_{\text{loc}}(\mathbb{R})$ and the following identity is valid:

$$(|\cdot|^{-z-1} * \hat{\psi})(v) = \int |v - w|^{-z-1} \hat{\psi}(w) \, dw \, .$$

The above is equal to

$$\begin{aligned} & |v|^{-z-1} \int_{|w| \le \frac{1}{2}|v|} \hat{\psi}(w) \, dw \\ & + \sum_{j=1}^{M} C_{z,j} |v|^{-z-1-j} \int_{|w| \le \frac{1}{2}|v|} w^j \hat{\psi}(w) \, dw \\ & + C_{z,M} \int_{|w| \le \frac{1}{2}|v|} |v - \theta w|^{-(M+1)-z-1} w^{M+1} \hat{\psi}(w) \, dw \\ & + \int_{|w| \ge \frac{1}{2}|v|} |v - w|^{-z-1} \hat{\psi}(w) \, dw \end{aligned} \tag{3.26}$$

for some θ in $(0, 1)$.

Using the properties of $\hat{\psi}$ we can write (3.26) as

$$
\begin{aligned}
& |v|^{-z-1} \\
& - \sum_{j=1}^{M} C_{z,j} |v|^{-z-i-j} \int_{|w| \geq \frac{1}{2}\,|v|} w^j \hat{\psi}(w)\, dw \\
& - |v|^{-z-1} \int_{|w| \geq \frac{1}{2}\,|v|} \hat{\psi}(w)\, dw \\
& + C_{z,M} \int_{|w| \leq \frac{1}{2}\,|v|} |v - \theta w|^{(-M+1)-z-1} w^{M+1} \hat{\psi}(w)\, dw \\
& + \int_{|w| \geq \frac{1}{2}\,|v|} |v - w|^{-z-1} \hat{\psi}(w)\, dw\,.
\end{aligned}
\tag{3.27}
$$

Because of the rapid decay of $\hat{\psi}$ at infinity, the second and third terms in (3.27) decay like $|v|^{-M}$ $\forall\ M > 0$ as $|v| \to \infty$. Since $\operatorname{Re} z < 0$, $|v|^{-z-1}$ is locally integrable and the fifth term in (3.27) is absolutely bounded by

$$
\begin{aligned}
& C_{z,M} \sup_{|w| \geq \frac{1}{2}\,|v|} |\hat{\psi}(w)| \int_{|w| \geq \frac{1}{2}\,|v|} |v - w|^{-\operatorname{Re} z - 1}\, dw \\
& \quad \leq C_{z,M} |v|^{-M} \quad \forall\ M > 0 \text{ as } |v| \to \infty\,.
\end{aligned}
$$

Finally, let's call $R_z(v)$ the fourth term in (3.27). First note that since $|w| \leq \frac{1}{2}\,|v|$, $|v - \theta w|$ and $|v - w|$ are comparable. We have

$$
\begin{aligned}
|R_z(v)| &\leq C_{z,M} \int_{|w| \leq \frac{1}{2}\,|v|} |w|^{M+1} |\hat{\psi}(w)|\, |v - w|^{-\operatorname{Re} z - M - 2}\, dw \\
&\leq C_{z,M} \int_{|w| \leq \frac{1}{2}\,|v|} |v - w|^{-\operatorname{Re} z - M - 2}\, dw \\
&\leq C_{z,M} |v|^{-\operatorname{Re} z - M - 1}\,.
\end{aligned}
$$

one can easily verify that every derivative of $R_z(v)$ is also $O(|v|^{-M})$ $\forall\ M$ as $|v| \to \infty$.

Since $L_z(v)$ is a nonzero multiple of (3.27), Lemma 3.1 is completely proved at least when $\operatorname{Re} z < 0$. The remaining z's can be treated similarly when we write an appropriate formula for the convolution $(|\cdot|^{-z-1} * \hat{\psi})(v)$, but we are not going to do this since we are only interested in the range $\operatorname{Re} z < 0$.

Proof of Lemma 3.3. Set $b(u) = (a(u) - a(0))/u$ and choose an R_0 such that the support of a is contained in $[-R_0, R_0]$. One can easily see that b is a C^∞ function. An application of Leibniz's rule shows that

$$b^{(j)}(u) = \frac{(-1)^{j+1} a(0)\, j!}{u^{j+1}} \quad \text{whenever } |u| \geq R_0 \,. \tag{3.28}$$

We may assume $\lambda > 0$. The case $\lambda < 0$ follows from the case $\lambda < 0$ and a change of variables $u \to -u$.

For any $R \geq R_0$ write

$$\begin{aligned} & pv \int a(u) e^{i\lambda u} \, \frac{du}{u} \\ &= \int_{-R}^{R} b(u) e^{i\lambda u} du + a(0) pv \int_{-R}^{R} \frac{e^{i\lambda u}}{u} \, du \\ &= \int_{-R}^{R} b(u) e^{i\lambda u} du + a(0) pv \int_{-R}^{R} \frac{e^{iu}}{u} \, du + a(0) \int_{R \leq |u| \leq R\lambda} \frac{e^{iu}}{u} \, du \end{aligned}$$

$N-1$ partial integrations by parts give:

$$\begin{aligned} & \sum_{j=0}^{N-1} (-1)^{j+1} \left[b^{(j)}(u) \frac{e^{i\lambda u}}{(i\lambda)^{j+1}} \right]_{-R}^{R} + (-1)^N \int_{-R}^{R} b^{(N)}(u) \, \frac{e^{i\lambda u}}{(i\lambda)^N} \, du \\ & + a(0) pv \int_{-R}^{R} \frac{e^{iu}}{u} \, du + a(0) \sum_{j=0}^{N-1} (-1)^{j+1} \left[\frac{e^{iu} (-1)^j \, j!}{(iu)^{j+1}} \right]_{-R\lambda}^{R\lambda} \\ & - a(0) \sum_{j=0}^{N-1} (-1)^{j+1} \left[\frac{e^{iu} (-1)^j j!}{(iu)^{j+1}} \right]_{-R}^{R} + (-1)^N \int_{R \leq |u| \leq R\lambda} \frac{(-1)^N N! e^{iu}}{u^{N+1} (iu)^N} \, du \,. \end{aligned} \tag{3.29}$$

Because of (3.28) the first and the fifth term in (3.29) cancel out. The fourth term in (3.29) is $O(R^{-1})$ as $R \to +\infty$. The sixth term in (3.29) is $O(R^{-N})$, as $R \to \infty$. Again because of (3.28) the second term in (3.29) can be written as

$$(i\lambda^{-1})^N \left\{ \int_{-R_0}^{R_0} b^{(N)}(u) e^{i\lambda u} du + \int_{R_0 \leq |u| \leq R} (-1)^{N+1} a(0) N! e^{i\lambda u} u^{(N+1)} \, du \right\} . \tag{3.30}$$

The first integral in (3.30) is independent of R and the second integral converges absolutely.

Clearly $\lim_{R\to\infty}$ (3.30) is $O(\lambda^{-N})$ as $\lambda \to \infty$. Letting $R \to \infty$ in (3.29) we get

$$pv \int a(u) e^{i\lambda u} \, \frac{du}{u} = a(0) \lim_{R \to \infty} pv \int_{-R}^{R} \frac{e^{iu}}{u} \, du + O(\lambda^{-N})$$

$$= a(0)2i \int_0^\infty \frac{\sin u}{u}\, du + O(\lambda^{-N}) = \pi i a(0) + O(\lambda^{-N}) \text{ as } \quad \lambda \to +\infty\,.$$

Lemma 3.3 is now proved.

4. Preliminaries for Theorem 2. By a cube we shall mean a closed rectangle Q in $\mathbb{R}^2$ with sides parallel to the axes, of horizontal sidelength 2^t and of vertical sidelength 2^{2t} for some t real. For each cube Q with sidelengths $(2^\sigma, 2^{2\sigma})$ we write $\sigma(Q) = \sigma$. Q is said to be dyadic if $\sigma(Q) = \sigma$ is an integer and if its lower left-hand vertex is located at a point of the form $(i2^\sigma, j2^{2\sigma})$ for some $i, j \in \mathbb{Z}$. Any two dyadic cubes that do not contain each other must have disjoint interiors.

Following [C1], for each σ, $\tau \in \mathbb{Z}$ with $\tau \geq \sigma$, let $R_{\sigma,\tau}$ denote the set of all closed rectangles with sides parallel to the axes, of horizontal dimension 2^σ, of vertical dimension $2^{\sigma+\tau}$ and with lower left-hand vertex at a point of the form $(i2^\sigma, i2^{\sigma+\tau})$ for some $i, j \in \mathbb{Z}$. According to our notation, $R_{\sigma,\sigma}$ denotes the set of all dyadic cubes Q with $\sigma(Q) = \sigma$. For each $q \in R_{\sigma,\tau}$ let $\sigma(q) = \sigma$ and $\tau(q) = \tau$. The triple q^* of q in $R_{\sigma,\tau}$ is the union of those nine rectangles in $R_{\sigma,\tau}$ which meet q. For each $q \in R_{\sigma,\tau}$ we denote by $T(q)$ the set $q^* + \{(t, t^2) : 0 \leq |t| \leq 2^{\tau(q)+2}\}$. The set $T(q)$ is called the (two-sided) tendril of q. For $q \in R_{\sigma,\tau}$, $|T(q)| \sim 2^{\sigma+2\tau}$ since $T(q)$ is essentially the union of $C2^{\tau-\sigma}$ rectangles in $R_{\sigma,\tau}$. ($|B|$ denotes the Lebesgue measure of the set B.)

An atom is a function a, supported in some cube Q which satisfies $|a(x)| \leq |Q|^{-1}\chi_Q(x)$ and $\int a_Q(x)\, dx = 0$. By χ_A we denote the characteristic function of the set A.

The parabolic real variable Hardy space $H^1(\mathbb{R}^2)$, henceforth H^1, is the subspace of $L^1(\mathbb{R}^2)$ consisting of all f which admit representations of the form $\sum_Q \lambda_Q a_Q$, where each Q is a cube, each a_Q is an atom supported in Q and $\{\lambda_Q\}$ is a sequence of complex numbers in ℓ^1. $\|f\|_{H^1}$ is defined to be the infimum of $\sum |\lambda_Q|$ over all representations of f as $\sum \lambda_Q a_Q$. H^1_{dyadic} is the subspace of H^1 consisting of all $f = \sum \lambda_Q a_Q$ in which every cube Q is dyadic. Our basic result is

Theorem 2. *For* $\operatorname{Re} z = -1$, H_z *maps* H^1 *to* $L^{1,\infty}$ *with a bound which grows at most exponentially in* $|\operatorname{Im} z|$, *as* $|\operatorname{Im} z| \to \infty$.

We are given an $\alpha > 0$, an $f \in H^1$ and a $z \in \mathbb{C}$ with $\operatorname{Re} z = -1$. α, f and z will be fixed until the end of the proof (end of Section 6). We can assume that f is a *finite* sum $\sum \lambda_Q a_Q$ and $\sum |\lambda_Q| \leq 2\|f\|_{H^1}$. Once the theorem is proved for such f, the general case will follow by a limiting argument. We can also assume that each λ_Q in the representation of f is positive, since we can always multiply by a scalar of modulus one to achieve this. Finally, we will assume that $f \in H^1_{\text{dyadic}}$. This is because of the following proposition whose proof we postpone until the end of this section.

Proposition 2. *If* T *is a convolution operator and* T *maps* H^1_{dyadic} *to* $L^{1,\infty}$ *then* T *maps* H^1 *to* $L^{1,\infty}$.

Let $\mathcal{F}$ denote the (finite) family of dyadic cubes appearing in the atomic decomposition of f. We state two lemmas which can be found in [C1].

Lemma 4.1. *For any $\alpha > 0$ and any finite collection $\mathcal{F}$ of dyadic cubes Q with associated scalars $\lambda_Q > 0$, there exists a collection $\mathcal{S}$ of pairwise disjoint cubes such that*:

(i) $\sum_{Q \subset S} \lambda_Q \leq 8\alpha|S|$ *for all* $S \in \mathcal{S}$

(ii) $\sum_{S \in \mathcal{S}} |S| \leq \alpha^{-1} \sum \lambda_Q$

(iii) $\| \sum_{Q \not\subset \, any \, S \in \mathcal{S}} \lambda_Q |Q|^{-1} \chi_Q \|_{L^\infty} \leq \alpha.$

Let $\mathcal{C}$ denote the collection of all $Q \in \mathcal{F}$ such that $Q \subset S$ for some $S \in \mathcal{S}$. For each $Q \in \mathcal{C}$ we denote by S_Q the unique $S \in \mathcal{S}$ that contains Q.

Lemma 4.2. *Let there be given an $\alpha > 0$, a finite collection of dyadic cubes $\mathcal{C}$ and a collection of pairwise disjoint dyadic cubes $\mathcal{S}$ such that each $Q \in \mathcal{C}$ is contained in some $S_Q \in \mathcal{S}$. Let there also be given for each $Q \in \mathcal{C}$ a positive scalar λ_Q. Then there exist a measurable set $E \subset \mathbb{R}^2$ and a function $\kappa : \mathcal{C} \to \mathbb{Z}$ such that*

(i) $|E| \leq C \left(\alpha^{-1} \sum_{Q \in \mathcal{C}} \lambda_Q + \sum_{S \in \mathcal{S}} |S| \right)$

(ii) *For all $Q \in \mathcal{C}$ and for all $j < \kappa(Q)$*

$$Q + \{(t, t^2) : 2^{j-1} \leq |t| \leq 2^{j+1}\} \subseteq E$$

(iii) $\kappa(Q) > \sigma(S_Q)$

(iv) *For any $\sigma, \tau \in \mathbb{Z}$, $\tau \geq \sigma$ and any $q \in R_{\sigma,\tau}$,* $\displaystyle\sum_{\substack{Q \subset q \\ \kappa(Q) \leq \tau}} \lambda_Q \leq 4\alpha 2^{\sigma + 2\tau}.$

C denotes a constant independent of $\alpha, \ \mathcal{S}, e, \ \{\lambda_Q\}$.

A combination of conditions (ii) in Lemma 4.1 and (i) in Lemma 4.2 give

$$|E| \leq C\alpha^{-1} \sum_{Q \in \mathcal{C}} \lambda_Q \leq C\alpha^{-1} \|f\|_{H^1} . \tag{4.1}$$

The definitions of κ and E will be relevant to us. $\mathcal{C}$ is the union of two disjoint classes $\mathcal{C}_1$ and $\mathcal{C}_2$. Each $Q \in \mathcal{C}_1$ is assigned to a unique $q_Q \in \bigcup_\sigma \bigcup_{\tau \geq \sigma} R_{\sigma,\tau}$ with $Q \subset q_Q$ and $\kappa(Q)$ is by definition $\max(1 + \sigma(S_Q), 1 + \tau(q_Q))$. For $Q \in \mathcal{C}_2$, $\kappa(Q)$ is by definition $1 + \sigma(S_Q)$.

E is the union of $T(q_Q)$ over all $Q \in \mathcal{C}_1$ together with the union of the triples S^* over all $S \in \mathcal{S}$.

We now decompose the given $f \in H^1$ as $g + b$ where

$$g = \sum_{Q \in \mathcal{F} - \mathcal{C}} \lambda_Q a_Q, \qquad b = \sum_{Q \in \mathcal{C}} \lambda_Q a_Q \, .$$

Then

$$\|g\|_{L^2}^2 \le \|g\|_{L^1}\|g\|_{L^\infty} \le \alpha \sum_{Q\in\mathcal{F}-\mathcal{C}} \lambda_Q \le 2\alpha\|f\|_{H^1}\,.$$

We now have

$$\begin{aligned}\left|\left\{x : |(H_z g)(x)| > \frac{\alpha}{2}\right\}\right| &\le \frac{4}{\alpha^2}\,\|H_z g\|_{L^2}^2\\ &\le \frac{C_z}{\alpha^2}\,\|g\|_{L^2}^2 \le \frac{C_z}{\alpha}\,\|f\|_{H^1}\,.\end{aligned}$$

Next, we need to prove that

$$\left|\left\{x : |(H_z b)(x)| > \frac{\alpha}{2}\right\}\right| \le \frac{C_z}{\alpha}\,\|f\|_{H^1}\,. \tag{4.2}$$

Fix $\eta \in C_0^\infty(\mathbb{R})$, even, supported in $\frac{1}{2} \le |t| \le 2$ and such that $\sum_{j\in\mathbb{Z}} \eta(2^{-j}t) = 1$ for all $t \ne 0$. Let $\phi(t) = \eta(t)/t$. Define distributions $\mu_{z,j}$, $j \in \mathbb{Z}$ acting on test functions h by

$$\langle \mu_{z,j}, h\rangle = \int \langle D_z(u), h(t, ut^2)\rangle 2^{-j}\phi(2^{-j}t)\;dt\,.$$

Write $b * K_z = F_0 + F_1 + F_2$ where

$$\begin{aligned} F_0 &= \sum_{Q\in\mathcal{C}} \left(\lambda_Q a_Q * \sum_{j\le\sigma(Q)} \mu_{z,j}\right)\\ F_1 &= \sum_{Q\in\mathcal{C}} \left(\lambda_Q a_Q * \sum_{\sigma(Q)<j\le\kappa(Q)} \mu_{z,j}\right)\\ F_2 &= \sum_{Q\in\mathcal{C}} \left(\lambda_Q a_Q * \sum_{j\ge\kappa(Q)} \mu_{z,j}\right)\,. \end{aligned}$$

We will show that

$$F_0 \quad \text{is supported in} \quad E\,, \tag{4.3}$$

$$\|F_1\|_{L^1(\mathbb{R}^2\setminus E)} \le C_z \sum_{Q\in\mathcal{C}} \lambda_Q\,, \tag{4.4}$$

$$\|F_2\|_{L^2(\mathbb{R}^2)}^2 \le C_z\alpha \sum_{Q\in\mathcal{C}} \lambda_Q\,. \tag{4.5}$$

A combination of (4.1), (4.3), (4.4) and (4.5) with the aid of Chebychev's inequality will establish (4.2).

Assertion (4.3) is the easiest to prove. Write F_0 as

$$\sum_{S\in\mathcal{S}}\left[\sum_{Q\subset S}\lambda_Q a_Q * \sum_{j\leq\sigma(Q)}\mu_{z,j}\right].$$

For any fixed S the expression inside the brackets above is supported in

$$S+[-2^{\sigma(S)+1},2^{\sigma(S)+1}]\times[0,2^{\sigma(S)+1}]\subseteq S^*.$$

Therefore, F_0 is supported in $\bigcup_{S\in\mathcal{S}} S^*\subseteq E$.

Estimates (4.4) and (4.5) will be proved in Sections 5 and 6 respectively.

We end this section by proving Proposition 2.

Proof. We assume that for some constant K, $\|Th\|_{L^{1,\infty}}\leq K\|h\|_{H^1_{\text{dyadic}}}$ holds for all $h\in H^1_{\text{dyadic}}$. We are given $f\in H^1$ given as a finite sum $\sum\lambda_Q a_Q$ where each $\lambda_Q>0$ and where $\sum\lambda_Q\leq 2\|f\|_{H^1}$. Let $\mathcal{F}$ be the collection of all Q appearing in the decomposition of f. Choose M integer such that $\sigma(Q)<M$ for all $Q\in\mathcal{F}$. For each $j\in\mathbb{Z}$ consider the grids G^1_j, G^2_j, G^3_j defined as follows: G^1_j consists of all dyadic cubes of dimensions $(2^j,2^{2j})$, G^2_j consists of all elements of G^1_j translated by $(\frac{1}{3}\,2^M,\frac{1}{3}\,2^{2M})$ and G^3_j consists of all elements of G^1_j translated by $(\frac{2}{3}\,2^M,\ \frac{2}{3}\,2^{2M})$. Given any $Q\in\mathcal{F}$ we find an m_Q integer such that $m_Q-1\leq\sigma(Q)<m_Q$. It is easy to verify that every $Q\in\mathcal{F}$ is contained in some Q_d where $Q_d\in G^1_{m_Q}\cup G^2_{m_Q}\cup G^3_{m_Q}$. We now split $\mathcal{F}$ as a union of three disjoint sets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ where $\mathcal{F}_j\subseteq\{Q\in\mathcal{F}: Q_d\in G^j_{m_Q}\}$. It is immediate that $\frac{1}{8}\,a_Q$ is an atom on Q_d. We set $F_j=\sum_{Q\in\mathcal{F}_j}\lambda_Q(\frac{1}{8}\,a_Q)$.

Using that T is translation invariant and that it maps H^1_{dyadic} to $L^{1,\infty}$ we get that

$$\|TF_j\|_{L^{1,\infty}}\leq K\sum_{Q\in\mathcal{F}_j}\lambda_Q\qquad j=1,2,3.$$

Summing over j we get that $\|\frac{1}{8}\,Tf\|_{L^{1,\infty}}\leq K\sum_{Q\in\mathcal{F}}\lambda_Q$, hence

$$\|Tf\|_{L^{1,\infty}}\leq 8K\sum_{Q\in\mathcal{F}}\lambda_Q\leq 16\,K\|f\|_{H^1}.$$

5. An L^1 estimate. *Until the end of Section 6 all Q considered are in $\mathcal{C}$. In all sums below this restriction is assumed to hold.*

To prove (4.4) it will suffice to show that for any Q we have

$$\left\| a_Q * \sum_{\sigma(Q)\le j<\kappa(Q)} \mu_{z,j} \right\|_{L^1(\mathbb{R}^2\setminus E)} \le C_z \, . \tag{5.1}$$

Suppose that (5.1) has been proved for all Q with $\sigma(Q) = 0$. We describe a rescaling argument that will yield the general case. Let r_j, $j \in \mathbb{Z}$ be the following family of dilations of $\mathbb{R}^2$: $r_j(x_1, x_2) = (2^j x_1, 2^{2j} x_2)$. For any cube Q, let $r_j Q = \{r_j x : x \in Q\}$. It follows from the definition of $\kappa(Q)$ in [C1] that $\sigma(Q)$ and $\kappa(Q)$ scale accordingly, i.e. $\kappa(r_j Q) - \kappa(Q) = j = \sigma(r_j Q) - \sigma(Q)$. A simple change of variables shows that for all $j, k \in \mathbb{Z}$, $\mu_{z,j} * (h \circ r_{-k}) = (\mu_{z,j-k} * h) \circ r_{-\kappa}$ where $(f \circ g)(x) = f(g(x))$. Assume now that (5.1) holds for cubes Q with $\sigma(Q) = 0$. Fix $Q \in \mathcal{C}$ and a_Q an atom supported in Q. Let $\sigma(Q) = \sigma$.

Let $Q_0 = r_{-\sigma} Q$ and define an atom $a_{Q_0} = 2^{3\sigma}(a_Q \circ r_\sigma)$ supported in Q_0 . Since $\sigma(Q_0) = 0$, (5.1) holds for Q_0. We then have

$$\begin{aligned}
&\left\| a_Q * \sum_{\sigma(Q)\le j<\kappa(Q)} \mu_{z,j} \right\|_{L^1(\mathbb{R}^2\setminus E)} \\
&= 2^{3\sigma} \left\| (a_{Q_0} \circ r_{-\sigma}) * \sum_{\sigma(Q)\le j<\kappa(Q)} \mu_{z,j} \right\|_{L^1(\mathbb{R}^2\setminus E)} \\
&= 2^{-3\sigma} \left\| \left(a_{Q_0} * \sum_{\sigma\le j<\kappa(Q_0)+\sigma} \mu_{z,j-\sigma} \right) \circ r_{-\sigma} \right\|_{L^1(\mathbb{R}^2\setminus E)} \\
&= \left\| a_{Q_0} * \sum_{0\le j<\kappa(Q_0)} \mu_{z,j} \right\|_{L^1(\mathbb{R}^2\setminus E)} \le C_z
\end{aligned}$$

and hence (5.1) is true for all Q.

We now prove (5.1) for all $Q \in \mathcal{C}$ with $\sigma(Q) = 0$. For such a Q, let S_Q be as in Lemma 4.2. If $Q \in \mathcal{C}_2$ then $\kappa(Q) = 1 + \sigma(S_Q)$ and $a_Q * \sum_{0\le j\le\sigma(S_Q)} \mu_{z,j}$ is supported in $S_Q^* \subseteq E$. Therefore only cubes $Q \in \mathcal{C}_2$ give nonzero left hand side in (5.1). Fix $Q \in \mathcal{C}_2$ and let $q = q_Q$ be the unique rectangle in $R_{\sigma,\kappa-1}$ assigned to Q as in Lemma 4.2. Set $\sigma(Q) = \sigma$, $\kappa(Q) = \kappa$. Let $\gamma(t)$, $t \in \mathbb{R}$ be a C_0^∞ function supported on the set $[2^{-1}, 2]$ such that $\sum_{m\in\mathbb{Z}} \gamma(2^{-m}t) = 1$ for all $t > 0$. Define

$$\gamma_m(x_1, x_2) = \gamma(2^{-m-\kappa}|x_2 - x_1^2|), \quad m = 1, 2, 3, \dots$$

$$\gamma_0(x_1,x_2)=\sum_{m\le 0}\gamma(2^{-m-\kappa}|x_2-x_1^2|)\,.$$

Then $\gamma_0(\sum_{0\le j<\kappa}\mu_{z,j})$ is a distribution supported in the set of all points in $\mathbb{R}^2$ of vertical distance at most $2^{\kappa+1}$ from the piece of the parabola $\{(t,t^2):0\le|t|\le 2^{\kappa+1}\}$ and therefore its convolution with a_Q is supported in $T(q)\subseteq E$.

Note that if m is bigger that $2j-\kappa+C$ then γ_m and $\mu_{z,j}$ have disjoint supports. These observations show that (5.1) will follow from

$$\sum_{0\le j<\kappa}\sum_{m=1}^{2j-\kappa+C}\|a_Q*\gamma_m\mu_{z,j}\|_{L^1}\le C_z\,. \tag{5.2}$$

We will need the following lemma whose proof we postpone until the end of this section.

Lemma 5.1. *For $m=1,2,\dots,2j-\kappa+C$*

$$\|\nabla(\gamma_m\mu_{z,j})\|_{L^\infty}\le C_z 2^{-2(\kappa+m)}\,.$$

Assuming the lemma we prove (5.2). We first compute $|\sup(\gamma_m\mu_{z,j}*a_Q)|$. The support of $\gamma_m\mu_{z,j}$ is the set of all points in $\mathbb{R}^2$ whose vertical distance from the piece of the parabola $\{(t,t^2):|t|\sim 2^j\}$ is about $2^{\kappa+m}$. It follows that $|\sup(\gamma_m\mu_{z,j})|\sim 2^{\kappa+m+j}$. Adding a cube Q of side lengths $(1,1)$ doesn't affect the size of the support of $\gamma_m\mu_{z,j}$ by more than a constant factor. Therefore

$$|\mathrm{supp}(\gamma_m\mu_{z,j}*a_Q)|\le C2^{\kappa+m+j}\,. \tag{5.3}$$

Using the fact that a_Q has mean value 0, is supported in a cube of sidelength 1 and has L^1 norm ≤ 1, we get that

$$\|\gamma_m\mu_{z,j}*a_Q\|_{L^\infty}\le C\|\nabla(\gamma_m\mu_{z,j})\|_{L^\infty}\,.$$

Lemma 5.1 gives

$$\|\gamma_m\mu_{z,j}*a_Q\|_{L^\infty}\le C_z2^{-2\kappa-2m}\,.$$

We use this estimate to prove (5.2). We have:

$$\begin{aligned}\|\gamma_m\mu_{z,j}*a_Q\|_{L^1}&\le\|\gamma_m\mu_{z,j}*a_Q\|_{L^\infty}|\mathrm{supp}(\gamma_m\mu_{z,j}*a_Q)|\\&\le C_z2^{-2\kappa-2m}2^{\kappa+m+j}\le C_z2^{-\kappa-m+j}\,.\end{aligned}$$

A summation over m $(1\le m\le 2j-\kappa+2)$ followed by a summation over j $(0\le j<\kappa)$ proves (5.2).

It remains to prove Lemma 5.1.

PROOF. For any $h \in C_0^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned}\mu_{z,j}(h) &= \int \langle D_z(u), h(t, ut^2)\rangle 2^{-j}\phi(2^{-j}t)\ dt \\ &= \int \left\langle \frac{1}{x_1^2}\ D_z\left(\frac{x_2}{x_1^2}\right), h(x_1,x_2)\right\rangle 2^{-j}\phi(2^{-j}x_1)\ dx_1\end{aligned}$$

which gives that

$$\mu_{z,j}(x_1,x_2) = 2\Gamma\left(\frac{z+1}{2}\right)^{-1} x_1^{-2}\left|\frac{x_2}{x_1^2}-1\right|^z \psi\left(\frac{x_2}{x_1^2}-1\right)2^{-j}\phi(2^{-j}x_1)\,.$$

Certainly $\gamma_m\mu_{z,j}$ is a C_0^∞ function. To estimate $\nabla(\gamma_m\mu_{z,j})$ we use Leibniz's rule. For $\alpha = 1, 2$ we have:

$$\begin{aligned}&\left\|\left[\frac{\partial}{\partial x_\alpha}\left(|x_2-x_1^2|^z\,\gamma_m(x_1,x_2)\right)\right]\psi\left(\frac{x_2}{x_1^2}-1\right)2^{-j}\phi(2^{-j}x_1)x_1^{-2z-2}\right\|_{L^\infty} \\ &\qquad\le C_2 2^{-2\kappa-2m-(\alpha-1)j} \le C_z 2^{-2\kappa-2m} \\ &\left\|\left[\frac{\partial}{\partial x_\alpha}\left(\psi\left(\frac{x_2}{x_1^2}-1\right)\right)\right]2^{-j}\phi(2^{-j}x_1)x_1^{-2z-2}|x_2-x_1^2|^z\gamma_m(x_1,x_2)\right\|_{L^\infty} \\ &\qquad\le C_z 2^{-\kappa-m-j-\alpha j} \le C_z 2^{-2\kappa-2m}\end{aligned}$$

where in the last estimate we used the fact that on the support of $\gamma_m\mu_{z,j}$

$$\frac{x_2}{x_1^2} = \frac{x_2-x_1^2}{x_1^2}+1 \le C\frac{2^{m+\kappa}}{2^{2j}}+1 \le C$$

and that $m \le 2j-\kappa+C$. Finally

$$\begin{aligned}&\left\|\left[\frac{\partial}{\partial x_1}(2^{-j}\phi(2^{-j}x_1)x_1^{-2z-2})\right]\psi\left(\frac{x_2}{x_1^2}-1\right)|x_2-x_1^2|^z\,\gamma_m(x_1,x_2)\right\|_{L^\infty} \\ &\qquad\le C_z 2^{-2j-\kappa-m} \le C_z 2^{-2\kappa-2m} \quad .\end{aligned}$$

The last estimate follows by our assumption on m. Our lemma is now proved.

6. An L^2 estimate. We remind the reader that all Q considered in this section are in $\mathcal{C}$ and that z is fixed with $\operatorname{Re} z = -1$. We begin by writing F_2 as

$$\sum_{s\ge 0}\sum_{j\in\mathbb{Z}} B_{j-s} * \mu_{z,j} \tag{6.1}$$

where

$$B_k = \sum_{\kappa(Q)=k} \lambda_Q a_Q, \quad k \in \mathbb{Z}\,.$$

If we have that

$$\text{(6.2)} \qquad \text{for } s = 0, 1, \ldots \qquad \left\| \sum_{j \in \mathbb{Z}} B_{j-s} * \mu_{z,j} \right\|_{L^2}^2 \leq C_z \alpha 2^{-s} \sum \lambda_Q$$

(4.5) will be a consequence of (6.1) and (6.2). Expanding the square out we find that the left hand side of (6.2) is equal to

$$\text{(6.3)} \qquad \sum_{j \in \mathbb{Z}} \Bigg[\|B_{j-s} * \mu_{z,j}\|_{L^2}^2 + 2 \text{ Re} \sum_{j-3<i<j} \int (B_{j-s} * \mu_{z,j}) \, (\overline{B_{i-s} * \mu_{z,i}}) \; dx + 2 \text{ Re} \sum_{i \leq j-3} \int (B_{j-s} * \mu_{z,j}) \, (\overline{B_{i-s} * \mu_{z,i}}\,) \; dx \Bigg].$$

If we can show that the expression inside the brackets in (6.3) at most

$$\text{(6.4)} \qquad C_z \alpha 2^{-s} \left[\sum_{i=-2}^{0} \sum_{\kappa(Q)=i+j-s} \lambda_Q \right]$$

then the conclusion will follow by simple summation on j. To prove this it suffices to show that the expression inside the brackets in (6.3) for $j = 0$ is less than (6.4) for $j = 0$. The general case will follow by a rescaling argument similar to the one in Section 5.

Define a singular measure $\nu_{z,0}$ supported on the parabola (t, t^2) as follows:

$$v_{z,0}(h) = \int h(t, t^2) \phi(t) t^{-2z-2} \; dt, \quad h \in C_0^\infty\,.$$

For any function h, let $\tilde{h}(x)$ denote the function $\tilde{h}(x) = h(-x)$. For any distribution D, let $\tilde{D}$ denote the distribution $\langle \tilde{D}, h \rangle = \langle D, \tilde{h} \rangle$. The complex conjugate $\bar{D}$ of a distribution D is defined by $\langle \bar{D}, h \rangle = \langle \overline{D, \bar{h}} \rangle$.

Let h_z be the distribution

$$2\Gamma\left(\tfrac{z+1}{2}\right)^{-1} |x_2|^z \psi(x_2) \delta_{x_1=0}\,.$$

Note that h_z is even, i.e. $\tilde{h}_z = h_z$.

We will need the following lemma:

Lemma 6.1. *There exists a $C_0^\infty(\mathbb{R}^2)$ function ζ_0 supported in $|x| < 20$ such that $\mu_{z,0} = \nu_{z,0} * h_z + \zeta_0$. Moreover the function ζ_0 and its C^k norms are all bounded above by constants which grow at most exponentially in $|\operatorname{Im} z|$ as $|\operatorname{Im} z| \to \infty$.*

Proof. Let $g \in \mathcal{S}(\mathbb{R}^2)$. We compute $\langle \mu_{z,0}, g\rangle - \langle \nu_{z,0} * h_z, g\rangle$. We have

$$\begin{aligned}\langle \mu_{z,0}, g\rangle &= \int \langle D_z(u), g(t, ut^2)\rangle \phi(t)\, dt \\ &= \int\int 2\Gamma\left(\tfrac{z+1}{2}\right)^{-1} \frac{1}{x_1^2}\left|\frac{x_2}{x_1^2} - 1\right|^z \psi\left(\frac{x_2}{x_1^2} - 1\right) g(x_1, x_2)\, dx_2 \phi(x_1)\, dx_1\,.\end{aligned}$$

Also

$$\begin{aligned}\langle \nu_{z,0} * h_z, g\rangle &= \langle h_z, \tilde{\nu}_{z,0} * g\rangle \\ &= \langle h_z(x_1, x_2), \int g(x_1 + t, x_2 + t^2)\phi(t) t^{-2z-2}\, dt\,\rangle \\ &= \int 2\Gamma\left(\tfrac{z+1}{2}\right)^{-1} |x_2|^z \psi(x_2) \int g(t, x_2 + t^2)\phi(t) t^{-2z-2}\, dt\, dx_2 \\ &= \int\int 2\Gamma\left(\tfrac{z+1}{2}\right)^{-1} |x_2 - x_1^2|^z \psi(x_2 - x_1^2) g(x_1, x_2)\, dx_2 \phi(x_1) x_1^{-2z-2}\, dx_1\,.\end{aligned}$$

By taking the difference we get

$$\langle \mu_{z,0} - \nu_{z,0} * h_z, g\rangle = \iint \zeta_0(x_1, x_2) g(x_1, x_2)\, dx_1\, dx_2$$

where

$$\zeta_0(x_1, x_2) = 2\Gamma\left(\tfrac{z+1}{2}\right)^{-1} x_1^{-2z-2} |x_2 - x_1^2|^z \phi(x_1)\{\psi\left(\frac{x_2}{x_1^2} - 1\right) - \psi(x_2 - x_1^2)\}\,. \tag{6.5}$$

The singularity of $|x_2 - x_1^2|^z$ at $x_2 = x_1^2$ is cut away by the expression inside the curly brackets in (6.5) which vanishes when $|x_2 - x_1^2| \leq 1/10$. Therefore $\zeta_0(x_1, x_2)$ is in $C_0^\infty(\mathbb{R}^2)$ and is clearly supported in some fixed compact set. The lemma is now proved.

As we remarked before our proof will be complete if we show the following:

$$\begin{aligned}&\|B_{-s} * \mu_{z,0}\|_{L^2}^2 + 2\ \mathrm{Re} \sum_{-3<i<0} \int (B_{-s} * \mu_{z,0})(\overline{B_{i-s} * \mu_{z,i}})\, dx \\ &\quad + 2\ \mathrm{Re} \sum_{i\leq -3} \int (B_{-s} * \mu_{z,0})\,(\overline{B_{i-s} * \mu_{z,i}})\, dx \\ &\quad \leq C_z \alpha 2^{-s} \left(\sum_{i=-2}^{0} \sum_{\kappa(Q)=i-s} \lambda_Q\right)\,.\end{aligned} \tag{6.6}$$

We first show

$$\|B_{-s} * \mu_{z,0}\|_{L^2}^2 \le C_z \alpha 2^{-s} \sum_{\kappa(Q)=-s} \lambda_Q \,. \tag{6.7}$$

We use Lemma 6.1 to write $\mu_{z,0} = \nu_{z,0} * h_z + \zeta_0$. By a formula in [GS] page 359, we get that

$$\hat{h}_z(\xi_1, \xi_2) = \hat{\delta}_{x_1=0}(D_z(x_2+1))^\wedge = C \frac{2^z |\xi_2|^{-z-1}}{\Gamma\left(-\frac{z}{2}\right)} * \hat{\psi}(\xi_2)\,.$$

Clearly $\|\hat{h}_z\|_{L^\infty} \le C_z$ and thus convolution with h_z gives a bounded operator on $L^2(\mathbb{R}^2)$ with a bound C_z that grows at most exponentially in $|\operatorname{Im} z|$. Now we get

$$\|B_{-s} * \nu_{z,0} * h_z\|_{L^2}^2 \le C_z \|B_{-s} * \nu_{z,0}\|_{L^2}^2 \,. \tag{6.8}$$

In [C1] Theorem 3, it has been shown that

$$\|B_{-s} * \mu_0\|_{L^2}^2 \le C\alpha \sum_{\kappa(Q)=-s} \lambda_Q \tag{6.9}$$

where μ_0 is the measure: $\mu_0(h) = \int h(t,t^2)\phi_0(t)\; dt$ and ϕ_0 is a fixed C_0^∞ function. A careful examination of the argument given there shows that the constant C in (6.9) comes from Lemmas 6.2 and 6.3 in [C1] and grows at most polynomially in $\|\phi_0\|_{L^\infty}$, $\|\phi_0'\|_{L^\infty}$. Setting $\phi_0(t) = \phi(t)t^{-2z-2}$ we get that

$$\|B_{-s} * \nu_{z,0}\|_{L^2}^2 \le C_z \alpha \sum_{\kappa(Q)=-s} \lambda_Q \tag{6.10}$$

with a constant C_z which grows at most polynomially in $|\operatorname{Im} z|$.

(6.8) and (6.10) give

$$\|B_{-s} * \nu_{z,0} * h_z\|_{L^2}^2 \le C_z \alpha \sum_{\kappa(Q)=-s} \lambda_Q \,. \tag{6.11}$$

(6.7) will be proved if we also show

$$\|B_{-s} * \zeta_0\|_{L^2}^2 \le C_z \alpha \sum_{\kappa(Q)=-s} \lambda_Q \,. \tag{6.12}$$

In the sequel we will use the following simple lemma whose proof we omit.

Lemma 6.2. *For every $h \in C^1(\mathbb{R}^2)$ and every Q we have*

$$\|a_Q * h\|_{L^\infty} \le 2^{\max(\sigma(Q),2\sigma(Q))}\|\nabla h\|_{L^\infty}\,.$$

To prove (6.12) we argue as follows:

$$\begin{aligned}\|B_{-s} * \zeta_0\|_{L^2}^2 &= \int (B_{-s} * \zeta_0)\,(\,\overline{B_{-s} * \zeta_0}\,) = \int \bar{B}_{-s}(B_{-s} * \zeta_0 * \bar{\tilde{\zeta}}_0)\;dx \\ &= \|B_{-s}\|_{L^1}\|\bar{B}_{-s} * \zeta_0 * \bar{\tilde{\zeta}}_0\|_{L^\infty} \le \|B_{-s}\|_{L^1}\sup_x \sum_x \lambda_Q |(a_Q * \zeta_0 * \bar{\tilde{\zeta}}_0)(x)|\end{aligned} \tag{6.13}$$

where the sum $\sum_x$ in (6.13) is taken over all $Q \in \mathcal{C}$ with $\kappa(Q) = -s$ that satisfy $Q \cap (-x + \text{ support}(\zeta_0 * \bar{\tilde{\zeta}}_0)) \ne \emptyset$.

A combination of Lemmas 6.1 and 6.2 gives that the last term in (6.13) is bounded above by

$$C_z\|B_{-s}\|_{L^1}\,\sup_x \sum_x \lambda_Q 2^{\sigma(Q)}\,. \tag{6.14}$$

The sum $\sum_x$ in (6.14) is taken over the same Q's as in (6.13). These Q's are contained in the union of a finite number of fixed cubes q of sidelengths C translated by the amount $-x$. By Lemma 5.2 (iv), $\sum_x \lambda_Q \le C\alpha$ independently of x. We use $\sigma(Q) \le \kappa(Q) < -s$ to get

$$(6.14)\ \le C_z\|B_{-s}\|_{L^1}2^{-s}\sup_x \sum_x \lambda_Q \le C_z\alpha 2^{-s}\sum_{\kappa(Q)=-s}\lambda_Q\,.$$

(6.12) is now proved and so is (6.7).

We now continue proving (6.6). Next we need to show that

$$\mathrm{Re}\sum_{-3<i<0}\int (B_{-s} * \mu_{z,0})\,(\,\overline{B_{i-s} * \mu_{z,i}}\,)\;dx \le C_z\alpha 2^{-s}\sum_{i=-2}^{0}\ \sum_{\kappa(Q)=i-s}\lambda_Q\,. \tag{6.15}$$

Apply the Cauchy-Schwartz inequality to bound the ith term in the left hand side of (6.15) by

$$\begin{aligned}&\|B_{-s} * \mu_{z,0}\|_{L^2}\|B_{i-s} * \mu_{z,i}\|_{L^2} \le \\ &\frac{1}{2}\,[\|B_{-s} * \mu_{z,0}\|_{L^2}^2 + \|B_{i-s} * \mu_{z,i}\|_{L^2}^2] \quad \text{for } i = -1, -2\,.\end{aligned}$$

We have shown that the first term above is bounded by $C_z\alpha 2^{-s}\sum_{\kappa(Q)=-s}\lambda_Q$. Rescaling shows that the second term above is bounded by $C_z\alpha 2^{-s}\sum_{\kappa(Q)=i-s}\lambda_Q$. (6.15) now follows by summing the results for $i=-1$ and $i=-2$.

The proof of (6.6) will be complete if we can establish

$$\text{Re}\sum_{i\le -3}\int (B_{-s}*\mu_{z,0})(\overline{B_{i-s}*\mu_{z,i}}\,)\le C_z\alpha 2^{-s}\sum_{\kappa(Q)=-s}\lambda_Q\,. \tag{6.16}$$

The case $i\le -3$ is different from the case $-3<i\le 0$, because when $i\le -3$ the distributions $\mu_{z,i}$ and $\mu_{z,0}$ have disjoint supports. It will turn out that in this case, the smoothness of $\mu_{z,i}*\bar{\tilde{\mu}}_{z,0}$ as well as the smoothness of $\mu_{z,i}$ away from the parabola will be crucial in the proof of (6.16).

We will use again Lemma 6.1. We have

$$\mu_{z,0}=\nu_{z,0}*h_z+\zeta_0\,.$$

For simplicity call $\zeta=\bar{\tilde{\zeta}}_0$, $\nu_z=\bar{\tilde{\nu}}_{z,0}$. Then

$$\bar{\tilde{\mu}}_{z,0}=\bar{\tilde{\nu}}_{z,0}*\bar{\tilde{h}}_z+\bar{\tilde{\zeta}}_0=\nu_z*\bar{h}_z+\zeta\,.$$

The identity

$$\int A(\,\overline{B*C}\,)\;dx=\int (A*\bar{\tilde{B}})\bar{C}\;dx$$

shows that (6.16) will follow from

$$\sum_{i\le -3}\left|\int (B_{i-s}*\mu_{z,i}*\bar{\tilde{\mu}}_{z,0})\bar{B}_{-s}\;dx\right|\le C_z\alpha 2^{-s}\sum_{\kappa(Q)=-s}\lambda_Q$$

which will be a consequence of (6.17) and (6.18).

$$\sum_{i\le -3}\left|\int (B_{i-s}*\nu_z*\bar{h}_z*\mu_{z,i})\bar{B}_{-s}\;dx\right|\le C_z\alpha 2^{-s}\sum_{\kappa(Q)=-s}\lambda_Q \tag{6.17}$$

$$\sum_{i\le -3}\left|\int (B_{i-s}*\zeta*\mu_{z,i})\bar{B}_{-s}\;dx\right|\le C_z\alpha 2^{-s}\sum_{\kappa(Q)=-s}\lambda_Q\,. \tag{6.18}$$

The proof of (6.18) is based on the following lemma:

Lemma 6.3. (i) $\zeta * \mu_{z,i}$ *is a* C_0^∞ *function supported in* $\{x : |x| < 20\}$,
(ii) $\|\zeta * \mu_{z,i}\|_{L^\infty} \le C_z$,
(iii) $\|\nabla(\zeta * \mu_{z,i})\|_{L^\infty} \le C_z$.

PROOF. By the definition of $\mu_{z,i}$ it follows that

$$(\zeta * \mu_{z,i})(x_1, x_2) = \int \langle D_z(u), \zeta(x_1 - t, x_2 - ut^2)\rangle 2^{-i}\phi(2^{-i}t)\; dt\,. \tag{6.19}$$

Assertion (i) of the lemma can be easily checked. Differentiation of (6.19) gives

$$\nabla(\zeta * \mu_{z,i})(x_1, x_2) = \int \langle D_z(u), (\nabla\zeta)(x_1 - t, x_2 - ut^2)\rangle 2^{-i}\phi(2^{-i}t)\; dt\,. \tag{6.20}$$

Assertions (ii) and (iii) will be an immediate consequence of (6.19), (6.20) and of

$$\begin{cases} \sup\limits_{t\sim 2^i} \|\langle D_z(u), \zeta(x_1 - t, x_2 - ut^2)\rangle\|_{L^\infty} \le C_z \\ \sup\limits_{t\sim 2^i} \|\langle D_z(u), (\nabla\zeta)(x_1 - t, x_2 - ut^2)\rangle\|_{L^\infty} \le C_z\,. \end{cases} \tag{6.21}$$

To prove (6.21) we simply use that $\zeta \in C_0^\infty$ and that for every $h \in \mathcal{S}(\mathbb{R})$

$$|\langle D_z, h\rangle| \le C_z(\|h\|_{L^\infty} + \|h'\|_{L^\infty})\,.$$

The proof of the lemma is now complete.

We now prove (6.18). The left hand side of (6.18) is hounded above by

$$\begin{aligned} &\|\bar{B}_{-s}\|_{L^1} \sum_{i\le -3} \|B_{i-s} * \zeta * \mu_{z,i}\|_{L^\infty} \\ &\quad \le \|B_{-s}\|_{L^1} \sum_{i\le -3}\ \sum_{\kappa(Q')=i-s} \lambda_{Q'} \|a_{Q'} * \zeta * \mu_{z,i}\|_{L^\infty}\,. \end{aligned} \tag{6.22}$$

By Lemmas 6.1 and 6.2, (6.22) is bounded above by

$$C_z \|B_{-s}\|_{L^1} \sum_{i\le -3} \sup_x \sum_x \lambda_{Q'} 2^{\sigma(Q')} \|a_{Q'}\|_{L^1} \tag{6.23}$$

where the sum $\sum_x$ is (6.23) is taken over all $Q' \in \mathcal{C}$ such that $\kappa(Q') = i - s$ and $Q' \cap (-x + \operatorname{supp}(\zeta * \mu_{z,i})) \ne \phi$.

To estimate (6.23) we first use that $2^{\sigma(Q')} \leq 2^{\kappa(Q')} = 2^{i-s}$ and that $\|a_{Q'}\|_{L^1} \leq 1$. Then the same reasoning as in the proof of (6.14) shows that $\sum_x \lambda_{Q'} \leq C\alpha$ uniformly in x. It follows that

$$(6.23) \ \leq C_z \|B_{-s}\|_{L^1} \sum_{i \leq -3} 2^{i-s} \sup_x \sum_x \lambda_{Q'} \leq C_z \alpha 2^{-s} \sum_{\kappa(Q)=-s} \lambda_Q \,.$$

This finishes the proof of (6.18).

We now turn to the proof of (6.17).

Let $\beta(t) \geq 0$ be a fixed $C_0^\infty(\mathbb{R})$ even function supported in $2^{-1} \leq |t| \leq 2$ and satisfying $\sum_{m \in \mathbb{Z}} \beta(2^{-m}t) = 1$ for $t \neq 0$.

Fix $i \leq -3$ and let

$$\beta_m(t) = \beta(2^{-i-m}t) \quad \text{for} \quad m = 1, 2, 3, \dots \quad \text{and}$$
$$\beta_0(t) = \sum_{m \leq 0} \beta(2^{-i-m}t)\,.$$

We decompose the distribution h_z as

$$\sum_{m=0}^{-i+2} \beta_m(x_2) h_z \,.$$

For simplicity call $g_{i,m} = \nu_z * \beta_m \bar{h}_z * \mu_{z,i}$ and $S_{i,m} = \text{support}(g_{i,m})$, $m = 0, 1, 2, \dots, -i+2$. Recall that $\nu = \bar{\tilde{\nu}}_0$ and ν_0 is supported in $\{(x_1, x_2) : x_2 = x_1^2, |x_1| \sim 1\}$. Thus ν is supported in $J = \{(x_1, x_2) : x_2 = -x_1^2, |x_1| \sim 1\}$. It follows that the support of $\nu * \beta_m \bar{h}_z$ is the set of all points whose vertical distance from J is about 2^{i+m}. Also, the support of $\mu_{z,i}$ is the set of points whose vertical distance from the piece of the parabola $\{(x_1, x_2) : x_2 = x_1^2,\ |x_1| \sim 2^i\}$ is less than $C2^{2i}$. It follows that $S_{i,m}$ is the union of four "curved rectangles" of constant length and width at most $C2^{i+m}$.

The following two lemmas give us the size estimates for the derivatives of $g_{i,m}$.

Lemma 6.4. *For $r = 0, 1, 2, \dots$* $\quad \|\nabla^r g_{i,0}\|_{L^\infty} \leq C_z 2^{-(r+1)i}$.

Lemma 6.5. *For $r = 0, 1, 2, \dots$* $\quad \|\nabla^r g_{i,m}\|_{L^\infty} \leq C_z 2^{-(r+1)(i+m)}$, $m = 1, 2, \cdots -i+2$.

The case $m = 0$ is studied separately because of the singularity of $\beta_0 \bar{h}_z$ at $x_2 = x_1^2$.

Proof of Lemma 6.4. For $r = 0, 1, 2, \dots$ we have that

$$(6.24) \quad \begin{aligned} &\|\nabla^r g_{i,0}\|_{L^\infty} = \|\nabla^r(\beta_0 \bar{h}_z * \nu_z * \mu_{z,i})\|_{L^\infty} = \\ &\|\beta_0 \bar{h}_z * \nabla^r(\nu_z * \mu_{z,i})\|_{L^\infty} \leq \\ &\|\nabla^r(\nu_z * \mu_{z,i})\|_{L^\infty} \ + \ |\text{supp}(\beta_0 \bar{h}_z)|\, \|\nabla^{r+1}(\nu_z * \mu_{z,i})\|_{L^\infty} \leq \\ &\|\nabla^r(\nu_z * \mu_{z,i})\|_{L^\infty} + 2^i \|\nabla^{r+1}(\nu_z * \mu_{z,i})\|_{L^\infty} \end{aligned}$$

Because of (6.24) the lemma will be proved if we can show that

$$\|\nabla^r(\nu_z * \mu_{z,i})\|_{L^\infty} \le C2^{-(r+1)i}, \quad r \ge 0\,.$$

We first find a formula for $\nu_z * \mu_{z,i}$. For $h \in \mathcal{S}(\mathbb{R}^2)$

$$\langle \nu_z * \mu_{z,i}, h\rangle = \langle \mu_{z,i}, \tilde{\nu}_z * h\rangle = \langle \mu_{z,i}, \bar{\nu}_{z,0} * h\rangle =$$

$$\iint \langle D_z(u), \int h(t-s, ut^2 - s^2)\phi(s)s^{-s\bar{z}-2}\, ds\rangle 2^{-i}\phi(2^{-i}t)\, dt\,. \tag{6.25}$$

By changing variables

$$x_1 = t - s, \qquad x_2 = ut^2 - s^2$$

we get that

$$(6.25) \; = \iint \left\langle D_z(u),\, 2^{-i}\phi(2^{-i}t)\phi(s)\, \frac{s^{-s\bar{z}-2}}{s-ut} \right\rangle h(x_1, x_2)\, dx_1 dx_2\,.$$

In this lemma t, s are C^∞ functions of x_1, x_2 given implicitly by formulas $x_1 = t - s$, $x_2 = ut^2 - s^2$.

Let's call $\phi_z(s) = \phi(s)s^{-2\bar{z}-2}$. Then ϕ_z is a C_0^∞ function and $\|\phi_z^{(r)}\|_{L^\infty} \le C_{z,r}$ for any $r \ge 0$.

We set $G(x_1, x_2, u) = 2^{-i}\phi(2^{-i}t)\phi_2(s)(s-ut)^{-1}$ and we then have $(\nu_z * \mu_{z,i})(x_1, x_2) = \langle D_z(u), G(x_1, x_2, u)\rangle$. We bound

$$\|\nabla^r(\nu_z * \mu_{z,i})\|_{L^\infty}$$

by

$$\sup_{u\sim 1} \left[\|\nabla_x^r G(x,u)\|_{L^\infty} + \left\| \frac{\partial}{\partial u} \nabla_x^r G(x,u) \right\|_{L^\infty} \right]. \tag{6.26}$$

Computation gives

$$\frac{\partial t}{\partial u} = \frac{\partial s}{\partial u} = \frac{t^2}{2(s-ut)}\,, \qquad \frac{\partial}{\partial u}\;(s-ut)^{-1} = \frac{t}{(s-ut)^2}\,.$$

We must show that $(6.26) \le C_z 2^{-(r+1)i}$ $\;, r = 0, 1, 2, 3, \ldots$. This will follow from the following estimates:

$$\begin{cases} \sup\limits_{u\sim 1} \|\nabla_x^r(2^{-i}\phi(2^{-i}t)\phi_z(s)(s-ut)^{-1})\|_{L^\infty} \le C_z 2^{-(r+1)i} \\ \sup\limits_{u\sim 1} \|\nabla_x^r(2^{-2i}\phi'(2^{-i}t)t^2\phi_z(s)(s-ut)^{-2})\|_{L^\infty} \le C_z 2^{-(r+1)i} \\ \sup\limits_{u\sim 1} \|\nabla_x^r(2^{-i}t)\phi(2^{-i}t)\phi_z'(s)t^2(s-ut)^{-2})\|_{L^\infty} \le C_z 2^{-(r+1)i} \\ \sup\limits_{u\sim 1} \|\nabla_x^r(2^{-i}\phi(2^{-i}t)\phi_z(s)t(s-ut)^{-2})\|_{L^\infty} \le C_z 2^{-(r+1)i}\,. \end{cases} \tag{6.27}$$

Observe that $t \sim 2^i$, $s \sim 1$, $s - uv \sim 1$.

This observation proves (6.27) when $r = 0$.

For $r \geq 1$ we must differentiate with respect to x_1, x_2 and make use of the identities in (6.28) which follow from the change of variables formulas $x_1 = t - s$ and $x_2 = ut^2 - s^2$ after implicit differentiation.

$$\begin{cases} \dfrac{\partial s}{\partial x_1} = \dfrac{ut}{s - ut} & \dfrac{\partial s}{\partial x_2} = \dfrac{-1}{s - ut} \\ \dfrac{\partial t}{\partial x_1} = \dfrac{s}{s - ut} & \dfrac{\partial t}{\partial x_2} = \dfrac{-1}{s - ut}\,. \end{cases} \tag{6.28}$$

Let's prove for example the first of the four estimates in (6.27).

Differentiations of (6.28) and the observations $t \leq 1$, $s \sim 1$, $s - ut \sim 1$ show that

$$\sup_{u \sim 1} |\nabla_x^r t| + |\nabla_x^r s| \leq C_r\,, \quad r \geq 1\,. \tag{6.29}$$

Since $\nabla_x^r \phi_z$ is sum of products of derivatives of ϕ_z, t and s it follows that

$$\sup_{u \sim 1} |\nabla_x^r \phi_z| \leq c_{z,r}\,, \quad r \geq 1\,.$$

Similar argument shows that $\sup_{u \sim 1} |\nabla_x^r (s - ut)^{-1}| \leq C_r$.

An application of Leibniz's formula gives

$$\sup_{n \sim 1} |\nabla_x^r \left(\phi_z(u)(s - ut)^{-1}\right)| \leq C_{z,r}, \quad r \geq 1\,. \tag{6.30}$$

It suffices to show that

$$\sup_{u \sim 1} |\nabla_x^r (2^{-i} \phi(2^{-i} t))| \leq C 2^{-(r+1)i}\,. \tag{6.31}$$

We have

$$|\nabla_x^r (2^{-i} \phi(2^{-i} t))| \leq C_r \sum_{k=0}^{r} 2^{-i(k+1)} |\phi^{(k)}(2^{-i} t)|\, |A_k(x)|$$

where the $A_k(x)$ are products of derivatives of t and s. By (6.29), $|A_k| \leq C_{k,r}$ therefore

$$|\nabla_x^r (2^{-i} \phi(2^{-i} t))| \leq C_r \sum_{k=0}^{r} 2^{-i(k+1)} \leq C_r 2^{-i(r+1)}\,.$$

(6.30), (6.31) and Leibniz's formula prove the first of the four estimates in (6.27). Similarly we argue for the remaining three.

Proof of Lemma 6.5. We first compute $\nu_z * \beta_m \bar{h}_z$. Recall that $\nu_z = \tilde{\bar{\nu}}_{z,0}$ and $\phi_z(x_1) = \phi(x_1)x_1^{-2\bar{z}-2}$. Let $g \in \mathcal{S}(\mathbb{R}^2)$.

$$\begin{aligned}
\langle \nu_z * \beta_m \bar{h}_z, g\rangle &= \langle \tilde{\bar{\nu}}_{z,0} * \beta_m \bar{h}_z, g\rangle = \langle \beta_m \bar{h}_z, \bar{\nu}_{z,0} * g\rangle \\
&= \langle \beta_m \bar{h}_z, \int g(x_1 - t, x_2 - t^2)\, \phi_z(t)\ dt\rangle \\
&= \int \beta_m(x_2) \left(\overline{\frac{2|x_2|^z}{\Gamma\left(\frac{z+1}{2}\right)}}\right) \int g(-t, x_2 - t^2)\, \phi_z(t)\ dt\ dx_2 \\
&= \int \beta_m(x_2) \left(\overline{\frac{2|x_2|^z}{\Gamma\left(\frac{z+1}{2}\right)}}\right) \int g(x_1, x_2 - x_1^2)\phi_z(-x_1)\ dx_1\ dx_2 \\
&= \iint \beta_m(x_2 + x_1^2) \left(\overline{\frac{2|x_2 + x_1^2|^z}{\Gamma\left(\frac{z+1}{2}\right)}}\right) \phi_z(-x_1) g(x_1, x_2)\ dx_1\ dx_2\,.
\end{aligned}$$

Thus

$$(\nu_z * \beta_m \bar{h}_z)(x) = 2\Gamma(\tfrac{\bar{z}+1}{2})^{-1} \beta(2^{-i-m}|x_2 + x_1^2|)\, |x_2 + x_1^2|^{\bar{z}} \phi_z(-x_1)\,.$$

It follows easily that

$$\|\nu_z * \beta_m \bar{h}_z\|_{L^\infty} \le C_z 2^{-i-m}$$

and by Leibniz's rule

$$\|\nabla^r(\nu_z * \beta_m \bar{h}_z)\|_{L^\infty} \le C_{z,r} 2^{-(r+1)(i+m)}, \quad r \ge 1\,.$$

We bound

$$\|\nabla^r(\mu_{z,i} * \nu_z * \beta_m \bar{h}_z)\|_{L^\infty} = \left\| \int \langle D_z(u), \nabla^r(\nu_z * \beta_m \bar{h}_z)(x_1 - t, x_2 - ut^2)\rangle 2^{-i}\phi(2^{-i}t)\ dt \right\|_{L^\infty}$$

by

$$C_z \left[\|\nabla^r(\nu_z * \beta_m \bar{h}_z\|_{L^\infty} + \sup_{u\sim 1}\ \sup_{t\sim 2^i} \left\| \frac{\partial}{\partial u}\ \nabla^r(\nu_z * \beta_m \bar{h}_z)(x_1 - t, x_2 - ut^2) \right\|_{L^\infty} \right]. \tag{6.32}$$

The second term in (6.32) is bounded above by

$$\begin{aligned}
&C_z \sup_{t\sim 2^i} \|t^2 \nabla^{r+1}(\nu_z * \beta_m \bar{h}_z)(x_1 - t, x_2 - ut^2)\|_{L^\infty} \\
&\le C_{z,r} 2^{2i-(r+2)(i+m)} \le C_{z,r} 2^{-(r+1)(i+m)}\,.
\end{aligned}$$

It follows that (6.32) is at most $C_{z,r}2^{-(r+1)(i+m)}$.

Our lemmas are now proved.

We now introduce some notation. Recall that $S_{i,m}$ is the support of $g_{i,m}$. For any $x \in \mathbb{R}^2$ let $S_{i,m}(x) = -x + S_{i,m}$. Let $S^*_{i,m}(x)$ be the triple of $S_{i,m}(x)$. For $Q \in R_{\sigma,\sigma}$ with $\sigma > i+m$ set $S(Q) = \cup_{x\in Q} S^*_{i,m}(x)$. We will need the following two lemmas:

Lemma 6.6. *Let $Q \in \mathcal{C}$ satisfy $i+m \le \sigma(Q) = \sigma < 0$. Then*

(i) $\forall\ y \in \mathbb{R}^2\ \int_Q \chi_{S^*_{i,m}(x)}(y)\ dx \le C2^{i+m+2\sigma}$.

(ii) $\sum\limits_{\substack{Q' \subseteq S(Q) \\ \kappa(Q') \le \sigma}} \lambda_{Q'} \le C\alpha 2^{\sigma}$.

Lemma 6.7. *For any $x \in \mathbb{R}^2$* $\sum\limits_{\substack{Q' \subseteq S^*_{i,m}(x) \\ \kappa(Q') \le i+m}} \lambda_{Q'} \le C\alpha 2^{i+m}$.

Proof. Recall that $S_{i,m}$ is the union of 4 "curved" rectangles of constant length and width at most $C2^{i+m}$. The same is true for $S^*_{i,m}(x)$. To prove Lemma 6.7, cover $S^*_{i,m}(x)$ by $C2^{-2(i+m)}$ rectangles q in $R_{i+m,i+m}$ and for each q apply Lemma 5.2 (iv).

We now prove Lemma 6.6.

(i) Fix any $y \in \mathbb{R}^2$ and consider all x for which $y \in S_{i,m}(x)$. The union of all such $S_{i,m}(x)$ is contained in $S^*_{i,m}(x_0)$ where x_0 is some point in $\mathbb{R}^2$ such that $y \in S_{i,m}(x_0)$. Then

$$\int_Q \chi_{S^*_{i,m}(x)}(y)\ dx \le |Q \cap S^*_{i,m}(x_0)|\,.$$

$S^*_{i,m}(x_0)$ is a union of four "curved rectangles" of horizontal dimension $\sim 2^{i+m}$ and vertical dimension $\sim C$. Since $\sigma \ge i+m$ it is clear that $|Q \cap S^*_{i,m}(x_0)| \le C2^{i+m+2\sigma}$.

(ii) Since $\sigma \ge i+m$, $S(Q)$ is the union of four "curved rectangles" of horizontal dimension $\sim 2^{\sigma}$ and vertical dimension $\sim C$. $S(Q)$ can be covered by $C2^{-2\sigma}$ rectangles in $R_{\sigma,\sigma}$ and for each one of them apply Lemma 5.2 (iv).

We are now ready to prove (6.17).

Decompose the left hand side of (6.17) as I+ II + III where

$$\mathrm{I} = \sum_{\sigma<-s} \sum_{\substack{\kappa(Q)=-s \\ \sigma(Q)=\sigma}} \lambda_Q \sum_{i\le\sigma} \sum_{m<\sigma-i} |\textstyle\int (B_{i-s} * g_{i,m})\bar{B}_s\ dx|$$

$$\mathrm{II} = \sum_{\sigma<-s} \sum_{\substack{\kappa(Q)=-s \\ \sigma(Q)=\sigma}} \lambda_Q \sum_{i\le\sigma} \sum_{m\ge\sigma-i} |\textstyle\int (B_{i-s} * g_{i,m})\bar{B}_{-s}\ dx|$$

$$\text{III} = \sum_{\sigma<-s} \sum_{\substack{\kappa(Q)=-s \\ \sigma(Q)=\sigma}} \lambda_Q \sum_{i>\sigma} \sum_{m\geq 0} |\textstyle\int (B_{i-s} * g_{i,m})\bar{B}_{-s}\ dx|.$$

The first sum is on σ, the second on Q, the third on i and the fourth on m. From our construction m depends on i and therefore we cannot change the order or summation.

We start by proving that

$$\text{I} \leq C_z \alpha 2^{-s} \sum_{\kappa(Q)=-s} \lambda_Q\,.$$

Fix $\sigma < -s$, Q with $\sigma(Q) = \sigma$ and $\kappa(Q) = -s$ and also fix i and m.

$$\begin{aligned} \left|\int (B_{i-s} * g_{i,m})\bar{a}_Q\ dy\right| &\leq |Q|^{-1} \int_Q |(B_{i-s} * g_{i,m})(x)|\ dx \\ &\leq 2^{-3\sigma} \int_Q \sum_{\kappa(Q')=i-s} \lambda_{Q'} |\alpha_{Q'} * g_{i,m}|\ dx\,. \end{aligned}$$

By Lemmas 6.2 and 6.4 or 6.5 the above is majorized by

$$C_z 2^{-3\sigma} \int_Q {\sum_x}' \lambda_{Q'} 2^{\sigma(Q')} 2^{-2i-2m}\ dx \tag{6.33}$$

where the sum $\sum_x'$ in (6.33) is taken over all Q' with $\kappa(Q') = i - s$ that intersect $S_{i,m}(x)$ (hence contained in $S^*_{i,m}(x)$). Since $\sigma(Q') < \kappa(Q') = i - s$, (6.33) is majorized by

$$\begin{aligned} &C_z 2^{-s} 2^{-3\sigma-i-2m} \int_Q {\sum_x}' \lambda_{Q'}\ dx \leq \\ &C_z 2^{-s} 2^{-3\sigma-i-2m} \int_Q \int_{S(Q)} \left({\sum_x}' \lambda_{Q'} |Q'|^{-1} \chi_{Q'}(y)\right) \chi_{S^*_{i,m}(x)}(y)\ dy\ dx = \\ &C_z 2^{-s} 2^{-3\sigma-i-2m} \int_{S(Q)} \int_Q \left({\sum_x}' \lambda_{Q'} |Q'|^{-1} \chi_{Q'}(y)\right) \chi_{S^*_{i,m}(x)}(y)\ dx\ dy \leq \end{aligned}$$

$$C_z 2^{-s} 2^{-3\sigma-i-2m} \int_{S(Q)} {\sum}' \lambda_{Q'} |Q'|^{-1} \chi_{Q'}(y) \left[\int \chi_{S^*_{i,m}(x)}(y)\ dx\right] dy \tag{6.34}$$

where the sum $\sum'$ in (6.34) is taken over all Q' with $\kappa(Q') = i - s$ which are contained in $S(Q)$. By Lemma 6.6 (i) the expression inside the brackets in (6.34) is dominated by $C2^{i+m+2\sigma}$. Thus (6.34) is dominated by

$$\leq C_z 2^{-s} 2^{-\sigma} 2^{-m} \int_{S(Q)} {\sum}' \lambda_{Q'} |Q'|^{-1} \chi_{Q'}(y)\ dy$$

$$\leq C_z 2^{-s} 2^{-\sigma-m} \sum_{\substack{Q' \subseteq S(Q) \\ \kappa(Q')=i-s}} \lambda_{Q'} \,.$$

Now we sum the expressions above on $m < \sigma - i$ to get

$$C_z 2^{-s} 2^{-\sigma} \sum_{\substack{\kappa(Q')=i-s \\ Q' \subseteq S(Q)}} \lambda_{Q'} \,.$$

Next, a summation on $i \leq \sigma$ gives

$$C_z 2^{-s} 2^{-\sigma} \sum_{\substack{\kappa(Q') \leq \sigma \\ Q' \subseteq S(Q)}} \lambda_{Q'} \,.$$

Finally, we apply Lemma 6.6 (ii) to bound this expression by $C_z \alpha 2^{-s}$. Summing over all $Q \in \mathcal{C}$ with $\sigma(Q) = \sigma$ and $\kappa(Q) = -s$ and over all $\sigma < -s$ we get the desired conclusion for term I:

$$\mathrm{I} \leq C_z \alpha 2^{-s} \sum_{\kappa(Q)=-s} \lambda_Q \,.$$

We prove similar estimates for II and III. Fix Q, i and m as before. Set $\sigma = \sigma(Q)$. Two applications of Lemma 6.2 give:

$$\begin{aligned} \left| \int (B_{i-s} * g_{i,m}) \bar{a}_Q \, dx \right| &\leq 2^{\sigma} \| \nabla (B_{i-s} * g_{i,m}) \|_{L^\infty} \\ &\leq C 2^{\sigma} \| \nabla^2 g_{i,m} \|_{L^\infty} \sup_x \sum_x \lambda_{Q'} 2^{\sigma(Q')} \end{aligned} \tag{6.35}$$

where the sum $\sum_x$ is taken over all Q' with $\kappa(Q') = i - s$ that intersect $S_{i,m}(x)$. Since $i - s < i + m$, those Q' are contained in $S^*_{i,m}(x)$. First majorize $\sigma(Q')$ by $\kappa(Q') = i - s$. By Lemma 6.7, $\sum_x \lambda_{Q'} \leq C\alpha 2^{i+m}$ uniformly in x. Thus (6.35) is dominated by

$$C\alpha 2^{\sigma+i+m+i-s} \| \nabla^2 g_{i,m} \|_{L^\infty} \,.$$

Use Lemmas 6.4 and 6.5 to bound the above by

$$C_z \alpha \, 2^{-s} 2^{m+\sigma+2i} 2^{-3m-3i} = C_z \alpha 2^{-s} 2^{-2m} 2^{\sigma-i} \,. \tag{6.36}$$

To treat III, just sum (6.36) over m and then over i. Use $\sigma < i$ to get

$$\mathrm{III} \leq C_z \alpha 2^{-s} \sum_{\kappa(Q)=-s} \lambda_Q \,.$$

To treat II, rewrite (6.36) as $C_z \alpha 2^{-s} 2^{i-\sigma} 2^{-2(m-\sigma+i)}$.
First sum over m (use $m-\sigma+i>0$) and then over i (use $i-\sigma<0$) to get

$$\mathrm{II} \le C_z \alpha 2^{-s} \sum_{\kappa(Q)=-s} \lambda_Q .$$

This concludes the proof of (6.17) and hence of (6.6). Our theorem is now proved.

7. An interpolation theorem and an application. Let H^1 be the parabolic Hardy space of Calderón and Torchinsky ([CT1]) as defined in Section 4 and let $L^{p,q}$ be the usual Lorentz spaces as defined in [SWE]. Also let $S = \{z : 0 < \mathrm{Re}\, z < 1\}$ and $\bar{S} = \{z : 0 \le \mathrm{Re}\, z \le 1\}$. Fix a pair of Banach spaces X_0, X_1 continuously embedded in some Banach space V such that $X_0 \cap X_1$ contains a dense subspace $\mathcal{D}$ of both X_0, X_1 under the corresponding norms.

Following Calderón [CA], for the fixed pair X_0, X_1 we define $\mathcal{F}(X_0, X_1)$ to be the set of all functions F on $\bar{S}$ with values in $X_0 + X_1$, continuous and bounded in $\bar{S}$ with respect to the norm $X_0 + X_1$, analytic in S and such that $F(it) \in X_0$ is X_0-continuous and tends to 0 as $|t| \to \infty$ and $F(1+it) \in X_1$ is X_1-continuous and tends to 0 as $|t| \to \infty$. $\mathcal{F}(X_0, X_1)$ becomes a Banach space under the norm

$$\|F\|_{\mathcal{F}} = \sup_{t \in \mathbb{R}} \max(\|F(it)\|_{X_0}, \|F(1+it)\|_{X_1}) .$$

Given a real number θ, $0 < \theta < 1$, Calderón constructed a subspace $[X_0, X_1]_\theta$ of $X_0 + X_1$ as follows:

$$[X_0, X_1]_\theta = \{F(\theta) : F \in \mathcal{F}(X_0, X_1)\} .$$

By introducing the norm $\|F\|_{[X_0,X_1]_\theta} = \inf\{\|F\|_{\mathcal{F}} : F \in \mathcal{F}(X_0, X_1), F(\theta) = f\}$, $[X_0, X_1]_\theta$ becomes a Banach space continuously embedded in $X_0 + X_1$.

We next define analytic families of operators. Fix X_0, X_1 and $\mathcal{D}$ as before. Let $\{T_z\}$ be a family of linear operators indexed by $z \in \bar{S}$ so that for each z, T_z is a mapping of functions in $\mathcal{D}$ to measurable functions on $\mathbb{R}^n$. Following [SA], $\{T_z\}$ is called an analytic family if for any $g \in \mathcal{D}$ and for almost all $y \in \mathbb{R}^n$, $(T_z(g))(y)$ is analytic in S and continuous on $\bar{S}$. The analytic family $\{T_z\}$ is of admissible growth if for all $y \in \mathcal{D}$ there exists a constant C_g and a constant $a < \pi$ such that

$$\sup_{z \in \bar{S}} \log |(T_z g)(y)| \le C_g e^{a|\mathrm{Im}\, z|}$$

for almost all $y \in \mathbb{R}^n$. The main result of this section is the following:

Theorem 3. *Let X_0, X_1, $\mathcal{D}$ as before, $0 < p_0,\ q_0,\ p_1,\ q_1 \le \infty$, and let $\{T_z\}$ be an analytic family of linear operators which is of admissible growth. If for all $f \in \mathcal{D}$ $\|T_z f\|_{L^{p_j,q_j}} \le c_j(z)\|f\|_{X_j}$ when $\operatorname{Re} z = j$, $j = 0,1$ for some constants $c_j(z)$ that satisfy $\log c_j(z) \le Ae^{a|\operatorname{Im} z|}$, $A > 0$, $0 \le a < \pi$, then for all $z \in S$ there exists $A_z > 0$ such that for $f \in \mathcal{D}$*

(7.1) $$\|T_z f\|_{L^{p,q}} \le A_z \|f\|_{[X_0,X_1]_\theta} \qquad \text{when} \quad \operatorname{Re} z = \theta$$

it where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Theorem 3 is a result of Y. Sagher [SA] when $X_0 = L^{\bar{p}_0,\bar{q}_0}$, $X_1 = L^{\bar{p}_1,\bar{q}_1}$, $[X_0, X_1]_\theta = L^{\bar{p},\bar{q}}$ and

$$\frac{1}{\bar{p}} = \frac{1-\theta}{\bar{p}_0} + \frac{\theta}{\bar{p}_1}, \qquad \frac{1}{\bar{q}} = \frac{1-\theta}{\bar{q}_0} + \frac{\theta}{\bar{q}_1}.$$

Our only contribution is the remark that the proof given in [SA] applies to our abstract setting. For, let $f \in [X_0, X_1]_\theta$, $\|f\|_{[X_0,X_1]\theta} = 1$. We can find $F(x, z) \in \mathcal{F}(X_0, X_1)$ such that $f(x) = F(x, \theta)$. Then $F(\cdot\,, it) \in X_0$, $F(\cdot\,, 1 + it) \in X_1$ and for almost every $y \in \mathbb{R}^n$, $(T_2F(\cdot\,, z))(y)$ is an analytic function of $z \in S$, continuous on $\bar{S}$ and of admissible growth. In [SA] the function F was constructed explicitly (following [HU]). In our case the function F is given from the definition of the intermediate space. Sagher's proof goes through without using the domain spaces $L^{\bar{p}_0,\bar{q}_0}$, $L^{\bar{p}_1,\bar{q}_1}$. (Note that our pair (p, q) corresponds to the pair $(\bar{p}, \bar{q})$ in [SA] and vice-versa.) It follows from [SA] that

$$\|T_\theta F(\cdot\,, \theta)\|_{L^{p,q}} \le A_\theta\,.$$

Because of the identity $T_\theta F(\cdot, \theta) = f(\theta)$, (7.1) is now proved for $z = \theta$. To extend (7.1) for any z with $\operatorname{Re} z = \theta$, fix t and apply the theorem to the analytic family $\{T_{z+it}\}$.

We now turn to an application. Let $\mathbb{R}^n = \mathbb{R}^2$, $L^{p_0,q_0} = L^{1,\infty}$, $L^{p_1,q_1} = L^2$, $X_0 = H^1$, $X_1 = L^2$ and let $\mathcal{D}$ be the set of all smooth functions with compact support and integral zero. $\mathcal{D}$ is known to be dense in both H^1 and L^2. Let $T_z = H_z$ defined in Section 3. $\{H_z\}$ is an analytic family of admissible growth. Proposition 1, Theorem 2 and an application of Theorem 3 give that

$$H_z : [H^1, L^2]_\theta \to L^{p,p'} \qquad \text{when} \qquad \operatorname{Re} z = -\frac{\theta}{2} - 1, \quad \frac{1}{p} = 1 - \frac{\theta}{2}.$$

Finally, because of (7.2) which can be found in [CT2]

(7.2) $$[H^1, L^2]_\theta = L^p, \quad \frac{1}{p} = 1 - \frac{\theta}{2}.$$

the proof of theorem 4 is now complete.

8. Final remarks 1. It is not true that H_{-1} maps H^1 to L^1. The following counterexample is in [C1]. Consider the unit cube Q with lower left hand vertex the origin and define $a(x) = 1$ for $x \in Q$, $x_1 > 1/2$ and $a(x) = -1$ for $x \in Q$, $x_1 < 1/2$. A calculation shows that $|\{x : |H_{-1}a(x)| > \alpha\}| = C\alpha^{-1}$ as $\alpha \to 0$ and thus H_{-1} cannot be in L^1.

2. It is easy to prove that for $\operatorname{Re} z < |1/p - 1/2| - 3/2$, H_z doesn't map $L^p \to L^p$ for $1 < p < \infty$. To see this, call f_δ the characteristic function of the square with left hand vertex the origin and sidelength δ. Let

$$A_\delta = \left\{ x \in \mathbb{R}^2 : \frac{1}{2} \leq x_1 \leq 2,\ 0 \leq x_2 \leq 2 \quad \text{and} \quad |x_2 - x_1^2| \geq 2\delta \right\} .$$

Since

$$K_z(x) = C_z x_1^{-3-2z} |x_2 - x_1^2|^z \psi\left(\frac{x_2}{x_1^2} - 1\right)$$

away from the parabola, it follows that

$$|(H_z f_\delta)(x)| \sim |x_2 - x_1^2|^{\operatorname{Re} z} \delta^2 \quad \text{for } x \in A_\delta .$$

Therefore

$$\left(\int_{A_\delta} |(H_z f_\delta)(x)|^p \, dx \right)^{1/p} \approx \delta^2 \delta^{\operatorname{Re} z + 1/p}$$

and since $\|f_\delta\|_{L^p} = \delta^{2/p}$, as $\delta \to 0$, no inequality of the form $\|H_z f\|_{L^p} \leq C\|f\|_{L^p}$ is possible when $2 + \operatorname{Re} z + 1/p < 2/p$.

3. We do not know whether on the critical line $\operatorname{Re} z = |1/p - 1/2| - 3/2$, $1 < p < \infty$, H_z maps L^p to L^p. This problem is related to finding an H^1 such that for $\operatorname{Re} z = -1$, H_z maps H^1 to L^1. This is not known even for $z = -1$.

4. We point out that our result is sharp in the sense that for $\operatorname{Re} z < 1/p - 2$, $1 < p < 2$, H_z doesn't map L^p to $L^{p,\infty}$. For, if this were true for some pair (p_0^{-1}, z_0) below the critical line, interpolation with endpoints $(p_0^{-1}, \operatorname{Re} z_0)$, $(2^{-1}, \operatorname{Re} z_0)$ would give that H_{z_0} maps L^{p_1} to L^{p_1} for some $p_1 \in (p, 2)$ with $(p_1^{-1}, \operatorname{Re} z_0)$ below the critical line contradicting 2.

5. It remains an open problem whether the Hilbert transform along the parabola is weak type $(1,1)$. However we can show that the associated operators H_z are not of weak type $(1,1)$ when $z = -1 + i\theta$ and $\theta \neq 0$. To prove this, fix such a z and let f be the characteristic function of the unit square. Then for x away from the parabola

$$|(f * K_z)(x)| \sim |K_z(x)| \sim |x_1|^{-3-2\operatorname{Re} z} |x_2 - x_1^2|^{\operatorname{Re} z}$$

Since $\operatorname{Re} z = -1$, it suffices to prove that the measure of the set

$$\{x : |x_2 - x_1^2| \geq 10 \,\&\, |x_1|^{-1}|x_2 - x_1|^{-1} > \alpha\}$$

cannot be bounded by $C\alpha^{-1}$. An easy examination of this set gives that it has infinite measure for every $\alpha > 0$.

I would like to thank my advisor, Mike Christ, who gave me guidance, encouragement and inspiring suggestions during our stay at MSRI, where this work was completed.

References

[BL] J. Bergh, J. Löfström, *Interpolation spaces, an Introduction*, Springer Verlag, Berlin Heidelberg, New York, 1976.

[CA] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113-190.

[CT1] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1-64.

[CT2] A. P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution II, Advances in Math. **24** (1977), 101-171.

[CC] S. Chanillo and M. Christ, *Weak type* (1,1) *bounds for oscillatory integrals*, Duke Math J. **55** (1987), 141-155.

[C1] M. Christ, *Weak type* (1,1) *bounds for rough operators*, Annals of Math. **128** (1988), 19-42.

[C2] ____________, *Hilbert transforms along variable curves and related operators*, unpublished manuscript.

[C3] ____________, *Weak type endpoint bounds for Bochner-Riesz operators*, Revista Matematica Iberoamericana **3** (1987), 25-31.

[CR] M. Christ and J. L. Rubio de Francia, *Weak type* (1,1) *bounds for rough operators* II, Invent. Math. **93** (1988) 225-237.

[CS] M. Christ and C. D. Sogge, *The weak type L^1 convergence of eigenfunction expansions for pseudodifferential operators*, Invent. Math. **94** (1988) 421-453.

[FS] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **229** (1972), 137-193.

[GS] I. M. Gelfand and G. E. Shilov, *Generalized Functions* Vol. I, Academic Press, New York and London 1964.

[HO] L. Hörmander, *The Analysis of Linear PDO*, Vol. I, Springer-Verlag, Berlin Heidelberg, New York, Tokyo, 1983.

[HU] R. Hunt, *On $L(p,q)$ spaces*, L^1 Enseignement Math. **12** (1966), 249-276.

[NSW] A. Nagel, E. Stein and S. Wainger, *Hilbert transforms and maximal functions related to variable curves*, Proceeding of Symposia in Pure Math., Vol. XXXV, Part I (1979), 95-98.

[SA] Y. Sagher, *On analytic families of operators*, Israel **7** (1969), 350-356.

[ST1] E. M. Stein, *Problems in harmonic analysis related to curvature and oscillatory integrals*, Proceedings of the International Congress of Mathematicians, Berkeley, CA, USA 1986.

[ST2] ____________, *Some problems in harmonic analysis*, Proceedings of Symposia in Pure Mathematics, Vol. XXXV, Part I, 1979.

[ST3] ____________, *Beijing Lectures in harmonic analysis*, Annals of Mathematical Studies, Princeton Univ. Press, Princeton 1986.

[ST4] ____________, *Singular Integrals and Differentiability Properties*, Princeton Univ. Press, Princeton 1970.

[SWE] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton 1971.

[SWA] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bulletin of Amer. Math. Soc., Vol. I **84** (1978), 1239-1295.

[WA] S. Wainger, *Applications of Fourier transforms to averages over lower dimensional sets*, Proceedings of Symposia in Pure Math. Vol. XXXV, Part I, 1979.

[Z] A. Zygmund, *Trigonometric Series*, Vol. I 2nd ed London, Cambridge University Press 1959.

Department of Mathematics, University of California, Los Angeles, CA 90024-1555

Current address: Department of Mathematics, Yale University, Box 2155 Yale Station, New Haven, CT 06520-2155