THE DISC AS A BILINEAR MULTIPLIER

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Abstract. A classical theorem of C. Fefferman [3] says that the characteristic function of the unit disc is not a Fourier multiplier on $L^p(\mathbf{R}^2)$ unless $p = 2$. In this article we obtain a result that brings a contrast with the previous theorem. We show that the characteristic function of the unit disc in $\mathbf{R}^2$ is the Fourier multiplier of a bounded bilinear operator from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ into $L^p(\mathbf{R})$, when $2 \le p_1, p_2 < \infty$ and $1 < p = \frac{p_1 p_2}{p_1+p_2} \le 2$. The proof of this result is based on a new decomposition of the unit disc and delicate orthogonality and combinatorial arguments. This result implies norm convergence of bilinear Fourier series and strengthens the uniform boundedness of the bilinear Hilbert transforms, as it yields uniform vector-valued bounds for families of bilinear Hilbert transforms.

1. Introduction

The theory of multilinear singular integral operators has lately enjoyed a resurgence of activity. This activity started with the proof of boundedness of the bilinear Hilbert transform, obtained by Lacey and Thiele [7], [8]. The bilinear Hilbert transforms are multiplier operators whose bilinear symbols are the functions $(\xi, \eta) \to -i\pi \operatorname{sgn}(\alpha\xi + \beta\eta)$ in $\mathbf{R}^2$, where α, β are real parameters. The jump discontinuity of these symbols (or multipliers) along the lines $\alpha\xi + \beta\eta = 0$ present significant difficulties that require a careful analysis based on a sensitive time-frequency decomposition and delicate combinatorial arguments. More recent work in the area includes, and this list is by no means exhaustive, the uniform boundedness of the bilinear Hilbert transforms by Thiele [15], Grafakos and Li [5], and Li [9], and work on singular multiplier operators by Gilbert and Nahmod [4], Muscalu [10], and Muscalu, Thiele, and Tao [11], [12].

The boundedness of the bilinear Hilbert transform is equivalent to the fact that the characteristic function of a half-plane is a bilinear Fourier multiplier. It is natural to ask whether the characteristic function of other geometric shapes are also bounded bilinear Fourier multipliers on products of Lebesgue spaces. As it is well known, in the linear case, problems arise when the geometric shape has curvature. Results of this type are consequences of the (proof of the) classical theorem of C. Fefferman [3] which says that the disc is not a (linear) multiplier on $L^p(\mathbf{R}^2)$ unless $p = 2$. In this article we show that the bilinear case is quite different than the linear case. Our main result says that the characteristic function of the unit disc in $\mathbf{R}^2$ is a bounded bilinear multiplier on products $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ for a large range of exponents p_1, p_2. The unit disc is one of the most natural geometric objects and it arises in many problems in harmonic analysis. For example the L^p-norm convergence of two dimensional Fourier series is equivalent to the property that the characteristic function of the unit disc is an L^p Fourier multiplier on $\mathbf{R}^2$. An

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analogous fact is valid in the bilinear case; in fact the boundedness of the disc as a bilinear multiplier is equivalent to the statement that the bilinear Fourier series

$$S_R(f,g) = \sum_{m_1^2+m_2^2\le R^2} \widehat{f}(m_1)\widehat{g}(m_2)e^{2\pi i(m_1+m_2)x}$$

converges to $f(x)g(x)$ in $L^p(\mathbf{T})$ whenever f and g are functions in $L^{p_j}(\mathbf{T})$ for suitable p_1, p_2. ($\mathbf{T}$ here is the unit circle.) Details of this equivalence require a well-developed theory of multilinear transference and will appear elsewhere.

The fact that the characteristic function of the unit disc is the symbol of a bounded pseudodifferential operator has some remarkable consequences. For instance, as in the linear case, it implies nontrivial vector-valued inequalities for families of multiplier operators. Analogous inequalities hold for families of bilinear Hilbert transforms as shown below, and these inequalities are uniform in the parameters involved.

We begin our presentation by recalling that the bilinear Hilbert transform in the direction (α,β) is given by

$$H_{\alpha,\beta}(f_1,f_2)(x) = \text{p.v.}\int_{\mathbf{R}} f_1(x-\alpha t)f_2(x-\beta t)\,\frac{dt}{t}\,, \qquad x,\alpha\in\mathbf{R}. \tag{1.1}$$

Using the definition $\widehat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-2\pi i x\xi}dx$ for the Fourier transform, the operators in (1.1) can also be written in multiplier form as

$$H_{\alpha,\beta}(f_1,f_2)(x) = -i\pi\int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{2\pi i(\xi+\eta)x}\text{sgn}\,(\alpha\xi+\beta\eta)\,d\xi\,d\eta\,,$$

and they are easily related to the operators whose symbol are the characteristic functions of the half-planes $\alpha\xi+\beta\eta>0$. Let us denote the characteristic function of the set A by 1_A. By replacing the characteristic functions of such half-planes by the characteristic function of the unit disc D in $\mathbf{R}^2$, we introduce the bilinear disc operator

$$T_D(f_1,f_2)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{2\pi i(\xi+\eta)x}1_{\xi^2+\eta^2<1}\,d\xi\,d\eta\,, \qquad x\in\mathbf{R},$$

defined for Schwartz functions f_1, f_2 on the line. The study of the operator T_D turns out to be a very delicate issue since it requires sensitive orthogonality considerations combined with elements from the aforementioned study of uniform bounds for the bilinear Hilbert transforms. The theorem below is the main result of this article. Throughout, we will denote by $\|h\|_q$ the L^q norm of a function h over the whole real line.

Theorem 1. *Let $2\le p_1,p_2<\infty$ and $1<p=\frac{p_1p_2}{p_1+p_2}\le 2$. Then there is a constant $C=C(p_1,p_2)$ such that for all f_1, f_2 Schwartz functions on $\mathbf{R}$ we have*

$$\|T_D(f_1,f_2)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2}.$$

We would like to note that Theorem 1 above provides a strengthening of the main results in [5] (and also [12]). These claim that the operators $H_{1,\alpha}$ are bounded from $L^{p_1}\times L^{p_2}\to L^p$ uniformly in $\alpha\in[-\infty,+\infty]$ whenever $2<p_1,p_2<\infty$ and $1<p=\frac{p_1p_2}{p_1+p_2}<2$. To see this assertion, we use a classical idea due to Y. Meyer. At first we observe that translations and dilations of bilinear symbols preserve their multiplier norms. Also Fatou's lemma gives that a pointwise limit of a sequence of bounded bilinear symbols is bounded. Given any half-plane one can find a sequence of increasing discs converging pointwise to it. Thus the norm of the disc as a bilinear multiplier controls that of the indicator function of any such half-plane. Clearly this control is uniform in the slope of the half-planes, thus uniform bounds for the bilinear Hilbert transforms follow.

Carrying the same idea a bit further, we can obtain the following stronger result.

Corollary 1. *Let $2 \le p_1, p_2 < \infty$ and $1 < p = \frac{p_1 p_2}{p_1+p_2} \le 2$. Then there is a constant $C = C(p_1, p_2)$ such that for all sequences of Schwartz functions f_j, g_j on $\mathbf{R}$ we have*

$$\sup_{\alpha_j,\beta_j\in\mathbf{R}} \Big\| \Big(\sum_j |H_{\alpha_j,\beta_j}(f_j, g_j)|^2 \Big)^{1/2} \Big\|_p \le C \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_{p_1} \Big\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \Big\|_{p_2} .$$

We also have the following.

Corollary 2. *Let $2 \le p_1, p_2 < \infty$ and $1 < p = \frac{p_1 p_2}{p_1+p_2} \le 2$. Then there is a constant $C = C(p_1, p_2)$ such that for all sequences of Schwartz functions f_j, g_k on $\mathbf{R}$ we have*

$$\sup_{\alpha,\beta\in\mathbf{R}} \Big\| \Big(\sum_j \sum_k |H_{\alpha,\beta}(f_j, g_k)|^2 \Big)^{1/2} \Big\|_p \le C \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_{p_1} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_{p_2} .$$

2. Vector-valued inequalities for the bilinear Hilbert transforms

Let us denote by $r_j(t)$ the Rademacher functions on the interval $[0, 1]$. We begin by recalling Khintchine's inequality

$$A_r \Big(\sum_j |\lambda_j|^2 \Big)^{\frac{1}{2}} \le \Big(\int_0^1 \Big| \sum_j \lambda_j r_j(t) \Big|^r dt \Big)^{\frac{1}{r}} \le B_r \Big(\sum_j |\lambda_j|^2 \Big)^{\frac{1}{2}} ,$$

valid for $\lambda_j \in \mathbf{R}$ and $0 < r < \infty$, and its two-dimensional generalization

$$A_r^2 \Big(\sum_{j,k} |\lambda_{jk}|^2 \Big)^{\frac{1}{2}} \le \Big(\int_0^1 \int_0^1 \Big| \sum_{j,k} \lambda_{jk} r_j(t) r_k(s) \Big|^r dt ds \Big)^{\frac{1}{r}} \le B_r^2 \Big(\sum_{j,k} |\lambda_{jk}|^2 \Big)^{\frac{1}{2}} , \tag{2.1}$$

where $0 < A_r < B_r < \infty$, $0 < r < \infty$, and $\lambda_{jk} \in \mathbf{R}$. See Stein [14] Appendix D.

Using (2.1) and a simple calculation, one obtains that any bounded bilinear operator $T : L^{p_1} \times L^{p_2} \to L^p$ with $1/p_1 + 1/p_2 = 1/p$ and $0 < p_1, p_2, p < \infty$ admits an l^2-valued extension, i.e. it satisfies the estimate

$$\Big\| \Big(\sum_j \sum_k |T(f_j, g_k)|^2 \Big)^{1/2} \Big\|_p \le \frac{B_{p_1} B_{p_2}}{A_p^2} \|T\| \, \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_{p_1} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_{p_2} , \tag{2.2}$$

where $\|T\|$ is the norm of T from $L^{p_1} \times L^{p_2}$ into L^p.

We now turn to Corollary 1. Fix a sequence of real numbers $\{\alpha_j\}_{j\in\mathbf{Z}}, \{\beta_j\}_{j\in\mathbf{Z}}$ and define the half-planes

$$P_j = \{(\xi, \eta) \in \mathbf{R}^2 : \ (\xi, \eta) \cdot (-\beta_j, \alpha_j) = -\xi\beta_j + \eta\alpha_j > 0\} ,$$

for all $j \in \mathbf{Z}$. For $R > 0$ we define the disc of radius R

$$D_j(R) = \big\{ (\xi, \eta) \in \mathbf{R}^2 : \ \big| (\xi, \eta) - R(k_j, l_j) \big| < R \big\} ,$$

where $k_j = \frac{-\beta_j}{(\alpha_j^2+\beta_j^2)^{1/2}}$, $l_j = \frac{\alpha_j}{(\alpha_j^2+\beta_j^2)^{1/2}}$, and we observe that the characteristic functions of the discs $D_j(R)$ tend to 1_{P_j} as $R \to \infty$. Let us denote by $T_{D_j(R)}$ the bilinear operators whose symbols are the functions $1_{D_j(R)}$. A simple calculation gives

$$T_{D_j(R)}(f_j, g_j)(x) = e^{2\pi i(k_j+l_j)Rx} \, T_{D(R)}(f_j e^{-2\pi i k_j R(\cdot)}, g_j e^{-2\pi i l_j R(\cdot)})(x) , \tag{2.3}$$

where $T_{D(R)}$ is the bilinear operator whose symbol is the characteristic function of the disc with radius R centered at the origin. But if T_D is $L^{p_1} \times L^{p_2} \to L^p$ bounded, then so is $T_{D(R)}$ for the same range of p's and with the same norm. Using Theorem 1 and (2.2) (with $k = j$) we obtain

$$\Big\|\Big(\sum_j |T_{D(R)}(f_j, g_j)|^2\Big)^{1/2}\Big\|_p \le C'(p_1,p_2)\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{p_1}\Big\|\Big(\sum_j |g_j|^2\Big)^{1/2}\Big\|_{p_2}, \tag{2.4}$$

where p_1, p_2, p are as in Theorem 1. Using (2.3) and (2.4), it follows that

$$\Big\|\Big(\sum_j |T_{D_j(R)}(f_j, g_j)|^2\Big)^{1/2}\Big\|_p \le C'(p_1,p_2)\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{p_1}\Big\|\Big(\sum_j |g_j|^2\Big)^{1/2}\Big\|_{p_2}.$$

Letting $R \to \infty$ above and using Fatou's lemma, we obtain the vector-valued inequality

$$\Big\|\Big(\sum_j |T_{P_j}(f_j, g_j)|^2\Big)^{1/2}\Big\|_p \le C'(p_1,p_2)\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{p_1}\Big\|\Big(\sum_j |g_j|^2\Big)^{1/2}\Big\|_{p_2}, \tag{2.5}$$

where T_{P_j} are the bilinear operators whose symbols are the characteristic functions of the half-planes P_j. A similar estimate holds for the operators T_{-P_j} whose symbols are the characteristic functions of the half-planes $-P_j$. These two estimates suffice to give the same vector-valued inequality for the family of operators H_{α_j,β_j}. This finishes the proof of Corollary 1. Applying Corollary 1 with $\alpha_{j_0} = \alpha$, $\beta_{j_0} = \beta$ and $\alpha_j = \beta_j = 0$ for $j \neq j_0$ and using (2.2) we obtain Corollary 2.

3. Three useful lemmata

The following lemma will be useful in many occasions throughout this paper. It will allow us to obtain that the sum of an infinite sequence of bounded bilinear operators is also bounded (on the same product of Lebesgue spaces), provided certain orthogonality conditions hold. We state it in slight generality to cover several situations. Let us recall our inner product notation

$$\langle f, g\rangle = \int_{\mathbf{R}} f(x)\overline{g(x)}\, dx\,,$$

where the bar denotes complex conjugation. We have the following.

Lemma 1. *Suppose $2 \le p_1, p_2 < \infty$, $1 < p \le 2$, and $1/p_1 + 1/p_2 = 1/p$. Suppose that $\{L_m\}_{m\in\mathbf{Z}}$ is a family of uniformly bounded bilinear operators from $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ into $L^r(\mathbf{R})$. Furthermore, suppose that for all functions f, g, h on the line we have*

$$\langle L_m(f,g), h\rangle = \langle L_m(\Delta_m^1 f, \Delta_m^2 g), \Delta_m^3 h\rangle\,,$$

where $\widehat{\Delta_m^1 f} = \widehat{f}\chi_{A_m}$, $\widehat{\Delta_m^2 g} = \widehat{f}\chi_{B_m}$, $\widehat{\Delta_m^3 f} = \widehat{h}\chi_{C_m}$, and $\{A_m\}_m$, $\{B_m\}_m$, $\{C_m\}_m$ are sets of intervals such that the A_m's being pairwise disjoint, the B_m's being pairwise disjoint, and the C_m's being pairwise disjoint. Then there is a constant $C = C(p_1, p_2, p)$ such that for all functions f, g we have

$$\Big\|\sum_m L_m(f,g)\Big\|_p \le C\|f\|_{p_1}\|g\|_{p_2}\,.$$

Proof. Denoting $p' = p/(p-1)$ we have

$$\begin{aligned}
\big|\langle \sum_m L_m(f,g), h\rangle\big| =& \big|\sum_m \langle L_m(\Delta^1_m f, \Delta^2_m g), \Delta^3_m h\rangle\big| \\
\le& \int_{\mathbf{R}} \Big(\sum_m |L_m(\Delta^1_m f, \Delta^2_m g)|^2\Big)^{1/2}\Big(\sum_m |\Delta^3_m|^2\Big)^{1/2} dx \\
\le& \Big\|\Big(\sum_m |L_m(\Delta^1_m f, \Delta^2_m g)|^2\Big)^{1/2}\Big\|_p \Big\|\Big(\sum_m |\Delta^3_m h|^2\Big)^{1/2}\Big\|_{p'} \\
\le& \Big\|\Big(\sum_m |L_m(\Delta^1_m f, \Delta^2_m g)|^2\Big)^{1/2}\Big\|_p \big\|h\big\|_{p'},
\end{aligned}$$

where the last inequality follows from Rubio de Francia's Littlewood-Paley inequality for arbitrary disjoint intervals ($p' \ge 2$), see [13]. It suffices to estimate the square function above. We have

$$\begin{aligned}
&\Big\|\Big(\sum_m |L_m(\Delta^1_m f, \Delta^2_m g)|^2\Big)^{1/2}\Big\|_p^p \\
\le& \int_{\mathbf{R}} \sum_m |L_m(\Delta^1_m f, \Delta^2_m g)|^p \, dx && \text{(since } p/2 \le 1) \\
=& \sum_m \|L_m(\Delta^1_m f, \Delta^2_m g)\|_p^p \\
\le& C \sum_m \|\Delta^1_m f\|_{p_1}^p \|\Delta^2_m g\|_{p_2}^p && \text{(by unif. boundedness of } L_m) \\
\le& C\Big(\sum_m \|\Delta^1_m f\|_{p_1}^{p_1}\Big)^{p/p_1}\Big(\sum_m \|\Delta^2_m g\|_{p_2}^{p_2}\Big)^{p/p_2} && \text{(Hölder)} \\
\le& C\Big\|\Big(\sum_m |\Delta^1_m f|^2\Big)^{1/2}\Big\|_{p_1}^p \Big\|\Big(\sum_m |\Delta^2_m g|^2\Big)^{1/2}\Big\|_{p_2}^p && \text{(since } p_1, p_2 \ge 2) \\
\le& C\|f\|_{p_1}^p \|g\|_{p_2}^p,
\end{aligned}$$

where the last inequality also follows from Rubio de Francia's Littlewood-Paley inequality for arbitrary disjoint intervals ($p_1, p_2 \ge 2$), see [13]. $\square$

We observe that Lemma 1 holds even when the intervals A_m are not necessarily disjoint, provided the intervals $A_{m+100}, A_{m+200}, A_{m+300}, \dots$ are disjoint for all $m \in \mathbf{Z}$. We are going to use this lemma under similar conditions on the intervals A_m, B_m, and C_m.

We denote by $f^\vee$ the inverse Fourier transform of a function f defined by $f^\vee(\xi) = \widehat{f}(-\xi)$. We will also need the following trivial lemma.

Lemma 2. *Suppose T is a bilinear operator with symbol $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbf{R}$, i.e.*

$$T(f,g)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\sigma(\xi,\eta)e^{2\pi i(\xi+\eta)x}\, d\xi\, d\eta\,.$$

Assume that the inverse Fourier transform $\sigma^\vee$ (in $\mathbf{R}^2$) satisfies

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |\sigma^\vee(x,y)|\, dx\, dy = C_0 < \infty\,.$$

Then T maps $L^p(\mathbf{R}) \times L^q(\mathbf{R}) \to L^r(\mathbf{R})$ when $1 \le p,q,r \le \infty$ and $1/p + 1/q = 1/r$ with constant at most C_0.

The proof of Lemma 2 is omitted since it is an easy consequence of Minkowski's integral inequality and Hölder's inequality.

Finally, we have the following lemma, whose proof is standard and also omitted. We denote by ∂_{radial} and $\partial_{angular}$ differentiation in the radial and angular directions respectively. In other words, $\partial_{radial} = \frac{\partial}{\partial r}$ and $\partial_{angular} = \frac{\partial}{\partial \theta}$, where (r, θ) are polar coordinates in $\mathbf{R}^2$.

Lemma 3. *Let k, l be real numbers greater than or equal to 1.*
(a) Let $\widehat{\phi}(\xi, \eta)$ be a smooth function supported in a rectangle of dimensions $2^{-k} \times 2^{-l}$ with sides parallel to the axes in $\mathbf{R}^2$. Assume that $|\partial_\xi^\alpha \widehat{\phi}| \le C_\alpha 2^{k\alpha}$ and $|\partial_\eta^\beta \widehat{\phi}| \le C_\beta 2^{l\beta}$ for all $\alpha, \beta \ge 0$. Then we have the estimate

$$|\phi(x, y)| \le \frac{C_N\, 2^{-k-l}}{(1 + (2^{-k}|x|)^2 + (2^{-l}|y|)^2)^N}$$

for all $N \ge 0$. Thus ϕ has L^1 norm bounded by some constant independent of k and l.
(b) Let $\widehat{\phi}$ be a smooth function on $\mathbf{R}^2$ supported inside the intersection of an annulus of width 2^{-k} and a sector of angle 2^{-l}. Suppose that $|\partial^\alpha_{radial} \widehat{\phi}| \le C_\alpha 2^{k\alpha}$ and $|\partial^\beta_{angular} \widehat{\phi}| \le C_\beta 2^{l\beta}$ for all $\alpha, \beta \ge 0$. Then we have the estimate

$$|\phi(x, y)| \le \frac{C_N\, 2^{-k-l}}{(1 + (2^{-k}|(x, y) \cdot \mathbf{e}_r|)^2 + (2^{-l}|(x, y) \cdot \mathbf{e}_a|)^2)^N}$$

for all $N \ge 0$, where $\mathbf{e}_r$ is a unit vector in the radial direction of the support of $\widehat{\phi}$ and $\mathbf{e}_a$ is a unit vector perpendicular to $\mathbf{e}_r$, while $\cdot$ is the usual inner product in $\mathbf{R}^2$. Therefore ϕ has L^1 norm bounded by some constant independent of k and l.

4. The decomposition of the disc

Fix a nonnegative smooth function ζ on $[0, 1]$ with the following properties

(1) ζ is identically equal to 1 on $[0, \frac{1}{2} - \frac{1}{2^{10}}]$,
(2) ζ is supported in $[0, \frac{1}{2} + \frac{1}{2^{10}}]$,
(3) $\zeta(t) + \zeta(1 - t) = 1$ for all $0 \le t \le 1$,

and define

$$\zeta_k(t) = \zeta(2^{k-1}(1 - t)) - \zeta(2^k(1 - t))$$

for k in $\mathbf{Z}^+$. Then each function ζ_k is supported in the interval

$$1 - 2^{-k}(1 + 2^{-9}) \le t \le 1 - 2^{-(k+1)}(1 - 2^{-9})$$

and we have the identity

$$\zeta(t) + \sum_{k=1}^{\infty} \zeta_k(t) = 1_{[0,1]}(t)\,.$$

Let ψ_0, ψ_1, ψ_2, ... be radial Schwartz functions on $\mathbf{R}^2$ whose Fourier transforms are

$$\widehat{\psi_k}(\xi, \eta) = \zeta_k(|(\xi, \eta)|)\,.$$

It follows that for $k \ge 1$, each $\widehat{\psi_k}$ is supported in the annulus

$$1 - 2^{-k}(1+2^{-9}) \le |(\xi, \eta)| \le 1 - 2^{-(k+1)}(1-2^{-9})$$

and that

$$1_D = \widehat{\psi_0} + \sum_{k=1}^{\infty} \widehat{\psi_k}, \tag{4.1}$$

where $D = D(0,1)$ is the unit disc. This way we have a decomposition of the characteristic function of the unit disc as an infinite sum of smooth functions supported in annuli whose width becomes smaller as they get closer to the boundary of the disc.

We now introduce a smooth function χ on the real line supported in $[-\frac{\pi}{8} - \frac{1}{2^{10}}, \frac{\pi}{8} + \frac{1}{2^{10}}]$ and equal to 1 on the interval $[-\frac{\pi}{8} + \frac{1}{2^{10}}, \frac{\pi}{8} - \frac{1}{2^{10}}]$, such that

$$\sum_{j \in \mathbf{Z}} \chi(x + \tfrac{\pi}{4} j) = 1 \tag{4.2}$$

for all $x \in \mathbf{R}$. For each $\ell \in \{1,2,3,4,5,6,7,8\}$ we introduce a function ϕ_ℓ on $\mathbf{R}^2$ whose Fourier transform is defined by

$$\widehat{\phi_\ell}(\xi,\eta) = \chi\Big(\text{Argument}\Big(\frac{\xi + i\eta}{\sqrt{\xi^2+\eta^2}} - e^{i\frac{\pi}{4}(\ell-1)}\Big)\Big) \tag{4.3}$$

and we also define functions

$$\widehat{b_k^\ell} = \widehat{\psi_k}\widehat{\phi_\ell}.$$

Observe that each $\widehat{\phi_\ell}$ is a homogeneous of degree zero function and that each $\widehat{\psi_k}$ is a radial function whose α^{th} derivative (in the radial direction) blows up like $C_\alpha 2^{k\alpha}$. Using (4.3), it follows that for all $\alpha, \beta \geq 0$

$$\big|\partial^\alpha_{radial}\partial^\beta_{angular}(\widehat{b_k^\ell})\big| \leq C_{\alpha,\beta} 2^{k\alpha}\,. \tag{4.4}$$

For all $k \geq 1$ and $\ell \in \{1,2,3,4,5,6,7,8\}$ we now introduce bilinear operators

$$T_{D(\ell)}(f,g)(x) = \sum_{k=1}^{\infty} \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \widehat{b_k^\ell}(\xi,\eta)\, d\xi\, d\eta\,.$$

Because of (4.1) and (4.2) we have obtained the following decomposition

$$T_D = T_0 + \sum_{\ell=1}^{8} T_{D(\ell)}\,,$$

where T_0 is the bilinear operator whose symbol is $\widehat{\psi_0}$. Using Lemma 2, it follows that T_0 is a bounded bilinear operator and we therefore need to concentrate on the $T_{D(\ell)}$'s.

It is easy to see that if $\sigma(\xi,\eta)$ is a bounded bilinear symbol, then so is $\sigma(-\xi,-\eta)$. Therefore, it suffices to obtain estimates for the bilinear operators $T_{D(1)}$, $T_{D(2)}$, $T_{D(3)}$, and $T_{D(4)}$, since these imply the same estimates for $T_{D(5)}$, $T_{D(6)}$, $T_{D(7)}$, and $T_{D(8)}$ respectively. Moreover, the symbol of $T_{D(3)}$ can be obtained from that of $T_{D(1)}$ by interchanging ξ and η. Since the set of (p_1,p_2,p) for which we plan to obtain boundedness for $T_{D(1)}$ from $L^{p_1} \times L^{p_2}$ into L^p is symmetric in p_1 and p_2, the estimates for $T_{D(3)}$ can be obtained from those for $T_{D(1)}$ by symmetry. It therefore suffices to obtain estimates for $T_{D(1)}$, $T_{D(2)}$, and $T_{D(4)}$.

We will now describe the decomposition of the operator $T_{D(1)}$ whose symbol is essentially supported in a tiny neighborhood of the sector $D(1)$.

For $-\frac{\pi}{7} \leq a < b \leq \frac{\pi}{7}$ let us denote by $\Sigma\big(a,b\big)$ the sector

$$a \leq \theta \leq b.$$

For every $k \ge 1$ and $\mu \in \{1, 2, \dots, k+1\}$, we introduce functions ρ_k^μ on $\mathbf{R}^2$ such that $\widehat{\rho_k^\mu}$ are homogeneous of degree zero, smooth (away from the origin), satisfying for all $\beta \ge 0$

$$|\partial^\beta_{angular}\widehat{\rho_k^\mu}| \le C_\beta 2^{\beta\frac{\mu}{2}}\,, \qquad 1 \le \mu \le k+1\,, \tag{4.5}$$

$$\widehat{\rho_k^{k+1}} + \widehat{\rho_k^k} + \widehat{\rho_k^{k-1}} + \dots + \widehat{\rho_k^1} = 1 \quad \text{on} \quad \Sigma\big(-\tfrac{\pi}{8}(1+2^{-8}), \tfrac{\pi}{8}(1+2^{-8})\big), \tag{4.6}$$

and such that for any $2 \le \mu \le k$ the functions $\widehat{\rho_k^\mu}$ are supported in

$$\Sigma\big(-2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1+2^{-9})\big) \setminus \Sigma\big(-2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1-2^{-9})\big)$$

and are equal to 1 on

$$\Sigma\big(-2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1-2^{-9})\big) \setminus \Sigma\big(-2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1+2^{-9})\big)\,,$$

while for $\mu = 1$ the function $\widehat{\rho_k^1}$ is supported in

$$\Sigma\big(-\tfrac{\pi}{8}(1+2^{-9}), \tfrac{\pi}{8}(1+2^{-9})\big) \setminus \Sigma\big(-2^{-\frac{1}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{1}{2}}\tfrac{\pi}{8}(1-2^{-9})\big)$$

and is equal to 1 on

$$\Sigma\big(-\tfrac{\pi}{8}(1+2^{-8}), \tfrac{\pi}{8}(1+2^{-8})\big) \setminus \Sigma\big(-2^{-\frac{1}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{1}{2}}\tfrac{\pi}{8}(1+2^{-9})\big)\,,$$

and for $\mu = k+1$ the function $\widehat{\rho_k^{k+1}}$ is supported in

$$\Sigma\big(-2^{-\frac{k}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{k}{2}}\tfrac{\pi}{8}(1+2^{-9})\big)$$

and is equal to 1 on

$$\Sigma\big(-2^{-\frac{k}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{k}{2}}\tfrac{\pi}{8}(1-2^{-9})\big)\,.$$

In view of (4.6) we have the identity

$$\widehat{b_k^1} = \widehat{b_k^1}\big(\widehat{\rho_k^{k+1}} + \widehat{\rho_k^k} + \widehat{\rho_k^{k-1}} + \dots + \widehat{\rho_k^1}\big) \tag{4.7}$$

since the set $\Sigma\big(-\frac{\pi}{8}(1+2^{-8}), \frac{\pi}{8}(1+2^{-8})\big)$ (on which the sum inside the parenthesis in (4.7) is equal to 1) contains the support of $\widehat{b_k^1}$ for all $k \ge 1$.

It follows from estimates (4.4) (with $\beta = 0$) and (4.5) that

$$\begin{aligned} |\partial^\alpha_{radial}(\widehat{b_k^1}\widehat{\rho_k^\mu})| &\le C_\alpha 2^{\alpha k} \\ |\partial^\beta_{angular}(\widehat{b_k^1}\widehat{\rho_k^\mu})| &\le C_\beta 2^{\beta\frac{\mu}{2}} \end{aligned} \tag{4.8}$$

for all $1 \le \mu \le k+1$ and $\alpha, \beta \ge 0$. Moreover the function $\widehat{b_k^1}\widehat{\rho_k^\mu}$ is supported inside an annulus of width approximately 2^{-k} and inside a sector of "length" approximately $2^{-\frac{\mu}{2}}$.

For each $k, \mu \ge 1$, we introduce bilinear operators S_k, T_μ as follows

$$S_k(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{b_k^1}(\xi,\eta)\widehat{\rho_k^{k+1}}(\xi,\eta)\, e^{2\pi i(\xi+\eta)x} d\xi\, d\eta\,,$$

$$T_\mu(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\sum_{k=\mu}^{\infty}\widehat{b_k^1}(\xi,\eta)\widehat{\rho_k^\mu}(\xi,\eta)\, e^{2\pi i(\xi+\eta)x} d\xi\, d\eta\,,$$

We have now achieved the following decomposition for $T_{D(1)}$:

$$T_{D(1)} = \sum_{k=1}^{\infty} S_k + \sum_{\mu=1}^{\infty} T_\mu\,,$$

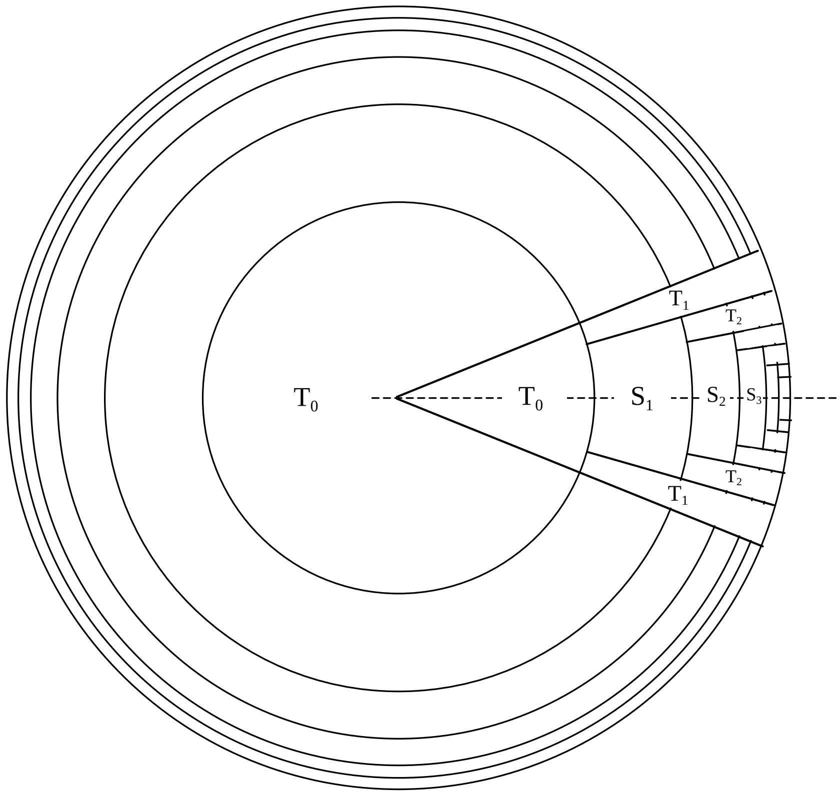

FIGURE 1. The decomposition of $D(1)$

and it will suffice to show that both sums above are bounded on the required product of L^p spaces. See Figure 1 for a pictorial representation of this decomposition.

5. THE BOUNDEDNESS OF $\sum_{k=1}^{\infty} S_k$

Let us denote the operator $\sum_{k=1}^{\infty} S_k(f_1, f_2)$ by $S(f_1, f_2)$. In this section we will prove the boundedness of S.

For each $k \geq 1$ we pick a Schwartz function $\Phi_{1,k}$ on the line whose Fourier transform $\widehat{\Phi_{1,k}}$ is supported in the interval $[-\frac{101}{100} \cdot 2^{-k}, -\frac{99}{100} \cdot 2^{-k-1}]$ and satisfies $|\frac{d^\alpha}{d\xi^\alpha}\widehat{\Phi_{1,k}}(\xi)| \leq C_\alpha 2^{k\alpha}$ for all $\alpha \geq 0$. Moreover we select these functions so that

$$\sum_{k=1}^{\infty} \widehat{\Phi_{1,k}}(\xi) = 1 \tag{5.1}$$

for all $-\frac{99}{200} < \xi < 0$. For each $k \geq 1$ we pick another Schwartz function $\Phi_{2,k}$ on the line whose Fourier transform $\widehat{\Phi_{2,k}}$ is equal to 1 on the interval $[-\frac{4}{5} \cdot 2^{-\frac{k}{2}}, \frac{4}{5} \cdot 2^{-\frac{k}{2}}]$, is supported in the interval $[-2^{-\frac{k}{2}}, 2^{-\frac{k}{2}}]$, and satisfies $|\frac{d^\alpha}{d\eta^\alpha}\widehat{\Phi_{2,k}}(\eta)| \leq C_\alpha 2^{\frac{k}{2}\alpha}$ for all $\alpha \geq 0$. We introduce a bilinear operator $\widetilde{S}'$ by setting

$$\widetilde{S}'(f_1, f_2)(x) = \sum_{k=1}^{\infty} \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)\widehat{\Phi_{1,k}}(\xi - 1)\widehat{\Phi_{2,k}}(\eta)e^{2\pi i(\xi+\eta)x} d\xi d\eta$$

and we prove the following result regarding it.

Lemma 4. *For all* $2 \le p_1, p_2 < \infty$ *and* $1 < p \le 2$ *satisfying* $1/p_1 + 1/p_2 = 1/p$, *there exists a constant* $C = C(p_1, p_2, p)$ *such that*

$$\|S(f_1, f_2) - \widetilde{S}'(f_1, f_2)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2}\,.$$

Proof. Let $L = S - \widetilde{S}'$. Using condition (5.1) we obtain that the symbol of the bilinear operator L consists of a smooth function with compact support plus a sum of smooth functions σ_k (k large) each supported in $A_k \times B_k$ union $A_k \times (-B_k)$, where

$$\begin{aligned} A_k &= [1 - 5 \cdot 2^{-k}, 1 - \tfrac{1}{100} \cdot 2^{-k}] \\ B_k &= [\tfrac{1}{8} \cdot 2^{-\frac{k}{2}}, 2 \cdot 2^{-\frac{k}{2}}]\,. \end{aligned}$$

Because of conditions (4.8), the support properties of $\widehat{\rho_k^{k+1}}$, and the properties of $\Phi_{1,k}$ and $\Phi_{2,k}$, Lemma 3 implies that the σ_k's have uniformly integrable inverse Fourier transforms. Lemma 2 implies that the bilinear operators with symbols σ_k are uniformly bounded from $L^p \times L^q \to L^r$ for all $1 < p, q, r < \infty$ satisfying $1/p + 1/q = 1/r$. Now observe that the intervals $A_k, A_{k+10}, A_{k+20}, \dots$ are pairwise disjoint and the same property holds for the intervals B_k, $A_k + B_k$, and $A_k - B_k$. Using Lemma 1 we obtain that L is bounded from $L^{p_1} \times L^{p_2} \to L^p$ where the exponents p_1, p_2, p are as in the announcement of Lemma 4. $\square$

We now turn our attention to the boundedness of $\widetilde{S}'$. Observe that

$$\widetilde{S}'(f_1, f_2)(x) = e^{2\pi i x}\widetilde{S}(f_1 e^{-2\pi i(\cdot)}, f_2)(x),$$

where

$$\begin{aligned} \widetilde{S}(f_1, f_2)(x) &= \sum_{k=1}^{\infty} \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)\widehat{\Phi_{1,k}}(\xi)\widehat{\Phi_{2,k}}(\eta)e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \sum_{k=1}^{\infty} (f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,k})(x). \end{aligned}$$

Therefore the boundedness of $\widetilde{S}'$ is equivalent to that of $\widetilde{S}$. We now have the following.

Lemma 5. *For each* $1 < p, q, r < \infty$ *satisfying* $1/p + 1/q = 1/r$, *there exists a constant* $C = C(p, q, r)$ *such that*

$$\|\widetilde{S}(f_1, f_2)\|_{L^r} \le C\|f_1\|_{L^p}\|f_2\|_{L^q}\,.$$

Proof. For each $k \ge 2$, we pick a third Schwartz function $\Phi_{3,k}$ whose Fourier transform is supported in the interval $[-2 \cdot 2^{-\frac{k}{2}}, 2 \cdot 2^{-\frac{k}{2}}]$, which is identically equal to 1 on the interval $[-\frac{8}{5} \cdot 2^{-\frac{k}{2}}, \frac{8}{5} \cdot 2^{-\frac{k}{2}}]$, and which satisfies $|\frac{d^\alpha}{d\xi^\alpha}\widehat{\Phi_{3,k}}(\xi)| \le C_\alpha 2^{\frac{k}{2}\alpha}$ for all $\alpha \ge 0$. For $k = 1$, pick $\Phi_{3,1}$ so that its Fourier transform is equal to 1 on the set $[-\frac{13}{10}, \frac{8}{5\sqrt{2}}]$ and supported in $[-\frac{13}{10} - \frac{1}{100}, \frac{8}{5\sqrt{2}} + \frac{1}{100}]$. It is easy to see that for all $k \ge 1$, the algebraic sum of the supports of $\widehat{\Phi_{1,k}}$ and $\widehat{\Phi_{2,k}}$ is contained in the interval $[-\frac{8}{5} \cdot 2^{-\frac{k}{2}}, \frac{8}{5} \cdot 2^{-\frac{k}{2}}]$ on which $\widehat{\Phi_{3,k}}$ is equal to one. It follows that

$$\widetilde{S}(f_1, f_2)(x) = \sum_{k=1}^{\infty} \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)\widehat{\Phi_{1,k}}(\xi)\widehat{\Phi_{2,k}}(\eta)\widehat{\Phi_{3,k}}(\xi+\eta)e^{2\pi i(\xi+\eta)x} d\xi d\eta\,,$$

and pairing with another function f_3 we write the inner product $\langle \widetilde{S}(f_1, f_2), f_3\rangle$ as

$$\int_{\mathbf{R}} \widetilde{S}(f_1, f_2)(x)\overline{f_3(x)}\, dx = \sum_{k=1}^{\infty} \int_{\mathbf{R}} (f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,k})(x)\overline{(f_3 * \Phi_{3,k})(x)}\, dx\,. \tag{5.2}$$

We now use a telescoping argument, inspired by [15], to write

$$\begin{aligned}
\int_{\mathbf{R}} \widetilde{S}(f_1, f_2)(x)\overline{f_3(x)}\, dx &= \sum_{k=1}^{\infty} \int_{\mathbf{R}} (f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,k})(x)\overline{(f_3 * \Phi_{3,k})(x)}\, dx \\
= \sum_{k=1}^{\infty} \int_{\mathbf{R}} (f_1 * \Phi_{1,k})(x) \sum_{m=0}^{k+5} \big\{ & (f_2 * \Phi_{2,k+m})(x)\overline{(f_3 * \Phi_{3,k+m})(x)} \\
& - (f_2 * \Phi_{2,k+m+1})(x)\overline{(f_3 * \Phi_{3,k+m+1})(x)} \big\}\, dx \\
+ \sum_{k=1}^{\infty} \int_{\mathbf{R}} & (f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,2k+6})(x)\overline{(f_3 * \Phi_{3,2k+6})(x)}\, dx\,.
\end{aligned}$$

We begin by claiming that the last sum above is identically equal to zero. Indeed, we have that the support of $\widehat{\Phi_{1,k}}$ is contained in $[-\frac{101}{100}\cdot 2^{-k}, -\frac{99}{100}\cdot 2^{-k-1}]$, the support of $\widehat{\Phi_{2,2k+6}}$ is contained in $[-2^{-k-3}, 2^{-k-3}]$, while the support of $\widehat{\Phi_{3,2k+6}}$ is contained in $[-2^{-k-2}, 2^{-k-2}]$. It follows that

$$\big(\mathrm{supp}\ \widehat{\Phi_{1,k}} + \mathrm{supp}\ \widehat{\Phi_{2,2k+6}}\big) \cap \mathrm{supp}\ \widehat{\Phi_{3,2k+6}} = \emptyset\,,$$

which establishes our claim. It follows that

$$\begin{aligned}
&\int_{\mathbf{R}} \widetilde{S}(f_1, f_2)(x)\overline{f_3(x)}\, dx \\
&= \sum_{k=1}^{\infty} \int_{\mathbf{R}} (f_1 * \Phi_{1,k})(x) \sum_{m=0}^{k+5} \big\{ (f_2 * \Phi_{2,k+m})(x)\overline{(f_3 * \Phi_{3,k+m})(x)} \\
&\qquad\qquad - (f_2 * \Phi_{2,k+m+1})(x)\overline{(f_3 * \Phi_{3,k+m+1})(x)} \big\}\, dx\,,
\end{aligned}$$

which by a change of variables, we write as

$$\begin{aligned}
&\int_{\mathbf{R}} \widetilde{S}(f_1, f_2)(x)\overline{f_3(x)}\, dx \\
&= \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{m=0}^{\frac{k+5}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} \Big\{ (f_2 * \Phi_{2,k})(x)\overline{(f_3 * \Phi_{3,k})(x)} \\
&\qquad\qquad - (f_2 * \Phi_{2,k+1})(x)\overline{(f_3 * \Phi_{3,k+1})(x)} \Big\}\, dx \\
&+ \sum_{k=1}^{6} \sum_{m=0}^{k-1} \int_{\mathbf{R}} (f_1 * \Phi_{1,k-m})(x) \Big\{ (f_2 * \Phi_{2,k})(x)\overline{(f_3 * \Phi_{3,k})(x)} \\
&\qquad\qquad - (f_2 * \Phi_{2,k+1})(x)\overline{(f_3 * \Phi_{3,k+1})(x)} \Big\}\, dx\,.
\end{aligned}$$

Now the last double sum above is indeed a finite sum which is easily controlled by a constant multiple of $\|Mf_1\|_p \|Mf_2\|_q \|Mf_3\|_{r'}$ and the required estimate easily follows for it. (M here denotes the Hardy-Littlewood maximal operator.) We therefore concentrate our attention to the sum over $k \geq 7$ above. We set

$$I = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{m=0}^{\frac{k+5}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} \Big\{ (f_2 * \Phi_{2,k})(x) \overline{(f_3 * \Phi_{3,k})(x)} - (f_2 * \Phi_{2,k+1})(x) \overline{(f_3 * \Phi_{3,k+1})(x)} \Big\} \, dx \,,$$

and we write $I = I_1 + I_2 + I_3$, where

$$I_1 = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{m=0}^{\frac{k+5}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} (f_2 * \Phi_{2,k})(x) \, \overline{(f_3 * (\Phi_{3,k} - \Phi_{3,k+1})(x)} \, dx \,,$$

$$I_2 = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{m=0}^{\frac{k+5}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} (f_2 * (\Phi_{2,k} - \Phi_{2,k+1}))(x) \, \overline{(f_3 * (\Phi_{3,k+1} - \Phi_{3,k+6}))(x)} \, dx \,,$$

$$I_3 = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{m=0}^{\frac{k+5}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} (f_2 * (\Phi_{2,k} - \Phi_{2,k+1}))(x) \, \overline{(f_3 * \Phi_{3,k+6})(x)} \, dx \,.$$

We write I_1 as $I_{11} + I_{12}$, where

$$I_{11} = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{m=0}^{\frac{k-4}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} (f_2 * \Phi_{2,k})(x) \, \overline{(f_3 * (\Phi_{3,k} - \Phi_{3,k+1})(x)} \, dx \,,$$

$$I_{12} = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{\frac{k-4}{2} < m \leq \frac{k+5}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} (f_2 * \Phi_{2,k})(x) \, \overline{(f_3 * (\Phi_{3,k} - \Phi_{3,k+1})(x)} \, dx \,.$$

We begin by observing that I_{11} is identically equal to zero. Indeed, let us calculate the supports of the functions the appear in I_{11}. We have

$$\operatorname{supp} \Big(\sum_{m=0}^{\frac{k-4}{2}} \widehat{\Phi_{1,k-m}} \Big) \subset \bigcup_{m=0}^{\frac{k-4}{2}} [-\tfrac{101}{100} \cdot 2^{-k+m}, -\tfrac{99}{100} \cdot 2^{-k+m-1}] \subset [-\tfrac{101}{400} \cdot 2^{-\frac{k}{2}}, -\tfrac{99}{200} \cdot 2^{-k}]$$

$$\operatorname{supp} \widehat{\Phi_{2,k}} \subset [-2^{-\frac{k}{2}}, 2^{-\frac{k}{2}}]$$

$$\operatorname{supp} \big(\widehat{\Phi_{3,k}} - \widehat{\Phi_{3,k+1}} \big) \subset [-2 \cdot 2^{-\frac{k}{2}}, -\tfrac{9}{5\sqrt{2}} \cdot 2^{-\frac{k}{2}}] \bigcup [\tfrac{9}{5\sqrt{2}} \cdot 2^{-\frac{k}{2}}, 2 \cdot 2^{-\frac{k}{2}}],$$

where in the last inclusion we used the fact that $\widehat{\Phi_{3,k}}$ is equal to one on a substantially large subset of its support. It is easy to see that

$$\operatorname{supp} \Big(\sum_{m=0}^{\frac{k-4}{2}} \widehat{\Phi_{1,k-m}} \Big) + \operatorname{supp} \widehat{\Phi_{2,k}} \subset [-\tfrac{501}{400} \cdot 2^{-\frac{k}{2}}, 2^{-\frac{k}{2}}],$$

from which it follows that

$$\Big(\operatorname{supp} \Big(\sum_{m=0}^{\frac{k-4}{2}} \widehat{\Phi_{1,k-m}} \Big) + \operatorname{supp} \widehat{\Phi_{2,k}} \Big) \bigcap \operatorname{supp} \big(\widehat{\Phi_{3,k}} - \widehat{\Phi_{3,k+1}} \big) = \emptyset,$$

since $\widehat{\Phi_{3,k}} - \widehat{\Phi_{3,k+1}}$ is supported in $[-2\cdot 2^{-\frac{k}{2}}, -\frac{8}{5\sqrt{2}}\cdot 2^{-\frac{k}{2}}]\bigcup[\frac{8}{5\sqrt{2}}\cdot 2^{-\frac{k}{2}}, 2\cdot 2^{-\frac{k}{2}}]$. We conclude that $I_{11}=0$ and we don't need to worry about this term. We proceed with term I_{12} which is equal to a finite sum of expressions of the form

$$\sum_{k=7}^{\infty}\int_{\mathbf{R}}(f_1*\Phi_{1,k-m(k)})(x)\,(f_2*\Phi_{2,k})(x)\,\overline{(f_3*(\Phi_{3,k}-\Phi_{3,k+1})(x)}\,dx\,, \tag{5.3}$$

where $m(k)$ is an integer in the interval $(\frac{k-4}{2}, \frac{k+5}{2}]$. We can now control I_{12} by

$$\int_{\mathbf{R}}\Big(\sum_{k=7}^{\infty}|f_1*\Phi_{1,k-m(k)}|^2\Big)^{\frac{1}{2}}\sup_k|f_2*\Phi_{2,k}|\Big(\sum_{k=7}^{\infty}|f_3*(\Phi_{3,k}-\Phi_{3,k+1})|^2\Big)^{\frac{1}{2}}\,dx$$

and this is easily seen to be bounded by a constant multiple of

$$\|f_1\|_p\|f_2\|_q\|f_3\|_{r'}$$

in view of the Littlewood-Paley theorem and the fact that $\sup_k|f_2*\Phi_{2,k}|$ is bounded by the Hardy-Littlewood maximal function of f_2.

We continue with term I_2. We first claim that the estimate below is valid:

$$\Big\|\sup_k\Big|\sum_{m=0}^{\frac{k+5}{2}}f_1*\Phi_{1,k-m}\Big|\Big\|_p\le C_p\|f_1\|_p\,. \tag{5.4}$$

To see this we bound the left side of (5.4) by

$$\Big\|\sup_k\Big|\sum_{m=\frac{k-5}{2}}^{k}f_1*\Phi_{1,m}\Big|\Big\|_p\le\Big\|\sup_k\Big|\sum_{\substack{m=\frac{k-5}{2}\\ m\text{ odd}}}^{k}f_1*\Phi_{1,m}\Big|\Big\|_p+\Big\|\sup_k\Big|\sum_{\substack{m=\frac{k-5}{2}\\ m\text{ even}}}^{k}f_1*\Phi_{1,m}\Big|\Big\|_p\,. \tag{5.5}$$

We let a_k (resp. c_k) be the infimum of the left points of the supports of the functions $\widehat{\Phi_{1,m}}$ with m odd (resp. even) in $[\frac{k-5}{2},k]$ and b_k (resp. d_k) is the supremum of the right points of the supports of the functions $\widehat{\Phi_{1,m}}$ with m odd (resp. even) in $[\frac{k-5}{2},k]$. Then the right hand side of (5.5) is equal to

$$\Big\|\sup_k\Big|\Big(\big(\sum_{m\text{ odd}}f_1*\Phi_{1,m}\big)\widehat{\ }\,1_{[a_k,b_k]}\Big)^{\vee}\Big|\Big\|_p+\Big\|\sup_k\Big|\Big(\big(\sum_{m\text{ even}}f_1*\Phi_{1,m}\big)\widehat{\ }\,1_{[c_k,d_k]}\Big)^{\vee}\Big|\Big\|_p$$

and this is bounded by

$$C_p\Big(\Big\|\sum_{k\text{ odd}}f_1*\Phi_{1,k}\Big\|_p+\Big\|\sum_{k\text{ even}}f_1*\Phi_{1,k}\Big\|_p\Big) \tag{5.6}$$

in view of the Carleson-Hunt theorem [1], [6]. The expression in (5.6) is easily controlled by $C_p'\|f_1\|_p$ via a simple orthogonality argument, and the proof of our claim in (5.4) is complete. It follows that I_2 is controlled by

$$\int_{\mathbf{R}}\sup_k\Big|\sum_{m=0}^{\frac{k+5}{2}}f_1*\Phi_{1,k-m}\Big|\Big(\sum_{k=7}^{\infty}|f_2*(\Phi_{2,k}-\Phi_{2,k+1})|^2\Big)^{\frac{1}{2}}\Big(\sum_{k=7}^{\infty}|f_3*(\Phi_{3,k}-\Phi_{3,k+6})|^2\Big)^{\frac{1}{2}}\,dx$$

and this is in turn bounded by a constant multiple of

$$\|f_1\|_p\|f_2\|_q\|f_3\|_{r'}$$

in view of the Littlewood-Paley theorem and the discussion above.

Before we turn our attention to I_3, we make a couple of observations regarding the supports of the Fourier transforms of the functions $\Phi_{1,k}$, $\Phi_{2,k}$, and $\Phi_{3,k}$. First we observe that

$$\begin{aligned}
&\operatorname{supp}\Big(\sum_{m=0}^{\frac{k-4}{2}}\widehat{\Phi_{1,k-m}}\Big)\subset\bigcup_{m=0}^{\frac{k-4}{2}}[-\tfrac{101}{100}\cdot 2^{-k+m},-\tfrac{99}{100}\cdot 2^{-k+m-1}]\subset[-\tfrac{101}{400}\cdot 2^{-\frac{k}{2}},-\tfrac{99}{200}\cdot 2^{-k}]\,,\\
&\operatorname{supp}\big(\widehat{\Phi_{2,k}}-\widehat{\Phi_{2,k+1}}\big)\subset[-2^{-\frac{k}{2}},-\tfrac{4}{5\sqrt{2}}\cdot 2^{-\frac{k}{2}}]\cup[\tfrac{4}{5\sqrt{2}}\cdot 2^{-\frac{k}{2}},2^{-\frac{k}{2}}]\,,\\
&\operatorname{supp}\widehat{\Phi_{3,k+6}}\subset[-\tfrac{1}{4}\cdot 2^{-\frac{k}{2}},\tfrac{1}{4}\cdot 2^{-\frac{k}{2}}]\,.
\end{aligned}$$

Therefore, the algebraic sum

$$\operatorname{supp}\Big(\sum_{m=0}^{\frac{k-4}{2}}\widehat{\Phi_{1,k-m}}\Big)+\operatorname{supp}\big(\widehat{\Phi_{2,k}}-\widehat{\Phi_{2,k+1}}\big)$$

is contained in the union of the intervals

$$[-\tfrac{501}{400}\cdot 2^{-\frac{k}{2}},-\tfrac{4}{5\sqrt{2}}\cdot 2^{-\frac{k}{2}}-\tfrac{99}{200}\cdot 2^{-k}]\cup[(\tfrac{4}{5\sqrt{2}}-\tfrac{101}{400})\cdot 2^{-\frac{k}{2}},\tfrac{101}{200}\cdot 2^{-\frac{k}{2}}]$$

from which it easily follows that

$$\left(\operatorname{supp}\Big(\sum_{m=0}^{\frac{k-4}{2}}\widehat{\Phi_{1,k-m}}\Big)+\operatorname{supp}\big(\widehat{\Phi_{2,k}}-\widehat{\Phi_{2,k+1}}\big)\right)\cap\operatorname{supp}\widehat{\Phi_{3,k+6}}=\emptyset\,.$$

Therefore I_3 reduces to the sum

$$\sum_{k=7}^{\infty}\int_{\mathbf{R}}\Big\{\sum_{\frac{k-4}{2}<m\le\frac{k+5}{2}}(f_1*\Phi_{1,k-m})(x)\Big\}(f_2*(\Phi_{2,k}-\Phi_{2,k+1}))(x)\,\overline{(f_3*\Phi_{3,k+6})(x)}\,dx\,,$$

in which m ranges only through a finite set (depending on k). For every such $m=m(k)$, we can estimate I_3 by

$$\int_{\mathbf{R}}\big(\sum_{k=7}^{\infty}|f_1*\Phi_{1,k-m(k)}|^2\big)^{\frac{1}{2}}\big(\sum_{k=7}^{\infty}|f_2*(\Phi_{2,k}-\Phi_{2,k+1})|^2\big)^{\frac{1}{2}}\sup_k\big|f_3*\Phi_{1,k+6}\big|\,dx$$

and this is clearly bounded by a constant multiple of $\|f_1\|_p\|f_2\|_q\|f_3\|_{r'}$ via the Littlewood-Paley theorem and the $L^{r'}$ boundedness of the Hardy-Littlewood maximal operator.

This estimate completes the proof of the boundedness of $\widetilde{S}$ and thus of Lemma 5. □

The proof of the boundedness of $S=\sum_{k=1}^{\infty}S_k$ is now complete.

6. The boundedness of $\sum_{\mu=1}^{\infty}T_\mu$

Throughout this section we will fix $2\le p_1,p_2<\infty$, $1<p\le 2$ with $1/p_1+1/p_2=1/p$. The main difficulty is to show that the operators T_μ are uniformly bounded from $L^{p_1}\times L^{p_2}\to L^p$. Once this is established we can use Lemma 1 to control the sum $\sum_{\mu=1}^{\infty}T_\mu$. To do so, it will suffice to check that the hypotheses of Lemma 1 apply for the operators

T_μ. Indeed, one can easily see that the support of the symbol of T_μ is contained in the set $A_\mu \times B_\mu$ union the set $A_\mu \times (-B_\mu)$, where

$$
\begin{aligned}
A_\mu =& \big[\big(1-(1+2^{-9})2^{-\mu}\big)\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac12-\frac{\mu}{2}}\big), \cos\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big)\big] \\
B_\mu =& \big[\big(1-(1+2^{-9})2^{-\mu}\big)\sin\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big), \sin\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac12-\frac{\mu}{2}}\big)\big]\,.
\end{aligned}
\tag{6.1}
$$

It is now elementary to check that the sets $A_\mu, A_{\mu+10}, A_{\mu+20}, \dots$ are disjoint, and similarly for the sets B_μ, $A_\mu + B_\mu$, and $A_\mu - B_\mu$. (To apply Lemma 1 one may consider the "upper" and the "lower" part of T_μ separately.)

We now turn our attention to the crucial issue of the uniform boundedness of the T_μ's. We fix a large μ and we break up the support of T_μ as the disjoint union of the "curved rectangles" $D_{\mu,k}$ for $k \ge \mu$, defined as follows

$$
\begin{aligned}
D_{\mu,\mu} =& \big\{(\xi,\eta):\ 1-2^{-\mu}(1+2^{-9}) \le |(\xi,\eta)| < 1-2^{-(\mu+1)}\big\} \\
& \bigcap \big\{(\xi,\eta):\ \tfrac{\pi}{8}2^{-\frac{\mu}{2}}(1-2^{-9}) \le |\mathrm{Argument}(\xi,\eta)| < \tfrac{\pi}{8}2^{-\frac{\mu-1}{2}}(1+2^{-9})\big\}\,,
\end{aligned}
$$

while for $k \ge \mu+1$

$$
\begin{aligned}
D_{\mu,k} =& \big\{(\xi,\eta):\ 1-2^{-k} \le |(\xi,\eta)| < 1-2^{-(k+1)}\big\} \\
& \bigcap \big\{(\xi,\eta):\ \tfrac{\pi}{8}2^{-\frac{\mu}{2}}(1-2^{-9}) \le |\mathrm{Argument}(\xi,\eta)| < \tfrac{\pi}{8}2^{-\frac{\mu-1}{2}}(1+2^{-9})\big\}\,.
\end{aligned}
$$

In the sequel, rectangles will be products of intervals of the form $[a,b)\times[c,d)$. The quantity $(b-a)\times(d-c)$ will be referred to as the size of a rectangle. We tile up the plane as the union of rectangles of size $2^{-\mu-5}\times 2^{-\frac{\mu}{2}-5}$ and we let $\mathcal{E}_\mu$ be the set of all such rectangles. Let $\mathcal{E}_\mu^{select}$ be the subset of $\mathcal{E}_\mu$ consisting of all rectangles that intersect $D_{\mu,\mu}$. We denote by $\mathcal{E}_{\mu+1}$ the set of all rectangles obtained by subdividing each rectangle in $\mathcal{E}_\mu \setminus \mathcal{E}_\mu^{select}$ into four rectangles, each of size $2^{-\mu-6}\times 2^{-\frac{\mu}{2}-6}$, by halving its sides. Let $\mathcal{E}_{\mu+1}^{select}$ be the subset of $\mathcal{E}_{\mu+1}$ consisting of all rectangles that intersect $D_{\mu,\mu+1}$. Next, we denote by $\mathcal{E}_{\mu+2}$ the set of all rectangles obtained by subdividing each rectangle in $\mathcal{E}_{\mu+1}\setminus\mathcal{E}_{\mu+1}^{select}$ into four rectangles, each of size $2^{-\mu-7}\times 2^{-\frac{\mu}{2}-7}$, by halving its sides. Continue this way by induction. Then we have "essentially covered" each $D_{\mu,k}$ by disjoint rectangles of size $2^{-k-5}\times 2^{\frac{\mu}{2}-k-5}$ and the set of all such rectangles is denoted by $\mathcal{E}_{\mu+k}^{select}$. Since each $D_{\mu,k}$ has area about $2^{-\frac{\mu}{2}-k}$, we have used approximately $2^{k-\mu}$ rectangles of size $2^{-k-5}\times 2^{\frac{\mu}{2}-k-5}$ to "cover" $D_{\mu,k}$. In other words, the cardinality of each set $\mathcal{E}_{\mu+k}^{select}$ is of the order of $2^{k-\mu}$.

Elements of $\mathcal{E}_{\mu+k}^{select}$ will be denoted by $R_{k,l,m}$; explicitly

$$
R_{k,l,m} = [\tfrac{1}{32}2^{-k}l, \tfrac{1}{32}2^{-k}(l+1)] \times [\tfrac{1}{32}2^{-k+\frac{\mu}{2}}m, \tfrac{1}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,.
$$

So the first index k indicates that the rectangle $R_{k,l,m}$ has size $2^{-k-5}\times 2^{\frac{\mu}{2}-k-5}$. The second index l indicates the horizontal location of the rectangle $R_{k,l,m}$, while the third index m indicates its vertical location. Furthermore, if $R_{k,l,m}$ is selected, then for any integer $k \ge \mu+1$, l ranges in the interval

$$
32\cdot(2^k-1)\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}+\frac12}\big)-1 \le l \le 32\cdot(2^k-\tfrac12)\cos\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big)\,,
\tag{6.2}
$$

with the left inequality above only slightly changed to

$$
32\cdot(2^k-(1+2^{-9}))\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}+\frac12}\big)-1 \le l
$$

when $k = \mu$. Moreover, for fixed k and l, the range of m is specified by the inequalities

$$2^{5+k-\frac{\mu}{2}}\sqrt{(1-2^{-k+1})^2-(\tfrac{2^{-k}(l+1)}{32})^2}-1<|m|<2^{5+k-\frac{\mu}{2}}\sqrt{(1-2^{-k-1})^2-(\tfrac{2^{-k}l}{32})^2}\,. \tag{6.3}$$

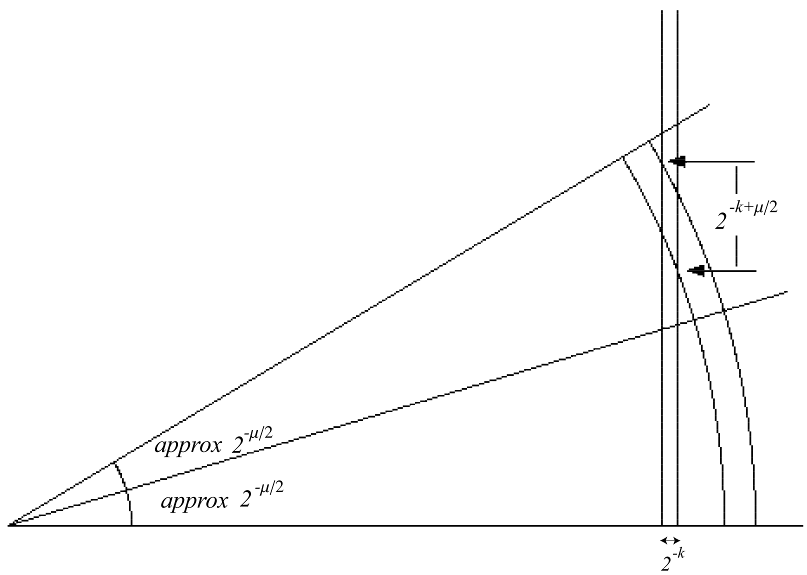

FIGURE 2. The intersection of $D_{\mu,k}$ with the smallest vertical strip that contains a fixed rectangle $R_{k,l,m}$ in $\mathcal{E}^{select}_{\mu+k}$ has vertical length at most a multiple of $2^{-k+\frac{\mu}{2}}$.

It is a very important geometric fact that given a fixed k and l, there exist at most 64 integers m such that the rectangles $R_{k,l,m}$ in $\mathcal{E}^{select}_{\mu+k}$ intersect $D_{\mu,k}$. The verification of this fact is a simple geometric exercise shown in Figure 2 and is left to the reader. Therefore in the sequel, we will think of $m = m(k,l,\mu)$ as a function of k, l, and μ whose range is a set of integers with at most 64 elements.

Let $\varepsilon > 0$ be a very small number. Pick a smooth function $\Psi_{k,l,m}(\xi,\eta)$ supported in a small neighborhood of $R_{k,l,m}$ such that

$$|D^\beta_\xi \Psi_{k,l,m}| \le C2^{\beta k} \qquad |D^\beta_\eta \Psi_{k,l,m}| \le C2^{\beta(k-\frac{\mu}{2})}\,.$$

and such that the function

$$\sum_{k=\mu}^{\infty} \sum_{\substack{l,m \text{ such that}\\ R_{k,l,m}\in\mathcal{E}^{select}_{\mu+k}}} \Psi_{k,l,m}(\xi,\eta)$$

is equal to 1 on the union of all $R_{k,l,m}$ in $\mathcal{E}^{select}_{\mu+k}$ that do not intersect the boundary of the support of T_μ. Using the Fourier series bilinear symbol expansion method of Coifman and

Meyer [2] (see also [5]), we write $\Psi_{k,l,m}(\xi,\eta)$ as

$$\Psi_{k,l,m}(\xi,\eta) = \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} C_{n_1,n_2}\widehat{\Phi_{1,k,l,n_1,n_2}}(\xi)\widehat{\Phi_{2,k,m,n_1,n_2}}(\eta)\,,$$

where $|C_{n_1,n_2}| \le C_M(1+n_1^2+n_2^2)^{-M}$ for all $M>0$, Φ_{1,k,l,n_1,n_2} and Φ_{2,k,m,n_1,n_2} are Schwartz functions which satisfy

$$\begin{aligned}\text{supp } \widehat{\Phi_{1,k,l,n_1,n_2}} &\subset (1+\varepsilon)[\tfrac{1}{32}2^{-k}l, \tfrac{1}{32}2^{-k}(l+1)]\,,\\ \text{supp } \widehat{\Phi_{2,k,m,n_1,n_2}} &\subset (1+\varepsilon)[\tfrac{1}{32}2^{-k+\frac{\mu}{2}}m, \tfrac{1}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,,\end{aligned}$$

$$\begin{aligned}\widehat{\Phi_{1,k,l,0,0}} = 1 \quad &\text{on} \quad (1-\varepsilon)[\tfrac{1}{32}2^{-k}l, \tfrac{1}{32}2^{-k}(l+1)]\,,\\ \widehat{\Phi_{2,k,m,0,0}} = 1 \quad &\text{on} \quad (1-\varepsilon)[\tfrac{1}{32}2^{-k+\frac{\mu}{2}}m, \tfrac{1}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,,\end{aligned}$$

and for all $\beta \ge 0$ there exist C_{β,n_1,n_2} bounded by $C_\beta(1+|n_1|+|n_2|)^\beta$ so that

$$\Big|\frac{d^\beta}{d\xi^\beta}\widehat{\Phi_{1,k,l,n_1,n_2}}(\xi)\Big| \le C_{\beta,n_1,n_2}2^{k\beta}, \qquad \Big|\frac{d^\beta}{d\eta^\beta}\widehat{\Phi_{2,k,m,n_1,n_2}}(\eta)\Big| \le C_{\beta,n_1,n_2}2^{(k-\frac{\mu}{2})\beta}\,. \tag{6.4}$$

Recall that the symbol of the bilinear operator T_μ is

$$\sigma_\mu(\xi,\eta) = \sum_{k=\mu}^{\infty}\widehat{b_k^1}(\xi,\eta)\widehat{\rho_k^\mu}(\xi,\eta)\,.$$

We now write σ_μ as

$$\sigma_\mu(\xi,\eta) = \sum_{n_1,n_2\in\mathbb{Z}} C_{n_1,n_2}\sum_{k=\mu}^{\infty}\sum_{\substack{l,m \text{ such that}\\ R_{k,l,m}\in\mathcal{E}_{\mu+k}^{select}}}\widehat{\Phi_{1,k,l,n_1,n_2}}(\xi)\widehat{\Phi_{2,k,m,n_1,n_2}}(\eta) + E_\mu^{(1)}(\xi,\eta)\,, \tag{6.5}$$

where $E_\mu^{(1)}$ is an error.

We start by studying the error $E_\mu^{(1)}$. Let (r,θ) denote polar coordinates in the (ξ,η) plane. The function $E_\mu^{(1)}$ consists of six pieces: The piece $E_{\mu,1}^{(1)}$ supported near the line $\theta = \frac{\pi}{8}2^{-\frac{\mu}{2}}$, the piece $E_{\mu,2}^{(1)}$ supported near the line $\theta = \frac{\pi}{8}2^{-\frac{\mu-1}{2}}$, the piece $E_{\mu,3}^{(1)}$ supported near the circle $r = 1-2^{-\mu}$ between these two lines, the piece $E_{\mu,4}^{(1)}$ supported near the line $\theta = -\frac{\pi}{8}2^{-\frac{\mu}{2}}$, the piece $E_{\mu,5}^{(1)}$ supported near the line $\theta = -\frac{\pi}{8}2^{-\frac{\mu-1}{2}}$, and the piece $E_{\mu,6}^{(1)}$ supported near the circle $r = 1-2^{-\mu}$ between these last two lines. The error $E_{\mu,3}^{(1)}+E_{\mu,6}^{(1)}$ is the easiest to control. Since $\mathcal{E}_\mu^{select}$ consists only of finitely many rectangles (independent of μ), $E_{\mu,3}^{(1)}+E_{\mu,6}^{(1)}$ is equal to a finite sum of smooth functions ϕ_μ which are supported in a small dilate of $D_{\mu,\mu}$ and which satisfy the estimates

$$\Big|\frac{d^\beta}{d\xi^\beta}\phi_\mu\Big| \le C_\beta 2^{\mu\beta}, \qquad \Big|\frac{d^\beta}{d\eta^\beta}\phi_\mu\Big| \le C_\beta 2^{\frac{\mu}{2}\beta}\,,$$

because of (4.8) and (6.4). It follows from Lemma 3 that the inverse Fourier transforms of the ϕ_μ's are in L^1 uniformly in μ. Using Lemma 2 we obtain the uniform (in μ) boundedness of the operators whose symbols are $E_{\mu,3}^{(1)}+E_{\mu,6}^{(1)}$.

We write $E_{\mu,1}^{(1)}+E_{\mu,2}^{(1)}+E_{\mu,4}^{(1)}+E_{\mu,5}^{(1)} = \sum_{k=\mu}^{\infty}E_{\mu,k}$, where each $E_{\mu,k}$ consists of the (smooth) piece of this function inside the annulus $1-2^{-k} \le r \le 1-2^{-k-1}$. An easy calculation shows that the support of $E_{\mu,k}$ is contained in the union of the sets $A_{\mu,k}\times B_{\mu,k}$, $A'_{\mu,k}\times B'_{\mu,k}$, $A_{\mu,k}\times(-B_{\mu,k})$, and $A'_{\mu,k}\times(-B'_{\mu,k})$, where

$$\begin{aligned}
A_{\mu,k} &= \big[(1-2^{-k+1})\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}}\big),(1-2^{-k-2})\cos\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big)\big]\\
B_{\mu,k} &= \big[(1-2^{-k+1})\sin\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big),(1-2^{-k-2})\sin\big(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}}\big)\big]\\
A'_{\mu,k} &= \big[(1-2^{-k+1})\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big),(1-2^{-k-2})\cos\big(\tfrac{\pi}{8}(1-2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big)\big]\\
B'_{\mu,k} &= \big[(1-2^{-k+1})\sin\big(\tfrac{\pi}{8}(1-2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big),(1-2^{-k-2})\sin\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big)\big]\,.
\end{aligned}$$

We now observe that the sets $A_{\mu,k}, A_{\mu,k+10}, A_{\mu,k+20},\dots$ are pairwise disjoint, and the same disjointness property also holds for the families $\{B_{\mu,k}\}_k$, $\{A_{\mu,k}+B_{\mu,k}\}_k$, $\{A_{\mu,k}-B_{\mu,k}\}_k$, $\{A'_{\mu,k}\}_k$, $\{B'_{\mu,k}\}_k$, $\{A'_{\mu,k}+B'_{\mu,k}\}_k$, and $\{A'_{\mu,k}-B'_{\mu,k}\}_k$. Using (4.8) and (6.4) we obtain that the inverse Fourier transforms of the functions $E_{\mu,k}$ are uniformly integrable in μ and k. Then Lemmata 1 and 2 imply that the operators with symbols $E^{(1)}_{\mu}$ are bounded from $L^p\times L^q$ into L^r uniformly in μ. (We apply Lemma 1 for each of the four pieces above separately.)

We now turn to the boundedness of the operator whose symbol is $\sigma_\mu - E^{(1)}_{\mu}$. In view of the controlled growth of the constant C_{β,n_1,n_2} in (6.4) and of the rapid decay of the constant C_{n_1,n_2} when $|n_1|+|n_2|$ is large, we may only consider the case $n_1=n_2=0$; in the remaining cases an extra decaying factor of $|n_1|+|n_2|$ is produced which allows us to sum the series in n_1,n_2. When $n_1=n_2=0$ we set $\Phi_{1,k,l,0,0}=\Phi_{1,k,l}$ and $\Phi_{2,k,m,0,0}=\Phi_{2,k,m}$. Let us consider the operator $\widetilde{T}_\mu$ defined by

$$\widetilde{T}_\mu(f,g)(x)=\sum_{k=\mu}^{\infty}\sum_{\substack{l,m\text{ such that}\\ R_{k,l,m}\in\mathcal{E}^{select}_{\mu+k}}}\int_{\mathbf{R}}\int_{\mathbf{R}}\widehat{f}(\xi)\widehat{g}(\eta)\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,m}}(\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta\,.$$

For every selected rectangle $R_{k,l,m}$ we now choose a third Schwartz function $\Phi_{3,k,l,m}$ such that

$$\Big|\frac{d^\beta}{d\xi^\beta}\widehat{\Phi_{3,k,l,m}}(\xi)\Big|\le C_\alpha 2^{(k-\frac{\mu}{2})\beta}$$

for all $\beta\ge 0$, $\widehat{\Phi_{3,k,l,m}}$ is equal to 1 on the interval

$$[\tfrac{1}{32}2^{-k}l+2^{-k+\frac{\mu}{2}}m,\,\tfrac{1}{32}2^{-k}(l+1)+2^{-k+\frac{\mu}{2}}(m+1)]$$

and also on the interval

$$[\tfrac{1}{32}2^{-k}l-2^{-k+\frac{\mu}{2}}m,\,\tfrac{1}{32}2^{-k}(l+1)-2^{-k+\frac{\mu}{2}}(m+1)]$$

and is supported in an $(1+\varepsilon)$-neighborhood of the union of the two intervals above. Because of the properties of the function $\Phi_{3,k,l,m}$, $\widetilde{T}_\mu(f,g)(x)$ is also equal to

$$\sum_{k=\mu}^{\infty}\sum_{\substack{l,m\text{ such that}\\ R_{k,l,m}\in\mathcal{E}^{select}_{\mu+k}}}\int_{\mathbf{R}}\int_{\mathbf{R}}\widehat{f}(\xi)\widehat{g}(\eta)\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,m}}(\eta)\widehat{\Phi_{3,k,l,m}}(\xi+\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta\,. \tag{6.6}$$

We now let $J_{k,\mu}$ be the set of all integers l satisfying (6.2) and for each k,l,μ we set

$$\lambda=\lambda(k,l,\mu)=\left[32\cdot 2^{k-\frac{\mu}{2}}\sqrt{(1-2^{-k-1})^2-(\tfrac{2^{-k}l}{32})^2}\;\right]+1\,,$$

where the square brackets above denote the integer part. We can therefore write $\widetilde{T}_\mu(f,g)(x)$ as a finite sum (over $1 \le s \le 64$) of operators of the form

$$I_\mu(f,g)(x) = \sum_{k=\mu}^{\infty} \sum_{l \in J_{k,\mu}} \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,\lambda-s}}(\eta)\widehat{\Phi_{3,k,l,\lambda-s}}(\xi+\eta)e^{2\pi i(\xi+\eta)x}d\xi\, d\eta\,.$$

The uniform boundedness of the I_μ's for the claimed range of exponents will be a consequence of the results in [5] and [9] once we have established Lemma 6 stated below. Then we can refer to Lemma 4 in [5] and Lemma 3 in [9] to obtain uniform boundedness for each I_μ from $L^{p_1} \times L^{p_2} \to L^p$, where p_1, p_2, p are as in Theorem 1. The results in [9] are only needed to cover the endpoint cases in which $p_1 = 2$, $p_2 = 2$, or $p = 2$.

For uniformity we replaced $\Phi_{2,k,\lambda(k,l,\mu)-s}$ by $\Phi_{2,k,l,\lambda(k,l,\mu)-s}$ in the Lemma below.

Lemma 6. *Let be $\Phi_{1,k,l}$, $\Phi_{2,k,l,\lambda-s}$, and $\Phi_{3,k,l,\lambda-s}$ be as above and let $|k-k'| \ge 100$.*
(1) If supp $\widehat{\Phi_{1,k,l}} \subsetneqq$ supp $\widehat{\Phi_{1,k',l'}}$, then for $j \in \{2,3\}$ we have

$$\text{supremum}\big(\textit{supp } \widehat{\Phi_{j,k,l,\lambda-s}}\big) < \text{infimum}\big(\textit{supp } \widehat{\Phi_{j,k',l',\lambda-s'}}\big)$$

and

$$\frac{1}{16} \cdot 2^{-k'+\frac{\mu}{2}} < \mathit{dist}\big(\textit{supp } \widehat{\Phi_{j,k,l,\lambda-s}}, \textit{supp } \widehat{\Phi_{j,k',l',\lambda-s'}}\big) < 5 \cdot 2^{-k'+\frac{\mu}{2}}\,. \tag{6.7}$$

(2) If supp $\widehat{\Phi_{j,k,l,\lambda-s}} \subsetneqq$ supp $\widehat{\Phi_{j,k',l',\lambda-s'}}$ for $j \in \{2,3\}$, then

$$\text{supremum}\big(\textit{supp } \widehat{\Phi_{1,k',l'}}\big) < \text{infimum}\big(\textit{supp } \widehat{\Phi_{1,k,l}}\big)$$

and

$$\frac{5}{64} \cdot 2^{-k'} < \mathit{dist}\big(\textit{supp } \widehat{\Phi_{1,k,l}}, \textit{supp } \widehat{\Phi_{1,k',l'}}\big) < 4 \cdot 2^{-k'}\,. \tag{6.8}$$

Proof. We first prove the assertion in part (1). The assumption supp $\widehat{\Phi_{1,k,l}} \subsetneqq$ supp $\widehat{\Phi_{1,k',l'}}$ implies that

$$2^{-k'}l' < 2^{-k}l < 2^{-k}(l+1) < 2^{-k'}(l'+1)$$

which gives $0 < 2^{-k}l - 2^{-k'}l' < 2^{-k'} - 2^{-k}$. Let $m = 32 \cdot 2^{k-\frac{\mu}{2}}\sqrt{(1-2^{-k-1})^2 - (\frac{2^{-k}l}{32})^2} - C$, where C is a real number between 0 and 64. Then

$$\begin{aligned}
&2^{-k+\frac{\mu}{2}}m - 2^{-k+\frac{\mu}{2}}m' \\
&= 32\sqrt{(1-2^{-k-1})^2 - (\tfrac{2^{-k}l}{32})^2} - 32\sqrt{(1-2^{-k'-1})^2 - (\tfrac{2^{-k'}l'}{32})^2} \\
&\qquad\qquad + C(2^{-k'+\frac{\mu}{2}} - 2^{-k+\frac{\mu}{2}}) - 2^{-k'+\frac{\mu}{2}} \\
&= \frac{16(2^{-k'}-2^{-k})(2-\frac{1}{2}2^{-k}-\frac{1}{2}2^{-k'}) + \frac{1}{32}(2^{-k'}l'-2^{-k}l)(2^{-k'}l'+2^{-k}l)}{\sqrt{(1-2^{-k-1})^2 - (\frac{2^{-k}l}{32})^2} + \sqrt{(1-2^{-k'-1})^2 - (\frac{2^{-k'}l'}{32})^2}} \\
&\qquad\qquad + C(2^{-k'+\frac{\mu}{2}} - 2^{-k+\frac{\mu}{2}}) - 2^{-k'+\frac{\mu}{2}}\,.
\end{aligned}$$

But observe that both radicals in the denominator above are between $2^{-\frac{\mu}{2}}$ and $2^{1-\frac{\mu}{2}}$. Since $0 < 2^{-k'}l' + 2^{-k}l < 2$, the last expression above can be estimated above and below in terms of $2^{-k'+\frac{\mu}{2}}$. Carrying out the algebra we find that

$$80 \cdot 2^{-k'+\frac{\mu}{2}} > 2^{-k+\frac{\mu}{2}}m - 2^{-k+\frac{\mu}{2}}m' > 2 \cdot 2^{-k'+\frac{\mu}{2}}\,.$$

This proves (6.7). We now turn our attention to part (2).

We only prove (6.8) for $j = 2$ since the proof for $j = 3$ is similar. Since the support of $\widehat{\Phi_{2,k,l,\lambda-s}}$ is properly contained in the support of $\widehat{\Phi_{2,k',l',\lambda-s'}}$, it follows that the number

$$\sqrt{(1-2^{-k'-1})^2-(\tfrac{2^{-k}l'}{32})^2}-\sqrt{(1-2^{-k-1})^2-(\tfrac{2^{-k}l}{32})^2}$$
$$=\frac{(2-2^{-k'-1}-2^{-k-1})(2^{-k-1}-2^{-k'-1})+\frac{1}{32}(2^{-k}l-2^{-k'}l')(2^{-k}l+2^{-k'}l')}{\sqrt{(1-2^{-k'-1})^2-(\frac{2^{-k}l'}{32})^2}+\sqrt{(1-2^{-k-1})^2-(\frac{2^{-k}l}{32})^2}}$$

lies between the numbers $\frac{C-1}{32}(2^{-k'+\frac{\mu}{2}}-2^{-k+\frac{\mu}{2}})$ and $\frac{C}{32}(2^{-k'+\frac{\mu}{2}}-2^{-k+\frac{\mu}{2}})$ for some $0 \le C \le 64$. But notice that the denominator of the fraction above is always between $2^{-\frac{\mu}{2}-4}$ and $2^{-\frac{\mu}{2}+1}$. This implies that the number

$$(2-2^{-k'-1}-2^{-k-1})(2^{-k-1}-2^{-k'-1})+\tfrac{1}{32}(2^{-k}l-2^{-k'}l')(2^{-k}l+2^{-k'}l')$$

lies between $-\frac{1}{16}(2^{-k'}-2^{-k})$ and $4(2^{-k'}-2^{-k})$, since $0 \le C \le 64$. It follows that

$$(\tfrac{1}{2}-\tfrac{1}{16})(2^{-k'}-2^{-k}) \le \tfrac{1}{32}(2^{-k}l-2^{-k'}l')(2^{-k}l+2^{-k'}l') \le 4(2^{-k'}-2^{-k})\,.$$

But since $1 < 2^{-k}l + 2^{-k'}l' < 2$, we obtain that

$$\tfrac{7}{64}2^{-k'} \le \tfrac{7}{32}(2^{-k'}-2^{-k}) \le \tfrac{1}{32}(2^{-k}l-2^{-k'}l') \le 4(2^{-k'}-2^{-k}) \le 4\cdot 2^{-k'}$$

from which it follows that

$$\tfrac{5}{64}2^{-k'} \le \frac{1}{32}2^{-k}l - \frac{1}{32}2^{-k'}(l'+1) \le 4\cdot 2^{-k'}\,.$$

This proves (6.8) for $j = 2$. As observed earlier the proof for $j = 3$ is similar. □

7. The boundedness of $T_{D(2)}$

We decompose the annular sector $D(2)$ in a way analogous to that we decomposed $D(1)$. We recall the functions ρ_k^μ introduced in section 4. We introduce functions γ_k^μ by rotating the functions ρ_k^μ by an angle $\pi/4$ as follows:

$$\gamma_k^\mu = \rho_k^\mu \circ R,$$

where R is the rotation

$$R(\xi,\eta) = (\tfrac{\sqrt{2}}{2}\xi + \tfrac{\sqrt{2}}{2}\eta, -\tfrac{\sqrt{2}}{2}\xi + \tfrac{\sqrt{2}}{2}\eta)\,. \tag{7.1}$$

For each $k, \mu \ge 1$, we introduce bilinear operators U_k, V_μ as follows

$$U_k(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{b_k^2}(\xi,\eta)\widehat{\gamma_k^{k+1}}(\xi,\eta)\, e^{2\pi i(\xi+\eta)x} d\xi\, d\eta\,,$$
$$V_\mu(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\sum_{k=\mu}^{\infty}\widehat{b_k^2}(\xi,\eta)\widehat{\gamma_k^{\mu}}(\xi,\eta)\, e^{2\pi i(\xi+\eta)x} d\xi\, d\eta\,.$$

We have now decomposed $T_{D(2)}$ as

$$T_{D(2)} = \sum_{k=1}^{\infty} U_k + \sum_{\mu=1}^{\infty} V_\mu\,.$$

Observe that the U_k's are analogous to the S_k's and the V_μ's are analogous to T_μ's. Precisely, the supports of U_k and V_μ are the images of the supports of S_k and T_μ under the rotation

R defined in (7.1). We will try to obtain estimates for U_k and V_μ by using the estimates we obtained for S_k and T_μ in the previous sections.

We will first obtain the boundedness of $U = \sum_{k=1}^{\infty} U_k$. For each $k \geq 1$, we fix a Schwartz function $\Psi_{1,k}$ whose Fourier transform is supported in the interval

$$I_1^k = [\sqrt{2} \cdot (1 - \tfrac{101}{100} \cdot 2^{-k}), \sqrt{2} \cdot (1 - \tfrac{99}{100} \cdot 2^{-k-1})]$$

and which satisfies

$$\sum_{k=1}^{\infty} \widehat{\Psi_{1,k}}(t) = 1$$

for all $\frac{\sqrt{2}}{2} \leq t < \sqrt{2}$. We also fix a Schwartz function $\Psi_{3,k}$ whose Fourier transform is equal to 1 on the set

$$I_3^k = [\tfrac{\sqrt{2}}{2} \cdot (1 - 2^{-k} - \tfrac{\pi}{8}(1 + 2^{-9})2^{-\frac{k}{2}}), \tfrac{\sqrt{2}}{2} \cdot (1 - 2^{-k} + \tfrac{\pi}{8}(1 + 2^{-9})2^{-\frac{k}{2}})] \,.$$

and is supported on $(1 + \varepsilon)I_3^k$ for some very small $\varepsilon > 0$. Moreover, we choose these functions so that

$$\Big|\frac{d^\alpha}{d\zeta^\alpha}\widehat{\Psi_{1,k}}(\zeta)\Big| \leq C_\alpha 2^{k\alpha} \qquad \Big|\frac{d^\beta}{d\xi^\beta}\widehat{\Psi_{3,k}}(\xi)\Big| \leq C_\beta 2^{\frac{k}{2}\beta}$$

for all $\alpha, \beta \geq 0$. Pick another Schwartz function $\Psi_{2,k}$ whose Fourier transform is equal to 1 on the set $I_2^k = I_1^k + (-(1 + \varepsilon)I_3^k)$ and supported in $(1 + \varepsilon)I_2^k$, where for an interval $[a, b]$ we set $-[a, b] = [-b, -a]$. Moreover, we select $\Psi_{2,k}$ so that it satisfies

$$\Big|\frac{d^\beta}{d\eta^\beta}\widehat{\Psi_{2,k}}(\eta)\Big| \leq C_\beta 2^{\frac{k}{2}\beta}$$

for all $\beta \geq 0$.

We now write $U = \sum_{k=1}^{\infty} U_k = \widetilde{U} + (U - \widetilde{U})$, where $U - \widetilde{U}$ is an error term which can be treated as in Lemma 4 and

$$\widetilde{U}(f, g)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x} \sum_{k=1}^{\infty} \widehat{\Psi_{1,k}}(\xi + \eta)\widehat{\Psi_{3,k}}(\xi)\, d\xi d\eta \,.$$

Because of the support properties of $\Psi_{2,k}$, we may write

$$\widetilde{U}(f, g)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x} \sum_{k=1}^{\infty} \widehat{\Psi_{1,k}}(\xi + \eta)\widehat{\Psi_{2,k}}(\eta)\widehat{\Psi_{3,k}}(\xi)\, d\xi d\eta.$$

Pairing with a function h we get

$$\langle \widetilde{U}(f, g), h\rangle = \int_{\mathbf{R}}\int_{\mathbf{R}} \sum_{k=1}^{\infty} \overline{\widehat{\Psi_{1,k} * h}}(\xi + \eta)\ \widehat{\Psi_{2,k} * g}(\xi)\ \widehat{\Psi_{3,k} * f}(\eta)\, d\xi d\eta \,,$$

But the above is equal to

$$\sum_{k=1}^{\infty} \int_{\mathbf{R}} \overline{(h * \Psi_{1,k})}(x)\, (g * \Psi_{2,k})(x)\, (f * \Psi_{3,k})(x)\, dx \,,$$

and this formula is entirely analogous to (5.2) except for the harmless complex bar. Moreover, the properties of the $\Psi_{j,k}$'s are similar to those of the $\Phi_{j,k}$'s in section 5. Using the same technique as in the proof of Lemma 5 we obtain the norm estimate

$$\|\widetilde{U}(f, g)\|_r \leq C\|f\|_p\|g\|_q$$

for all $1 < p, q, r < \infty$ with $1/p + 1/q = 1/r$.

It remains to prove the boundedness of $\sum_{\mu=1}^{\infty} V_\mu$. To achieve this we use similar techniques to those in section 6. For each $k \ge \mu$ we let $G_{\mu,k}$ be the image of $D_{\mu,k}$ (of section 6) under the rotation R. We tile up the plane into slanted rectangles of the form

$$\big\{(\xi,\eta):\ \tfrac{\sqrt{2}}{32}2^{-\mu}l \le \xi+\eta < \tfrac{\sqrt{2}}{32}2^{-\mu}(l+1),\quad \tfrac{\sqrt{2}}{32}2^{-\mu+\frac{\mu}{2}}m \le \xi < \tfrac{\sqrt{2}}{32}2^{-\mu+\frac{\mu}{2}}(m+1)\big\}$$

and among these we select the ones that intersect $G_{\mu,\mu}$. We call the set of these selected slanted rectangles $\mathcal{F}_{\mu}^{select}$. We subdivide the nonselected slanted rectangles into four slanted rectangles of one-quarter of their area by halving both sides. Among these we select those that intersect $G_{\mu,\mu+1}$ and we call this set of selected slanted rectangles $\mathcal{F}_{\mu+1}^{select}$. We similarly subdivide the nonselected slanted rectangles and among them we select those that intersect $G_{\mu,\mu+2}$. We call the set of these selected slanted rectangles $\mathcal{F}_{\mu+1}^{select}$. Continue this way by induction. We now have a family of selected slanted rectangles

$$Q_{k,l,m} = \big\{(\xi,\eta):\ \tfrac{\sqrt{2}}{32}2^{-k}l \le \xi+\eta < \tfrac{\sqrt{2}}{32}2^{-k}(l+1),\quad \tfrac{\sqrt{2}}{32}2^{-k+\frac{\mu}{2}}m \le \xi < \tfrac{\sqrt{2}}{32}2^{-k+\frac{\mu}{2}}(m+1)\big\}$$

of dimensions $2^{-k-5+\frac{1}{2}} \times 2^{-k-5+\frac{1}{2}+\frac{\mu}{2}}$. Given such a $Q_{k,l,m}$ we set

$$Q'_{k,l,m} = [\tfrac{\sqrt{2}}{32}2^{-k}l, \tfrac{\sqrt{2}}{32}2^{-k}(l+1)] \times [\tfrac{\sqrt{2}}{32}2^{-k+\frac{\mu}{2}}m, \tfrac{\sqrt{2}}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,.$$

As we did in the previous section, we find a smooth partition of unity adapted to these $Q'_{k,l,m}$ and then we use the Fourier series method of Coifman and Meyer to write the partition as a double sum of products of functions of the variables $\xi+\eta$ and ξ. In view of the fast decay of the coefficients of the sum at infinity, we only need to consider the $(0,0)$ term of the expansion. This is the bilinear operator V'_μ defined by

$$\sum_{k=\mu}^{\infty} \sum_{\substack{l,m \text{ such that} \\ Q_{k,l,m}\in\mathcal{F}_{k+\mu}^{select}}} \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Psi_{1,k,l}}(\xi+\eta)\widehat{\Psi_{3,k,m}}(\xi)d\xi\, d\eta + \text{Error}\,, \tag{7.2}$$

where $\Psi_{1,k,l}$ and $\Psi_{3,k,m}$ are Schwartz functions (unrelated to $\Psi_{1,k}$ and $\Psi_{3,k}$ used earlier in this section) which satisfy

$$\text{supp } \widehat{\Psi_{1,k,l}} \subset (1+\varepsilon)[\tfrac{\sqrt{2}}{32}2^{-k}l, \tfrac{\sqrt{2}}{32}2^{-k}(l+1)]\,,$$
$$\text{supp } \widehat{\Psi_{3,k,l,m}} \subset (1+\varepsilon)[\tfrac{\sqrt{2}}{32}2^{-k+\frac{\mu}{2}}m, \tfrac{\sqrt{2}}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,,$$

and

$$\Big|\frac{d^\beta}{d\zeta^\beta}\widehat{\Psi_{1,k,l}}(\zeta)\Big| \le C_\beta 2^{k\beta}, \qquad \Big|\frac{d^\beta}{d\xi^\beta}\widehat{\Psi_{3,k,m}}(\xi)\Big| \le C_\beta 2^{(k-\frac{\mu}{2})\beta}\,, \tag{7.3}$$

for all $\beta \ge 0$, and the function

$$\sum_{k=\mu}^{\infty}\sum_{l,m}\widehat{\Psi_{1,k,l}}(\zeta)\widehat{\Psi_{3,k,m}}(\xi)$$

is equal to 1 on the union of all the rectangles $Q'_{k,l,m}$ that do not lie near the boundary of this union.

Let $E_\mu^{(2)}$ be the symbol of the error that appears in (7.2). The uniform boundedness (in μ) of the operators with symbols $E_\mu^{(2)}$ is obtained in a way similar for those with symbols

$E_\mu^{(1)}$. One can easily see that the projections of the supports of $E_\mu^{(2)}$ on the ξ axis, η axis, and the line $\xi=\eta$ are contained in the sets A_μ, B_μ, and C_μ respectively, where

$$\begin{aligned}A_\mu =&[(1-(1+2^{-9})2^{-\mu})\cos(\tfrac{\pi}{4}-\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}),\cos(\tfrac{\pi}{4}-\tfrac{\pi}{8}(1+2^{-9})2^{\frac12-\frac{\mu}{2}})]\\ B_\mu =&[(1-(1+2^{-9})2^{-\mu})\sin(\tfrac{\pi}{4}-\tfrac{\pi}{8}(1+2^{-9})2^{\frac12-\frac{\mu}{2}}),\sin(\tfrac{\pi}{4}-\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}})]\\ C_\mu =&[(1-(1+2^{-9})2^{-\mu})\cos(\tfrac{\pi}{8}(1+2^{-9})2^{\frac12-\frac{\mu}{2}}),\cos(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}})]\,.\end{aligned}$$

For μ large enough, the sets above are contained in the following sets respectively

$$\begin{aligned}&[\tfrac{\sqrt2}{2}(1+2^{-\frac{\mu}{2}-10}),\tfrac{\sqrt2}{2}(1+2^{-\frac{\mu}{2}+10})]\\ &[\tfrac{\sqrt2}{2}(1-2^{-\frac{\mu}{2}+10}),\tfrac{\sqrt2}{2}(1-2^{-\frac{\mu}{2}-10})]\\ &[1-2^{-\mu+10},1-2^{-\mu-10}]\end{aligned}$$

and these sets satisfy the hypotheses of Lemma 1 if taken modulo 100. The boundedness of the operator with symbol $\sum_\mu E_\mu^{(2)}$ is therefore a consequence of Lemma 1.

Geometric considerations give that if $Q_{k,l,m}$ intersects $G_{\mu,k}$, then l must satisfy (6.2) and m must lie between the number

$$2^{-\frac{\mu}{2}}l-\tfrac{16}{\sqrt2}2^{-\frac{\mu}{2}}\sqrt{2(2^k-\tfrac12)^2-(\tfrac{\sqrt2}{32}l)^2} \tag{7.4}$$

and the number

$$2^{-\frac{\mu}{2}}(l+1)-\tfrac{16}{\sqrt2}2^{-\frac{\mu}{2}}\sqrt{2(2^k-1)^2-(\tfrac{\sqrt2}{32}(l+1))^2}-1 \tag{7.5}$$

or between the numbers

$$2^{-\frac{\mu}{2}}(l+1)+\tfrac{16}{\sqrt2}2^{-\frac{\mu}{2}}\sqrt{2(2^k-1)^2-(\tfrac{\sqrt2}{32}(l+1))^2} \tag{7.6}$$

and the number

$$2^{-\frac{\mu}{2}}l+\tfrac{16}{\sqrt2}2^{-\frac{\mu}{2}}\sqrt{2(2^k-\tfrac12)^2-(\tfrac{\sqrt2}{32}l)^2}-1\,. \tag{7.7}$$

Observe that for a fixed k and l, the number of integers m that lie between (7.4) and (7.5) or between (7.6) and (7.7) is at most 64. This means that once k and l are fixed, there are only finitely many m's such that $Q_{k,l,m}$ was selected.

We now let $\Psi_{2,k,l,m}$ be a Schwartz function whose Fourier transform is real-valued, is equal to 1 on supp $(\widehat{\Psi_{1,k,l}})+(-\text{supp }(\widehat{\Psi_{3,k,m}}))$, is supported in a small neighborhood of this set, and satisfies the derivative estimate

$$\Big|\frac{d^\beta}{d\eta^\beta}\widehat{\Psi_{2,k,l,m}}(\eta)\Big|\le C_\beta 2^{(k-\frac{\mu}{2})\beta} \tag{7.8}$$

for all $\beta\ge 0$. Because of the choice of this function we have that (7.2) is equal to

$$\sum_{k=\mu}^{\infty}\ \sum_{\substack{l,m\ \text{such that}\\ Q_{k,l,m}\in\mathcal{F}^{select}_{k+\mu}}}\int_{\mathbf R}\int_{\mathbf R}\widehat f(\xi)\widehat g(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Psi_{1,k,l}}(\xi+\eta)\widehat{\Psi_{2,k,l,m}}(\eta)\widehat{\Psi_{3,k,m}}(\xi)d\xi\,d\eta\,. \tag{7.9}$$

Denote the bilinear operator in (7.9) by $\widetilde V_\mu$. Let $(\widetilde V_\mu)^{*1}(h,g)$ be the bilinear operator defined via the identity

$$\langle \widetilde V_\mu(f,g),h\rangle=\langle f,(\widetilde V_\mu)^{*1}(h,g)\rangle\,,\qquad \text{for all } f,g,h. \tag{7.10}$$

It is easy to check that $(\widetilde{V}_\mu)^{*1}(h,g)(x)$ is equal to

$$\sum_{k=\mu}^{\infty} \sum_{\substack{l,m \text{ such that} \\ Q_{k,l,m}\in\mathcal{F}^{select}_{k+\mu}}} \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{h}(\xi)\widehat{\overline{g}}(\eta)\widehat{\Psi_{1,k,l}}(\xi)\widetilde{\widehat{\Psi_{2,k,l,m}}}(\eta)\widehat{\Psi_{3,k,m}}(\xi+\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta \tag{7.11}$$

where $\widetilde{F}(x) = F(-x)$ in the above identity and the bar denotes complex conjugation as usually. We now observe that $(\widetilde{V}_\mu)^{*1}(h,g)$ given in (7.11) is the same as $\widetilde{T}_\mu(h,\overline{g}\,)$, where $\widetilde{T}_\mu$ is as in (6.6) when $\widehat{\Psi_{1,k,l}}$ is replaced by $\widehat{\Phi_{1,k,l}}$, $\widetilde{\widehat{\Psi_{2,k,l,m}}}$ is replaced by $\widehat{\Phi_{2,k,m}}$, and $\widehat{\Psi_{3,k,l,m}}$ is replaced by $\widehat{\Phi_{3,k,l,m}}$. The procedure used in section 6 to give uniform bounds for the operators $\widetilde{T}_\mu$ can be also applied to give uniform bounds for the operators $(\widetilde{V}_\mu)^{*1}$. To achieve this we need a slight variation of Lemma 6 in which the Φ's are replaced by Ψ's. This can be easily obtained since the geometry of the selected cubes in this section for the V_μ's is essentially the same as the geometry of the selected cubes in section 6 for T_μ's. The details are left to the reader.

Using duality and identity (7.10) we can now obtain the uniform boundedness of the $\widetilde{V}_\mu$'s in the specified range of exponents. Finally we have that the operators V_μ satisfy the hypotheses of Lemma 1 which we use to obtain the boundedness of the sum $\sum_{\mu=1}^{\infty} V_\mu$. This concludes the boundedness of $T_{D(2)}$.

8. The boundedness of $T_{D(4)}$

Recall that the symbol of the operator $T_{D(4)}$ is the function $\sum_{k\ge 1}\widehat{b^4_k}$ which is supported in the intersection of the sector $\frac{5\pi}{8}(1-2^{-10}) \le \theta \le (1+2^{-10})\frac{7\pi}{8}$ with the outside of the disc of radius $\frac{1}{2}(1+2^{-9})$ centered at the origin. We decompose this set as a disjoint union of annuli H_k for $k \ge 1$, defined as follows

$$\begin{aligned} H_1 =& \big\{(\xi,\eta):\ 1-\tfrac{1}{2}(1+2^{-9}) \le |(\xi,\eta)| < 1-2^{-2}\big\} \\ &\bigcap \big\{(\xi,\eta):\ \tfrac{5\pi}{8}(1-2^{-10}) \le |\text{Argument}(\xi,\eta)| < \tfrac{7\pi}{8}(1+2^{-10})\big\}\,, \end{aligned}$$

while for $k \ge 2$

$$\begin{aligned} H_k =& \big\{(\xi,\eta):\ 1-2^{-k} \le |(\xi,\eta)| < 1-2^{-(k+1)}\big\} \\ &\bigcap \big\{(\xi,\eta):\ \tfrac{5\pi}{8}(1-2^{-10}) \le |\text{Argument}(\xi,\eta)| < \tfrac{7\pi}{8}(1+2^{-10})\big\}\,. \end{aligned}$$

We tile up the plane as the union of squares of size $2^{-6}\times 2^{-6}$ and we let $\mathcal{P}_1$ be the set of all such squares. Let $\mathcal{P}_1^{select}$ be the subset of $\mathcal{P}_1$ consisting of all squares that intersect H_1. We denote by $\mathcal{P}_2$ the set of all squares obtained by subdividing each rectangle in $\mathcal{P}_1 \setminus \mathcal{P}_1^{select}$ into four squares, each of size $2^{-7}\times 2^{-7}$, by halving its sides. Let $\mathcal{P}_2^{select}$ be the subset of $\mathcal{P}_2$ consisting of all squares that intersect H_2. Next, we denote by $\mathcal{P}_3$ the set of all squares obtained by subdividing each square in $\mathcal{P}_2\setminus\mathcal{P}_2^{select}$ into four squares, each of size $2^{-8}\times 2^{-8}$, by halving its sides. Continue this way by induction. Then we have "essentially covered" each H_k by disjoint squares of size $2^{-k-5}\times 2^{-k-5}$ and the set of all such squares is denoted by $\mathcal{P}_k^{select}$. Elements of $\mathcal{P}_k^{select}$ will be denoted by $S_{k,l,m}$ and they have the form

$$S_{k,l,m} = [2^{-k-5}l, 2^{-k-5}(l+1)] \times [2^{-k-5}m, 2^{-k-5}(m+1)]\,.$$

Observe that we have used approximately 2^k squares to cover H_k.

We now make a few observations about the range of l and m among the selected squares $S_{k,l,m}$. Geometric considerations give that for a fixed $k \geq 1$, the index l has range

$$-1 - 32 \cdot (2^k - \tfrac{1}{2}) \cos(\tfrac{\pi}{8}) \leq l \leq -32 \cdot (2^k - 1) \cos(\tfrac{3\pi}{8})$$

while the index m has range

$$32 \cdot 2^k \sqrt{(1-2^{-k})^2 - (\tfrac{2^{-k} l}{32})^2} - 1 \leq m \leq 32 \cdot 2^k \sqrt{(1-2^{-k-1})^2 - (\tfrac{2^{-k}(l+1)}{32})^2}\,.$$

It follows that given k and l, there exist at most finitely many indices m (which depend on k and l) such that $S_{k,l,m}$ was selected.

As we did in section 6, we fix a smooth partition of unity adapted to the squares $S_{k,l,m}$. Then we use the Fourier series method used earlier to transform this partition into a double sum of products of functions in each variable ξ, η. The $(0,0)$ term of this series is the bilinear operator $V_\mu^{(4)}$ whose symbol $\widehat{b_k^4}$ is

$$\sum_{k=1}^{\infty} \sum_{\substack{l,m \text{ such that} \\ S_{k,l,m} \in \mathcal{P}_k^{select}}} \widehat{\Theta_{1,k,l}}(\xi) \widehat{\Theta_{2,k,m}}(\eta) + \sum_{k=1}^{\infty} E_k^{(4)}(\xi, \eta)\,,$$

where $E_k^{(4)}(\xi,\eta)$ is the error of this approximation near the intersection of the lines $\theta = \frac{5\pi}{8}$ and $\theta = \frac{7\pi}{8}$ with the annulus $1 - 2^{-k} \leq |(\xi,\eta)| \leq 1 - 2^{-k-1}$, and the Fourier transforms of Schwartz functions $\Theta_{1,k,l}$, $\Theta_{2,k,m}$ are supported in small neighborhoods of the sets $[2^{-k-5} l, 2^{-k-5}(l+1)]$, $[2^{-k-5} m, 2^{-k-5}(m+1)]$ respectively, whose derivatives blow up inverse proportionally to the length of their supports, such that the function

$$\sum_{k=1}^{\infty} \sum_{\substack{l,m \text{ such that} \\ S_{k,l,m} \in \mathcal{P}_k^{select}}} \widehat{\Theta_{1,k,l}}(\xi) \widehat{\Theta_{2,k,m}}(\eta)$$

is equal to 1 on the union of all the selected squares $S_{k,l,m}$ that do not meet the boundary of the support of the function $\sum_{k \geq 1} \widehat{b_k^4}$.

Each $E_k^{(4)}$ consists of a finite sum of smooth functions satisfying the hypotheses of Lemma 3. It follows that each $E_k^{(4)}$ is the symbol of a bounded bilinear operator from $L^p \times L^q \to L^r$ for all $1 < p, q, r < \infty$ with $1/p + 1/q = 1/r$. Since the supports of $E_k^{(4)}$ satisfy the hypotheses of Lemma 1, it follows that the sum $\sum_{k=1}^{\infty} E_k^{(4)}$ is also the symbol of a bounded bilinear operator from $L^{p_1} \times L^{p_2} \to L^p$, where p_1, p_2, p are as in Theorem 1.

We now introduce a third Schwartz function $\Theta_{3,k,l,m}$ whose Fourier transform is equal to 1 on the set

$$[2^{-k-5} l, 2^{-k-5}(l+1)] + [2^{-k-5} m, 2^{-k-5}(m+1)]$$

and supported in a small neighborhood of this set, and whose derivatives satisfy the correct size estimates.

The main term of the operator $T_{D(4)}$ is then a finite (in m) sum of terms of the form

$$\sum_{k=1}^{\infty} \sum_{\substack{l,m \text{ such that} \\ S_{k,l,m} \in \mathcal{P}_k^{select}}} \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i (\xi+\eta) x} \widehat{\Theta_{1,k,l}}(\xi) \widehat{\Theta_{2,k,m}}(\eta) \widehat{\Theta_{3,k,l,m}}(\xi+\eta) d\xi\, d\eta\,.$$

Boundedness for the sum above is a consequence of the work in [5], [9], provided Lemma 7 below is proved. See also [7] in which a similar model is treated. For simplicity in its statement, for every k and l we fix an $m = m(k,l)$ (so that $S_{k,l,m}$ was selected) and we write $\Theta'_{1,k,l} = \Theta_{1,k,l}$, $\Theta'_{2,k,l} = \Theta_{2,k,m(k,l)}$, and $\Theta'_{3,k,l} = \Theta_{3,k,l,m(k,l)}$ in the lemma below.

Lemma 7. *Let be $\Theta'_{1,k,l}$, $\Theta'_{2,k,l}$, and $\Theta'_{3,k,l}$ be as above and let $|k-k'| \geq 100$. If supp $\widehat{\Theta'_{i,k,l}} \subsetneq$ supp $\widehat{\Theta'_{i,k',l'}}$ for some $i \in \{1,2,3\}$, then we have*

$$\operatorname{dist}\big(\text{supp } \widehat{\Theta'_{j,k,l}}, \text{supp } \widehat{\Theta'_{j,k',l'}}\big) \approx |\text{supp } \widehat{\Theta'_{i,k',l'}}| \tag{8.1}$$

for all $j \in \{1,2,3\} \setminus \{i\}$.

The proof of this lemma is similar to that of Lemma 6 and is omitted. This model is simpler than those discussed in the previous sections since this part of the disc implies bounds for the bilinear Hilbert transforms along directions that stay away from the three degenerate cases.

9. Final Remarks

A careful examination gives that the boundedness of the characteristic function of the sector $D(1)$ implies uniform bounds for the bilinear Hilbert transforms $H_{1,\alpha}$ as $\alpha \to 0$. Similarly the boundedness of the characteristic function of the sector $D(3)$ implies implies uniform bounds for the bilinear Hilbert transforms $H_{1,\alpha}$ as $\alpha \to \infty$. Also the boundedness of the characteristic function of the sector $D(2)$ implies implies uniform bounds for the bilinear Hilbert transforms $H_{1,\alpha}$ as $\alpha \to 1$. These are the three degenerate directions that appear in the study of the bilinear Hilbert transform. The boundedness of the characteristic function of the sector $D(4)$ implies bounds for the bilinear Hilbert transforms $H_{1,\alpha}$ as $\alpha \to 1$ which is not a degenerate direction. This explains why the model case studied in section 8 is much simpler than those in the previous sections. Also Lemma 7 is simpler than Lemma 6. The crucial and beautiful feature of the disc multiplier is that it captures all possible directions that appear in the study of the bilinear Hilbert transforms uniformly.

We note that duality considerations imply that the characteristic functions of the ellipses

$$(\xi+\eta)^2 + \eta^2 < 1$$
$$\xi^2 + (\xi+\eta)^2 < 1$$

are also bounded multipliers for the range of exponents claimed in Theorem 1. The authors know how to adapt the proof of Theorem 1 to replace the disc by the characteristic function of any of the following ellipses

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} < 1\,, \qquad a, b > 0\,.$$

To avoid unnecessary complications in the presentation, only the case $a = b = 1$ was treated in the article. The characteristic functions of certain other geometric figures have also been studied by C. Muscalu [10]. It is an interesting open problem to find a general description of geometric figures with smooth boundary in $\mathbf{R}^2$ whose characteristic functions are bounded bilinear multipliers.

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