The Bourgain–Brezis–Mironescu formula on ball Banach function spaces

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Abstract Let $p \in [1, \infty)$ and X be a ball Banach function space on $\mathbb{R}^n$ with an absolutely continuous norm for which the Hardy–Littlewood maximal operator is bounded on $(X^{1/p})'$, the associate (dual) space of its $1/p$-convexification. The purpose of this work is to establish the fundamental formula

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_X^p = \frac{2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{p+1}{2}\right)}{p\,\Gamma\left(\frac{p+n}{2}\right)}\,\|\,|\nabla f|\,\|_X^p$$

for any $f \in X$, where Γ is the Gamma function. This identity coincides with the celebrated classical formula of Bourgain, Brezis, and Mironescu [22], [12] when $X = L^p(\mathbb{R}^n)$, but it is new for general X, in particular for $X = L^q(\mathbb{R}^n)$ $(1 \le p < q < \infty)$. Translation invariance plays a vital role in the proof of this formula in its aforementioned standard proofs in [22], [12]. But translation invariance may not be valid for ball Banach function spaces, nor is there an explicit expression for the associated norm. The authors overcome these obstacles via a key weighted estimate, obtained using fine geometric properties of adjacent systems of dyadic cubes, and Poincaré's inequality. This estimate is then combined with harmonic analysis tools, such as extrapolation and the boundedness of the Hardy–Littlewood maximal operator, to derive the desired formula. Applications of this limiting identity yield new characterizations of ball Banach Sobolev spaces. Explicit spaces X for which these results apply are Morrey spaces, mixed-norm (resp., weighted or variable) Lebesgue spaces, Orlicz(-slice) (or generalized amalgam) spaces, and Lorentz spaces.

Contents

2020 *Mathematics Subject Classification*. Primary 46E35; Secondary 26D10, 42B25, 26A33.

Key words and phrases. Sobolev semi-norm, Gagliardo semi-norm, ball Banach function space, the BBM Formula.

The first author is supported by NSERC of Canada Discovery grant RGPIN-2020-03909, the second author is supported by a Simons Foundation Fellowship (No. 819503) and a Simons Grant (No. 624733), and this project is also supported by the National Natural Science Foundation of China (Grant Nos. 11971058, 12071197 and 12122102) and the National Key Research and Development Program of China (Grant No. 2020YFA0712900).

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1 Introduction

For a given $s \in (0,1)$ and $p \in [1,\infty)$, the *homogeneous fractional Sobolev space* $\dot{W}^{s,p}(\mathbb{R}^n)$, introduced by Gagliardo in [44], is defined as the set of all the measurable functions f on $\mathbb{R}^n$ for which the semi-norm

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right]^{\frac{1}{p}} \tag{1.1}$$

is finite. The spaces $\dot{W}^{s,p}(\mathbb{R}^n)$ measure smoothness in an essential way and play fundamental roles in harmonic analysis and partial differential equations; for instance we refer to [14, 15, 20] for recent applications in Gagliardo–Nirenberg inequalities, to [75, 81, 17, 18, 85, 39, 34] for applications in the theory of Sobolev spaces, to [48, 41] for applications in the theory of Lebesgue and Besov–Sobolev spaces, and to [74, 24, 25, 77] for applications in partial differential equations.

We denote the gradient (in the sense of distributions) of a weakly differentiable function f on $\mathbb{R}^n$ by $\nabla f := (\partial_1 f,\dots,\partial_n f)$ and, for any $j \in \{1,\dots,n\}$, $\partial_j f$ denotes the j-th weak derivative of f, namely, for any $\phi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ (the set of all infinitely differentiable functions with compact support),

$$\int_{\mathbb{R}^n} f(x)\partial_j\phi(x)\,dx = -\int_{\mathbb{R}^n} \partial_j f(x)\phi(x)\,dx.$$

A well-known *deficiency* of the Gagliardo semi-norm in (1.1) is that $\|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)}$ does not converge to the homogeneous Sobolev semi-norm $\|f\|_{\dot{W}^{1,p}(\mathbb{R}^n)} := \|\,|\nabla f|\,\|_{L^p(\mathbb{R}^n)}$ when $s \to 1^-$ (by $s \to 1^-$ we mean $s \in (0,1)$ and $s \to 1$). Indeed, if f is a non-constant measurable function on $\mathbb{R}^n$, then $\|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \to \infty$ as $s \to 1^-$; see [12, 21, 22]. This 'defect' was nicely amended by Bourgain, Brezis, and Mironescu in their fundamental work [22] on this topic. On general smooth and bounded domains Ω, the idea in [22] was to recover $\| |\nabla f| \|_{L^p(\Omega)}$ as the limit of the expressions $(1-s)\|f\|^p_{\dot{W}^{s,p}(\Omega)}$ as $s \to 1^-$. Later, Brezis [12] extended this formula to the entire $\mathbb{R}^n$. Today the following identity is referred to as the Bourgain–Brezis–Mironescu (BBM) formula on $\mathbb{R}^n$: precisely, for any $p \in (1,\infty)$ and any $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ one has

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dy\,dx = \frac{K(p,n)}{p}\,\| |\nabla f| \|^p_{L^p(\mathbb{R}^n)}, \tag{1.2}$$

where

$$K(p,n) := \int_{\mathbb{S}^{n-1}} |\xi\cdot e|^p\,d\sigma(\xi) = \frac{2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)} \tag{1.3}$$

with e being some unit vector in $\mathbb{R}^n$ and $d\sigma(\xi)$ the surface Lebesgue measure on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. The fact that the constant $K(p,n)$ is independent of the choice of $e \in \mathbb{S}^{n-1}$ can be seen by changing variables via an orthogonal transformation of the sphere. The precise value of the constant can be derived by the identity in Appendix D3 in [47]. This is done at the end of the paper.

We refer to [11] for applications of BBM formula on Triebel–Lizorkin spaces, to [23] concerning limiting embeddings of fractional Sobolev spaces, to [49, 72] on metric measure spaces, and to [35, 100, 76, 86, 82, 70, 40] concerning other applications in the theory of Sobolev spaces.

Next we discuss a few ideas from the proof of (1.2) in [12]. In view of the fact that the set of twice continuously differentiable functions with compact support, $C^2_{\mathrm{c}}(\mathbb{R}^n)$, is dense in $W^{1,p}(\mathbb{R}^n)$, it suffices to first prove (1.2) for any $f \in C^2_{\mathrm{c}}(\mathbb{R}^n)$ and then extend it to $W^{1,p}(\mathbb{R}^n)$. Here and thereafter, for any given $p \in [1,\infty)$, the *inhomogeneous Sobolev space* $W^{1,p}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ for which the following norm is finite

$$\|f\|_{W^{1,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \| |\nabla f| \|_{L^p(\mathbb{R}^n)}.$$

Since the norm of $W^{1,p}(\mathbb{R}^n)$ has an explicit expression, Brezis in [12] proved the validity of the BBM formula for $C^2_{\mathrm{c}}(\mathbb{R}^n)$ via a change of variables and Fubini's theorem. Then using a known classical characterization of $W^{1,p}(\mathbb{R}^n)$ (see [13, Proposition 9.3]), whose proof strongly depends on the translation invariance of the Lebesgue measure, one can extend the BBM formula from $C^2_{\mathrm{c}}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$. Thus, both translation invariance and the explicit expression of the $L^p(\mathbb{R}^n)$ norm play a vital role in the proof of the BBM formula (1.2) in [12].

Recently, Brezis, Van Schaftingen, and Yung [19] discovered an alternative way to amend the aforementioned continuity *deficiency* of the norm $\|\cdot\|_{\dot{W}^{s,p}(\mathbb{R}^n)}$ in (1.1) as $s \to 1^-$. They showed that replacing the $L^p(\mathbb{R}^n\times\mathbb{R}^n)$ norm in (1.1) by $L^{p,\infty}(\mathbb{R}^n\times\mathbb{R}^n)$ (the weak L^p quasi-norm) produces an expression equivalent to $\| |\nabla f| \|_{L^p(\mathbb{R}^n)}$ when $s = 1$. Later, Dai et al. [33] extended the result in [19] to ball Banach function spaces (Definition 2.1). Given these recent advances, it is quite natural to ask whether or not the BBM formula holds for general ball Banach function spaces. In this article we precisely answer this question in a positive way.

Ball (quasi-)Banach function spaces were introduced by Sawano et al. [93] in an attempt to unify the study of several important function spaces. In particular, ball (quasi-)Banach function spaces include Morrey spaces, mixed-norm Lebesgue spaces, weighted Lebesgue spaces, variable Lebesgue spaces, Orlicz spaces, Orlicz-slice spaces, and Lorentz spaces (see, respectively, Subsections 5.1 through 5.7 below for definitions and historical notes). Topics related to ball (quasi-)Banach function spaces can be found in [26, 90, 93, 105, 106] (concerning Hardy spaces associated with them), in [51, 101, 107] (on the boundedness of operators between them), and in [55, 59, 60, 98, 102] (for further applications).

As a somewhat surprising consequence of our work, the following BBM formula holds: for any given $q \in (1, \infty)$ and $p \in [1, q]$, and for any $f \in \dot{W}^{1,q}(\mathbb{R}^n)$,

$$\begin{aligned}&\lim_{s\to 1^-}(1-s)\left\{\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dy\right]^{\frac{q}{p}}\,dx\right\}^{\frac{p}{q}}\\&\quad=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{L^q(\mathbb{R}^n)}^p\end{aligned}\tag{1.4}$$

with $K(p, n)$ as in (1.3). So the appearance of the mixed norm on the left does not alter the limiting behavior of the expression as $s \to 1^-$. In the case of $q = p$, (1.4) just reduces to (1.2).

In this article we establish the following analogue of (1.2) for a given ball Banach function space X that satisfies some mild additional hypotheses (see Theorems 4.12): given $p \in [1, \infty)$ and $|\nabla f| \in X$ we have

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_X^p=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_X^p.\tag{1.5}$$

Formula (1.5) yields asymptotics for the fractional Sobolev-type semi-norm involving differences of the function f as $s \to 1^-$. Finding an appropriate way to study asymptotics and/or to characterize functions in terms of their finite differences is a notoriously difficult problem in approximation theory, even for certain simple weighted Lebesgue spaces in one dimension (see [66, 73] and the references therein). A major difficulty arises from the fact that the difference operators $\Delta_h f := f(\cdot + h) - f(\cdot)$ for $h \in \mathbb{R}^n$ may be unbounded on weighted Lebesgue spaces.

The main contribution of this work in proving (1.5) is to overcome the difficulties caused by the deficiency of the explicit expression and the translation invariance of the norm of X; both of these are quite crucial in the proof of the BBM formula on $L^p(\mathbb{R}^n)$. We bypass the issue of the deficiency of the explicit expression of the norm of X by exploiting the local doubling property of X (Definition 2.10); see Theorem 2.13. Secondly we extend the BBM formula to the entire ball Banach Sobolev space (Theorem 3.4), via extrapolation [31] and the boundedness of the Hardy–Littlewood maximal operator. To achieve this goal we establish an extension lemma on general ball Banach function spaces (Lemma 3.34) by first obtaining a key estimate on weighted Lebesgue spaces equipped with Muckenhoupt weights (Lemma 3.23). Note that our expression in (3.12) is no longer the integral $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cdots\,dx\,dy$, but the integral $\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\cdots\,dy\right]\omega(x)\,dx$, which is not symmetric with respect to x and y. This causes additional technical difficulties in the proof of this weighted estimate. Indeed, the proof of Lemma 3.23 is fairly nontrivial because the proof in the unweighted case in [12] heavily depends on the translation invariance of the $L^p(\mathbb{R}^n)$ norm, which seems to be inapplicable in the weighted case here. To overcome this obstacle, we make full use of fine geometric properties of systems of adjacent dyadic cubes in $\mathbb{R}^n$ (see, for instance, [69, Section 2.2]) and appeal to the Poincaré inequality. The main result of this article is precisely stated in Theorem 3.36.

This article is organized as follows: Section 2 is devoted to the proof of Theorem 2.7 which provides a special case of our main result for $C_c^2(\mathbb{R}^n)$ functions. This section begins with preliminaries regarding ball Banach function spaces. Then we introduce Sobolev spaces associated with ball Banach function spaces by defining a new Sobolev-type space $W^{1,X}(\mathbb{R}^n)$, called the ball Banach Sobolev space (see Definition 2.6). In this section, we also introduce the concept of the locally β-doubling condition (see Definition 2.10) to overcome the difficulty caused by the deficiency of the explicit expression of the norm of X and show the sufficiency of the locally β-doubling condition, namely, Lemma 2.18. Finally, we obtain the BBM formula on ball Banach function spaces for $C_c^2(\mathbb{R}^n)$ functions.

The purpose of Section 3 is to generalize the BBM formula (1.2) to ball Banach function spaces; this result is contained in Theorem 3.4. To achieve this, we first prove that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,X}(\mathbb{R}^n)$ under some mild additional hypotheses on X (see Corollary 3.19). Thus, to prove Theorem 3.4, it is sufficient to establish the key estimate (3.12) on weighted Lebesgue spaces (see Lemma 3.23), which characterizes the relation between the $\dot{W}_\omega^{s,p}(\mathbb{R}^n)$ semi-norm and the $\dot{W}_\omega^{1,p}(\mathbb{R}^n)$ semi-norm, and which plays an essential role in the process of extending (3.2) from $C_c^2(\mathbb{R}^n)$ functions to $W^{1,X}(\mathbb{R}^n)$ functions. Finally, we prove Theorem 3.4 via the approach of the extrapolation which connects ball Banach function spaces and weighted Lebesgue spaces.

In Section 4, borrowing some ideas from [22], we establish the lower estimate of (1.5) (see Theorem 4.1). Moreover, the upper estimate of (1.5) comes from the BBM formula in ball Banach function spaces, namely, Theorem 3.4. Combining these two estimates, we further establish the characterization of the ball Banach Sobolev space $W^{1,X}(\mathbb{R}^n)$ (see Theorem 4.8).

In Section 5, we apply the results obtained in Sections 2, 3, and 4, respectively, to $X := M_r^\alpha(\mathbb{R}^n)$ (the Morrey space), $X := L^{p(\cdot)}(\mathbb{R}^n)$ (the variable Lebesgue space), $X := L^{\vec{p}}(\mathbb{R}^n)$ (the mixed-norm Lebesgue space), $X := L_\omega^p(\mathbb{R}^n)$ (the weighted Lebesgue space), $X := L^\Phi(\mathbb{R}^n)$ (the Orlicz space), $X := (E_\Phi^r)_t(\mathbb{R}^n)$ (the Orlicz-slice space or the generalized amalgam space), and $X := L^{q,r}(\mathbb{R}^n)$ (the Lorentz space).

We outline the basics of our notation. We let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We denote by $C_c^k(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ the space of all k-order ($k \in \mathbb{N}$) continuously differentiable functions on $\mathbb{R}^n$ with compact support, and the space of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support, respectively. The space of continuous functions on $\mathbb{R}^n$ with compact support is denoted by $C_c(\mathbb{R}^n)$. We use the symbol $\mathscr{M}(\mathbb{R}^n)$ for the space of all measurable functions on $\mathbb{R}^n$. In addition, we denote by $L^1_{\mathrm{loc}}(0,\infty)$ [resp., $L^1_{\mathrm{loc}}(\mathbb{R}^n)$] the set of all locally integrable functions on $(0,\infty)$ (resp., on $\mathbb{R}^n$). For any function f on $\mathbb{R}^n$, we let $\mathrm{supp}\,(f) := \{x \in \mathbb{R}^n : f(x) \neq 0\}$. The letter C will represent a *positive constant* which is independent of the main parameters, but may vary from line to line. We also use $C_{(\alpha,\beta,\dots)}$ to indicate a positive constant depending on the underlying parameters $\alpha, \beta, \dots$. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. We use $\mathbf{0}$ to denote the *origin* of $\mathbb{R}^n$. If E is a subset of $\mathbb{R}^n$, we denote by $\mathbf{1}_E$ its *characteristic function* and, for any bounded measurable set $E \subset \mathbb{R}^n$ with $|E| \neq 0$, and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\fint_E f(x)\,dx := \frac{1}{|E|}\int_E f(x)\,dx =: f_E.$$

The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ is indicated by $|E|$. In addition, we denote by σ the surface Lebesgue measure on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. A *cube* in this article will always have its edges parallel to the coordinate axes, and need not be open or closed. We use $\ell(Q)$ to denote the side length of a cube Q of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, we let $B(x, r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ and

$$\mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\}. \tag{1.6}$$

For any $\alpha \in (0, \infty)$ and any ball $B := B(x_B, r_B)$ in $\mathbb{R}^n$, with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, let $\alpha B := B(x_B, \alpha r_B)$. Finally, for any $q \in [1, \infty]$, we denote by q' its *conjugate exponent*, which satisfies $1/q + 1/q' = 1$.

2 Asymptotics of $C^2_{\mathrm{c}}(\mathbb{R}^n)$ functions in terms of ball Banach Sobolev norms

In this section, we prove identity (1.5) on ball Banach function spaces for functions $f \in C^2_{\mathrm{c}}(\mathbb{R}^n)$. This space is an important subclass of functions for which Theorem 3.4 is valid.

We review the basics related to ball quasi-Banach function spaces as introduced in [93].

Definition 2.1. A quasi-Banach space X of complex-valued functions defined on $\mathbb{R}^n$, equipped with a quasi-norm $\|\cdot\|_X$ [which is defined on the entire $\mathscr{M}(\mathbb{R}^n)$], is called a *ball quasi-Banach function space* (in short, BQBF *space*) if it satisfies

(i) $f \in \mathscr{M}(\mathbb{R}^n)$ and $\|f\|_X = 0$ imply that $f = 0$ almost everywhere;

(ii) $f, g \in \mathscr{M}(\mathbb{R}^n)$ and $|g| \le |f|$ almost everywhere imply that $\|g\|_X \le \|f\|_X$;

(iii) $\{f_m\}_{m\in\mathbb{N}} \subset \mathscr{M}(\mathbb{R}^n)$, $f \in \mathscr{M}(\mathbb{R}^n)$, and $0 \le f_m \uparrow f$ almost everywhere as $m \to \infty$ imply that $\|f_m\|_X \uparrow \|f\|_X$ as $m \to \infty$;

(iv) $B \in \mathbb{B}$ implies that $\mathbf{1}_B \in X$, where $\mathbb{B}$ is as in (1.6).

Moreover, a ball quasi-Banach function space X is called a *ball Banach function space* (in short, BBF *space*) if the quasi-norm of X satisfies the triangle inequality: for any $f,\ g \in X$,

$$\|f + g\|_X \le \|f\|_X + \|g\|_X,$$

and that, for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$ so that for any $f \in X$ we have

$$\int_B |f(x)|\, dx \le C_{(B)} \|f\|_X.$$

Remark 2.2. (i) Let X be a ball quasi-Banach function space. By [103, Remark 2.6(i)] (see also [104]), we conclude that, for any $f \in \mathscr{M}(\mathbb{R}^n)$, $\|f\|_X = 0$ if and only if $f = 0$ almost everywhere.

(ii) As was mentioned in [103, Remark 2.6(ii)] (see also [104]), we obtain an equivalent formulation of Definition 2.1 via replacing any ball B therein by any bounded measurable set E.

(iii) We point out that, in Definition 2.1(iv), if we replace the balls B by arbitrary measurable sets E with $|E| < \infty$, we obtain the definition of (quasi-)Banach function spaces which were originally introduced in [10, Definition 1.1 and 1.3]. Thus, a (quasi-)Banach function space is also a ball (quasi-)Banach function space.

(iv) By [36, Theorem 2], we conclude that both (ii) and (iii) of Definition 2.1 imply that any ball quasi-Banach function space is complete.

(v) From both (iv) and (ii) of Definition 2.1, it is easy to deduce that $C_{\mathrm{c}}(\mathbb{R}^n) \subset X$.

We now introduce the concept of the approximation of the identity of radial type on $\mathbb{R}^n$. In the sequel, $\epsilon \to 0^+$ means $\epsilon \in (0, \infty)$ and $\epsilon \to 0$.

Definition 2.3. A family $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ of locally integrable functions on $(0, \infty)$ is called an *approximation of the identity of radial type on* $\mathbb{R}^n$ (for short, a *radial*-ATI) if, for any $\epsilon \in (0, \infty)$, ρ_ϵ is nonnegative, it satisfies

$$\int_0^\infty \rho_\epsilon(r) r^{n-1}\, dr = 1, \tag{2.1}$$

and, for any $\delta \in (0, \infty)$,

$$\lim_{\epsilon\to 0^+} \int_\delta^\infty \rho_\epsilon(r) r^{n-1}\, dr = 0. \tag{2.2}$$

Moreover, an approximation of the identity of radial type on $\mathbb{R}^n$, $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$, is called a *decreasing approximation of the identity of radial type on* $\mathbb{R}^n$ (for short, a *decreasing-radial*-ATI) if, for any given $\epsilon \in (0, \infty)$, ρ_ϵ is decreasing on $(0, \infty)$.

For the convenience of the reader, we present two decreasing-radial-ATIs here. We refer the reader to [12, 22] for more examples of (decreasing-)radial-ATIs. These two examples are as follows, which can be found, for instance, in [47, Example 1.2.17] or [95, p. 111].

Example 2.4. *Let ϕ be a nonnegative bounded decreasing radial integrable function on $\mathbb{R}^n$ such that* $\operatorname{supp}(\phi) \subset B(\mathbf{0}, 1)$ *and* $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$, *and let*

$$\rho_\epsilon(r) := \frac{1}{\sigma(\mathbb{S}^{n-1})} \epsilon^{-n} \phi(\epsilon^{-1} r)$$

for any $r \in (0, \infty)$ and for any given $\epsilon \in (0, \infty)$. Then $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ is a decreasing-radial-ATI.

Example 2.5. *For any given $\epsilon \in (0, \infty)$ and for any $r \in (0, \infty)$, let*

$$\rho_\epsilon(r) := \frac{1}{\sigma(\mathbb{S}^{n-1})} \frac{1}{(4\pi\epsilon)^{n/2}} e^{-r^2/4\epsilon}.$$

Then $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ is a decreasing-radial-ATI.

Next, we extend the concept of Sobolev spaces to ball Banach function spaces.

Definition 2.6. Let X be a ball Banach function space. The *ball Banach Sobolev space* $W^{1,X}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{W^{1,X}(\mathbb{R}^n)} := \|f\|_X + \| |\nabla f| \|_X < \infty,$$

where $\nabla f := (\partial_1 f, \ldots, \partial_n f)$ is the distributional gradient of f.

The main result of this section is the following theorem, which provides a generalization of the BBM formula in $C_{\mathrm{c}}^2(\mathbb{R}^n)$; this plays a vital role in the proof of Theorem 3.4 below.

Theorem 2.7. *Let X be a ball Banach function space, $p \in [1,\infty)$, and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a radial-*ATI. *Assume that X and $\{\rho_\epsilon\}$ satisfy that, for any given $M \in (0,\infty)$, there exists an $N \in (0,\infty)$ such that*

$$\lim_{\epsilon\to 0^+} \left\| \left[\int_{B(\mathbf{0},M)} \frac{\rho_\epsilon(|\cdot - y|)}{|\cdot - y|^p} \, dy \right]^{\frac{1}{p}} \mathbf{1}_{\mathbb{R}^n \setminus B(\mathbf{0},N)} \right\|_X = 0. \tag{2.3}$$

Then, for any $f \in C_{\mathrm{c}}^2(\mathbb{R}^n)$,

$$\lim_{\epsilon\to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|) \, dy \right]^{\frac{1}{p}} \right\|_X^p = K(p,n) \, \| \, |\nabla f| \, \|_X^p, \tag{2.4}$$

where $K(p,n)$ is as in (1.3).

Proof. We first focus our attention to proving that

$$\limsup_{\epsilon\to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|) \, dy \right]^{\frac{1}{p}} \right\|_X^p \le K(p,n) \, \| \, |\nabla f| \, \|_X^p. \tag{2.5}$$

Since $f \in C_{\mathrm{c}}^2(\mathbb{R}^n)$, it follows that there exists a ball $B(\mathbf{0},M)$ with $M \in (0,\infty)$ such that $\operatorname{supp}(f)$ is contained in $B(\mathbf{0},M)$. Let $N \in (2M+1,\infty)$ and $\delta \in (0,1)$. Then

$$\begin{aligned} J_\epsilon &:= \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|) \, dy \right]^{\frac{1}{p}} \right\|_X \\ &\le J_\epsilon^{(1)}(\delta) + J_\epsilon^{(2)}(\delta, N) + J_\epsilon^{(3)}(N), \end{aligned} \tag{2.6}$$

where

$$J_\epsilon^{(1)}(\delta) := \left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot - y|<\delta\}} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|) \, dy \right]^{\frac{1}{p}} \right\|_X,$$

$$J_\epsilon^{(2)}(\delta, N) := \left\| \left[\int_{\{y\in\mathbb{R}^n:\ \delta\le|\cdot - y|<2N\}} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|) \, dy \right]^{\frac{1}{p}} \right\|_X,$$

and

$$J_\epsilon^{(3)}(N) := \left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot - y|\ge 2N\}} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|) \, dy \right]^{\frac{1}{p}} \right\|_X.$$

We first consider $J_\epsilon^{(3)}(N)$. Observe that, for any $x, y \in [B(\mathbf{0},M)]^\complement$, $|f(x) - f(y)| = 0$. Moreover, for any $x \in \mathbb{R}^n$ satisfying $|x| \in [M,N)$, we have $B(\mathbf{0},M) \cap \{y \in \mathbb{R}^n : \ |x-y| \ge 2N\} = \emptyset$. From these facts and (2.1), we deduce that, for any given $N \in (2M+1,\infty)$ and for any $\epsilon \in (0,\infty)$ we have

$$J_\epsilon^{(3)}(N)$$

$$
\begin{aligned}
&\le \left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\ge 2N\}} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p} \rho_\epsilon(|\cdot-y|)\,dy \right]^{\frac{1}{p}} \mathbf{1}_{B(\mathbf{0},M)} \right\|_X \\
&\quad + \left\| \left[\int_{B(\mathbf{0},M)\cap\{y\in\mathbb{R}^n:\ |\cdot-y|\ge 2N\}} \frac{|f(y)|^p}{|\cdot-y|^p} \rho_\epsilon(|\cdot-y|)\,dy \right]^{\frac{1}{p}} \mathbf{1}_{B(\mathbf{0},N)\setminus B(\mathbf{0},M)} \right\|_X \\
&\quad + \left\| \left[\int_{B(\mathbf{0},M)} \frac{|f(y)|^p}{|\cdot-y|^p} \rho_\epsilon(|\cdot-y|)\,dy \right]^{\frac{1}{p}} \mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)} \right\|_X \\
&\lesssim N^{-1}\|f\|_{L^\infty(\mathbb{R}^n)}\|\mathbf{1}_{B(\mathbf{0},M)}\|_X \\
&\quad + \|f\|_{L^\infty(\mathbb{R}^n)} \left\| \left[\int_{B(\mathbf{0},M)} \frac{\rho_\epsilon(|\cdot-y|)}{|\cdot-y|^p}\,dy \right]^{\frac{1}{p}} \mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)} \right\|_X \\
&\sim N^{-1} + \left\| \left[\int_{B(\mathbf{0},M)} \frac{\rho_\epsilon(|\cdot-y|)}{|\cdot-y|^p}\,dy \right]^{\frac{1}{p}} \mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)} \right\|_X .
\end{aligned}
$$

By this and (2.3), we conclude that, for any given sufficiently large $N \in (2M+1,\infty)$,

$$
\limsup_{\epsilon\to 0^+} J_\epsilon^{(3)}(N) \lesssim N^{-1}.
$$

This further implies that, for any given $\zeta \in (0,\infty)$, there exists an $N \in (2M+1,\infty)$ such that

$$
\limsup_{\epsilon\to 0^+} J_\epsilon^{(3)}(N) < \zeta. \tag{2.7}
$$

Then we fix an $N \in (2M+1,\infty)$ such that (2.7) holds and we estimate $J_\epsilon^{(2)}(\delta,N)$. Observe that, for any $x \in [B(\mathbf{0},2N+M)]^\complement$ and $y \in \mathbb{R}^n$ satisfying $|x-y| \in [\delta,2N)$, $f(x)=f(y)=0$ from the support condition of f. Moreover, since $f \in C_c^2(\mathbb{R}^n)$, from the mean value theorem and the Cauchy–Schwarz inequality, it follows that, for any $x, y \in \mathbb{R}^n$,

$$
|f(x)-f(y)| \le \|\,|\nabla f|\,\|_{L^\infty(\mathbb{R}^n)}\,|x-y|.
$$

Using these and (2.2), we conclude that, for any $\delta \in (0,1)$,

$$
\begin{aligned}
&J_\epsilon^{(2)}(\delta,N) \\
&\quad = \left\| \left[\int_{\{y\in\mathbb{R}^n:\ \delta\le|\cdot-y|<2N\}} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p} \rho_\epsilon(|\cdot-y|)\,dy \right]^{\frac{1}{p}} \mathbf{1}_{B(\mathbf{0},2N+M)} \right\|_X \\
&\quad \le \|\,|\nabla f|\,\|_{L^\infty(\mathbb{R}^n)} \left\| \left[\int_{\{y\in\mathbb{R}^n:\ \delta\le|\cdot-y|<2N\}} \rho_\epsilon(|\cdot-y|)\,dy \right]^{\frac{1}{p}} \mathbf{1}_{B(\mathbf{0},2N+M)} \right\|_X \\
&\quad \le \|\,|\nabla f|\,\|_{L^\infty(\mathbb{R}^n)} \|\mathbf{1}_{B(\mathbf{0},2N+M)}\|_X \left[\sigma(\mathbb{S}^{n-1}) \int_\delta^\infty \rho_\epsilon(r) r^{n-1}\,dr \right]^{\frac{1}{p}} \to 0
\end{aligned}
$$

as $\epsilon \to 0^+$, which implies that

$$
\limsup_{\epsilon\to 0^+} J_\epsilon^{(2)}(\delta,N) = 0. \tag{2.8}
$$

Finally, we deal with term $J_\epsilon^{(1)}(\delta)$. Notice that, for any $x, y \in \mathbb{R}^n$, if $|f(x)-f(y)| \neq 0$ and $|x-y| < \delta$, then $x \in B(\mathbf{0}, M+1)$. By this, together with Minkowski's inequality and the assumption that X is a BBF space, we obtain, for any $\delta \in (0,1)$ and $\epsilon \in (0,\infty)$,

$$
\begin{aligned}
J_\epsilon^{(1)}(\delta) &= \left\| \left[\int_{\mathbb{S}^{n-1}} \int_0^\delta \frac{|f(\cdot + r\xi) - f(\cdot)|^p}{r^p} \rho_\epsilon(r) r^{n-1}\, dr\, d\sigma(\xi) \right]^{\frac{1}{p}} \mathbf{1}_{B(\mathbf{0},M+1)} \right\|_X \\
&\leq J_\epsilon^{(1,1)}(\delta) + J_\epsilon^{(1,2)}(\delta), \qquad (2.9)
\end{aligned}
$$

where

$$
J_\epsilon^{(1,1)}(\delta) := \left\| \left[\int_{\mathbb{S}^{n-1}} \int_0^\delta |\xi \cdot \nabla f(\cdot)|^p \rho_\epsilon(r) r^{n-1}\, dr\, d\sigma(\xi) \right]^{\frac{1}{p}} \mathbf{1}_{B(\mathbf{0},M+1)} \right\|_X
$$

and

$$
\begin{aligned}
J_\epsilon^{(1,2)}(\delta) : = \Bigg\| \Bigg[\int_{\mathbb{S}^{n-1}} \int_0^\delta & \left| \frac{f(\cdot + r\xi) - f(\cdot)}{r} - \xi \cdot \nabla f(\cdot) \right|^p \\
& \times \rho_\epsilon(r) r^{n-1}\, dr\, d\sigma(\xi) \Bigg]^{\frac{1}{p}} \mathbf{1}_{B(\mathbf{0},M+1)} \Bigg\|_X .
\end{aligned}
$$

Since $f \in C_c^2(\mathbb{R}^n)$, from Taylor's remainder theorem, it follows that, for any $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $r \in (0, \delta]$, and $\xi := (\xi_1, \ldots, \xi_n) \in \mathbb{S}^{n-1}$, there exists a $\theta \in (0,1)$ such that

$$
f(x + r\xi) = f(x) + r\xi \cdot \nabla f(x) + r^2 \sum_{i,j=1}^n \frac{\partial^2 f(x + \theta r\xi)}{2\partial x_i \partial x_j} \xi_i \xi_j.
$$

This, combined with the Cauchy–Schwarz inequality, implies that there exists a positive constant $C_{(f)}$, depending only on f, such that, for any $x \in \mathbb{R}^n$, $r \in (0, \delta]$, and $\xi \in \mathbb{S}^{n-1}$,

$$
\left| \frac{f(x + r\xi) - f(x)}{r} - \xi \cdot \nabla f(x) \right| \leq n^2 C_{(f)} r \leq n^2 C_{(f)} \delta.
$$

Using this and (2.1), we conclude that, for any given $\delta \in (0,1)$ and for any $\epsilon \in (0,\infty)$,

$$
\begin{aligned}
J_\epsilon^{(1,2)}(\delta) &\leq n^2 C_{(f)} \delta \left\| \mathbf{1}_{B(\mathbf{0},M+1)} \right\|_X \left[\sigma(\mathbb{S}^{n-1}) \int_0^\delta \rho_\epsilon(r) r^{n-1}\, dr \right]^{\frac{1}{p}} \\
&\leq n^2 C_{(f)} \delta \left\| \mathbf{1}_{B(\mathbf{0},M+1)} \right\|_X [\sigma(\mathbb{S}^{n-1})]^{1/p},
\end{aligned}
$$

which further implies that

$$
J_\epsilon^{(1,2)}(\delta) \leq C_{(f,n)} \delta, \qquad (2.10)
$$

where $C_{(f,n)}$ is a positive constant depending only on both f and n. For $J_\epsilon^{(1,1)}(\delta)$, by the assumption that $\operatorname{supp}(f) \subset B(\mathbf{0}, M)$ and (2.1), we have, for any $\epsilon \in (0,\infty)$,

$$
J_\epsilon^{(1,1)}(\delta) = \left[K(p,n) \int_0^\delta \rho_\epsilon(r) r^{n-1}\, dr \right]^{\frac{1}{p}} \left\| |\nabla f| \mathbf{1}_{B(\mathbf{0},M+1)} \right\|_X
$$

$$
\begin{aligned}
&= \left[K(p,n)\int_0^\delta \rho_\epsilon(r) r^{n-1}\,dr\right]^{\frac1p} \|\,|\nabla f|\,\|_X \\
&\le [K(p,n)]^{1/p} \|\,|\nabla f|\,\|_X
\end{aligned}
$$

with $K(p,n)$ as in (1.3). This, together with (2.6) and (2.9), yields that, for any $\epsilon \in (0,\infty)$,

$$
J_\epsilon \le [K(p,n)]^{1/p} \|\,|\nabla f|\,\|_X + J_\epsilon^{(1,2)}(\delta) + J_\epsilon^{(2)}(\delta, N) + J_\epsilon^{(3)}(N).
$$

Combining this inequality with (2.7), (2.8), and (2.10), it follows that, for any $\zeta \in (0,\infty)$ and $\delta \in (0,1)$ one has

$$
\limsup_{\epsilon\to 0^+} J_\epsilon \le [K(p,n)]^{1/p} \|\,|\nabla f|\,\|_X + C_{(f,n)}\delta + \zeta.
$$

By this and the arbitrariness of both δ and ζ, we finally obtain

$$
\limsup_{\epsilon\to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac1p} \right\|_X^p \le K(p,n)\ \|\,|\nabla f|\,\|_X^p,
$$

which implies the validity of (2.5).

Based on (2.5), to prove (2.4), it suffices to show that

$$
K(p,n)\ \|\,|\nabla f|\,\|_X^p \le \liminf_{\epsilon\to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac1p} \right\|_X^p. \tag{2.11}
$$

Since $f \in C_c^2(\mathbb{R}^n)$, from the Taylor expansion and the mean value theorem, it follows that there exists a positive constant L such that, for any $x, h \in \mathbb{R}^n$,

$$
|f(x+h) - f(x) - h\cdot\nabla f(x)| \le L|h|^2.
$$

Using this, we find that, for any $x, h \in \mathbb{R}^n$,

$$
|h\cdot\nabla f(x)| \le |f(x+h) - f(x)| + L|h|^2. \tag{2.12}
$$

Recall that, for any $\theta \in (0,1)$, there is a positive constant C_θ such that, for any $a, b \in (0,\infty)$,

$$
(a+b)^p \le (1+\theta)a^p + C_\theta b^p
$$

(see, for instance, [12, p. 699]). By this and (2.12), we conclude that, for any $x, h \in \mathbb{R}^n$, and $\theta \in (0,1)$,

$$
|h\cdot\nabla f(x)|^p \le (1+\theta)|f(x+h) - f(x)|^p + C_\theta L^p |h|^{2p},
$$

which further implies that

$$
\begin{aligned}
&\left\| \left[\int_{B(\mathbf{0},1)} \frac{|h\cdot\nabla f(\cdot)|^p}{|h|^p} \rho_\epsilon(|h|)\,dh\right]^{\frac1p} \mathbf{1}_{B(\mathbf{0},M)} \right\|_X \\
&\quad \le (1+\theta)^{1/p} \left\| \left[\int_{B(\mathbf{0},1)} \frac{|f(\cdot+h)-f(\cdot)|^p}{|h|^p} \rho_\epsilon(|h|)\,dh\right]^{\frac1p} \mathbf{1}_{B(\mathbf{0},M)} \right\|_X
\end{aligned}
$$

$$+ (C_\theta L^p)^{1/p}\left[\int_{B(\mathbf{0},1)} |h|^p \rho_\epsilon(|h|)\,dh\right]^{\frac{1}{p}} \left\|\mathbf{1}_{B(\mathbf{0},M)}\right\|_X. \tag{2.13}$$

Observe that, for any $x \in \mathbb{R}^n$,

$$\int_{B(\mathbf{0},1)} \frac{|h\cdot\nabla f(x)|^p}{|h|^p}\rho_\epsilon(|h|)\,dh = K(p,n)|\nabla f(x)|^p \int_0^1 \rho_\epsilon(r)r^{n-1}dr$$

and, by (2.1) and (2.2), we conclude that

$$\lim_{\epsilon\to 0^+}\int_{B(\mathbf{0},1)} |h|^p\rho_\epsilon(|h|)\,dh = 0$$

(see, for instance, [12, p. 700]). Using these, the assumption that $\operatorname{supp}(f) \subset B(\mathbf{0}, M)$, (2.1), (2.2), and (2.13), we conclude that, for any given $\theta \in (0,1)$,

$$\begin{aligned}
&K(p,n)\,\|\,|\nabla f|\,\|_X^p\\
&\quad = K(p,n)\left\|\,|\nabla f|\mathbf{1}_{B(\mathbf{0},M)}\,\right\|_X^p \liminf_{\epsilon\to 0^+}\int_0^1 \rho_\epsilon(r)r^{n-1}dr\\
&\quad = \liminf_{\epsilon\to 0^+}\left\|\left[K(p,n)|\nabla f|\int_0^1 \rho_\epsilon(r)r^{n-1}dr\right]^{\frac{1}{p}}\mathbf{1}_{B(\mathbf{0},M)}\right\|_X^p\\
&\quad = \liminf_{\epsilon\to 0^+}\left\|\left[\int_{B(\mathbf{0},1)}\frac{|h\cdot\nabla f(\cdot)|^p}{|h|^p}\rho_\epsilon(|h|)\,dh\right]^{\frac{1}{p}}\mathbf{1}_{B(\mathbf{0},M)}\right\|_X^p\\
&\quad \le (1+\theta)\liminf_{\epsilon\to 0^+}\left\|\left[\int_{B(\mathbf{0},1)}\frac{|f(\cdot+h)-f(\cdot)|^p}{|h|^p}\rho_\epsilon(|h|)\,dh\right]^{\frac{1}{p}}\mathbf{1}_{B(\mathbf{0},M)}\right\|_X^p.
\end{aligned}$$

From this, we deduce that

$$\begin{aligned}
&K(p,n)\,\|\,|\nabla f|\,\|_X^p\\
&\quad \le \liminf_{\epsilon\to 0^+}\left\|\left[\int_{B(\mathbf{0},1)}\frac{|f(\cdot+h)-f(\cdot)|^p}{|h|^p}\rho_\epsilon(|h|)\,dh\right]^{\frac{1}{p}}\mathbf{1}_{B(\mathbf{0},M)}\right\|_X^p\\
&\quad \le \liminf_{\epsilon\to 0^+}\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\le 1\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|h|)\,dh\right]^{\frac{1}{p}}\mathbf{1}_{B(\mathbf{0},M)}\right\|_X^p\\
&\quad \le \liminf_{\epsilon\to 0^+}\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|h|)\,dh\right]^{\frac{1}{p}}\mathbf{1}_{B(\mathbf{0},M)}\right\|_X^p.
\end{aligned}$$

This implies that (2.11) holds and then finishes the proof of Theorem 2.7. □

Now, we compare the results of Theorem 2.7 with the classical BBM formula in [12, Theorems 2 and 3].

Remark 2.8. If $X := L^p(\mathbb{R}^n)$ and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ is a radial-ATI, then (2.4) is valid. Indeed, for any given $M \in (0,\infty)$, let $N \in (2M+1,\infty)$. By some simple calculations, we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\left\|\left[\int_{B(\mathbf{0},M)}\frac{\rho_\epsilon(|\cdot-y|)}{|\cdot-y|^p}\,dy\right]^{\frac{1}{p}}\mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)}\right\|_{L^p(\mathbb{R}^n)}^p\\
&\quad\le \lim_{\epsilon\to 0^+}\int_{B(\mathbf{0},M)}\int_{\mathbb{R}^n}\frac{\rho_\epsilon(|h|)}{|h|^p}\mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N/2)}(h)\,dh\,dy\\
&\quad\lesssim \lim_{\epsilon\to 0^+}\int_{N/2}^\infty \rho_\epsilon(r)r^{n-1}\,dr = 0.
\end{aligned}$$

In this case, Theorem 2.7 is just [12, Theorems 2 and 3] restricted to $f \in C_{\mathrm{c}}^2(\mathbb{R}^n)$.

Definition 2.9. [93, Definition 2.6] Assume that X is a ball quasi-Banach function space and let $p \in (0,\infty)$. The *p-convexification* of X is defined as the space $X^p := \{f \in \mathscr{M}(\mathbb{R}^n) :\ |f|^p \in X\}$ equipped with the quasi-norm $\|f\|_{X^p} := \|\,|f|^p\|_X^{1/p}$.

For instance the $1/p$-convexification of $L^p(\mathbb{R}^n)$ is $L^1(\mathbb{R}^n)$ and of weak $L^p(\mathbb{R}^n)$ is weak $L^1(\mathbb{R}^n)$. We now introduce the following locally β-doubling concept of ball quasi-Banach function spaces.

Definition 2.10. Let X be a ball quasi-Banach function space and $\beta \in (0,\infty)$. Then X is said to be *locally β-doubling* if there exists a positive constant C such that, for any $B := B(\mathbf{0},r) \in \mathbb{B}$ with $r \in (0,\infty)$ and $\alpha \in [1,\infty)$,

$$\|\mathbf{1}_{\alpha B}\|_X \le C\alpha^\beta\,\|\mathbf{1}_B\|_X\,.$$

The following proposition is a direct conclusion of both Definitions 2.9 and 2.10.

Proposition 2.11. *Let X be a ball quasi-Banach function space, $\beta \in (0,\infty)$, and $p \in (0,\infty)$. Then X is locally β-doubling if and only if X^p is locally β/p-doubling.*

Proof. From the definition of X^p, we deduce that, for any $B := B(\mathbf{0},r) \in \mathbb{B}$ with $r \in (0,\infty)$, and for any $\alpha \in [1,\infty)$,

$$\|\mathbf{1}_{\alpha B}\|_{X^p} = \|\mathbf{1}_{\alpha B}\|_X^{1/p} \lesssim \alpha^{\beta/p}\|\mathbf{1}_B\|_X^{1/p} \sim \alpha^{\beta/p}\,\|\mathbf{1}_B\|_{X^p}$$

if and only if

$$\|\mathbf{1}_{\alpha B}\|_X = \|\mathbf{1}_{\alpha B}\|_{X^p}^p \lesssim \alpha^\beta\,\|\mathbf{1}_B\|_{X^p}^p \sim \alpha^\beta\,\|\mathbf{1}_B\|_X\,.$$

This finishes the proof of Proposition 2.11. □

The following lemma shows the sufficiency of the condition (2.3) in Theorem 2.7.

Lemma 2.12. *Let X be a ball Banach function space, $p \in [1,\infty)$, and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a decreasing-radial-*ATI. *Assume that X is locally β-doubling with $\beta \in (0, 1+n/p)$. Then given $M \in (0,\infty)$ there exists an $N \in (0,\infty)$ such that* (2.3) *is valid.*

Proof. Given $M > 0$ we pick $N \in (2M+1,\infty)$. Observe that, for any $x,\ y \in \mathbb{R}^n$ satisfying $|x| \in [N,\infty)$ and $|y| \in [0,M)$, we must have $|x-y| > |x|/2$. Using this fact and the assumption that ρ_ϵ is decreasing on $(0,\infty)$ for any $\epsilon \in (0,\infty)$, we conclude that, for any $\epsilon \in (0,\infty)$,

$$\left\|\left[\int_{B(\mathbf{0},M)}\frac{1}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)}\right\|_X$$

$$\lesssim \left\| \frac{[\rho_\epsilon(|\cdot|/2)]^{1/p}}{|\cdot|} \mathbf{1}_{\mathbb{R}^n \setminus B(\mathbf{0},N)} \right\|_X .$$

Using this estimate, the assumption that X is locally β-doubling when $\beta \in (0, 1 + n/p)$, that ρ_ϵ is decreasing on $(0, \infty)$ for any $\epsilon \in (0, \infty)$, Hölder's inequality for series, and (2.2), we deduce that

$$\begin{aligned}
&\left\| \left[\int_{B(\mathbf{0},M)} \frac{1}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|)\, dy \right]^{\frac{1}{p}} \mathbf{1}_{\mathbb{R}^n \setminus B(\mathbf{0},N)} \right\|_X \\
&\quad \lesssim \sum_{j=1}^{\infty} \left\| \frac{[\rho_\epsilon(|\cdot|/2)]^{1/p}}{|\cdot|} \mathbf{1}_{B(\mathbf{0},2^j N) \setminus B(\mathbf{0},2^{j-1}N)} \right\|_X \\
&\quad \lesssim \sum_{j=1}^{\infty} (2^j N)^{-1} [\rho_\epsilon(2^{j-2}N)]^{1/p} \left\| \mathbf{1}_{B(\mathbf{0},2^j N)} \right\|_X \\
&\quad \lesssim \sum_{j=1}^{\infty} (2^j N)^{\beta - 1} [\rho_\epsilon(2^{j-2}N)]^{1/p} \left\| \mathbf{1}_{B(\mathbf{0},1)} \right\|_X \\
&\quad \lesssim \left[\sum_{j=1}^{\infty} (2^j N)^n \rho_\epsilon(2^{j-2}N) \right]^{\frac{1}{p}} \left[\sum_{j=1}^{\infty} (2^j N)^{(\beta - 1 - n/p)p'} \right]^{\frac{1}{p'}} \\
&\quad \lesssim \left[\sum_{j=1}^{\infty} \int_{B(\mathbf{0},2^{j-3}N) \setminus B(\mathbf{0},2^{j-2}N)} \rho_\epsilon(|x|)\, dx \right]^{\frac{1}{p}} \\
&\quad \sim \left[\int_{N/4}^{\infty} \rho_\epsilon(r) r^{n-1}\, dr \right]^{\frac{1}{p}} \to 0
\end{aligned}$$

as $\epsilon \to 0^+$. This finishes the proof of Lemma 2.12. □

As a consequence of both Theorem 2.7 and Lemma 2.12, we derive the following conclusion.

Theorem 2.13. *Let X be a ball Banach function space, $p \in [1, \infty)$, and $\{\rho_\epsilon\}_{\epsilon \in (0,\infty)}$ be a decreasing-radial-*ATI. *Assume that X is locally β-doubling with $\beta \in (0, 1 + n/p)$. Then* (2.4) *holds for any function $f \in C_c^2(\mathbb{R}^n)$.*

Remark 2.14. Let $q \in [1, \infty)$, $X := L^q(\mathbb{R}^n)$, and $\{\rho_\epsilon\}_{\epsilon \in (0,\infty)}$ be a decreasing-radial-ATI. In this case, X is locally β-doubling with $\beta = n/q$. Thus, when $p \in [1, \infty)$, X is locally β-doubling with $\beta \in (0, 1 + n/p)$ if and only if $n(1/q - 1/p) < 1$. From this we deduce that if $n(1/q - 1/p) < 1$, then for any $f \in C_c^2(\mathbb{R}^n)$ we have

$$\begin{aligned}
&\lim_{\epsilon \to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|)\, dy \right]^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^n)}^p \\
&\quad = K(p, n) \, \| |\nabla f| \|_{L^q(\mathbb{R}^n)}^p . \qquad (2.14)
\end{aligned}$$

As a consequence of Theorem 2.13, we have the following conclusion.

Theorem 2.15. *Let X be a ball Banach function space and $p \in [1,\infty)$. Assume that X is locally β-doubling with $\beta \in (0, 1+n/p)$. Then, for any function $f \in C_c^2(\mathbb{R}^n)$ we have*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_X^p = \frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_X^p, \tag{2.15}$$

with $K(p,n)$ as in (1.3).

Before we prove Theorem 2.15, we provide two observations related to both Theorems 2.13 and 2.15. In what follows, for a given $p \in (0,\infty)$, we denote by $L^p_{\rm loc}(\mathbb{R}^n)$ the space of all the measurable functions whose p-th power is integrable over compact subsets of $\mathbb{R}^n$.

Remark 2.16. (i) Let $q \in [1,\infty)$ and $X := L^q(\mathbb{R}^n)$. In this case, X is locally n/q-doubling. Thus, when $p \in [1,\infty)$, X is locally n/q-doubling with $n/q \in (0, 1+n/p)$ if and only if $n(1/q-1/p) < 1$. From this and Theorem 2.15 we deduce that, if $n(1/q-1/p) < 1$, then for any function $f \in C_c^2(\mathbb{R}^n)$ we have

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^q(\mathbb{R}^n)}^p = \frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{L^q(\mathbb{R}^n)}^p. \tag{2.16}$$

Thus, Theorem 2.15 with $X = L^q(\mathbb{R}^n)$ and $p = q$ is exactly the classical BBM formula (1.2) for $C_c^2(\mathbb{R}^n)$ functions, while the case $p \neq q$ is new.

(ii) Let $p, q \in [1,\infty)$ satisfy $n\max\{0, 1/q-1/p\} < s < 1$. By [54, Theorem 1.3], we conclude that, if $f \in L^{\min\{q,p\}}_{\rm loc}(\mathbb{R}^n)$, then $f \in F^s_{q,p}(\mathbb{R}^n)$ if and only if

$$\mathrm{I} := \|f\|_{L^q(\mathbb{R}^n)} + \left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^q(\mathbb{R}^n)} < \infty$$

and, moreover, in this case,

$$\mathrm{I} \sim \|f\|_{F^s_{q,p}(\mathbb{R}^n)} \tag{2.17}$$

with the positive equivalence constants independent of f, where $F^s_{q,p}(\mathbb{R}^n)$ denotes the classical Triebel–Lizorkin spaces (see [99, Section 2.3] for the precise definition). Using (2.17) we have that, for any given $p \in [1,\infty)$ and $s \in (0,1)$, $F^s_{p,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ with equivalent norms. Also, from [54, Theorem 1.5], we deduce that, under the assumptions that p, $q \in [1,\infty)$, then (2.17) is valid only if $n\max\{0, 1/q-1/p\} \leq s < 1$. Moreover, using both (i) and (ii) of [33, Theorem 3.2] we conclude that, when $s = 1$, then

$$\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+p}}\,dy\right]^{\frac{1}{p}}\right\|_{L^q(\mathbb{R}^n)} = \infty$$

unless f is a constant. Thus, the Gagliardo semi-norm in (2.17) does *not* recover the Triebel–Lizorkin semi-norm $\|\cdot\|_{F^1_{q,p}(\mathbb{R}^n)}$ as $s \to 1^-$. (Recall the well-known identity $F^1_{q,2}(\mathbb{R}^n) =$

$W^{1,q}(\mathbb{R}^n)$ with equivalent norms; see [99]). In this sense, the assumption $n(1/q-1/p)<1$ in (i) is optimal. This indicates that the requirement $\beta\in(0,1+n/p)$ in both Theorems 2.13 and 2.15 is also optimal.

The next lemma provides an essential tool that plays a vital role in the proof of Theorem 2.15.

Lemma 2.17. *Let X be a ball Banach function space and $p\in[1,\infty)$. Assume that X is locally β-doubling with $\beta\in(0,1+n/p)$. Then, for any $f\in X\cap L^p_{\rm loc}(\mathbb{R}^n)$ with compact support,*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\ge(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\right\|_X^p=0. \tag{2.18}$$

Proof. Let $f\in X\cap L^p_{\rm loc}(\mathbb{R}^n)$ have compact support. Then there is a ball $B(\mathbf{0},M)$ with $M\in(0,\infty)$ such that $\operatorname{supp}(f)\subset B(\mathbf{0},M)$. Let $N\in(2M+1,\infty)$. Observe that, for any $y\in[B(\mathbf{0},N)]^\complement$ and $x\in[B(\mathbf{0},M)]^\complement$, $f(y)=f(x)=0$. From this and the Minkowski inequality, we deduce that, for any $s\in(0,1)$,

$$\begin{aligned}
&(1-s)^{1/p}\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\ge(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\right\|_X\\
&\quad\le(1-s)^{1/p}\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\ge(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\mathbf{1}_{B(\mathbf{0},N)}\right\|_X\\
&\qquad+(1-s)^{1/p}\left\|\left[\int_{B(\mathbf{0},M)}\frac{|f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)}\right\|_X\\
&\quad\le(1-s)^{1/p}\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\ge(1-s)^{-1/2}\}}\frac{|f(\cdot)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\mathbf{1}_{B(\mathbf{0},N)}\right\|_X\\
&\qquad+(1-s)^{1/p}\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\ge(1-s)^{-1/2}\}}\frac{|f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\mathbf{1}_{B(\mathbf{0},N)}\right\|_X\\
&\qquad+(1-s)^{1/p}\left\|\left[\int_{B(\mathbf{0},M)}\frac{|f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)}\right\|_X\\
&\quad=:I_s^{(1)}(N)+I_s^{(2)}(N)+I_s^{(3)}(N).
\end{aligned}\tag{2.19}$$

We first consider $I_s^{(1)}(N)$. Since $f\in X$ with $\operatorname{supp}(f)\subset B(\mathbf{0},M)$, the definition of $I_s^{(1)}(N)$ and some simple calculations yield that

$$\begin{aligned}
I_s^{(1)}(N)&\lesssim(1-s)^{(2+sp)/(2p)}(sp)^{-1/p}\left\|f\mathbf{1}_{B(\mathbf{0},N)}\right\|_X\\
&\sim(1-s)^{(2+sp)/(2p)}(sp)^{-1/p}\|f\|_X\to 0
\end{aligned}\tag{2.20}$$

as $s\to1^-$. As for $I_s^{(2)}(N)$, observe that, for any $s\in(1-(2N)^{-2},1)$, $x\in B(\mathbf{0},N)$, and $y\in\mathbb{R}^n$ satisfying $|x-y|\ge(1-s)^{-1/2}$,

$$|y|\ge|x-y|-|x|>(1-s)^{-1/2}-N>N.$$

From this, the assumption that $\operatorname{supp}(f)\subset B(\mathbf{0},M)$, and the definition of $I_s^{(2)}(N)$, it follows that, for any $s\in(1-(2N)^{-2},1)$,

$$I_s^{(2)}(N)=0. \tag{2.21}$$

Finally, we deal with term $I_s^{(3)}(N)$. Observe that, for any $x\in[B(\mathbf{0},N)]^{\complement}$ and $y\in B(\mathbf{0},M)$, $|x-y|>|x|/2$. Using this and the assumptions that $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and X is locally β-doubling with $\beta\in(0,1+n/p)$, we conclude that, for any $s\in(0,1)$,

$$\begin{aligned}
I_s^{(3)}(N)&\lesssim(1-s)^{1/p}\left[\int_{B(\mathbf{0},M)}|f(y)|^p\,dy\right]^{\frac1p}\left\|\frac{\mathbf{1}_{\mathbb{R}^n\setminus B(\mathbf{0},N)}}{|\cdot|^{s+n/p}}\right\|_X\\
&\lesssim(1-s)^{1/p}\sum_{j=1}^{\infty}(2^jN)^{-(s+n/p)}\left\|\mathbf{1}_{B(\mathbf{0},2^jN)\setminus B(\mathbf{0},2^{j-1}N)}\right\|_X\\
&\lesssim(1-s)^{1/p}\sum_{j=1}^{\infty}(2^jN)^{-(s+n/p)}\left\|\mathbf{1}_{B(\mathbf{0},2^jN)}\right\|_X\\
&\lesssim(1-s)^{1/p}\left\|\mathbf{1}_{B(\mathbf{0},1)}\right\|_X\sum_{j=1}^{\infty}(2^jN)^{\beta-(s+n/p)}\\
&\sim(1-s)^{1/p}\sum_{j=1}^{\infty}(2^jN)^{\beta-(s+n/p)},
\end{aligned}$$

where the implicit positive constants depend on f. By this and the assumption that $\beta\in(0,1+n/p)$, we have

$$\lim_{s\to1^-}I_s^{(3)}(N)=0.$$

Combining this estimate with (2.20), (2.21), and (2.19) yields

$$\lim_{s\to1^-}(1-s)\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\ge(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\right\|_X^p=0,$$

and this completes the proof of Lemma 2.17. □

Proof of Theorem 2.15. For any $\epsilon\in(0,1/p)$, let

$$\rho_\epsilon(r):=\begin{cases}\dfrac{1}{C_\epsilon}\dfrac{\epsilon}{r^{n-\epsilon p}}, & r\in(0,\epsilon^{-1/2}),\\ 0, & r\in[\epsilon^{-1/2},\infty),\end{cases} \tag{2.22}$$

where

$$C_\epsilon:=\epsilon\int_0^{\epsilon^{-1/2}}r^{\epsilon p-1}\,dr=\frac{1}{p\epsilon^{\epsilon p/2}}.$$

Observe that $\lim_{\epsilon\to0^+}C_\epsilon=\frac1p$, $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}\subset L^1_{\mathrm{loc}}(0,\infty)$ satisfy that, for any given $\epsilon\in(0,1/p)$, ρ_ϵ is nonnegative and decreasing,

$$\int_0^\infty\rho_\epsilon(r)r^{n-1}\,dr=1,$$

and, for any $\delta \in (0, \epsilon^{-1/2})$,

$$\lim_{\epsilon\to 0^+} \int_\delta^\infty \rho_\epsilon(r) r^{n-1}\,dr = \lim_{\epsilon\to 0^+} \frac{\epsilon}{C_\epsilon} \int_\delta^{\epsilon^{-1/2}} r^{\epsilon p-1}\,dr \le \lim_{\epsilon\to 0^+} \frac{\epsilon}{C_\epsilon} \frac{\delta^{\epsilon p-1}}{\epsilon^{1/2}} = 0.$$

Thus, $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ is a decreasing-radial-ATI. From this and Theorem 2.13, it follows that

$$\begin{aligned}
&\lim_{s\to 1^-} (1-s) \left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|<(1-s)^{-1/2}\}} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy \right]^{\frac{1}{p}} \right\|_X^p \\
&= \lim_{s\to 1^-} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p} \frac{1-s}{|\cdot-y|^{n-(1-s)p}} \mathbf{1}_{\{y\in\mathbb{R}^n:\ |\cdot-y|<(1-s)^{-1/2}\}}(y)\,dy \right]^{\frac{1}{p}} \right\|_X^p \\
&= \lim_{\epsilon\to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p} \frac{\epsilon}{|\cdot-y|^{n-\epsilon p}} \mathbf{1}_{\{y\in\mathbb{R}^n:\ |\cdot-y|<\epsilon^{-1/2}\}}(y)\,dy \right]^{\frac{1}{p}} \right\|_X^p \\
&= \lim_{\epsilon\to 0^+} C_\epsilon \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p} \rho_\epsilon(|\cdot-y|)\,dy \right]^{\frac{1}{p}} \right\|_X^p \\
&= \frac{K(p,n)}{p} \,\|\,|\nabla f|\,\|_X^p,
\end{aligned}$$

which, together with the fact that $f \in C_{\mathrm{c}}^2(\mathbb{R}^n)$ and Lemma 2.17, further implies that

$$\lim_{s\to 1^-} (1-s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy \right]^{\frac{1}{p}} \right\|_X^p = \frac{K(p,n)}{p} \,\|\,|\nabla f|\,\|_X^p.$$

This finishes the proof of Theorem 2.15. □

Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}$ is defined for any f in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$\mathcal{M}(f)(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\,dy, \qquad \forall\, x \in \mathbb{R}^n,$$

where the supremum is taken over all the balls $B \subset \mathbb{R}^n$ containing x. The following lemma gives a sufficient condition for the locally β-doubling property of X.

Lemma 2.18. *Let X be a ball quasi-Banach function space. Assume that the Hardy–Littlewood maximal operator $\mathcal{M}$ is weakly bounded on X, namely, there exists a positive constant C such that, for any $f \in X$,*

$$\sup_{\lambda\in(0,\infty)} \left[\lambda \|\mathbf{1}_{\{x\in\mathbb{R}^n:\ \mathcal{M}(f)(x)>\lambda\}}\|_X \right] \le C\|f\|_X. \tag{2.23}$$

Then X is locally n-doubling.

Proof. Observe that for any $B := B(\mathbf{0}, r) \in \mathbb{B}$ with $r \in (0,\infty)$, $\alpha \in [1,\infty)$, and $x \in \alpha B$,

$$\mathcal{M}(\mathbf{1}_B)(x) \ge ⨍_{\alpha B} \mathbf{1}_B(y)\,dy = \frac{|B|}{|\alpha B|} = \frac{1}{\alpha^n}.$$

From this, the assumption that X is a BQBF space, and (2.23), it follows that, for any $B \in \mathbb{B}$ and $\alpha \in [1, \infty)$,

$$\|\mathbf{1}_{\alpha B}\|_X \le \|\mathbf{1}_{\{x\in\mathbb{R}^n:\ \mathcal{M}(\mathbf{1}_B)(x) > \frac{1}{2\alpha^n}\}}\|_X \le 2C\alpha^n \|\mathbf{1}_B\|_X.$$

This implies that X is locally n-doubling, and hence finishes the proof of Lemma 2.18. □

Remark 2.19. Lemma 2.18 yields that the locally β-doubling assumption on X is weaker than the assumption that the Hardy–Littlewood maximal operator $\mathcal{M}$ is weakly bounded on X.

As a consequence of both Theorem 2.13 and Lemma 2.18, we obtain the following crucial fact.

Theorem 2.20. *Let X be a ball Banach function space and let $p \in [1, \infty)$, $q \in (0, \infty)$ satisfy $n(1/q - 1/p) < 1$. Let $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a decreasing-radial-*ATI. *Assume that the Hardy–Littlewood maximal operator $\mathcal{M}$ is weakly bounded on $X^{1/q}$. Then* (2.4) *is valid for any $f \in C_c^2(\mathbb{R}^n)$.*

Proof. By Lemma 2.18 we conclude that $X^{1/q}$ is locally n-doubling. This fact combined with Proposition 2.11 implies that X is locally β-doubling with $\beta := n/q \in (0, 1 + n/p)$. Thus, all the assumptions of Theorem 2.13 are satisfied. Using Theorem 2.13, we conclude that (2.4) holds for any $f \in C_c^2(\mathbb{R}^n)$. This finishes the proof of Theorem 2.20. □

Remark 2.21. Let $p \in [1, \infty)$, $q \in (0, \infty)$ satisfy $n(1/q - 1/p) < 1$, and $X := L^q(\mathbb{R}^n)$. In this case, $X^{1/q} = L^1(\mathbb{R}^n)$ and the Hardy–Littlewood maximal operator $\mathcal{M}$ is weakly bounded on $X^{1/q}$. This, together with Theorem 2.20, implies that, for any $f \in C_c^2(\mathbb{R}^n)$, (2.14) is valid for $X = L^q(\mathbb{R}^n)$.

As a consequence of Theorem 2.15, we obtain the following extension of the famous classical BBM formula (1.2) to ball Banach Sobolev spaces. The proof is similar to that used in the proof of Theorem 2.20 and we omit the details.

Theorem 2.22. *Let X be a ball Banach function space and let $p \in [1, \infty)$, $q \in (0, \infty)$ satisfy $n(1/q - 1/p) < 1$. Assume that the Hardy–Littlewood maximal operator $\mathcal{M}$ is weakly bounded on $X^{1/q}$. Then* (2.15) *is valid for any $f \in C_c^2(\mathbb{R}^n)$.*

Remark 2.23. Let p, q, and X be as in Remark 2.21. Then it follows Theorem 2.22 that (2.16) holds for any $f \in C_c^2(\mathbb{R}^n)$.

3 Asymptotics of $W^{1,X}(\mathbb{R}^n)$ functions in terms of ball Banach Sobolev norms

In this section, we establish the main result of this work, Theorem 3.36. This is achieved via a secondary main result, Theorem 3.4. We begin by recalling the definition of ball Banach function spaces with absolutely continuous norm; see [10, Definition 3.1], [101, Definition 3.2].

Definition 3.1. A ball Banach function space X is said to have an *absolutely continuous norm* if, for any $f \in X$ and any sequence of measurable sets $\{E_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ satisfying that $\mathbf{1}_{E_j} \to 0$ a. e. as $j \to \infty$, we have $\|f\mathbf{1}_{E_j}\|_X \to 0$ as $j \to \infty$.

We also recall the following definition of the associate space of a ball Banach function space, which can be found, for instance, in [10, Chapter 1, Definitions 2.1 and 2.3].

Definition 3.2. For any ball Banach function space X, the *associate space* (also called the *Köthe dual*) X' is defined by

$$X' := \left\{ f \in \mathscr{M}(\mathbb{R}^n) : \ \|f\|_{X'} := \sup_{\{g \in X:\ \|g\|_X = 1\}} \|fg\|_{L^1(\mathbb{R}^n)} < \infty \right\}, \tag{3.1}$$

where $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$.

Remark 3.3. By [93, Proposition 2.3] we have that the associate space X' of a ball Banach function space X is also a ball Banach function space.

Theorem 3.4. *Let X be a ball Banach function space, $p \in [1, \infty)$, and $\{\rho_\epsilon\}_{\epsilon \in (0,\infty)}$ be a decreasing-radial-*ATI. *Assume that X has an absolutely continuous norm, $X^{1/p}$ is a ball Banach function space, and the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$. Then for any function $f \in W^{1,X}(\mathbb{R}^n)$ we have*

$$\lim_{\epsilon \to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|)\, dy \right]^{\frac{1}{p}} \right\|_X^p = K(p,n)\, \| \, |\nabla f| \, \|_X^p . \tag{3.2}$$

Remark 3.5. (i) Let $1 \le p \le q < \infty$, $X := L^q(\mathbb{R}^n)$, and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a decreasing-radial-ATI. In this case for any $f \in W^{1,q}(\mathbb{R}^n)$ (2.14) holds. Thus, Theorem 3.4 with $X = L^q(\mathbb{R}^n)$ coincides with [12, Theorems 2 and 3] when $p = q$ while the case $p < q$ is new.

(ii) In contrast to Theorems 2.7, 2.13, and 2.22, we need certain minor assumptions on the underlying space X in Theorem 3.4 in order to extend the identity to $W^{1,X}(\mathbb{R}^n)$.

The proof of Theorem 3.4 is given in Subsection 3.3. In Subsection 3.1 we prove that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,X}(\mathbb{R}^n)$ under some mild assumptions, while Subsection 3.2 is devoted to establishing the key estimate (3.12) in weighted Lebesgue spaces.

3.1 Density in $W^{1,X}(\mathbb{R}^n)$

In what follows, we use the *symbol* $W_c^{1,X}(\mathbb{R}^n)$ to denote the set of all the functions in $W^{1,X}(\mathbb{R}^n)$ with compact support. The main objective of this subsection is to exhibit a good dense subspaces of $W^{1,X}(\mathbb{R}^n)$. Our approach consists of two parts: we first prove that $W_c^{1,X}(\mathbb{R}^n)$ is dense in $W^{1,X}(\mathbb{R}^n)$, and further show that $C_c^\infty(\mathbb{R}^n)$ is dense in $W_c^{1,X}(\mathbb{R}^n)$ under a stronger hypothesis (see Propositions 3.6 and 3.8 below). We begin with the following technical proposition.

Proposition 3.6. *Let X be a ball Banach function space. Assume that X has an absolutely continuous norm. Then $W_c^{1,X}(\mathbb{R}^n)$ is dense in $W^{1,X}(\mathbb{R}^n)$.*

Proof. Let $f \in W^{1,X}(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $\phi(x) = 1$ for any x with $|x| \le 1$, and $\phi(x) = 0$ for any x with $|x| > 2$. Moreover, for any $l \in \mathbb{N}$, let

$$f_l(\cdot) := f(\cdot)\phi\left(\frac{\cdot}{l}\right). \tag{3.3}$$

Then, for any $l \in \mathbb{N}$, $f_l \in X$ with compact support. From (3.3) and [42, p. 216, Theorem 1(iv)], it follows that, for any $j \in \{1, \dots, n\}$ and $l \in \mathbb{N}$,

$$\partial_j(f_l)(\cdot) = \partial_j f(\cdot)\phi\left(\frac{\cdot}{l}\right) + f(\cdot)\frac{1}{l}\partial_j\phi\left(\frac{\cdot}{l}\right)$$

and $\partial_j(f_l) \in X$ with compact support. Then we claim that

$$\lim_{l\to\infty} \|f_l - f\|_{W^{1,X}(\mathbb{R}^n)} = 0. \tag{3.4}$$

Indeed, for any $x \in \mathbb{R}^n$,

$$|f_l(x) - f(x)| \le \left[\|\phi\|_{L^\infty(\mathbb{R}^n)} + 1\right] |f(x)| \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(\mathbf{0},l)}}(x).$$

From this and the assumption that X has an absolutely continuous norm, it follows that

$$\limsup_{l\to\infty} \|f_l - f\|_X \le \left[\|\phi\|_{L^\infty(\mathbb{R}^n)} + 1\right] \limsup_{l\to\infty} \left\| |f| \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(\mathbf{0},l)}} \right\|_X = 0. \tag{3.5}$$

Moreover, observe that, for any $j \in \{1, \dots, n\}$, $l \in \mathbb{N}$, and $x \in \mathbb{R}^n$,

$$\begin{aligned} &\partial_j(f_l)(x) - \partial_j f(x) \\ &\quad = \left[\phi\left(\frac{x}{l}\right) - 1\right] \partial_j f(x) \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(\mathbf{0},l)}}(x) + \frac{1}{l}\partial_j\phi\left(\frac{x}{l}\right) f(x). \end{aligned}$$

By this, we have, for any $j \in \{1, \dots, n\}$, $l \in \mathbb{N}$, and $x \in \mathbb{R}^n$,

$$\begin{aligned} &|\partial_j(f_l)(x) - \partial_j f(x)| \\ &\quad \le \left[\|\phi\|_{L^\infty(\mathbb{R}^n)} + 1\right] |\partial_j f(x)| \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(\mathbf{0},l)}}(x) + \frac{\|\partial_j\phi\|_{L^\infty(\mathbb{R}^n)}}{l} |f(x)|. \end{aligned}$$

Using this and the assumption that X has an absolutely continuous norm, we conclude that, for any $j \in \{1, \dots, n\}$,

$$\limsup_{l\to\infty} \|\partial_j(f_l) - \partial_j f\|_X \lesssim \limsup_{l\to\infty} \left\| |\partial_j f| \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(\mathbf{0},l)}} \right\|_X + \limsup_{l\to\infty} \frac{1}{l}\|f\|_X = 0.$$

Thus, from this and (3.5) it follows that

$$\|f_l - f\|_X + \| |\nabla f_l - \nabla f| \|_X \lesssim \|f_l - f\|_X + \sum_{j=1}^n \|\partial_j(f_l) - \partial_j f\|_X \to 0$$

as $l \to \infty$. This yields the validity of (3.4), hence $W^{1,X}_{\mathrm{c}}(\mathbb{R}^n)$ is dense in $W^{1,X}(\mathbb{R}^n)$, and this finishes the proof of Proposition 3.6. □

Now, we introduce the concept of centered ball average operators.

Definition 3.7. Let $r \in (0, \infty)$. The *centered ball average operator B_r* is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$B_r(f)(x) := \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| \, dy.$$

Next we address the density of $C_c^\infty(\mathbb{R}^n)$ in $W_c^{1,X}(\mathbb{R}^n)$ under some assumptions on X.

Proposition 3.8. *Let X be a ball Banach function space. Assume that X has an absolutely continuous norm and that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X, namely, there exists a positive constant C such that, for any $f \in X$ we have*

$$\sup_{r\in(0,\infty)} \|B_r(f)\|_X \le C\|f\|_X.$$

Then $C_c^\infty(\mathbb{R}^n)$ is dense in $W_c^{1,X}(\mathbb{R}^n)$.

Proof. Let $f \in W_c^{1,X}(\mathbb{R}^n)$. Then there is a ball $B(\mathbf{0}, M)$ with $M > 0$ such that $\operatorname{supp}(f) \subset B(\mathbf{0}, M)$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}(\eta) \subset B(\mathbf{0}, 1)$, $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$, and $\eta_k(\cdot) := k^n\eta(k\cdot)$ for any $k \in \mathbb{N}$. From this and [42, p. 714, Theorem 7(i)], it follows that $f * \eta_k \in C_c^\infty(\mathbb{R}^n)$. Then we claim

$$\lim_{k\to\infty} \|f * \eta_k - f\|_X = 0. \tag{3.6}$$

Indeed, since X has an absolutely continuous norm, from [98, Proposition 3.8], it follows that $C_c(\mathbb{R}^n)$ is dense in X. Thus, for any $\zeta \in (0, \infty)$ there is a $g \in C_c(\mathbb{R}^n)$ such that

$$\|f - g\|_X < \zeta. \tag{3.7}$$

By this and the assumption that X is a BBF space, we conclude that, for any $k \in \mathbb{N}$,

$$\|f * \eta_k - f\|_X \le \|f * \eta_k - g * \eta_k\|_X + \|g * \eta_k - g\|_X + \|g - f\|_X. \tag{3.8}$$

From the definition of η_k and the assumption that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X, we deduce that, for any $k \in \mathbb{N}$,

$$\begin{aligned} \|f * \eta_k - g * \eta_k\|_X &= \left\| \int_{\mathbb{R}^n} (f - g)(y)\eta_k(\cdot - y)\,dy \right\|_X \\ &\lesssim \left\| \fint_{B(\cdot,k^{-1})} |(f - g)(y)|\,dy \right\|_X \\ &\sim \left\| B_{k^{-1}}(|f - g|) \right\|_X \lesssim \|f - g\|_X \lesssim \zeta. \end{aligned} \tag{3.9}$$

Moreover, since $g \in C_c(\mathbb{R}^n)$, it must be uniformly continuous, thus there is a ball $B(\mathbf{0}, N)$ such that $\operatorname{supp}(g) \subset B(\mathbf{0}, N)$. This fact combined with the definition of η_k implies that

$$\begin{aligned} \|g * \eta_k - g\|_X &\lesssim \left\| \fint_{B(\cdot,k^{-1})} |g(y) - g(\cdot)|\,dy \right\|_X \\ &\le \sup_{|x-y|\le k^{-1}} |g(x) - g(y)| \left\| \mathbf{1}_{B(\mathbf{0},N+1)} \right\|_X \to 0 \end{aligned}$$

as $k \to \infty$. By this, (3.7), (3.8), and (3.9), we have, for any $\zeta \in (0, \infty)$,

$$\lim_{k\to\infty} \|f * \eta_k - f\|_X \lesssim \zeta,$$

which, together with the arbitrariness of ζ, implies that (3.6) is valid. This completes the proof of the preceding claim.

Observe that, for any $j \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$, $\partial_j(f * \eta_k) = (\partial_j f) * \eta_k$ (see, for instance, [43, Proposition 8.10]). Applying this and an argument similar to that used in the proof of (3.6) (with f replaced by $\partial_j f$), we find that, for any $j \in \{1, \ldots, n\}$,

$$\lim_{k\to\infty} \|\partial_j(f * \eta_k) - \partial_j f\|_X = 0.$$

This fact and (3.6) yield that $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is dense in $W_{\mathrm{c}}^{1,X}(\mathbb{R}^n)$. Proposition 3.8 is now proved. □

The following corollary is a consequence of both Propositions 3.6 and 3.8; we omit the details.

Corollary 3.9. *Let X be a ball Banach function space. Assume that X has an absolutely continuous norm and the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X. Then $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is dense in $W^{1,X}(\mathbb{R}^n)$.*

The proofs of both Propositions 3.6 and 3.8 yield the following; we also omit the details.

Corollary 3.10. *Let X be a ball Banach function space. Assume that X has an absolutely continuous norm and the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X. Then $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is dense in X.*

The following lemma gives a sufficient condition for the uniform boundedness of centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ on X.

Lemma 3.11. *Let X be a ball Banach function space and $p \in [1,\infty)$. Assume that $X^{1/p}$ is a ball Banach function space and the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$. Then the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X; moreover, there exists a positive constant $C_{(n,p)}$, depending only on both n and p, such that for any $r \in (0,\infty)$ and $f \in X$ we have*

$$\|B_r(f)\|_X \le C_{(n,p)} \|\mathcal{M}\|^{2/p}_{(X^{1/p})'\to(X^{1/p})'} \|f\|_X,$$

where $\|\mathcal{M}\|_{(X^{1/p})'\to(X^{1/p})'}$ denotes the operator norm of $\mathcal{M}$ from $(X^{1/p})'$ to $(X^{1/p})'$.

To prove Lemma 3.11, we need to borrow some ideas from the proof of the extrapolation theorem in [31, Theorem 1.4]. To this end, we first recall the definition of Muckenhoupt weights $A_p(\mathbb{R}^n)$ (see, for instance, [47, Definitions 7.1.2 and 7.1.3]).

Definition 3.12. An $A_p(\mathbb{R}^n)$-*weight* ω, with $p \in [1,\infty)$, is a nonnegative locally integrable function on $\mathbb{R}^n$ satisfying that, when $p \in (1,\infty)$,

$$[\omega]_{A_p(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \left[\frac{1}{|Q|}\int_Q \omega(x)\,dx\right]\left\{\frac{1}{|Q|}\int_Q \omega(x)^{\frac{1}{1-p}}\,dx\right\}^{p-1} < \infty,$$

and

$$[\omega]_{A_1(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q \omega(x)\,dx \left[\|\omega^{-1}\|_{L^\infty(Q)}\right] < \infty,$$

where the suprema are taken over all cubes $Q \subset \mathbb{R}^n$.

Moreover, we set

$$A_\infty(\mathbb{R}^n) := \bigcup_{p\in[1,\infty)} A_p(\mathbb{R}^n).$$

We also recall the definition of weighted Lebesgue spaces.

Definition 3.13. Let $p \in [0, \infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. The *weighted Lebesgue space* $L^p_\omega(\mathbb{R}^n)$ is defined to be the space of all the measurable functions f on $\mathbb{R}^n$ with the property

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx\right]^{\frac{1}{p}} < \infty.$$

For any measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ it is customary in this area to use the notation

$$f(E) := \int_E f(x)\,dx.$$

We recall a few facts about Muckenhoupt weights $A_p(\mathbb{R}^n)$. The following lemma is a part of [47, Proposition 7.1.5].

Lemma 3.14. *Let $p \in [1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Then the following statements are valid.*

(i)

$$[\omega]_{A_p(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} \sup_{\substack{f\mathbf{1}_Q \in L^p_\omega(\mathbb{R}^n) \\ \int_Q |f(t)|^p \omega(t)\,dt \in (0,\infty)}} \frac{\left[\frac{1}{|Q|}\int_Q |f(t)|\,dt\right]^p}{\frac{1}{\omega(Q)}\int_Q |f(t)|^p \omega(t)\,dt},$$

where the first supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

(ii) *For any $\lambda \in (1, \infty)$ and any cube $Q \subset \mathbb{R}^n$, one has $\omega(\lambda Q) \le [\omega]_{A_p(\mathbb{R}^n)} \lambda^{np} \omega(Q)$.*

(iii) *For any $q \in [p, \infty)$, one has $\omega \in A_q(\mathbb{R}^n)$ and $[\omega]_{A_q(\mathbb{R}^n)} \le [\omega]_{A_p(\mathbb{R}^n)}$.*

The following lemma shows that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on weighted Lebesgue spaces.

Lemma 3.15. *For any given $p \in [1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$, the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on $L^p_\omega(\mathbb{R}^n)$; moreover, there exists a positive constant $C_{(n,p)}$, depending only on both n and p, such that for any $r \in (0, \infty)$ and $f \in L^p_\omega(\mathbb{R}^n)$ we have*

$$\|B_r(f)\|_{L^p_\omega(\mathbb{R}^n)} \le C_{(n,p)} [\omega]^{2/p}_{A_p(\mathbb{R}^n)} \|f\|_{L^p_\omega(\mathbb{R}^n)}.$$

Proof. Since $\omega \in A_p(\mathbb{R}^n)$, from Lemma 3.14(i) and Tonelli's theorem, it follows that for any $r \in (0, \infty)$ and $f \in L^p_\omega(\mathbb{R}^n)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |B_r(f)(x)|^p \omega(x)\,dx \\
&\quad \le \int_{\mathbb{R}^n} \left[⨍_{B(x,r)} |f(y)|\,dy\right]^p \omega(x)\,dx \\
&\quad \le [\omega]_{A_p(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{1}{\omega(B(x,r))} \left[\int_{B(x,r)} |f(y)|^p \omega(y)\,dy\right] \omega(x)\,dx \\
&\quad = [\omega]_{A_p(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left[\int_{B(y,r)} \frac{\omega(x)}{\omega(B(x,r))}\,dx\right] |f(y)|^p \omega(y)\,dy. \qquad (3.10)
\end{aligned}$$

Observe that, for any $r \in (0,\infty)$, $y \in \mathbb{R}^n$, and $x \in B(y,r)$ we have $B(y,r) \subset B(x,2r)$ which, combined with Lemma 3.14(ii), gives

$$\omega(B(y,r)) \le \omega(B(x,2r)) \lesssim [\omega]_{A_p(\mathbb{R}^n)}\omega(B(x,r)).$$

Using this fact and (3.10) we conclude that for any $r \in (0,\infty)$ and $f \in L^p_\omega(\mathbb{R}^n)$ we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |B_r(f)(x)|^p \omega(x)\,dx \\ &\quad \lesssim [\omega]^2_{A_p(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left[\int_{B(y,r)} \frac{\omega(x)}{\omega(B(y,r))}\,dx\right] |f(y)|^p \omega(y)\,dy \\ &\quad \sim [\omega]^2_{A_p(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(y)|^p \omega(y)\,dy.\end{aligned}$$

This implies that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on $L^p_\omega(\mathbb{R}^n)$. This completes the proof of Lemma 3.15. □

The next lemma can be found in [33, Lemma 3.6] and its proof is modeled after [31, p. 18].

Lemma 3.16. *Let X be a ball Banach function space. Assume that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded from X to itself with norm $\|\mathcal{M}\|_{X\to X}$. For any $g \in X$ and $x \in \mathbb{R}^n$ let*

$$R_X g(x) := \sum_{k=0}^{\infty} \frac{\mathcal{M}^k(g)(x)}{2^k \|\mathcal{M}\|^k_{X\to X}},$$

where for any $k \in \mathbb{N}$, $\mathcal{M}^k := \mathcal{M} \circ \cdots \circ \mathcal{M}$ is the k-fold iteration of the Hardy–Littlewood maximal operator and $\mathcal{M}^0(g)(x) := |g(x)|$. Then, for any $g \in X$ and $x \in \mathbb{R}^n$ one has

(i) $|g(x)| \le R_X g(x)$;

(ii) $R_X g \in A_1(\mathbb{R}^n)$ *and* $[R_X g]_{A_1(\mathbb{R}^n)} \le 2\|\mathcal{M}\|_{X\to X}$;

(iii) $\|R_X g\|_X \le 2\|g\|_X$.

To prove Lemma 3.11, we also need two key lemmas about ball Banach function spaces, namely, Lemmas 3.17 and 3.18. The following lemma is just [107, Lemma 2.6].

Lemma 3.17. *Let X be a ball Banach function space. Then X coincides with its second associate space X''. In other words, $f \in X$ if and only if $f \in X''$ and in that case,*

$$\|f\|_X = \|f\|_{X''}.$$

The following version of Hölder's inequality is a direct consequence of both Definition 2.1(i) and (3.1) (see also [10, Theorem 2.4]).

Lemma 3.18. *Let X be a ball Banach function space and X' its associate space. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \|f\|_X \|g\|_{X'}.$$

Proof of Lemma 3.11. Using the definition of $X^{1/p}$, Lemma 3.17 (with X replaced by $X^{1/p}$), and Lemma 3.16(i) [with X replaced by $(X^{1/p})'$], it follows that, for any $r > 0$ and $f \in X$ we have

$$\begin{aligned}\|B_r(f)\|_X^p &= \left\| |B_r(f)|^p \right\|_{X^{1/p}} = \left\| |B_r(f)|^p \right\|_{(X^{1/p})''} \\ &= \sup_{\|g\|_{(X^{1/p})'}=1} \int_{\mathbb{R}^n} |B_r(f)(x)|^p g(x)\,dx \\ &\leq \sup_{\|g\|_{(X^{1/p})'}=1} \int_{\mathbb{R}^n} |B_r(f)(x)|^p R_{(X^{1/p})'} g(x)\,dx.\end{aligned}$$

By this, both (i) and (iii) of Lemma 3.14 [with ω replaced by $R_{(X^{1/p})'}g(x)$], both (ii) and (iii) of Lemma 3.16 [with X replaced by $(X^{1/p})'$], both Lemmas 3.15 and 3.18, and the definition of $X^{1/p}$, we obtain

$$\begin{aligned}\|B_r(f)\|_X^p &\lesssim \sup_{\|g\|_{(X^{1/p})'}=1} [R_{(X^{1/p})'}g]_{A_1(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} |f(x)|^p R_{(X^{1/p})'} g(x)\,dx \\ &\lesssim \|\mathcal{M}\|_{(X^{1/p})' \to (X^{1/p})'}^2 \sup_{\|g\|_{(X^{1/p})'}=1} \left\| |f|^p \right\|_{X^{1/p}} \|R_{(X^{1/p})'}g\|_{(X^{1/p})'} \\ &\lesssim \|f\|_X^p\end{aligned}$$

for any $r > 0$ and $f \in X$. This implies that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X and completes the proof of Lemma 3.11. □

As a consequence of Corollaries 3.9, 3.10 and Lemma 3.11 we obtain the following corollary.

Corollary 3.19. *Let X be a ball Banach function space and $p \in [1,\infty)$. Assume that X has an absolutely continuous norm, $X^{1/p}$ is a ball Banach function space, and the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in both X and $W^{1,X}(\mathbb{R}^n)$.*

Moreover, at the end of this subsection, we show that $C_c^\infty(\mathbb{R}^n)$ is dense in the weighted Sobolev space $W_\omega^{1,p}(\mathbb{R}^n)$ (see Corollary 3.21 below); this plays a vital role in the next subsection. The next definition of the weighted Sobolev space can be found, for instance, in [68, Definition 3.5].

Definition 3.20. Let $p \in [1,\infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. The *weighted Sobolev space* $W_\omega^{1,p}(\mathbb{R}^n)$ is defined as the set of all the measurable functions f on $\mathbb{R}^n$ whose distributional gradient $\nabla f := (\partial_1 f, \ldots, \partial_n f)$ is an $L_\omega^p(\mathbb{R}^n)$ function and

$$\|f\|_{W_\omega^{1,p}(\mathbb{R}^n)} := \|f\|_{L_\omega^p(\mathbb{R}^n)} + \|\,|\nabla f|\,\|_{L_\omega^p(\mathbb{R}^n)} < \infty.$$

Let $\omega \in A_1(\mathbb{R}^n)$. In the sequel we denote by the $L_{\omega^{-1}}^\infty(\mathbb{R}^n)$ the set of all the measurable functions f on $\mathbb{R}^n$ with

$$\|f\|_{L_{\omega^{-1}}^\infty(\mathbb{R}^n)} := \left\| f\omega^{-1} \right\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

As a consequence of Propositions 3.6 and 3.8, we obtain the following corollary.

Corollary 3.21. *Let $p \in [1,\infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $W_\omega^{1,p}(\mathbb{R}^n)$.*

Proof. The fact that $L^p_\omega(\mathbb{R}^n)$ is a BBF space is contained in [93, Section 7.1]. In view of this and Lemma 3.15, we conclude that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on $L^p_\omega(\mathbb{R}^n)$; Moreover, by [89, Theorem 3.14], we find that $L^p_\omega(\mathbb{R}^n)$ has an absolutely continuous norm. Thus, all the assumptions of Corollary 3.9 are satisfied and its conclusion is that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}_\omega(\mathbb{R}^n)$. Corollary 3.21 is now proved. □

Remark 3.22. When $\omega = 1$ Corollary 3.21 is contained in [2, Theorem 7.38].

3.2 A key estimate in $W^{1,p}_\omega(\mathbb{R}^n)$

The subsequent lemma provides a key estimate in $W^{s,p}_\omega(\mathbb{R}^n)$.

Lemma 3.23. *Let $p \in [1,\infty)$ and $\rho \in L^1_{\mathrm{loc}}(0,\infty)$ be a nonnegative and decreasing function with*

$$\int_0^\infty \rho(r) r^{n-1}\, dr < \infty. \tag{3.11}$$

Assume that $\omega \in A_1(\mathbb{R}^n)$. Then there exists a positive constant $C_{(n,p)}$, depending only on both n and p, such that for any $f \in W^{1,p}_\omega(\mathbb{R}^n)$ we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\,dy\right]\omega(x)\,dx\\&\quad\le C_{(n,p)}[\omega]^2_{A_1(\mathbb{R}^n)}\,\|\,|\nabla f|\,\|^p_{L^p_\omega(\mathbb{R}^n)}\int_0^\infty \rho(r)r^{n-1}\,dr.\end{aligned} \tag{3.12}$$

Remark 3.24. For simplicity we only considered $A_1(\mathbb{R}^n)$-weights here. However, our proof actually works for general $A_p(\mathbb{R}^n)$-weights. For instance, a slight modification of the proofs in this section shows that (3.12) holds for any given $\omega \in A_p(\mathbb{R}^n)$ when $p \in [1, n/(n-1))$.

We first prove Lemma 3.23 for any $f \in C^1_c(\mathbb{R}^n)$ (Lemma 3.30). To accomplish this we need two technical results, both Lemmas 3.26 and 3.29, which depend on the following variant of the Poincaré inequality. In what follows, for any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, the $C^1(B(x,r))$ denotes the set of all first-order continuously differentiable functions on $B(x,r)$.

Lemma 3.25. *([33, Lemma 2.20]) Let $B = B(x,r) \in \mathbb{B}$ be a ball with $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, and $B_1 \in \mathbb{B}$ such that $x \in B_1 \subset B$. Then there exists a positive constant $C_{(n)}$, depending only on n, such that for any $f \in C^1(B)$ we have*

$$|f(x) - f_{B_1}| \le C_{(n)} r \sum_{j=0}^\infty 2^{-j} \fint_{2^{-j}B} |\nabla f(z)|\, dz.$$

The first technical lemma is as follows.

Lemma 3.26. *Let $p \in [1,\infty)$ and $\rho \in L^1_{\mathrm{loc}}(0,\infty)$ be a nonnegative function satisfying* (3.11). *Assume that $\omega \in A_1(\mathbb{R}^n)$. Then there exists a positive constant $C_{(n,p)}$, depending only on both n and p, such that, for any $f \in C^1_c(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{\left|f(x)-f_{B(y,2|x-y|)}\right|^p}{|x-y|^p}\rho(|x-y|)\,dy\right]\omega(x)\,dx$$

$$\leq C_{(n,p)}[\omega]_{A_1(\mathbb{R}^n)}^2 \| |\nabla f| \|_{L^p_\omega(\mathbb{R}^n)}^p \int_0^\infty \rho(r) r^{n-1}\, dr.$$

Proof. By a change of variables we write

$$\begin{aligned}
\mathrm{I} :&= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \frac{\left|f(x)-f_{B(y,2|x-y|)}\right|^p}{|x-y|^p}\rho(|x-y|)\,dy\right]\omega(x)\,dx\\
&= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \frac{\left|f(x)-f_{B(x+h,2|h|)}\right|^p}{|h|^p}\rho(|h|)\,dh\right]\omega(x)\,dx. \qquad (3.13)
\end{aligned}$$

Observe that, for any $x, h \in \mathbb{R}^n$,

$$B(x+h, 2|h|) \subset B(x, 4|h|).$$

By this, applying Lemma 3.25 [with $B(x+h, 2|h|)$ in place of B and $B(x, 4|h|)$ in place of B_1], the assumption that $\omega \in A_1(\mathbb{R}^n)$, and parts (i), (iii) of Lemma 3.14, for any $x, h \in \mathbb{R}^n$, we find that

$$\begin{aligned}
&\left|f(x)-f_{B(x+h,2|h|)}\right|\\
&\quad\lesssim \sum_{j=0}^\infty 2^{-j}|h| \fint_{B(x,2^{-j+2}|h|)} |\nabla f(z)|\,dz\\
&\quad\lesssim \{[\omega]_{A_1(\mathbb{R}^n)}\}^{\frac{1}{p}} \sum_{j=0}^\infty 2^{-j}|h|\\
&\qquad\times\left[\frac{1}{\omega(B(x,2^{-j+2}|h|))}\int_{B(x,2^{-j+2}|h|)} |\nabla f(z)|^p \omega(z)\,dz\right]^{\frac{1}{p}}.
\end{aligned}$$

From this and Hölder's inequality for series it follows that

$$\begin{aligned}
&\left|f(x)-f_{B(x+h,2|h|)}\right|^p\\
&\quad\lesssim [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^\infty \frac{2^{-j}|h|^p}{\omega(B(x,2^{-j+2}|h|))}\int_{B(x,2^{-j+2}|h|)} |\nabla f(z)|^p \omega(z)\,dz,
\end{aligned}$$

which, combined with (3.13), yields that

$$\begin{aligned}
\mathrm{I} &\lesssim [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^\infty 2^{-j} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{\omega(B(x,2^{-j+2}|h|))}\\
&\quad\times\left[\int_{B(x,2^{-j+2}|h|)} |\nabla f(z)|^p\omega(z)\,dz\right]\rho(|h|)\omega(x)\,dx\,dh\\
&\sim [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^\infty 2^{-j} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left[\int_{B(z,2^{-j+2}|h|)} \frac{\omega(x)}{\omega(B(x,2^{-j+2}|h|))}\,dx\right]\\
&\quad\times |\nabla f(z)|^p \omega(z)\rho(|h|)\,dz\,dh. \qquad (3.14)
\end{aligned}$$

Observe that for any $z, h \in \mathbb{R}^n$, $j \in \mathbb{Z}_+$, and $x \in B(z, 2^{-j+2}|h|)$ we have

$$B(z, 2^{-j+2}|h|) \subset B(x, 2^{-j+3}|h|).$$

Using this inclusion, the assumption that $\omega \in A_1(\mathbb{R}^n)$, and Lemma 3.14(ii) we conclude that

$$\omega(B(z, 2^{-j+2}|h|)) \le \omega(B(x, 2^{-j+3}|h|)) \lesssim [\omega]_{A_1(\mathbb{R}^n)}\omega(B(x, 2^{-j+2}|h|)),$$

which implies that

$$\begin{aligned}&\int_{B(z,2^{-j+2}|h|)} \frac{\omega(x)}{\omega(B(x, 2^{-j+2}|h|))}\,dx\\ &\quad\lesssim [\omega]_{A_1(\mathbb{R}^n)} \int_{B(z,2^{-j+2}|h|)} \frac{\omega(x)}{\omega(B(z, 2^{-j+2}|h|))}\,dx \sim [\omega]_{A_1(\mathbb{R}^n)}.\end{aligned}$$

Combining this estimate with (3.14) we finally conclude that

$$\begin{aligned}\mathrm{I} &\lesssim [\omega]^2_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z)\,dz\right]\rho(|h|)\,dh\\ &\sim [\omega]^2_{A_1(\mathbb{R}^n)} \| |\nabla f| \|^p_{L^p_\omega(\mathbb{R}^n)} \int_0^\infty \rho(r) r^{n-1}\,dr.\end{aligned}$$

This finishes the proof of Lemma 3.26. □

To prove the second technical lemma (Lemma 3.29 below), we make full use of a basic ingredient concerning 3^n adjacent systems of dyadic cubes, stated below.

Lemma 3.27. *([69, Section 2.2]) For any $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, let*

$$\mathcal{D}^\alpha := \left\{Q^\alpha_{j,k} := 2^{-j}(k + [0, 1)^n + (-1)^j\alpha) : \ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\right\}.$$

Then

(i) *for any $Q, Q' \in \mathcal{D}^\alpha$ with $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, $Q \bigcap Q' \in \{\emptyset, Q, Q'\}$;*

(ii) *for any ball $B \subset \mathbb{R}^n$, there exists an $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ and a $Q \in \mathcal{D}^\alpha$ such that $B \subset Q \subset c_{(n)}B$, where the positive constant $c_{(n)}$ depends only on n.*

Fix an $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$. For any $j \in \mathbb{Z}$, let $\mathcal{D}^\alpha_j := \{Q^\alpha_{j,k} : \ k \in \mathbb{Z}^n\}$ and fix a $j \in \mathbb{Z}$. For any $Q^\alpha_{j,k} \in \mathcal{D}^\alpha_j$ with $k \in \mathbb{Z}^n$ let

$$\mathfrak{N}(Q^\alpha_{j,k}) := \left\{Q^\alpha_{j,k+l} \in \mathcal{D}^\alpha_j : \ l := (l_1, \ldots, l_n) \in \mathbb{Z}^n,\ \max_{i\in\{1,\ldots,n\}} |l_i| \le 1\right\}$$

and

$$N_{Q^\alpha_{j,k}} := \bigcup_{Q^\alpha_{j,l}\in\mathfrak{N}(Q^\alpha_{j,k})} Q^\alpha_{j,l}. \tag{3.15}$$

Then by the definitions of both $\mathfrak{N}(Q^\alpha_{j,k})$ and $N_{Q^\alpha_{j,k}}$, we have

$$\sharp\,\mathfrak{N}(Q^\alpha_{j,k}) \le 3^n$$

and

$$|N_{Q^\alpha_{j,k}}| \sim |Q^\alpha_{j,k}|.$$

Here and thereafter, for a finite set E, the *symbol* $\sharp E$ denotes its cardinality.

Lemma 3.27 is crucial in the next result which plays a key role in the proof of Lemma 3.29.

Lemma 3.28. *Let $p \in [1,\infty)$, $x, y \in \mathbb{R}^n$ with $x \neq y$, and for $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ let $Q_{x,y} \in \mathcal{D}^\alpha$ be a dyadic cube as in Lemma 3.27(ii) with B replaced by $B(y, 2|x-y|)$, namely,*

$$B(y, 2|x-y|) \subset Q_{x,y} \subset c_{(n)} B(y, 2|x-y|). \tag{3.16}$$

Then, for any given $\varepsilon \in (0, p)$, there exists a positive constant $C_{(n,p,\varepsilon)}$, depending only on n, p, and ε, such that for any $f \in C^1_{\mathrm{c}}(\mathbb{R}^n)$ we have

$$\frac{\left|f(y) - f_{B(y,2|x-y|)}\right|^p}{|x-y|^p} \le C_{(n,p,\varepsilon)} \sum_{\substack{Q \in \mathcal{D}^\alpha \\ y \in Q \subset Q_{x,y}}} \left[\frac{\ell(Q)}{\ell(Q_{x,y})}\right]^{p-\varepsilon} \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p.$$

Proof. Without loss of generality, we may assume that $Q_{x,y} \in \mathcal{D}^\alpha_{j_0}$ for some $j_0 \in \mathbb{Z}$. First by (3.16) we conclude that

$$2^{-j_0} \lesssim 2|x-y| \le 2^{-j_0-1}. \tag{3.17}$$

From this and Lemma 3.25 (with $B(y, 2|x-y|)$ in place of B and $B(y, 2^{-j_0-1})$ in place of B_1) we deduce that

$$\left|f(y) - f_{B(y,2|x-y|)}\right| \lesssim \sum_{j=j_0}^\infty 2^{-j} \fint_{B(y,2^{-j-1})} |\nabla f(z)|\, dz.$$

Let $\varepsilon \in (0, p)$. Combining this estimate with (3.17) and Hölder's inequality for series yields

$$\begin{aligned}
\frac{\left|f(y) - f_{B(y,2|x-y|)}\right|^p}{|x-y|^p} &\sim \frac{|f(y) - f_{B(y,2|x-y|)}|^p}{2^{-j_0 p}} \\
&\sim \left[\sum_{j=j_0}^\infty 2^{-(j-j_0)} \fint_{B(y,2^{-j-1})} |\nabla f(z)|\, dz\right]^p \\
&\lesssim \sum_{j=j_0}^\infty 2^{-(j-j_0)(p-\varepsilon)} \left[\fint_{B(y,2^{-j-1})} |\nabla f(z)|\, dz\right]^p. \qquad (3.18)
\end{aligned}$$

By Lemma 3.27(i), for any $j \ge j_0$, there exists a unique dyadic cube $Q^\alpha_j(y)$ in $\mathcal{D}^\alpha_j$ containing y. From this and Lemma 3.27(i), it follows that $Q^\alpha_j(y) \subset Q_{x,y}$. Moreover, by the definition of $N_{Q^\alpha_j(y)}$ as in (3.15), we conclude that $B(y, 2^{-j-1}) \subset N_{Q^\alpha_j(y)}$. From this and (3.18), we infer

$$\begin{aligned}
\frac{\left|f(y) - f_{B(y,2|x-y|)}\right|^p}{|x-y|^p} &\lesssim \sum_{j=j_0}^\infty 2^{-(j-j_0)(p-\varepsilon)} \left[\fint_{N_{Q^\alpha_j(y)}} |\nabla f(z)|\, dz\right]^p \\
&\sim \sum_{j=j_0}^\infty \sum_{\substack{Q \in \mathcal{D}^\alpha_j \\ y \in Q}} 2^{-(j-j_0)(p-\varepsilon)} \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p \\
&\sim \sum_{\substack{Q \in \mathcal{D}^\alpha \\ y \in Q \subset Q_{x,y}}} \left[\frac{\ell(Q)}{\ell(Q_{x,y})}\right]^{p-\varepsilon} \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p.
\end{aligned}$$

This implies that (3.16) holds and hence finishes the proof of Lemma 3.28. □

The second technical lemma of this section is as follows:

Lemma 3.29. *Let $p \in [1,\infty)$ and $\rho \in L^1_{\mathrm{loc}}(0,\infty)$ be a nonnegative and decreasing function satisfying* (3.11)*. Assume that $\omega \in A_1(\mathbb{R}^n)$. Then there exists a positive constant $C_{(n,p)}$, depending only on both n and p, such that for any $f \in C^1_{\mathrm{c}}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{\left|f(y)-f_{B(y,2|x-y|)}\right|^p}{|x-y|^p}\rho(|x-y|)\,dy\right]\omega(x)\,dx$$
$$\leq C_{(n,p)}[\omega]^2_{A_1(\mathbb{R}^n)}\left\|\,|\nabla f|\,\right\|^p_{L^p_\omega(\mathbb{R}^n)}\int_0^\infty \rho(r)r^{n-1}\,dr. \tag{3.19}$$

Proof. Let $c_{(n)}$ be as in Lemma 3.27. We claim that for any given $\varepsilon \in (0,p)$ we have

$$\mathrm{J} := \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{\left|f(y)-f_{B(y,2|x-y|)}\right|^p}{|x-y|^p}\rho(|x-y|)\,dy\right]\omega(x)\,dx$$
$$\lesssim \sum_{\alpha\in\{0,\frac13,\frac23\}^n}\sum_{P\in\mathcal{D}^\alpha}\sum_{\substack{Q\in\mathcal{D}^\alpha\\ Q\subset P}}\left[\frac{\ell(Q)}{\ell(P)}\right]^{p-\varepsilon}\left[⨍_{N_Q}|\nabla f(z)|\,dz\right]^p$$
$$\times\int_P\left[\int_{Q\cap[B(x,C_{(n)}\ell(P))]^\complement}\rho(|x-y|)\,dy\right]\omega(x)\,dx, \tag{3.20}$$

where

$$C_{(n)} := [4c_{(n)}]^{-1}\sqrt{n}.$$

Indeed, from Lemma 3.27(ii), it follows that, for any $x, y \in \mathbb{R}^n$ with $x \neq y$, there exists a dyadic cube $Q_{x,y} \in \mathcal{D}^\alpha$ with $\alpha \in \{0, \frac13, \frac23\}^n$ such that $B(y, 2|x-y|) \subset Q_{x,y} \subset c_{(n)}B(y, 2|x-y|)$. Using this, for any $x, y \in \mathbb{R}^n$ with $x \neq y$, we obtain

$$C_{(n)}\ell(Q_{x,y}) = [4c_{(n)}]^{-1}\sqrt{n}\,\ell(Q_{x,y}) \leq |x-y| \leq 4^{-1}\ell(Q_{x,y}). \tag{3.21}$$

Fixing an $\varepsilon \in (0,p)$, by Lemmas 3.28 and 3.27(i), and (3.21), we conclude that, for any $x, y \in \mathbb{R}^n$ with $x \neq y$ we have

$$\frac{\left|f(y)-f_{B(y,2|x-y|)}\right|^p}{|x-y|^p}$$
$$\lesssim \sum_{\substack{Q\in\mathcal{D}^\alpha\\ y\in Q\subset Q_{x,y}}}\left[\frac{\ell(Q)}{\ell(Q_{x,y})}\right]^{p-\varepsilon}\left[⨍_{N_Q}|\nabla f(z)|\,dz\right]^p$$
$$\sim \sum_{\substack{Q\in\mathcal{D}^\alpha\\ Q\subset Q_{x,y}}}\left[\frac{\ell(Q)}{\ell(Q_{x,y})}\right]^{p-\varepsilon}\left[⨍_{N_Q}|\nabla f(z)|\,dz\right]^p\mathbf{1}_Q(y)$$
$$\lesssim \sum_{\alpha\in\{0,\frac13,\frac23\}^n}\sum_{P\in\mathcal{D}^\alpha}\sum_{\substack{Q\in\mathcal{D}^\alpha\\ Q\subset P}}\mathbf{1}_{\{(x,y)\in P\times Q:\ P\subset B(y,2c_{(n)}|x-y|)\}}(x,y)$$

$$
\begin{aligned}
&\times \left[\frac{\ell(Q)}{\ell(P)}\right]^{p-\varepsilon} \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p \\
\sim &\sum_{\alpha\in\{0,\frac13,\frac23\}^n} \sum_{P\in\mathcal{D}^\alpha} \sum_{\substack{Q\in\mathcal{D}^\alpha\\ Q\subset P}} \mathbf{1}_P(x)\mathbf{1}_{Q\cap[B(x,C_{(n)}\ell(P))]^\complement}(y) \\
&\times \left[\frac{\ell(Q)}{\ell(P)}\right]^{p-\varepsilon} \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p,
\end{aligned}
$$

which establishes the validity of (3.20).

Using (3.20) and the assumption that ρ is decreasing on $(0,\infty)$ we find that

$$
\begin{aligned}
\mathrm{J} \lesssim &\sum_{\alpha\in\{0,\frac13,\frac23\}^n} \sum_{P\in\mathcal{D}^\alpha} \omega(P)\rho(C_{(n)}\ell(P)) \\
&\times \sum_{\substack{Q\in\mathcal{D}^\alpha\\ Q\subset P}} \left[\frac{\ell(Q)}{\ell(P)}\right]^{p-\varepsilon} |Q| \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p
\end{aligned}
\tag{3.22}
$$

for any $\varepsilon\in(0,p)$. Now fix an $\alpha\in\{0,\frac13,\frac23\}^n$. Since $P\in\mathcal{D}^\alpha$, we may assume that $P\in\mathcal{D}^\alpha_{j_0}$ for some $j_0\in\mathbb{Z}$. From this, the properties of dyadic cubes, the assumption that $\omega\in A_1(\mathbb{R}^n)$, and both (i) and (iii) of Lemma 3.14, it follows that

$$
\begin{aligned}
&\sum_{\substack{Q\in\mathcal{D}^\alpha\\ Q\subset P}} \left[\frac{\ell(Q)}{\ell(P)}\right]^{p-\varepsilon} |Q| \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p \\
&\quad= \sum_{j=j_0}^\infty 2^{-(j-j_0)(p-\varepsilon)} \sum_{\substack{Q\in\mathcal{D}^\alpha_j\\ Q\subset P}} |Q| \left[\fint_{N_Q} |\nabla f(z)|\, dz\right]^p \\
&\quad\le [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=j_0}^\infty 2^{-(j-j_0)(p-\varepsilon)} \sum_{\substack{Q\in\mathcal{D}^\alpha_j\\ Q\subset P}} \frac{|Q|}{\omega(N_Q)} \int_{\mathbb{R}^n} \mathbf{1}_{N_Q}(z)|\nabla f(z)|^p\omega(z)\, dz \\
&\quad= [\omega]_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} \sum_{j=j_0}^\infty 2^{-(j-j_0)(p-\varepsilon)} \\
&\qquad\times \sum_{\substack{Q\in\mathcal{D}^\alpha_j\\ Q\subset P}} \frac{|Q|}{\omega(N_Q)} \mathbf{1}_{N_Q}(z)|\nabla f(z)|^p\omega(z)\, dz.
\end{aligned}
\tag{3.23}
$$

Moreover, by Lemma 3.14(ii), we have, for any given $P\in\mathcal{D}^\alpha$ and for any $Q\subset P$,

$$
\frac{|Q|}{\omega(Q)} \le [\omega]_{A_1(\mathbb{R}^n)} \frac{|P|}{\omega(P)}.
$$

Using this and the definition of N_Q as in (3.15), for any given $\varepsilon\in(0,p)$, any $\alpha\in\{0,\frac13,\frac23\}^n$, $P\in\mathcal{D}^\alpha$, and $z \in \mathbb{R}^n$ we conclude that

$$\begin{aligned}
&\sum_{j=j_0}^{\infty} 2^{-(j-j_0)(p-\varepsilon)} \sum_{\substack{Q\in\mathcal{D}_j^\alpha \\ Q\subset P}} \frac{|Q|}{\omega(Q)} \mathbf{1}_{N_Q}(z) \\
&\qquad \leq [\omega]_{A_1(\mathbb{R}^n)} \frac{|P|}{\omega(P)} \sum_{j=j_0}^{\infty} 2^{-(j-j_0)(p-\varepsilon)} \sum_{\substack{Q\in\mathcal{D}_j^\alpha \\ Q\subset P}} \mathbf{1}_{N_Q}(z) \\
&\qquad \lesssim [\omega]_{A_1(\mathbb{R}^n)} \frac{|P|}{\omega(P)} \mathbf{1}_{N_Q}(z).
\end{aligned}$$

Combining this estimate with (3.23), implies that

$$\begin{aligned}
&\sum_{\substack{Q\in\mathcal{D}^\alpha \\ Q\subset P}} \left[\frac{\ell(Q)}{\ell(P)}\right]^{p-\epsilon} |Q| \left[\fint_{N_Q} |\nabla f(z)|\, dz \right]^p \\
&\qquad \lesssim [\omega]_{A_1(\mathbb{R}^n)}^2 \frac{|P|}{\omega(P)} \int_{\mathbb{R}^n} \mathbf{1}_{N_P}(z) |\nabla f(z)|^p \omega(z)\, dz.
\end{aligned}$$

By this, (3.22), and the definition of N_Q as in (3.15), we obtain

$$\begin{aligned}
\mathrm{J} &\lesssim [\omega]_{A_1(\mathbb{R}^n)}^2 \sum_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} \sum_{P\in\mathcal{D}^\alpha} \rho(C_{(n)}\ell(P))|P| \int_{\mathbb{R}^n} \mathbf{1}_{N_P}(z) |\nabla f(z)|^p \omega(z)\, dz \\
&\sim [\omega]_{A_1(\mathbb{R}^n)}^2 \sum_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} \int_{\mathbb{R}^n} \sum_{P\in\mathcal{D}^\alpha} \rho(C_{(n)}\ell(P))|P| \mathbf{1}_{N_P}(z) |\nabla f(z)|^p \omega(z)\, dz \\
&\sim [\omega]_{A_1(\mathbb{R}^n)}^2 \sum_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} \int_{\mathbb{R}^n} \sum_{P\in\mathcal{D}^\alpha} \rho(C_{(n)}\ell(P))|P| \mathbf{1}_{Q}(z) |\nabla f(z)|^p \omega(z)\, dz. \qquad (3.24)
\end{aligned}$$

Moreover, using Lemma 3.27(i) and the assumption that ρ is decreasing on $(0, \infty)$, for any $z \in \mathbb{R}^n$, we conclude that

$$\begin{aligned}
\sum_{Q\in\mathcal{D}^\alpha} \rho(C_{(n)}\ell(Q))|Q|\mathbf{1}_Q(z) &= \sum_{j\in\mathbb{Z}} \rho(C_{(n)}2^{-j})2^{-jn} \\
&\lesssim \sum_{j\in\mathbb{Z}} \int_{B(\mathbf{0},C_{(n)}2^{-j+1})\setminus B(\mathbf{0},C_{(n)}2^{-j})} \rho(|y|)\, dy \\
&\sim \int_0^\infty \rho(r) r^{n-1}\, dr.
\end{aligned}$$

From this and (3.24) we finally deduce (3.19) and conclude the proof of Lemma 3.29. □

Combining Lemmas 3.26 and 3.29, we now derive the following conclusion.

Lemma 3.30. *Let $p \in [1, \infty)$ and $\rho \in L^1_{\mathrm{loc}}(0, \infty)$ be a nonnegative and decreasing function satisfying* (3.11)*. Assume that $\omega \in A_1(\mathbb{R}^n)$. Then there exists a positive constant $C_{(n,p)}$, depending only on both n and p, such that* (3.12) *is valid for any $f \in C^1_c(\mathbb{R}^n)$.*

Proof. Let f, ω, and ρ be as stated. Observe that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\omega(x)\,dx\,dy \le 2^p(\mathrm{I}+\mathrm{J}) \sim \mathrm{I}+\mathrm{J},$$

where I and J are as in (3.13) and (3.20), respectively. This estimate combined with Lemmas 3.26 and 3.29 implies that (3.12) holds for any $f \in C_{\mathrm{c}}^1(\mathbb{R}^n)$, and finishes the proof of Lemma 3.30. □

Now, we are in a position to prove Lemma 3.23.

Proof of Lemma 3.23. Following the notation in the statement we set $C_{(\rho)} := \int_0^\infty \rho(r)r^{n-1}\,dr$. It follows from Corollary 3.21 that for any $f \in W_\omega^{1,p}(\mathbb{R}^n)$ there exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset C_{\mathrm{c}}^1(\mathbb{R}^n)$ such that

$$\limsup_{k\to\infty}\|f-f_k\|_{L^p_\omega(\mathbb{R}^n)} = 0 = \limsup_{k\to\infty}\|\,|\nabla f-\nabla f_k|\,\|_{L^p_\omega(\mathbb{R}^n)}. \tag{3.25}$$

Since $\{f_k\}_{k\in\mathbb{N}} \subset C_{\mathrm{c}}^1(\mathbb{R}^n)$, from Lemma 3.30 (with f replaced by f_k), we deduce that, for any $k \in \mathbb{N}$,

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f_k(x)-f_k(y)|^p}{|x-y|^p}\rho(|x-y|)\omega(x)\,dy\,dx\\ &\quad\lesssim C_{(\rho)}[\omega]^2_{A_1(\mathbb{R}^n)}\|\,|\nabla f_k|\,\|^p_{L^p_\omega(\mathbb{R}^n)}.\end{aligned} \tag{3.26}$$

For any $N \in \mathbb{N}$, let

$$\begin{aligned}E_N := \Big\{&(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\ |x-y|\in(N^{-1},\infty),\\ &\omega(x)\in(N^{-1},N),\ \omega(y)\in(N^{-1},N)\Big\}.\end{aligned}$$

By this, Tonelli's theorem, and (3.26) we obtain, for any fixed $N \in \mathbb{N}$ and for any $k \in \mathbb{N}$, that

$$\begin{aligned}&\iint_{E_N}\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\omega(x)\,dy\,dx\\
&\quad\lesssim \iint_{E_N}\frac{|f(x)-f_k(x)|^p}{|x-y|^p}\rho(|x-y|)\omega(x)\,dy\,dx\\
&\qquad+\iint_{E_N}\frac{|f_k(x)-f_k(y)|^p}{|x-y|^p}\rho(|x-y|)\omega(x)\,dy\,dx\\
&\qquad+\iint_{E_N}\frac{|f_k(y)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\frac{\omega(x)}{\omega(y)}\omega(y)\,dx\,dy\\
&\quad\lesssim C_{(\rho)}N^p\int_{\mathbb{R}^n}|f(x)-f_k(x)|^p\omega(x)\,dx + C_{(\rho)}[\omega]^2_{A_1(\mathbb{R}^n)}\|\,|\nabla f_k|\,\|^p_{L^p_\omega(\mathbb{R}^n)}\\
&\qquad+C_{(\rho)}N^{p+2}\int_{\mathbb{R}^n}|f(y)-f_k(y)|^p\omega(y)\,dy\\
&\quad\lesssim C_{(\rho)}\left[N^{p+2}\|f-f_k\|^p_{L^p_\omega(\mathbb{R}^n)}+[\omega]^2_{A_1(\mathbb{R}^n)}\|\,|\nabla f_k|\,\|^p_{L^p_\omega(\mathbb{R}^n)}\right].\end{aligned}$$

Using this and (3.25), we conclude that, for any fixed $N \in \mathbb{N}$ and for any $k \in \mathbb{N}$,

$$\iint_{E_N}\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\omega(x)\,dy\,dx$$

$$
\begin{aligned}
&\lesssim \limsup_{k\to\infty} C_{(\rho)}\left[N^{p+2}\|f-f_k\|_{L^p_\omega(\mathbb{R}^n)}^p+[\omega]_{A_1(\mathbb{R}^n)}^2\|\,|\nabla f_k|\,\|_{L^p_\omega(\mathbb{R}^n)}^p\right]\\
&\sim C_{(\rho)}[\omega]_{A_1(\mathbb{R}^n)}^2\|\,|\nabla f|\,\|_{L^p_\omega(\mathbb{R}^n)}^p .
\end{aligned}
$$

Combining the preceding estimate with Levi's lemma implies

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\,dy\right]\omega(x)\,dx\\
&\quad=\lim_{N\to\infty}\iint_{E_N}\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\omega(x)\,dy\,dx\\
&\quad\lesssim C_{(\rho)}[\omega]_{A_1(\mathbb{R}^n)}^2\|\,|\nabla f|\,\|_{L^p_\omega(\mathbb{R}^n)}^p
\end{aligned}
$$

and hence finishes the proof of Lemma 3.23. □

3.3 Proof of Theorem 3.36. The main result.

In this subsection we prove the main result of this article, Theorem 3.36. This is derived from Theorem 3.4 whose proof is based on three lemmas. We begin with the following lemma which gives another sufficient condition for the locally β-doubling property of X.

Lemma 3.31. *Let X be a ball Banach function space. Assume that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on X'. Then X is locally n-doubling.*

Proof. By Lemma 3.17 and the definition of X'', for any $B\in\mathbb{B}$ and $\alpha\in[1,\infty)$, we have

$$
\|\mathbf{1}_{\alpha B}\|_X=\|\mathbf{1}_{\alpha B}\|_{X''}=\sup_{\|g\|_{X'}=1}\int_{\mathbb{R}^n}\mathbf{1}_{\alpha B}(x)|g(x)|\,dx.
$$

From this, Lemma 3.16 (with X replaced by X'), Lemma 3.14(ii) (with ω replaced by $R_{X'}g$), and Lemma 3.18, we deduce that

$$
\begin{aligned}
\|\mathbf{1}_{\alpha B}\|_X&\le\sup_{\|g\|_{X'}=1}\int_{\mathbb{R}^n}\mathbf{1}_{\alpha B}(x)R_{X'}g(x)\,dx\\
&\le\alpha^n\sup_{\|g\|_{X'}=1}[R_{X'}g]_{A_1(\mathbb{R}^n)}\int_{\mathbb{R}^n}\mathbf{1}_B(x)R_{X'}g(x)\,dx\\
&\lesssim\alpha^n\|\mathcal{M}\|_{X'\to X'}\|\mathbf{1}_B\|_X\sup_{\|g\|_{X'}=1}\|R_{X'}g\|_{X'}\sim\alpha^n\|\mathbf{1}_B\|_X,
\end{aligned}
$$

which implies that X is locally n-doubling. This completes the proof of Lemma 3.31. □

Remark 3.32. By Lemma 3.31, we conclude that the assumption that X is locally β-doubling is weaker than the assumption that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on X'.

As a consequence of Theorem 2.13 and Lemma 3.31, we have the following conclusion.

Lemma 3.33. *Let X be a ball Banach function space, $p\in[1,\infty)$, and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a decreasing-radial-*ATI*. Assume that $X^{1/p}$ is a ball Banach function space with the property that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$. Then* (3.2) *is valid for any $f\in C_c^2(\mathbb{R}^n)$.*

Proof. By Lemma 3.31 we conclude that $X^{1/p}$ is locally n-doubling. This, together with Proposition 2.11, implies that X is locally β-doubling with $\beta := n/p \in (0, 1+n/p)$. Thus, all the assumptions of Theorem 2.13 are satisfied. Applying this theorem we deduce the validity of (3.2) for any $f \in C_c^2(\mathbb{R}^n)$. This finishes the proof of Lemma 3.33. □

Based on Lemma 3.33, in order to prove Theorem 3.4, it is sufficient to show the following

Lemma 3.34. *Let X be a ball Banach function space satisfying the same assumptions as in Lemma 3.33, $p \in [1,\infty)$, and $\rho \in L^1_{\rm loc}(0,\infty)$ be a nonnegative and decreasing function satisfying* (3.11). *Then there exists a positive constant $C_{(n,p)}$, depending only on both n and p, such that for any $f \in W^{1,X}(\mathbb{R}^n)$ we have*

$$\left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p \\ \le C_{(n,p)}\|\mathcal{M}\|^2_{(X^{1/p})'\to(X^{1/p})'}\,\|\,|\nabla f|\,\|_X^p \int_0^\infty \rho(r)r^{n-1}\,dr.$$

Proof. Let $C_{(\rho)} := \int_0^\infty \rho(r)r^{n-1}\,dr$. By the definition of $X^{1/p}$, Lemma 3.17 (with X replaced by $X^{1/p}$), and the definition of $(X^{1/p})''$, for any $f \in W^{1,X}(\mathbb{R}^n)$, we have

$$\begin{aligned}&\left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p \\ &\quad= \left\|\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right\|_{X^{1/p}} \\ &\quad= \left\|\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right\|_{(X^{1/p})''} \\ &\quad= \sup_{\|g\|_{(X^{1/p})'}=1}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\epsilon(|x-y|)\,dy\right]|g(x)|\,dx.\end{aligned}$$

From this, Lemma 3.16 [with X replaced by $(X^{1/p})'$], Lemma 3.23 [with ω replaced by $R_{(X^{1/p})'}g$], and Lemma 3.18 (with X replaced by $X^{1/p}$), we deduce that, for any $f \in W^{1,X}(\mathbb{R}^n)$,

$$\begin{aligned}&\left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p \\ &\quad\le \sup_{\|g\|_{(X^{1/p})'}=1}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(|x-y|)\,dy\right]R_{(X^{1/p})'}g(x)\,dx \\ &\quad\lesssim C_{(\rho)}\sup_{\|g\|_{(X^{1/p})'}=1}[R_{(X^{1/p})'}g]^2_{A_1(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\nabla f(x)|^p R_{(X^{1/p})'}g(x)\,dx \\ &\quad\lesssim C_{(\rho)}\|\mathcal{M}\|^2_{(X^{1/p})'\to(X^{1/p})'}\sup_{\|g\|_{(X^{1/p})'}=1}\left\|\,|\nabla f|^p\right\|_{X^{1/p}}\|R_{(X^{1/p})'}g\|_{(X^{1/p})'} \\ &\quad\lesssim C_{(\rho)}\|\mathcal{M}\|^2_{(X^{1/p})'\to(X^{1/p})'}\,\|\,|\nabla f|\,\|_X^p.\end{aligned}$$

This finishes the proof of Lemma 3.34. □

Remark 3.35. In the case where $X = L^p(\mathbb{R}^n)$, the translation invariance of the Lebesgue measure and the explicit expression of the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$ provide straightforward proofs of Lemma 3.34; see, for instance, [13, Proposition 9.3]. Our proof does not rely on these properties.

Proof of Theorem 3.4. In view of Corollary 3.19, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,X}(\mathbb{R}^n)$. This implies that, for any $\zeta \in (0,\infty)$, there exists a $g \in C_c^\infty(\mathbb{R}^n)$ such that

$$\| |\nabla g - \nabla f| \|_X < \zeta.$$

Using this and Lemma 3.34 (with f and ρ replaced, respectively, by $f - g$ and ρ_ϵ), we conclude that, for any given $\zeta \in (0,\infty)$ and for any $\epsilon \in (0,\infty)$,

$$\left\|\left[\int_{\mathbb{R}^n} \frac{|(f-g)(\cdot)-(f-g)(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X \lesssim \| |\nabla g - \nabla f| \|_X \lesssim \zeta,$$

which, combined with the Minkowski inequality, further implies that

$$\begin{aligned}
&\left|\left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X - [K(p,n)]^{1/p}\| |\nabla f| \|_X\right| \\
&\quad \le \left|\left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X \right. \\
&\qquad \left. - \left\|\left[\int_{\mathbb{R}^n} \frac{|g(\cdot)-g(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X\right| \\
&\qquad + \left|\left\|\left[\int_{\mathbb{R}^n} \frac{|g(\cdot)-g(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X - [K(p,n)]^{1/p}\| |\nabla g| \|_X\right| \\
&\qquad + \left|[K(p,n)]^{1/p}\| |\nabla g| \|_X - [K(p,n)]^{1/p}\| |\nabla f| \|_X\right| \\
&\quad \le \left\|\left[\int_{\mathbb{R}^n} \frac{|(f-g)(\cdot)-(f-g)(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X \\
&\qquad + \left|\left\|\left[\int_{\mathbb{R}^n} \frac{|g(\cdot)-g(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X - [K(p,n)]^{1/p}\| |\nabla g| \|_X\right| \\
&\qquad + [K(p,n)]^{1/p}\| |\nabla g - \nabla f| \|_X \\
&\quad \lesssim \zeta + \left|\left\|\left[\int_{\mathbb{R}^n} \frac{|g(\cdot)-g(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X - [K(p,n)]^{1/p}\| |\nabla g| \|_X\right|. \qquad (3.27)
\end{aligned}$$

As $g \in C_c^\infty(\mathbb{R}^n)$, Lemma 3.33 (with f replaced by g) implies that

$$\lim_{\epsilon\to 0^+}\left\|\left[\int_{\mathbb{R}^n} \frac{|g(\cdot)-g(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p = K(p,n)\ \| |\nabla g| \|_X^p.$$

By this and (3.27), for any given $\zeta \in (0,\infty)$, we conclude that

$$\limsup_{\epsilon\to 0^+}\left|\left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^p}\rho_\epsilon(|\cdot - y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p - K(p,n)\ \| |\nabla f| \|_X^p\right| \lesssim \zeta.$$

As $\zeta > 0$ is arbitrary, we obtain the validity of (3.2) and then complete the proof of Theorem 3.4. □

As an application of Theorem 3.4, we now prove the main result of this article (Theorem 3.36), namely, the generalization of the classical BBM formula (1.2) for ball Banach Sobolev spaces.

Theorem 3.36. *Let X be a ball Banach function space and $p \in [1, \infty)$. Assume that X has an absolutely continuous norm, $X^{1/p}$ is a ball Banach function space, and the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$. Then* (2.15) *is valid for any $f \in W^{1,X}(\mathbb{R}^n)$.*

Before proving Theorem 3.36, we show the following key lemma.

Lemma 3.37. *Let X be a ball Banach function space and $p \in [1, \infty)$. Assume that $X^{1/p}$ is a ball Banach function space and the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on $X^{1/p}$. Then* (2.18) *is valid for any $f \in X$.*

Proof. From both the definition of $X^{1/p}$ and the assumption that $X^{1/p}$ is a BBF space, for any $s \in (0, 1)$, it follows that

$$\begin{aligned}
&(1-s)\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\geq(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_X^p\\
&\quad=(1-s)\left\|\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\geq(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right\|_{X^{1/p}}\\
&\quad\leq(1-s)\left\|\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\geq(1-s)^{-1/2}\}}\frac{|f(\cdot)|^p}{|\cdot-y|^{n+sp}}\,dy\right\|_{X^{1/p}}\\
&\qquad+(1-s)\left\|\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\geq(1-s)^{-1/2}\}}\frac{|f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right\|_{X^{1/p}}\\
&\quad=:I_s^{(1)}+I_s^{(2)}.
\end{aligned}\tag{3.28}$$

We first consider $I_s^{(1)}$. Using the definition of $X^{1/p}$ we write

$$I_s^{(1)}\lesssim(1-s)^{\frac{3+sp}{2}}\left\||f|^p\right\|_{X^{1/p}}=(1-s)^{\frac{3+sp}{2}}\|f\|_X^p\to0\tag{3.29}$$

as $s \to 1^-$. As for $I_s^{(2)}$, from the assumptions that $X^{1/p}$ is a BBF space and the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on $X^{1/p}$, and the definition of $X^{1/p}$, it follows that

$$\begin{aligned}
I_s^{(2)}&\lesssim(1-s)\sum_{j=1}^{\infty}\left[2^{-j}(1-s)^{1/2}\right]^{n+sp}\\
&\quad\times\left\|\int_{\{y\in\mathbb{R}^n:\ 2^{j-1}(1-s)^{-1/2}\leq|\cdot-y|<2^j(1-s)^{-1/2}\}}|f(y)|^p\,dy\right\|_{X^{1/p}}\\
&\lesssim(1-s)\sum_{j=1}^{\infty}\left[2^{-j}(1-s)^{1/2}\right]^{n+sp}\left\|\int_{B(\cdot,2^j(1-s)^{-1/2})}|f(y)|^p\,dy\right\|_{X^{1/p}}
\end{aligned}$$

$$
\begin{aligned}
&\sim (1-s)\sum_{j=1}^{\infty}\left[2^{-j}(1-s)^{1/2}\right]^{sp}\left\|B_{2^j(1-s)^{-1/2}}(|f|^p)\right\|_{X^{1/p}}\\
&\lesssim (1-s)^{\frac{2+sp}{2}}\sum_{j=1}^{\infty}2^{-jsp}\left\|\,|f|^p\,\right\|_{X^{1/p}}\\
&\sim (1-s)^{\frac{2+sp}{2}}\|f\|_X^p\sum_{j=1}^{\infty}2^{-jsp}\to 0
\end{aligned}
$$

as $s\to 1^-$. Combining this fact with (3.28) and (3.29) implies that (2.18) holds for any $f\in X$, and this finishes the proof of Lemma 3.37. □

Proof of Theorem 3.36. Let ρ_ϵ be as in (2.22) for any $\epsilon\in(0,1/p)$. By the proof of Theorem 2.15 and the assumptions of the present theorem, we conclude that all the assumptions of Theorem 3.4 are satisfied. Using this and Theorem 3.4, for any $f\in W^{1,X}(\mathbb{R}^n)$, we obtain

$$
\begin{aligned}
&\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|<(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_X^p\\
&\quad=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_X^p\,.
\end{aligned}
\tag{3.30}
$$

Moreover, from Lemma 3.11 with the assumption that $\mathcal{M}$ is bounded on $(X^{1/p})'$, it follows that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on $X^{1/p}$. This fact and Lemma 3.37 imply that

$$
\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\{y\in\mathbb{R}^n:\ |\cdot-y|\geq(1-s)^{-1/2}\}}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_X^p=0.
$$

By this and (3.30) we obtain the validity of (2.15) for any $f\in W^{1,X}(\mathbb{R}^n)$ and hence complete the proof of Theorem 3.36. □

4 New characterizations of ball Banach Sobolev spaces

In this section we apply the BBM formula for ball Banach Sobolev spaces (Theorems 3.4 and 3.36) to obtain new characterizations of $W^{1,X}(\mathbb{R}^n)$. The main such characterizations are the contents of Theorems 4.8 and 4.12 below.

Theorem 4.1. *Let X be a ball Banach function space, $p\in[1,\infty)$, and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a decreasing-radial-*ATI. *Assume that $C_c^\infty(\mathbb{R}^n)$ is dense in X', and X' has an absolutely continuous norm and is locally β-doubling with $\beta\in(0,n+1)$. If $f\in X$ satisfies*

$$
\liminf_{\epsilon\to 0^+}\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X<\infty,
\tag{4.1}
$$

then $f \in W^{1,X}(\mathbb{R}^n)$ *and*

$$\|\,|\nabla f|\,\|_X \lesssim \liminf_{\epsilon\to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|)\,dy \right]^{\frac{1}{p}} \right\|_X^p$$

with the implicit positive constant independent of f.

Remark 4.2. When $X = L^p(\mathbb{R}^n)$, this result can be found in [12, Theorem 2]; the proof there strongly depends on the translation invariance of the $L^p(\mathbb{R}^n)$ norm. But ball Banach function spaces may not be translation invariant, for instance this is the case for weighted Lebesgue spaces and variable Lebesgue spaces (Subsections 5.3 and 5.4, respectively). We overcome this essential obstacle via an elegant use of the Funk–Hecke formula for spherical harmonics (Lemmas 4.3–4.7).

The proof of Theorem 4.1 relies on the five lemmas below.

Lemma 4.3. *Let* X *be a ball Banach function space. Assume that* X *has an absolutely continuous norm. Then, for any bounded linear functional* L *on* X*, there exists a unique* $g \in X'$ *such that* $\|L\|_{X^*} = \|g\|_{X'}$ *and, for any* $f \in X$,

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx,$$

where $\|L\|_{X^*} := \sup\{|L(f)| : f \in X, \|f\|_X \le 1\}$.

The proof of this lemma can be obtained by a repetition of an argument similar to that used in the proof of [10, Corollary 4.3] and is omitted.

The next tool establishes an equivalent characterization of $W^{1,X}(\mathbb{R}^n)$ under certain assumptions.

Lemma 4.4. *Let* X *be a ball Banach function space and* $f \in X$*. Assume that* $C_c^\infty(\mathbb{R}^n)$ *is dense in* X'*, and that* X' *has an absolutely continuous norm. Then* $|\nabla f| \in X$ *if and only if for any* $j \in \{1,\dots,n\}$ *we have*

$$B_j(f) := \sup\left\{ \left| \int_{\mathbb{R}^n} f(x)\partial_j\phi(x)\,dx \right| : \phi \in C_c^\infty(\mathbb{R}^n),\ \|\phi\|_{X'} \le 1 \right\} < \infty; \tag{4.2}$$

moreover,

$$\|\,|\nabla f|\,\|_X \sim \sum_{j=1}^n B_j(f) \tag{4.3}$$

with the positive equivalence constants independent of f.

Proof. We first prove the necessity. Let $f \in W^{1,X}(\mathbb{R}^n)$. For any $j \in \{1,\dots,n\}$, by the definition of $\partial_j f$ and Lemma 3.18, we have, for any $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\|\phi\|_{X'} \le 1$,

$$\begin{aligned}\left| \int_{\mathbb{R}^n} f(x)\partial_j\phi(x)\,dx \right| &= \left| \int_{\mathbb{R}^n} \partial_j f(x)\phi(x)\,dx \right| \\ &\le \|\partial_j f\|_X \|\phi\|_{X'} \le \|f\|_{W^{1,X}(\mathbb{R}^n)} < \infty,\end{aligned}$$

which, combined with the definition of $B_j(f)$, implies that (4.2) holds and

$$\sum_{j=1}^n B_j(f) \le n\, \|\,|\nabla f|\,\|_X .$$

This finishes the proof of the necessity.

It remains to show the sufficiency. To achieve this, assume that $B_j(f) < \infty$ for any $j \in \{1, \dots, n\}$. For any $j \in \{1, \dots, n\}$ let

$$L_j(\phi) := \int_{\mathbb{R}^n} f(x)\partial_j\phi(x)\,dx, \quad \forall\,\phi \in C_{\rm c}^\infty(\mathbb{R}^n). \tag{4.4}$$

By the definition of $B_j(f)$, for any $\phi \in C_{\rm c}^\infty(\mathbb{R}^n)$ satisfying $\|\phi\|_{X'} \le 1$, we obtain

$$|L_j(\phi)| = \left|\int_{\mathbb{R}^n} f(x)\partial_j\phi(x)\,dx\right| \le B_j(f) < \infty, \tag{4.5}$$

which implies that L_j is a bounded linear functional on $(C_{\rm c}^\infty(\mathbb{R}^n), \|\cdot\|_{X'})$. From this, the assumption that $C_{\rm c}^\infty(\mathbb{R}^n)$ is dense in X', [13, Theorem 1.1], and (4.5), we deduce that L_j can be extended to a unique bounded linear operator on $(X', \|\cdot\|_{X'})$, which is still denoted by L_j and which satisfies

$$\|L_j\|_{(X')^*} \le B_j(f). \tag{4.6}$$

By Lemma 4.3 (with X replaced by X'), using the assumption that X' has an absolutely continuous norm, we conclude that, for any $j \in \{1, \dots, n\}$, there exists a unique $g_j \in X''$ such that

$$\|g_j\|_{X''} = \|L_j\|_{(X')^*} \tag{4.7}$$

and, for any $\phi \in C_{\rm c}^\infty(\mathbb{R}^n)$,

$$L_j(\phi) = \int_{\mathbb{R}^n} g_j(x)\phi(x)\,dx;$$

this fact combined with (4.4) implies that $\partial_j f = -g_j$. Moreover, from this and the assumption that $g_j \in X''$, it follows that $\partial_j f \in L^1_{\rm loc}(\mathbb{R}^n)$. Using this, Lemma 3.18, (4.7), and (4.6), for any $j \in \{1, \dots, n\}$, we obtain

$$\|\partial_j f\|_X = \|g_j\|_X = \|g_j\|_{X''} = \|L_j\|_{(X')^*} \le B_j(f) < \infty,$$

which, together with the assumption that $f \in X$, implies that $f \in W^{1,X}(\mathbb{R}^n)$ and

$$\|\,|\nabla f|\,\|_X \le \sum_{j=1}^n B_j(f).$$

This finishes the proof of the sufficiency, and hence of Lemma 4.4. □

The following Funk–Hecke formula for spherical harmonics is a part of [6, Theorem 2.22].

Lemma 4.5. *Let $e, \eta \in \mathbb{S}^{n-1}$ with $n \ge 2$, and f be a measurable functions on $(-1, 1)$ satisfying that*

$$\int_{-1}^1 |f(t)|(1-t^2)^{(n-3)/2}dt < \infty.$$

Then

$$\int_{\mathbb{S}^{n-1}} [\xi\cdot\eta]f(\xi\cdot e)\,d\sigma(\xi) = \lambda(f)[e\cdot\eta],$$

where

$$\lambda(f) := \sigma(\mathbb{S}^{n-2})\int_{-1}^1 tf(t)(1-t^2)^{(n-3)/2}\,dt.$$

Borrowing a few ideas from [22, (9)], we prove the following technical lemma.

Lemma 4.6. *Let* $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ *be a radial*-ATI. *Then, for any* $\phi \in C_c^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, *and* $e \in \mathbb{S}^{n-1}$ *we have*

$$\lim_{\epsilon\to 0^+} \int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0\}} \frac{\phi(x+h)-\phi(x)}{|h|}\rho_\epsilon(|h|)\,dh = \frac{K(1,n)}{2}[e\cdot\nabla\phi(x)], \tag{4.8}$$

where the convergence is uniform in $x \in \mathbb{R}^n$*; here* $K(1,n)$ *as in* (1.3) *with* $p = 1$.

Proof. Let $\delta \in (0,1)$. Then for any given $\epsilon > 0$ write

$$K_\epsilon := \int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0\}} \frac{\phi(x+h)-\phi(x)}{|h|}\rho_\epsilon(|h|)\,dh = K_\epsilon^{(1)}(\delta) + K_\epsilon^{(2)}(\delta), \tag{4.9}$$

where

$$K_\epsilon^{(1)}(\delta) := \int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0,\ |h|\geq\delta\}} \frac{\phi(x+h)-\phi(x)}{|h|}\rho_\epsilon(|h|)\,dh$$

and

$$K_\epsilon^{(2)}(\delta) := \int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0,\ |h|<\delta\}} \frac{\phi(x+h)-\phi(x)}{|h|}\rho_\epsilon(|h|)\,dh.$$

We first consider $K_\epsilon^{(1)}(\delta)$. Since $\phi \in C_c^2(\mathbb{R}^n)$, applying the mean value theorem and the Cauchy–Schwarz inequality for any $x, h \in \mathbb{R}^n$, we obtain

$$|\phi(x+h)-\phi(x)| \leq \|\,|\nabla\phi|\,\|_{L^\infty(\mathbb{R}^n)}\,|h|.$$

Combining this estimate with (2.2), for any given $\delta \in (0,1)$, implies that

$$\begin{aligned} |K_\epsilon^{(1)}(\delta)| &\leq \int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0,\ |h|\geq\delta\}} \frac{|\phi(x+h)-\phi(x)|}{|h|}\rho_\epsilon(|h|)\,dh \\ &\leq \|\,|\nabla\phi|\,\|_{L^\infty(\mathbb{R}^n)} \int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0,\ |h|\geq\delta\}} \rho_\epsilon(|h|)\,dh \\ &\lesssim \int_\delta^\infty \rho_\epsilon(r) r^{n-1}\,dr \to 0 \end{aligned} \tag{4.10}$$

as $\epsilon \to 0^+$.

Next, we examine $K_\epsilon^{(2)}(\delta)$. Consider first the case $n = 1$ in which we may take $e = 1$. Observe that $K(1,1) = 2$. From this and (2.1), for any $\delta \in (0,1)$ and $\epsilon > 0$, it follows that

$$\begin{aligned} K_\epsilon^{(2)}(\delta) &= \int_0^\delta \frac{\phi(x+r)-\phi(x)}{r}\rho_\epsilon(r)\,dr \\ &= \int_0^\delta \left[\frac{\phi(x+r)-\phi(x)}{r} - \phi'(x)\right]\rho_\epsilon(r)\,dr + \phi'(x)\int_0^\delta \rho_\epsilon(r)\,dr \\ &= \int_0^\delta \left[\frac{\phi(x+r)-\phi(x)}{r} - \phi'(x)\right]\rho_\epsilon(r)\,dr \\ &\quad + \phi'(x)\left[1 - \int_\delta^\infty \rho_\epsilon(r)\,dr\right] \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\delta \left[\frac{\phi(x+r)-\phi(x)}{r} - \phi'(x)\right]\rho_\epsilon(r)\,dr \\
&\quad + \phi'(x)\left[\frac{K(1,1)}{2} - \int_\delta^\infty \rho_\epsilon(r)\,dr\right]. \qquad (4.11)
\end{aligned}
$$

Since $\phi \in C_c^2(\mathbb{R})$, applying Taylor's remainder theorem and the Cauchy–Schwarz inequality, we deduce that there exists a positive constant $C_{(\phi)}$ such that for any $x \in \mathbb{R}$ and $r \in (0, \delta]$,

$$
\left|\frac{\phi(x+r)-\phi(x)}{r} - \phi'(x)\right| \le C_{(\phi)} r \le C_{(\phi)}\delta.
$$

By this, (4.11), (2.1), and (2.2), for any given $\delta \in (0, 1)$, we conclude that

$$
\begin{aligned}
&\limsup_{\epsilon\to 0^+}\left|K_\epsilon^{(2)}(\delta) - \frac{K(1,1)}{2}\phi'(x)\right| \\
&\quad \le \limsup_{\epsilon\to 0^+}\int_0^\delta \left|\frac{\phi(x+r)-\phi(x)}{r} - \phi'(x)\right|\rho_\epsilon(r)\,dr \\
&\qquad + |\phi'(x)|\limsup_{\epsilon\to 0^+}\int_\delta^\infty \rho_\epsilon(r)\,dr \\
&\quad \lesssim \delta \limsup_{\epsilon\to 0^+}\int_0^\delta \rho_\epsilon(r)\,dr + \|\phi'\|_{L^\infty(\mathbb{R})}\limsup_{\epsilon\to 0^+}\int_\delta^\infty \rho_\epsilon(r)\,dr \\
&\quad \lesssim \delta. \qquad (4.12)
\end{aligned}
$$

Now consider the case $n \ge 2$. Observe that, for any $\delta \in (0, 1)$ and $\epsilon > 0$, one has

$$
\begin{aligned}
K_\epsilon^{(2)}(\delta) &= \int_0^\delta \left[\int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\ge 0\}} \frac{\phi(x+r\xi)-\phi(x)}{r}\,d\sigma(\xi)\right]\rho_\epsilon(r)r^{n-1}\,dr \\
&= \int_0^\delta \left\{\int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\ge 0\}} \left[\frac{\phi(x+r\xi)-\phi(x)}{r} - \xi\cdot\nabla\phi(x)\right]d\sigma(\xi)\right\} \\
&\quad \times \rho_\epsilon(r)r^{n-1}\,dr \\
&\quad + \int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\ge 0\}} \xi\cdot\nabla\phi(x)\,d\sigma(\xi)\int_0^\delta \rho_\epsilon(r)r^{n-1}\,dr \\
&= \int_0^\delta \left\{\int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\ge 0\}} \left[\frac{\phi(x+r\xi)-\phi(x)}{r} - \xi\cdot\nabla\phi(x)\right]d\sigma(\xi)\right\} \\
&\quad \times \rho_\epsilon(r)r^{n-1}\,dr \\
&\quad + \int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\ge 0\}} \xi\cdot\nabla\phi(x)\,d\sigma(\xi)\left[1 - \int_\delta^\infty \rho_\epsilon(r)r^{n-1}\,dr\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left|K_\epsilon^{(2)}(\delta) - \int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\ge 0\}} \xi\cdot\nabla\phi(x)\,d\sigma(\xi)\right| \\
&\quad \le \int_0^\delta \left[\int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\ge 0\}} \left|\frac{\phi(x+r\xi)-\phi(x)}{r} - \xi\cdot\nabla\phi(x)\right|d\sigma(\xi)\right]
\end{aligned}
$$

$$\times \rho_\epsilon(r) r^{n-1}\, dr + \left| \int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\geq 0\}} \xi\cdot\nabla\phi(x)\, d\sigma(\xi) \right| \int_\delta^\infty \rho_\epsilon(r) r^{n-1}\, dr. \tag{4.13}$$

Since $\phi \in C_c^2(\mathbb{R}^n)$, applying Taylor's remainder theorem and the Cauchy–Schwarz inequality, it follows that there exists a positive constant $C_{(\phi,n)}$ such that, for any $x\in\mathbb{R}^n$, $r\in(0,\delta]$, and $\xi\in\mathbb{S}^{n-1}$,

$$\left| \frac{\phi(x+r\xi)-\phi(x)}{r} - \xi\cdot\nabla\phi(x) \right| \leq C_{(\phi,n)} r \leq C_{(\phi,n)}\delta. \tag{4.14}$$

Moreover, using the Funk–Hecke formula for spherical harmonics [with f and η in Lemma 4.5 replaced, respectively, by $\mathbf{1}_{[0,1]}$ and $\nabla\phi(x)/|\nabla\phi(x)|$], and [16, (1.6)], we obtain

$$\begin{aligned} \int_{\{\xi\in\mathbb{S}^{n-1}:\ \xi\cdot e\geq 0\}} \xi\cdot\nabla\phi(x)\, d\sigma(\xi) &= \int_{\mathbb{S}^{n-1}} \xi\cdot\nabla\phi(x)\mathbf{1}_{[0,1]}(\xi\cdot e)\, d\sigma(\xi) \\ &= \frac{\sigma(\mathbb{S}^{n-2})}{n-1}[e\cdot\nabla\phi(x)] \\ &= \frac{K(1,n)}{2}[e\cdot\nabla\phi(x)]. \end{aligned}$$

From this, (4.13), (4.14), (2.1), and (2.2), for any given $n\geq 2$ and $\delta\in(0,1)$, it follows that

$$\begin{aligned} &\limsup_{\epsilon\to 0^+} \left| K_\epsilon^{(2)}(\delta) - \frac{K(1,n)}{2}[e\cdot\nabla\phi(x)] \right| \\ &\quad\lesssim \delta \limsup_{\epsilon\to 0^+} \int_0^\delta \rho_\epsilon(r) r^{n-1}\, dr \\ &\qquad + \frac{K(1,n)}{2}|e\cdot\nabla\phi(x)| \limsup_{\epsilon\to 0^+} \int_\delta^\infty \rho_\epsilon(r) r^{n-1}\, dr \\ &\quad\lesssim \delta + \frac{K(1,n)}{2} \| |\nabla\phi| \|_{L^\infty(\mathbb{R}^n)} \limsup_{\epsilon\to 0^+} \int_\delta^\infty \rho_\epsilon(r) r^{n-1}\, dr \sim \delta. \end{aligned}$$

This estimate together with (4.12) implies that for any given $n\in\mathbb{N}$ and for any $\delta\in(0,1)$, we have

$$\limsup_{\epsilon\to 0^+} \left| K_\epsilon^{(2)}(\delta) - \frac{K(1,n)}{2}[e\cdot\nabla\phi(x)] \right| \lesssim \delta.$$

By this, (4.9), and (4.10), for any given $\delta\in(0,1)$, we finally conclude that

$$\begin{aligned} &\limsup_{\epsilon\to 0^+} \left| \int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0\}} \frac{\phi(x+h)-\phi(x)}{|h|}\rho_\epsilon(|h|)\, dh - \frac{K(1,n)}{2}[e\cdot\nabla\phi(x)] \right| \\ &\quad\leq \limsup_{\epsilon\to 0^+} |K_\epsilon^{(1)}(\delta)| + \limsup_{\epsilon\to 0^+} \left| K_\epsilon^{(2)}(\delta) - \frac{K(1,n)}{2}[e\cdot\nabla\phi(x)] \right| \\ &\quad= \limsup_{\epsilon\to 0^+} \left| K_\epsilon^{(2)}(\delta) - \frac{K(1,n)}{2}[e\cdot\nabla\phi(x)] \right| \lesssim \delta. \end{aligned}$$

Let $\delta\to 0^+$. Then we conclude that (4.8) holds, and this finishes the proof of Lemma 4.6. □

Next we provide a modification of [22, Lemma 1].

Lemma 4.7. *Let $\rho \in L^1_{\mathrm{loc}}(0,\infty)$ be a nonnegative function satisfying (3.11). Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\phi \in C_{\mathrm{c}}(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|\phi(y)-\phi(x)|}{|y-x|}\rho(|y-x|)\,dy\right]|f(x)|\,dx<\infty. \tag{4.15}$$

Then, for any $e \in \mathbb{S}^{n-1}$ we have

$$\left|\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\ge 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho(|y-x|)\,dy\right]f(x)\,dx\right| \le \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|}\rho(|y-x|)\,dy\right]|\phi(x)|\,dx.$$

Proof. For any given $\delta \in (0,\infty)$ define

$$\rho_\delta(r) := \begin{cases} 0, & r\in(0,\delta),\\ \rho(r), & r\in[\delta,\infty).\end{cases}$$

Then for any $x, y \in \mathbb{R}^n$ we certainly have

$$\frac{|\phi(y)-\phi(x)|}{|y-x|}\rho_\delta(|y-x|)\mathbf{1}_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\ (y-x)\cdot e\ge 0\}}(x,y)|f(x)| \le \frac{|\phi(y)-\phi(x)|}{|y-x|}\rho(|y-x|)|f(x)|. \tag{4.16}$$

In view of (4.15) this implies that

$$\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\ge 0\}}\frac{|\phi(y)-\phi(x)|}{|y-x|}\rho_\delta(|y-x|)\,dy\right]|f(x)|\,dx \le \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|\phi(y)-\phi(x)|}{|y-x|}\rho(|y-x|)\,dy\right]|f(x)|\,dx<\infty. \tag{4.17}$$

As $\phi \in C_{\mathrm{c}}(\mathbb{R}^n)$, there exists a ball $B(\mathbf{0}, M)$ with $M \in (0,\infty)$ such that $\operatorname{supp}(\phi) \subset B(\mathbf{0}, M)$. Using this, the assumption that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and (3.11), we obtain, for any $\delta \in (0,\infty)$,

$$\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\ge 0\}}\frac{|\phi(x)|}{|y-x|}\rho_\delta(|y-x|)\,dy\right]|f(x)|\,dx \le \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{\rho_\delta(|h|)}{|h|}\,dh\right]|\phi(x)||f(x)|\,dx \lesssim \frac{\|\phi\|_{L^\infty(\mathbb{R}^n)}}{\delta}\int_{B(\mathbf{0},M)}|f(x)|\,dx\int_\delta^\infty \rho(r)r^{n-1}\,dr<\infty. \tag{4.18}$$

This, combined with (4.17), further implies that, for any $\delta \in (0,\infty)$,

$$\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\ge 0\}}\frac{|\phi(y)|}{|y-x|}\rho_\delta(|y-x|)\,dy\right]|f(x)|\,dx$$

$$\leq \int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{|\phi(y)-\phi(x)|}{|y-x|}\rho_\delta(|y-x|)\,dy\right]|f(x)|\,dx$$
$$+\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{|\phi(x)|}{|y-x|}\rho_\delta(|y-x|)\,dy\right]|f(x)|\,dx<\infty. \tag{4.19}$$

Let $\delta>0$. We claim that

$$\left|\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]f(x)\,dx\right|$$
$$\leq \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|}\rho(|y-x|)\,dy\right]|\phi(x)|\,dx. \tag{4.20}$$

Indeed, by (4.17), (4.18), (4.19), and Fubini's theorem, we conclude that

$$\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]f(x)\,dx$$
$$=\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(y)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]f(x)\,dx$$
$$-\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(x)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]f(x)\,dx$$
$$=\int_{\mathbb{R}^n}\left[\int_{\{x\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{f(x)}{|y-x|}\rho_\delta(|y-x|)\,dx\right]\phi(y)\,dy$$
$$-\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{f(x)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]\phi(x)\,dx$$
$$=\int_{\mathbb{R}^n}\left[\int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0\}}\frac{f(y-h)}{|h|}\rho_\delta(|h|)\,dh\right]\phi(y)\,dy$$
$$-\int_{\mathbb{R}^n}\left[\int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0\}}\frac{f(x)}{|h|}\rho_\delta(|h|)\,dh\right]\phi(x)\,dx$$
$$=\int_{\mathbb{R}^n}\left[\int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0\}}\frac{f(x-h)-f(x)}{|h|}\rho_\delta(|h|)\,dh\right]\phi(x)\,dx,$$

which implies that

$$\left|\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]f(x)\,dx\right|$$
$$\leq \int_{\mathbb{R}^n}\left[\int_{\{h\in\mathbb{R}^n:\ h\cdot e\geq 0\}}\frac{|f(x-h)-f(x)|}{|h|}\rho_\delta(|h|)\,dh\right]|\phi(x)|\,dx$$
$$\leq \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|}\rho(|y-x|)\,dy\right]|\phi(x)|\,dx.$$

Thus, the claim (4.20) is valid. From this, (4.15), (4.16), and the Lebesgue dominated convergence theorem, it follows that

$$\left|\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho(|y-x|)\,dy\right]f(x)\,dx\right|$$

$$
\begin{aligned}
&= \left|\int_{\mathbb{R}^n}\left[\lim_{\delta\to 0}\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]f(x)\,dx\right|\\
&= \lim_{\delta\to 0}\left|\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\delta(|y-x|)\,dy\right]f(x)\,dx\right|\\
&\leq \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|}\rho(|y-x|)\,dy\right]|\phi(x)|\,dx.
\end{aligned}
$$

This finishes the proof of Lemma 4.7. □

Proof of Theorem 4.1. By Lemma 4.4, it suffices to show that

$$B_j(f) < \infty, \tag{4.21}$$

for any $j \in \{1,\dots,n\}$, where $B_j(f)$ is as in (4.2). So let $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\|\phi\|_{X'} \leq 1$. Using this and Lemma 4.6 [with e replaced by e_j (the j-th unit vector in $\mathbb{R}^n$)], we find that, for any $j \in \{1,\dots,n\}$ and $x \in \mathbb{R}^n$,

$$
\begin{aligned}
\partial_j\phi(x) &= e_j\cdot\nabla\phi(x)\\
&= \frac{2}{K(1,n)}\lim_{\epsilon\to 0^+}\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e_j\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\epsilon(|y-x|)\,dy,
\end{aligned}\tag{4.22}
$$

where the convergence is uniform in x on $\mathbb{R}^n$, and $K(1,n)$ as in (1.3) with $p=1$. Since $\phi\in C_c^\infty(\mathbb{R}^n)$, it follows that there is a ball $B(\mathbf{0},M)$ with $M>0$ such that $\operatorname{supp}(\phi)\subset B(\mathbf{0},M)$. By this, $f\in X$ [which implies that $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$], and (4.22), we conclude that, for any $j\in\{1,\dots,n\}$,

$$
\begin{aligned}
&\frac{K(1,n)}{2}\left|\int_{\mathbb{R}^n}f(x)\partial_j\phi(x)\,dx\right|\\
&\quad= \frac{K(1,n)}{2}\left|\int_{B(\mathbf{0},M)}f(x)\partial_j\phi(x)\,dx\right|\\
&\quad= \lim_{\epsilon\to 0^+}\left|\int_{B(\mathbf{0},M)}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e_j\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\epsilon(|y-x|)\,dy\right]f(x)\,dx\right|\\
&\quad= \lim_{\epsilon\to 0^+}\left|\int_{\mathbb{R}^n}\left[\int_{\{y\in\mathbb{R}^n:\ (y-x)\cdot e_j\geq 0\}}\frac{\phi(y)-\phi(x)}{|y-x|}\rho_\epsilon(|y-x|)\,dy\right]f(x)\,dx\right|.
\end{aligned}\tag{4.23}
$$

Then we claim that, for any $\epsilon\in(0,\infty)$,

$$\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|\phi(y)-\phi(x)|}{|y-x|}\rho_\epsilon(|y-x|)\,dy\right]|f(x)|\,dx \lesssim \|f\|_X < \infty. \tag{4.24}$$

Indeed, since $\phi\in C_c^\infty(\mathbb{R}^n)$, from the mean value theorem and the Cauchy–Schwarz inequality, it follows that, for any $x, y\in\mathbb{R}^n$ we have

$$|\phi(x)-\phi(y)| \leq \|\,|\nabla\phi|\,\|_{L^\infty(\mathbb{R}^n)}\,|x-y|. \tag{4.25}$$

Observe that when $x\in\mathbb{R}^n$ satisfies $|x|\geq 2M$, and $y\in B(\mathbf{0},M)$, then we must have $|y-x|\geq 2^{-1}|x|$. Using this, (4.25), and Lemma 3.18, we conclude that

$$\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\frac{|\phi(y)-\phi(x)|}{|y-x|}\rho_\epsilon(|y-x|)\,dy\right]|f(x)|\,dx$$

$$
\begin{aligned}
&= \int_{B(\mathbf{0},2M)} \left[\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x)|}{|y-x|}\rho_\epsilon(|y-x|)\,dy\right]|f(x)|\,dx \\
&\quad + \int_{\mathbb{R}^n\setminus B(\mathbf{0},2M)}\int_{B(\mathbf{0},M)}\cdots + \int_{\mathbb{R}^n\setminus B(\mathbf{0},2M)}\int_{\mathbb{R}^n\setminus B(\mathbf{0},M)}\cdots \\
&\lesssim \|\,|\nabla\phi|\,\|_{L^\infty(\mathbb{R}^n)} \int_{B(\mathbf{0},2M)}\left[\int_{\mathbb{R}^n}\rho_\epsilon(|y-x|)\,dy\right]|f(x)|\,dx \\
&\quad + \|\phi\|_{L^\infty(\mathbb{R}^n)}|B(\mathbf{0},M)|\int_{\mathbb{R}^n\setminus B(\mathbf{0},2M)}\frac{|f(x)|}{|x|}\rho_\epsilon(2^{-1}|x|)\,dx \\
&\sim \int_{B(\mathbf{0},2M)}|f(x)|\,dx + \int_{\mathbb{R}^n\setminus B(\mathbf{0},2M)}\frac{|f(x)|}{|x|}\rho_\epsilon(2^{-1}|x|)\,dx \\
&\lesssim \|f\|_X\|\mathbf{1}_{B(\mathbf{0},2M)}\|_{X'} + \int_{\mathbb{R}^n\setminus B(\mathbf{0},2M)}\frac{|f(x)|}{|x|}\rho_\epsilon(2^{-1}|x|)\,dx \\
&\lesssim \|f\|_X + \int_{\mathbb{R}^n\setminus B(\mathbf{0},2M)}\frac{|f(x)|}{|x|}\rho_\epsilon(2^{-1}|x|)\,dx. \qquad (4.26)
\end{aligned}
$$

Moreover, from the assumption that ρ_ϵ is decreasing for any $\epsilon\in(0,\infty)$, the assumption that X' is locally β-doubling with $\beta\in(0,n+1)$, and (2.1), it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(\mathbf{0},2M)}\frac{|f(x)|}{|x|}\rho_\epsilon(2^{-1}|x|)\,dx \\
&\quad= \sum_{j=1}^{\infty}\int_{B(\mathbf{0},2^{j+1}M)\setminus B(\mathbf{0},2^jM)}\frac{|f(x)|}{|x|}\rho_\epsilon(2^{-1}|x|)\,dx \\
&\quad\lesssim \sum_{j=1}^{\infty}\frac{\rho_\epsilon(2^{j-1}M)}{2^jM}\int_{B(\mathbf{0},2^{j+1}M)}|f(x)|\,dx \\
&\quad\lesssim \|f\|_X\sum_{j=1}^{\infty}\frac{\rho_\epsilon(2^{j-1}M)}{2^jM}\|\mathbf{1}_{B(\mathbf{0},2^{j+1}M)}\|_{X'} \\
&\quad\lesssim \|f\|_X\|\mathbf{1}_{B(\mathbf{0},1)}\|_{X'}\sum_{j=1}^{\infty}\rho_\epsilon(2^{j-1}M)(2^jM)^{\beta-1} \\
&\quad\lesssim \|f\|_X\sum_{j=1}^{\infty}\rho_\epsilon(2^{j-1}M)(2^jM)^n \\
&\quad\lesssim \|f\|_X\sum_{j=1}^{\infty}\int_{B(\mathbf{0},2^{j+1}M)\setminus B(\mathbf{0},2^jM)}\rho_\epsilon(2^{j-1}|x|)\,dx \\
&\quad\lesssim \|f\|_X\int_M^\infty \rho_\epsilon(r)r^{n-1}\,dr \lesssim \|f\|_X<\infty,
\end{aligned}
$$

which, combined with (4.26), implies that the claim (4.24) holds. By (4.23), (4.24), and Lemma 4.7 (with ρ replaced by ρ_ϵ), for any $j\in\{1,\dots,n\}$, we find that

$$
\left|\int_{\mathbb{R}^n} f(x)\partial_j\phi(x)\,dx\right|
$$

$$\lesssim \liminf_{\epsilon\to 0^+} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|}\rho_\epsilon(|x-y|)\,dy\right] |\phi(x)|\,dx.$$

From this, Lemma 3.18, Hölder's inequality, (2.1), and (4.1), for any $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\|\phi\|_{X'} \le 1$ we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)\partial_j\phi(x)\,dx\right| \\
&\quad\lesssim \liminf_{\epsilon\to 0^+} \left\|\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|}{|\cdot-y|}\rho_\epsilon(|\cdot-y|)\,dy\right\|_X \|\phi\|_{X'} \\
&\quad\lesssim \liminf_{\epsilon\to 0^+} \left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}^n} \rho_\epsilon(|\cdot-y|)\,dy\right]^{1/p'}\right\|_X \\
&\quad\lesssim \liminf_{\epsilon\to 0^+} \left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X < \infty.
\end{aligned}$$

This fact, together with (4.3), yields the validity of (4.21) and

$$\|\,|\nabla f|\,\|_X \lesssim \sum_{j=1}^n B_j(f) \lesssim \liminf_{\epsilon\to 0^+} \left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p.$$

This finishes the proof of Theorem 4.1. □

As a direct consequence of Theorems 3.4 and 4.1, Corollary 3.19, and Lemma 3.31, we obtain the following new characterization of $W^{1,X}(\mathbb{R}^n)$. Indeed, both Corollary 3.19 and Lemma 3.31 are used to show that the BBF space X under consideration satisfies all the hypotheses of both Theorems 3.4 and 4.1.

Theorem 4.8. *Let X be a ball Banach function space, $p \in [1,\infty)$, and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a decreasing-radial-*ATI. *Assume that both X and X' have absolutely continuous norms, $X^{1/p}$ is a ball Banach function space, and the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$. Then $f \in W^{1,X}(\mathbb{R}^n)$ if and only if $f \in X$ and*

$$\lim_{\epsilon\to 0^+} \left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X < \infty; \tag{4.27}$$

moreover,

$$\lim_{\epsilon\to 0^+} \left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p = K(p,n)\,\|\,|\nabla f|\,\|_X^p.$$

To show Theorem 4.8, we first establish the following two lemmas.

Lemma 4.9. *Let X be a ball Banach function space. Assume that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X. Then $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X'.*

Proof. Let $r > 0$ and $f \in X$. By Definition 3.2 and Fubini's theorem, we find that

$$\begin{aligned}
\|B_r(f)\|_{X'} &= \sup_{\|g\|_X=1} \int_{\mathbb{R}^n} B_r(f)(x)g(x)\,dx \\
&\leq \sup_{\|g\|_X=1} \int_{\mathbb{R}^n} \frac{1}{|B(x,r)|}\left[\int_{B(x,r)} |f(y)|\,dy\right]|g(x)|\,dx \\
&= \sup_{\|g\|_X=1} \int_{\mathbb{R}^n} \frac{1}{|B(y,r)|}\left[\int_{B(y,r)} |g(x)|\,dx\right]|f(y)|\,dy \\
&= \sup_{\|g\|_X=1} \int_{\mathbb{R}^n} |f(y)|B_r(g)(y)\,dy.
\end{aligned}$$

From this, Lemma 3.18, and the assumption that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X, it follows that

$$\|B_r(f)\|_{X'} \leq \sup_{\|g\|_X=1} \|B_r(g)\|_X\|f\|_{X'} \lesssim \sup_{\|g\|_X=1} \|g\|_X\|f\|_{X'} \sim \|f\|_{X'}.$$

This finishes the proof of Lemma 4.9. □

The following provides the third sufficient condition for the locally β-doubling property of X.

Lemma 4.10. *Let X be a ball quasi-Banach function space. Assume that the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X. Then X is locally n-doubling.*

Proof. Observe that for any $B := B(\mathbf{0}, r) \in \mathbb{B}$ with $r \in (0,\infty)$, and for any $\alpha \in [1,\infty)$ and $x \in \alpha B$ we have

$$B_{2\alpha r}(\mathbf{1}_B)(x) = \fint_{B(x,2\alpha r)} \mathbf{1}_B(y)\,dy = \frac{|B \cap B(x,2\alpha r)|}{|B(x,2\alpha r)|} = \frac{1}{(2\alpha)^n}.$$

From this and the assumptions that X is a BQBF space and that $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X, it follows that, for any $B \in \mathbb{B}$ and $\alpha \in [1,\infty)$,

$$\|\mathbf{1}_{\alpha B}\|_X \leq 2^n\alpha^n\|B_{2\alpha r}(\mathbf{1}_B)\|_X \lesssim \alpha^n\|\mathbf{1}_B\|_X.$$

This implies that X is locally n-doubling, and hence finishes the proof of Lemma 4.10. □

Proof of Theorem 4.8. Let $f \in W^{1,X}(\mathbb{R}^n)$. By Theorem 3.4, we obtain

$$\lim_{\epsilon\to 0^+} \left\|\left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^p}\rho_\epsilon(|\cdot-y|)\,dy\right]^{\frac{1}{p}}\right\|_X^p = K(p,n)\,\|\,|\nabla f|\,\|_X^p < \infty,$$

which completes the proof of the necessity.

Next, we show the sufficiency. To this end, assume that (4.27) holds. Using Corollary 3.19, Lemma 3.11, and the assumptions that $X^{1/p}$ is a ball Banach function space and that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$, we find that $C_c^\infty(\mathbb{R}^n)$ is dense in X' and the centered ball average operators $\{B_r\}_{r\in(0,\infty)}$ are uniformly bounded on X. This, combined with Lemmas 4.9 and 4.10, implies that X' is locally β-doubling with $\beta = n \in (0, n+1)$. Thus, all the assumptions of Theorem 4.1 are satisfied. Using Theorem 4.1, we conclude that $f \in W^{1,X}(\mathbb{R}^n)$. This finishes the proof of the sufficiency, and hence of Theorem 4.8. □

Remark 4.11. Let $q \in (1,\infty)$, $p \in [1,q]$, $X := L^q(\mathbb{R}^n)$, and $\{\rho_\epsilon\}_{\epsilon\in(0,\infty)}$ be a decreasing-radial-ATI. Then, by Theorem 4.8, we find that $f \in W^{1,q}(\mathbb{R}^n)$ if and only if $f \in L^q(\mathbb{R}^n)$ and (4.1) holds with $X = L^q(\mathbb{R}^n)$; moreover, if (4.1) holds for some $f \in L^q(\mathbb{R}^n)$, then we have

$$\lim_{\epsilon\to 0^+} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^p} \rho_\epsilon(|\cdot - y|)\, dy \right]^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^n)}^p = K(p,n)\, \| |\nabla f| \|_{L^q(\mathbb{R}^n)}^p .$$

If $X := L^q(\mathbb{R}^n)$, Theorem 4.8 is just [12, Theorem 2] when $p = q$ but is new when $p < q$.

Theorems 3.36 and 4.1 yield the following conclusion in a way similar to Theorem 4.8.

Theorem 4.12. *Let X be a ball Banach function space, and $p \in [1,\infty)$. Assume that both X and X' have absolutely continuous norms, $X^{1/p}$ is a ball Banach function space, and the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $(X^{1/p})'$. Then $f \in W^{1,X}(\mathbb{R}^n)$ if and only if $f \in X$ and*

$$\lim_{s\to 1^-} (1-s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|_X < \infty;$$

moreover, if this holds for a function $f \in X$, then we have

$$\lim_{s\to 1^-} (1-s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|_X^p = \frac{K(p,n)}{p}\, \| |\nabla f| \|_X^p .$$

Remark 4.13. Let $q \in (1,\infty)$, $p \in [1,q]$, and $X := L^q(\mathbb{R}^n)$. Then, by Theorem 4.12, we conclude that $f \in W^{1,q}(\mathbb{R}^n)$ if and only if $f \in L^q(\mathbb{R}^n)$ and

$$\lim_{s\to 1^-} (1-s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^n)} < \infty;$$

moreover,

$$\lim_{s\to 1^-} (1-s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^n)}^p = \frac{K(p,n)}{p}\, \| |\nabla f| \|_{L^q(\mathbb{R}^n)}^p .$$

If $X := L^q(\mathbb{R}^n)$, Theorem 4.12 is just [12, (44)] when $p = q$ but is new when $p < q$.

5 Applications to specific spaces

In this section, we apply Theorems 2.13, 2.15, 2.20, 2.22, 3.4, 3.36, 4.8, and 4.12 to seven concrete examples of ball Banach function spaces, namely, Morrey spaces (Subsection 5.1), mixed-norm Lebesgue spaces (Subsection 5.2), weighted Lebesgue spaces (Subsection 5.3), variable Lebesgue spaces (Subsection 5.4), Orlicz spaces (Subsection 5.5), Orlicz-slice spaces (Subsection 5.6), and Lorentz spaces (Subsection 5.7).

5.1 Morrey spaces

For any given $0 < r \le \alpha < \infty$, the *Morrey space* $M_r^\alpha(\mathbb{R}^n)$ is defined as the set of all the measurable functions f on $\mathbb{R}^n$ for which the quasi-norm

$$\|f\|_{M_r^\alpha(\mathbb{R}^n)} := \sup_{B\in\mathbb{B}} |B|^{1/\alpha-1/r}\|f\|_{L^r(B)}$$

is finite. These spaces were introduced in 1938 by Morrey [78] in order to study the regularity of solutions to certain equations. They find important applications in the theory of elliptic partial differential equations, potential theory, and harmonic analysis (see, for instance, [3, 27, 61, 91, 92, 97]). As indicated in [93, p. 87], the Morrey space $M_r^\alpha(\mathbb{R}^n)$ for any given $r \in [1,\infty)$ is a ball Banach function space, but not a Banach function space in the terminology of Bennett and Sharpley [10].

The following theorem is a corollary of Theorem 2.15.

Theorem 5.1. *Let* $1 \le r \le \alpha < \infty$ *and* $p \in [1,\infty)$ *satisfy* $n(1/\alpha - 1/p) < 1$. *Let* $K(p,n)$ *be as in* (1.3). *Then, for any* $f \in C_c^2(\mathbb{R}^n)$,

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{M_r^\alpha(\mathbb{R}^n)}^p = \frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{M_r^\alpha(\mathbb{R}^n)}^p. \tag{5.1}$$

Proof. From the conclusion in [93, p. 87], it follows that the Morrey space $M_r^\alpha(\mathbb{R}^n)$ is a BBF space. Using this fact and Theorem 2.15, we find that in order to prove the required conclusion, it would suffice to show that $M_r^\alpha(\mathbb{R}^n)$ is locally n/α-doubling with $n/\alpha \in (0, 1+n/p)$. Indeed, from the definition of $M_r^\alpha(\mathbb{R}^n)$, for any $B_0 := B(\mathbf{0}, R) \in \mathbb{B}$, we deduce that

$$\begin{aligned}\|\mathbf{1}_{B_0}\|_{M_r^\alpha(\mathbb{R}^n)} &= \max\left\{\sup_{\substack{B\in\mathbb{B}\\ r_B\le R}}|B|^{\frac{1}{\alpha}-\frac{1}{r}}\|\mathbf{1}_{B_0}\|_{L^r(B)},\ \sup_{\substack{B\in\mathbb{B}\\ r_B> R}}|B|^{\frac{1}{\alpha}-\frac{1}{r}}\|\mathbf{1}_{B_0}\|_{L^r(B)}\right\}\\ &= \max\left\{\sup_{\substack{B\in\mathbb{B}\\ r_B\le R}}|B|^{\frac{1}{\alpha}-\frac{1}{r}}|B_0\cap B|^{\frac{1}{r}},\ \sup_{\substack{B\in\mathbb{B}\\ r_B> R}}|B|^{\frac{1}{\alpha}-\frac{1}{r}}|B_0\cap B|^{\frac{1}{r}}\right\}\\ &= |B_0|^{1/\alpha}.\end{aligned}$$

This implies that, for any $\lambda \in [1,\infty)$,

$$\|\mathbf{1}_{\lambda B_0}\|_{M_r^\alpha(\mathbb{R}^n)} = |\lambda B_0|^{1/\alpha} = \lambda^{n/\alpha}|B_0|^{1/\alpha} = \lambda^{n/\alpha}\|\mathbf{1}_{B_0}\|_{M_r^\alpha(\mathbb{R}^n)}.$$

Thus, $M_r^\alpha(\mathbb{R}^n)$ is locally n/α-doubling. Observe that the assumption that $n(1/\alpha - 1/p) < 1$ gives $n/\alpha \in (0, 1+n/p)$ and this completes the proof of Theorem 5.1. □

By the proof of Theorem 5.1, we conclude that the assumptions of Theorem 2.13 are satisfied for $M_r^\alpha(\mathbb{R}^n)$. Using Theorem 2.13, we obtain the following corollary; we omit the details here.

Corollary 5.2. *Let* $1 \le r \le \alpha < \infty$ *and* $p \in [1,\infty)$ *satisfy* $n(1/\alpha - 1/p) < 1$. *Then Theorem 2.13 remains valid when* $X = M_r^\alpha(\mathbb{R}^n)$.

Proposition 5.3. *Let $1 < r < \alpha < \infty$ and $p \in [1, r)$. Then $f \in W^{1,M_r^\alpha(\mathbb{R}^n)}(\mathbb{R}^n)$ if and only if $f \in M_r^\alpha(\mathbb{R}^n)$ and*

$$\liminf_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{M_r^\alpha(\mathbb{R}^n)}^p<\infty; \tag{5.2}$$

moreover, there exists positive constants C_1 and C_2 such that

$$\begin{aligned}C_1\,\|\,|\nabla f|\,\|_{M_r^\alpha(\mathbb{R}^n)}^p&\le\liminf_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{M_r^\alpha(\mathbb{R}^n)}^p\\&\le\limsup_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{M_r^\alpha(\mathbb{R}^n)}^p\\&\le C_2\,\|\,|\nabla f|\,\|_{M_r^\alpha(\mathbb{R}^n)}^p.\end{aligned}$$

Proof. We first prove that ∇f exists and

$$\|\,|\nabla f|\,\|_{M_r^\alpha(\mathbb{R}^n)}^p\lesssim\liminf_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{M_r^\alpha(\mathbb{R}^n)}^p. \tag{5.3}$$

Indeed, when $1 < r < \alpha < \infty$ and $\theta \in (0, r/\alpha)$, according to [91, Proposition 285], we have the following norm equivalence: for any measurable function f on $\mathbb{R}^n$,

$$\|f\|_{M_r^\alpha(\mathbb{R}^n)}\sim\sup_{Q\subset\mathbb{R}^n}|Q|^{1/\alpha-1/r}\|f\|_{L^r_{[\mathcal{M}(\mathbf{1}_Q)]^{1/(1-\theta)}}(\mathbb{R}^n)}, \tag{5.4}$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$,

$$\|f\|_{L^r_{[\mathcal{M}(\mathbf{1}_Q)]^{1/(1-\theta)}}(\mathbb{R}^n)}:=\left\{\int_{\mathbb{R}^n}|f(x)|^r\,[\mathcal{M}(\mathbf{1}_Q)(x)]^{\frac{1}{1-\theta}}\,dx\right\}^{\frac{1}{r}},$$

and the positive equivalence constants are independent of f. From (5.4) and (5.2), it follows that

$$\begin{aligned}&\lim_{s\to 1^-}(1-s)|Q|^{p/\alpha-p/r}\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^r_{[\mathcal{M}(\mathbf{1}_Q)]^{1/(1-\theta)}}(\mathbb{R}^n)}^p\\&\quad\lesssim\liminf_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{M_r^\alpha(\mathbb{R}^n)}^p<\infty.\end{aligned} \tag{5.5}$$

On the other hand, by [91, Theorem 281], we know that $[\mathcal{M}(\mathbf{1}_Q)]^{\frac{1}{1-\theta}} \in A_1(\mathbb{R}^n)$ for any cube $Q \subset \mathbb{R}^n$. From this, (5.5), and Theorem 4.12 with $X := L^r_{[\mathcal{M}(\mathbf{1}_Q)]^{1/(1-\theta)}}(\mathbb{R}^n)$ (see also Theorem 5.10 below with $\omega := [\mathcal{M}(\mathbf{1}_Q)]^{\frac{1}{1-\theta}}$), we infer that ∇f exists and, for any cube $Q \subset \mathbb{R}^n$,

$$\frac{K(p,n)}{p}|Q|^{p/\alpha-p/r}\,\|\,|\nabla f|\,\|_{L^r_{[\mathcal{M}(\mathbf{1}_Q)]^{1/(1-\theta)}}(\mathbb{R}^n)}^p$$

$$
\begin{aligned}
&= \lim_{s\to 1^-} (1-s)|Q|^{p/\alpha - p/r} \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|^p_{L^r_{[\mathcal{M}(\mathbf{1}_Q)]^{1/(1-\theta)}}(\mathbb{R}^n)} \\
&\lesssim \liminf_{s\to 1^-} (1-s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|^p_{M^\alpha_r(\mathbb{R}^n)},
\end{aligned}
$$

which, combined with (5.4) again, further implies that (5.3) holds true.

On the other hand, by the conclusion in [93, p. 87], we find that the Morrey space $M^\alpha_r(\mathbb{R}^n)$ is a BBF space. From [94, Theorem 4.1], it follows that the Hardy–Littlewood maximal function $\mathcal{M}$ is bounded on $\{[M^\alpha_r(\mathbb{R}^n)]^{1/p}\}'$. By this, we conclude that all the assumptions of Lemma 3.34 are satisfied for $X := M^\alpha_r(\mathbb{R}^n)$ [with ρ in Lemma 3.34 replaced by ρ_ϵ in (2.22) for any $\epsilon \in (0, 1/p)$], which implies that

$$
\begin{aligned}
(1-s) &\left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot - y|<(1-s)^{-1/2}\}} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|^p_{M^\alpha_r(\mathbb{R}^n)} \\
&= \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{p}} \rho_\epsilon(|\cdot - y|)\, dy \right]^{\frac{1}{p}} \right\|^p_{M^\alpha_r(\mathbb{R}^n)} \\
&\lesssim \| |\nabla f| \|^p_{M^\alpha_r(\mathbb{R}^n)}. \qquad (5.6)
\end{aligned}
$$

Moreover, using Lemma 3.37 and the fact the Hardy–Littlewood maximal function $\mathcal{M}$ is bounded on $\{[M^\alpha_r(\mathbb{R}^n)]^{1/p}\}'$ (see, for instance, [94, Theorem 4.1]), we find that

$$
\limsup_{s\to 1^-} (1-s) \left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot - y|\ge(1-s)^{-1/2}\}} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|^p_{M^\alpha_r(\mathbb{R}^n)} = 0.
$$

This, together with (5.6), further implies that

$$
\begin{aligned}
\limsup_{s\to 1^-} (1-s) &\left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|^p_{M^\alpha_r(\mathbb{R}^n)} \\
&\le \limsup_{s\to 1^-} (1-s) \left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot - y|<(1-s)^{-1/2}\}} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|^p_{M^\alpha_r(\mathbb{R}^n)} \\
&\quad + \limsup_{s\to 1^-} (1-s) \left\| \left[\int_{\{y\in\mathbb{R}^n:\ |\cdot - y|\ge(1-s)^{-1/2}\}} \frac{|f(\cdot)-f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|^p_{M^\alpha_r(\mathbb{R}^n)} \\
&\lesssim \| |\nabla f| \|^p_{M^\alpha_r(\mathbb{R}^n)}.
\end{aligned}
$$

This finishes the proof of Proposition 5.3. □

Remark 5.4. (i) By [94, Example 5.1], we conclude that the Morrey space $M^\alpha_r(\mathbb{R}^n)$ has no absolutely continuous norm if $1 < r < \alpha < \infty$. Thus, it is still unknown whether or not (5.1) is valid for any $f \in W^{1,M^\alpha_r(\mathbb{R}^n)}(\mathbb{R}^n)$.

(ii) We point out that the upper estimate (5.3) of Proposition 5.3 is attributed to the referee.

5.2 Mixed-norm Lebesgue spaces

For a given vector $\vec{r} := (r_1, \dots, r_n) \in (0, \infty]^n$, the *mixed-norm Lebesgue space* $L^{\vec{r}}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ for which the quasi-norm

$$\|f\|_{L^{\vec{r}}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{r_1} \, dx_1 \right]^{\frac{r_2}{r_1}} \cdots \, dx_n \right\}^{\frac{1}{r_n}},$$

is finite; the usual modifications are made when $r_i = \infty$ for some $i \in \{1, \dots, n\}$. In the sequel let

$$r_- := \min\{r_1, \dots, r_n\}. \tag{5.7}$$

The study of mixed-norm Lebesgue spaces can be traced back to Hörmander [53] and Benedek and Panzone [9]. Important developments of mixed-norm function spaces can be found in [28, 46, 56, 57, 58, 71, 83, 84]. When $\vec{r} \in (0, \infty)^n$ the set $L^{\vec{r}}(\mathbb{R}^n)$ is a ball quasi-Banach function space, but it may not be a quasi-Banach function space (see, for instance, [107, Remark 7.20]).

The following theorem is a corollary of Theorem 2.15.

Theorem 5.5. *Let $\vec{r} := (r_1, \dots, r_n) \in [1, \infty)^n$ and $p \in [1, \infty)$ satisfy $1/r_1 + \cdots + 1/r_n < 1 + n/p$. Let $K(p, n)$ be as in* (1.3)*. Then, for any $f \in C_c^2(\mathbb{R}^n)$,*

$$\lim_{s \to 1^-} (1 - s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^{n+sp}} \, dy \right]^{\frac{1}{p}} \right\|_{L^{\vec{r}}(\mathbb{R}^n)}^p = \frac{K(p, n)}{p} \, \| \, |\nabla f| \, \|_{L^{\vec{r}}(\mathbb{R}^n)}^p .$$

Proof. From [56, Remark 2.8(iii)], it follows that the mixed-norm Lebesgue space $L^{\vec{r}}(\mathbb{R}^n)$ is a BBF space. Using this and Theorem 2.15 we conclude that in order to prove the required conclusion it suffices to show that $L^{\vec{r}}(\mathbb{R}^n)$ is locally β-doubling with some $\beta \in (0, 1 + n/p)$. Indeed, by [83, Proposition 2.1], we conclude that, for any $B := B(\mathbf{0}, r) \in \mathbb{B}$ with $r \in (0, \infty)$, and any $\alpha \in [1, \infty)$ we have

$$\|\mathbf{1}_{\alpha B}\|_{L^{\vec{r}}(\mathbb{R}^n)} = \alpha^{\frac{1}{r_1} + \cdots + \frac{1}{r_n}} \|\mathbf{1}_B\|_{L^{\vec{r}}(\mathbb{R}^n)}.$$

Combining this estimate with the assumption that $1/r_1 + \cdots + 1/r_n < 1 + n/p$, implies that $L^{\vec{r}}(\mathbb{R}^n)$ is locally β-doubling with $\beta = 1/r_1 + \cdots + 1/r_n \in (0, 1 + n/p)$. Theorem 5.5 is now proved. □

The following result is a consequence of Theorem 4.12.

Theorem 5.6. *Let $\vec{r} := (r_1, \dots, r_n) \in (1, \infty)^n$ and let $p \in [1, r_-)$ with r_- defined in* (5.7)*. Then $f \in W^{1, L^{\vec{r}}(\mathbb{R}^n)}(\mathbb{R}^n)$ if and only if $f \in L^{\vec{r}}(\mathbb{R}^n)$ and in this case we have*

$$\lim_{s \to 1^-} (1 - s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^{n+sp}} \, dy \right]^{\frac{1}{p}} \right\|_{L^{\vec{r}}(\mathbb{R}^n)}^p < \infty;$$

moreover,

$$\lim_{s \to 1^-} (1 - s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^{n+sp}} \, dy \right]^{\frac{1}{p}} \right\|_{L^{\vec{r}}(\mathbb{R}^n)}^p = \frac{K(p, n)}{p} \, \| \, |\nabla f| \, \|_{L^{\vec{r}}(\mathbb{R}^n)}^p$$

with $K(p, n)$ as in (1.3).

Proof. By the definition of $L^{\vec{r}}(\mathbb{R}^n)$ we obtain that

$$[L^{\vec{r}}(\mathbb{R}^n)]^{1/p} = L^{\vec{r}/p}(\mathbb{R}^n)$$

and thus $[L^{\vec{r}}(\mathbb{R}^n)]^{1/p}$ is a BBF space (see, for instance, [56, Remark 2.8(iii)]). Then, from this and [9, Theorems 1 and 2], it follows that

$$\left[L^{\vec{r}}(\mathbb{R}^n)\right]' = L^{\vec{r}'}(\mathbb{R}^n) \quad \text{and} \quad \left(\left[L^{\vec{r}}(\mathbb{R}^n)\right]^{1/p}\right)' = L^{(\vec{r}/p)'}(\mathbb{R}^n),$$

where $\vec{r}' := (r_1', \ldots, r_n')$ with $1/r_i + /r_i' = 1$, and $(\vec{r}/p)' := (r_1^*, \ldots, r_n^*)$ with $p/r_i + 1/r_i^* = 1$ for any $i \in \{1, \cdots, n\}$. Moreover, in view of [45, Lemma 4.1] and the assumption that $\vec{r} := (r_1, \ldots, r_n) \in (1, \infty)^n$, we conclude that $L^{\vec{r}}(\mathbb{R}^n)$ and $[L^{\vec{r}}(\mathbb{R}^n)]'$ both have absolutely continuous norms. Furthermore, by [56, Lemma 3.5] and the assumptions that $\vec{r} = (r_1, \ldots, r_n) \in (1, \infty)^n$ and $p \in [1, r_-)$, we find that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $([L^{\vec{r}}(\mathbb{R}^n)]^{1/p})'$. Thus, all the assumptions of Theorem 4.12 are satisfied for $X := L^{\vec{r}}(\mathbb{R}^n)$ with $\vec{r} = (r_1, \ldots, r_n) \in (1, \infty)^n$ and $p \in [1, r_-)$. From this and Theorem 4.12 with $X := L^{\vec{r}}(\mathbb{R}^n)$, we deduce the claimed conclusions. This finishes the proof of Theorem 5.6. □

From the proof of Theorem 5.6, we deduce that all the assumptions of Theorems 2.13, 3.4, 3.36, and 4.8 with $X := L^{\vec{r}}(\mathbb{R}^n)$ are satisfied. Thus we obtain the following results and we omit the details.

Corollary 5.7. *Let $\vec{r} := (r_1, \ldots, r_n) \in (1, \infty)^n$, and $p \in [1, r_-)$. Then Theorems 2.13, 3.4, 3.36, and 4.8 are valid for $X = L^{\vec{r}}(\mathbb{R}^n)$.*

Remark 5.8. We point out that the Sobolev-type space $W^{1, L^{\vec{r}}(\mathbb{R}^n)}(\mathbb{R}^n)$ associated with mixed-norm Lebesgue space has been previously studied in [62, 63].

5.3 Weighted Lebesgue spaces

In this section, we apply Theorems 2.20, 2.22, 3.4, 3.36, 4.8, and 4.12 to weighted Lebesgue spaces (see Definition 3.13). It is worth pointing out that a weighted Lebesgue space with an $A_\infty(\mathbb{R}^n)$-weight may not be a Banach function space; see [93, Section 7.1].

The following theorem is a consequence of Theorem 2.22.

Theorem 5.9. *Let $r, p \in [1, \infty)$ satisfy that $n(1/r - 1/p) < 1$, and $\omega \in \bigcup_{q \in [1, r(1/n+1/p))} A_q(\mathbb{R}^n)$. Let $K(p, n)$ be as in* (1.3). *Then, for any $f \in C_c^2(\mathbb{R}^n)$,*

$$\lim_{s \to 1^-} (1 - s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^{n+sp}} \, dy \right]^{\frac{1}{p}} \right\|_{L^r_\omega(\mathbb{R}^n)}^p = \frac{K(p, n)}{p} \, \| |\nabla f| \|_{L^r_\omega(\mathbb{R}^n)}^p .$$

Proof. It is known that $L^r_\omega(\mathbb{R}^n)$ is a BBF space (see, e.g., [93, p. 86]). Since $q \in (pn/(p + n), r]$, it follows that $n(1/q - 1/p) < 1$ and $q \leq r$. Moreover, by the definition of $L^r_\omega(\mathbb{R}^n)$, we have

$$[L^r_\omega(\mathbb{R}^n)]^{1/q} = L^{r/q}_\omega(\mathbb{R}^n).$$

Using this, the assumption that $\omega \in A_{r/q}(\mathbb{R}^n)$, and [47, Theorem 7.1.9 (a)] we conclude that the Hardy–Littlewood maximal operator $\mathcal{M}$ is weakly bounded on $[L^r_\omega(\mathbb{R}^n)]^{1/q}$. Thus all the assumptions of Theorem 2.22 with $X := L^r_\omega(\mathbb{R}^n)$ are satisfied. The conclusion of Theorem 2.22 yields the claimed assertion and then completes the proof of Theorem 5.9. □

The following result is a corollary of Theorem 4.12.

Theorem 5.10. *Let $r \in (1,\infty)$, $p \in [1,r]$, and $\omega \in A_{r/p}(\mathbb{R}^n)$. Then $f \in W^{1,L^r_\omega(\mathbb{R}^n)}(\mathbb{R}^n)$ if and only if $f \in L^r_\omega(\mathbb{R}^n)$ and*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|^p_{L^r_\omega(\mathbb{R}^n)}<\infty;$$

moreover,

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|^p_{L^r_\omega(\mathbb{R}^n)}=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|^p_{L^r_\omega(\mathbb{R}^n)}$$

with $K(p,n)$ as in (1.3).

Proof. By [96, Lemma 4.2] and [89, Theorem 1.34], we conclude that $L^r_\omega(\mathbb{R}^n)$ and $[L^r_\omega(\mathbb{R}^n)]' = L^{r'}_{\omega^{1-r'}}(\mathbb{R}^n)$ both have absolutely continuous norms. From the definition of $L^r_\omega(\mathbb{R}^n)$, it follows that

$$[L^r_\omega(\mathbb{R}^n)]^{1/p} = L^{r/p}_\omega(\mathbb{R}^n)$$

and $[L^r_\omega(\mathbb{R}^n)]^{1/p}$ is a BBF space.

Then we consider two cases based on the size of p. If $p \in [1,r)$, from the assumption that $\omega \in A_{r/p}(\mathbb{R}^n)$ and [47, Proposition 7.1.5(4)], it follows that

$$\omega^{1-(r/p)'} \in A_{(r/p)'}(\mathbb{R}^n). \tag{5.8}$$

Moreover, using [96, Lemma 4.2], we conclude that

$$\left([L^r_\omega(\mathbb{R}^n)]^{1/p}\right)' = L^{(r/p)'}_{\omega^{1-(r/p)'}}(\mathbb{R}^n).$$

By this, (5.8), and [5, Theorem 3.1(b)], we find that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $([L^r_\omega(\mathbb{R}^n)]^{1/p})'$. Thus, all the assumptions of Theorem 4.12 are satisfied for $X := L^r_\omega(\mathbb{R}^n)$ with $p \in [1,r)$ and $\omega \in A_{r/p}(\mathbb{R}^n)$.

If $p = r$ we apply the conclusion in [59, p. 9] and the assumption that $\omega \in A_{r/p}(\mathbb{R}^n)$ to obtain

$$\left([L^r_\omega(\mathbb{R}^n)]^{1/p}\right)' = L^\infty_{\omega^{-1}}(\mathbb{R}^n).$$

This combined with [5, Theorem 3.1(b)] and [59, p. 9] yields that $\mathcal{M}$ is bounded on $([L^r_\omega(\mathbb{R}^n)]^{1/p})'$. Thus, all the assumptions of Theorem 4.12 are also satisfied for $X := L^r_\omega(\mathbb{R}^n)$ when $p = r$ and $\omega \in A_{r/p}(\mathbb{R}^n)$. The conclusion of this theorem yields the claimed assertion and then finishes the proof of Theorem 5.10. □

From the proof of Theorem 5.10, it follows that all the assumptions of Theorems 2.20, 3.4, 3.36, and 4.8 with $X := L^r_\omega(\mathbb{R}^n)$ are satisfied. As a consequence of these theorems with $X := L^r_\omega(\mathbb{R}^n)$, we obtain the following results; we omit the details here.

Corollary 5.11. *Let $r \in (1,\infty)$, $p \in [1,r]$, and $\omega \in A_{r/p}(\mathbb{R}^n)$. Then Theorems 2.20, 3.4, 3.36, and 4.8 are valid for $X = L^r_\omega(\mathbb{R}^n)$.*

5.4 Variable Lebesgue spaces

Let $r:\ \mathbb{R}^n\to(0,\infty)$ be a nonnegative measurable function. Let

$$\widetilde{r}_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} r(x) \text{ and } \widetilde{r}_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} r(x).$$

A function $r:\ \mathbb{R}^n\to(0,\infty)$ is said to be *globally log-Hölder continuous* if there exist an $r_\infty\in\mathbb{R}$ and a positive constant C such that for any $x,\ y\in\mathbb{R}^n$ we have

$$|r(x)-r(y)|\le\frac{C}{\log(e+1/|x-y|)}\quad\text{and}\quad|r(x)-r_\infty|\le\frac{C}{\log(e+|x|)}.$$

The *variable Lebesgue space* $L^{r(\cdot)}(\mathbb{R}^n)$ associated with the function $r:\ \mathbb{R}^n\to(0,\infty)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{L^{r(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda\in(0,\infty):\ \int_{\mathbb{R}^n}\left[\frac{|f(x)|}{\lambda}\right]^{r(x)}dx\le 1\right\}.$$

By the definition of $L^{r(\cdot)}(\mathbb{R}^n)$, whenever $r:\ \mathbb{R}^n\to(0,\infty)$, it is easy to show that $L^{r(\cdot)}(\mathbb{R}^n)$ is a ball quasi-Banach function space. If $1\le\widetilde{r}_-\le\widetilde{r}_+<\infty$, then $(L^{r(\cdot)}(\mathbb{R}^n),\|\cdot\|_{L^{r(\cdot)}(\mathbb{R}^n)})$ is a Banach function space and hence also a ball Banach function space (see, for instance, [93, p. 94]). For more results on variable Lebesgue spaces, we refer the reader to [30, 32, 38, 67, 79, 80].

We begin with the following consequence of Theorem 2.22.

Theorem 5.12. *Let $r:\ \mathbb{R}^n\to(0,\infty)$ be globally log-Hölder continuous. Assume that both $1\le\widetilde{r}_-\le\widetilde{r}_+<\infty$ and $p\in[1,\infty)$ satisfy $n(1/\widetilde{r}_- - 1/p)<1$. Let $K(p,n)$ be as in* (1.3)*. Then, for any $f\in C_c^2(\mathbb{R}^n)$,*

$$\lim_{s\to1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\right\|^p_{L^{r(\cdot)}(\mathbb{R}^n)}=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|^p_{L^{r(\cdot)}(\mathbb{R}^n)}.$$

Proof. It follows from [93, p. 94] that the variable Lebesgue space $L^{r(\cdot)}(\mathbb{R}^n)$ is a BBF space. Moreover, by the definition of $L^{r(\cdot)}(\mathbb{R}^n)$, we have

$$\left[L^{r(\cdot)}(\mathbb{R}^n)\right]^{1/\widetilde{r}_-}=L^{r(\cdot)/\widetilde{r}_-}(\mathbb{R}^n).$$

Using this and [37, Corollary 4.4.12], together with the assumption that r is globally log-Hölder continuous, we obtain that the Hardy–Littlewood maximal operator $\mathcal{M}$ is weakly bounded on $[L^{r(\cdot)}(\mathbb{R}^n)]^{1/\widetilde{r}_-}$. Thus, the assumptions of Theorem 2.22 are satisfied for $X:=L^{r(\cdot)}(\mathbb{R}^n)$ with $1\le\widetilde{r}_-\le\widetilde{r}_+<\infty$ and $p\in[1,\infty)$ satisfying $n(1/\widetilde{r}_- - 1/p)<1$. The conclusion of this theorem then completes the proof of Theorem 5.12. □

The following theorem is a corollary of Theorem 4.12.

Theorem 5.13. *Let $r:\ \mathbb{R}^n\to(0,\infty)$ be globally log-Hölder continuous. Assume that $1<\widetilde{r}_-\le\widetilde{r}_+<\infty$ and $p\in[1,\widetilde{r}_-)$. Then $f\in W^{1,L^{r(\cdot)}(\mathbb{R}^n)}(\mathbb{R}^n)$ if and only if $f\in L^{r(\cdot)}(\mathbb{R}^n)$ and*

$$\lim_{s\to1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac1p}\right\|^p_{L^{r(\cdot)}(\mathbb{R}^n)}<\infty;$$

moreover,

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)}^p=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{L^{r(\cdot)}(\mathbb{R}^n)}^p$$

with $K(p,n)$ as in (1.3).

Proof. By the proof of Theorem 5.12, we obtain that

$$\left[L^{r(\cdot)}(\mathbb{R}^n)\right]^{1/p}=L^{r(\cdot)/p}(\mathbb{R}^n)$$

and $[L^{r(\cdot)}(\mathbb{R}^n)]^{1/p}$ is a BBF space (see, for instance, [93, p. 94]). Then, from [30, Theorem 2.80], we deduce that

$$\left[L^{r(\cdot)}(\mathbb{R}^n)\right]'=L^{r(\cdot)'}(\mathbb{R}^n)\quad\text{and}\quad\left(\left[L^{r(\cdot)}(\mathbb{R}^n)\right]^{1/p}\right)'=L^{(r(\cdot)/p)'}(\mathbb{R}^n),$$

where $r(\cdot)':=[r(\cdot)-1]/r(\cdot)$ and $(r(\cdot)/p)':=[r(\cdot)-p]/r(\cdot)$. Moreover, by [30, p. 73] and the assumption that $1<\widetilde{r}_-\le\widetilde{r}_+<\infty$, we conclude that $L^{r(\cdot)}(\mathbb{R}^n)$ and $[L^{r(\cdot)}(\mathbb{R}^n)]'$ both have absolutely continuous norms. Furthermore, from [1, Theorem 1.7], it follows that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $([L^{r(\cdot)}(\mathbb{R}^n)]^{1/p})'$ with $1<\widetilde{r}_-\le\widetilde{r}_+<\infty$ and $p\in[1,\widetilde{r}_-)$. Thus, all the assumptions of Theorem 4.12 are satisfied for $X:=L^{r(\cdot)}(\mathbb{R}^n)$ with $1<\widetilde{r}_-\le\widetilde{r}_+<\infty$ and $p\in[1,\widetilde{r}_-)$. From this and Theorem 4.12 with $X:=L^{r(\cdot)}(\mathbb{R}^n)$, we deduce the desired conclusions of the present theorem. This finishes the proof of Theorem 5.13. □

From the proof of Theorem 5.13, we deduce that all the assumptions of Theorems 2.20, 3.4, 3.36, and 4.8 with $X:=L^{r(\cdot)}(\mathbb{R}^n)$ are satisfied. Using this, we obtain the following corollary.

Corollary 5.14. *Let $r:\ \mathbb{R}^n\to(0,\infty)$ be globally log-Hölder continuous. Assume that $1<\widetilde{r}_-\le\widetilde{r}_+<\infty$ and $p\in[1,\widetilde{r}_-)$. Then Theorems 2.20, 3.4, 3.36, and 4.8 hold for $X=L^{r(\cdot)}(\mathbb{R}^n)$.*

Remark 5.15. Sobolev-type spaces $W^{1,L^{r(\cdot)}(\mathbb{R}^n)}(\mathbb{R}^n)$ associated with variable Lebesgue space were introduced in [38].

5.5 Orlicz spaces

We discuss a few basics on Orlicz spaces. A non-decreasing function $\Phi:\ [0,\infty)\to[0,\infty)$ is called an *Orlicz function* if $\Phi(0)=0$, $\Phi(t)>0$ for any $t\in(0,\infty)$, and $\lim_{t\to\infty}\Phi(t)=\infty$. An Orlicz function Φ is said to be of *lower* (resp., *upper*) *type* r for some $r\in\mathbb{R}$ if there exists a positive constant $C_{(r)}$ such that, for any $t\in[0,\infty)$ and $s\in(0,1)$ [resp., $s\in[1,\infty)$],

$$\Phi(st)\le C_{(r)}s^r\Phi(t).$$

In the remainder of this subsection, we always assume that $\Phi:\ [0,\infty)\to[0,\infty)$ is an Orlicz function with positive lower type r_Φ^- and positive upper type r_Φ^+. The Orlicz norm $\|f\|_{L^\Phi(\mathbb{R}^n)}$ of a measurable function f on $\mathbb{R}^n$ is then defined by setting

$$\|f\|_{L^\Phi(\mathbb{R}^n)}:=\inf\left\{\lambda\in(0,\infty):\ \int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\right\}.$$

Accordingly, the *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ with finite norm $\|f\|_{L^\Phi(\mathbb{R}^n)}$. It is known that $L^\Phi(\mathbb{R}^n)$ is a Banach function space when $1 \le r_\Phi^- \le r_\Phi^+ < \infty$ (see [87, p. 67, Theorem 10]).

The following theorem is a corollary of Theorem 2.22.

Theorem 5.16. *Let Φ be an Orlicz function with positive lower type r_Φ^- and positive upper type r_Φ^+. Assume that both $1 \le r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1,\infty)$ satisfy $n(1/r_\Phi^- - 1/p) < 1$. Let $K(p,n)$ be as in* (1.3)*. Then, for any $f \in C_c^2(\mathbb{R}^n)$,*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^\Phi(\mathbb{R}^n)}^p = \frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{L^\Phi(\mathbb{R}^n)}^p .$$

Proof. For $t \in [0,\infty)$ define

$$\Phi_{r_\Phi^-}(t) := \Phi(t^{1/r_\Phi^-}). \tag{5.9}$$

First, from [93, p. 94], it follows that the Orlicz space $L^\Phi(\mathbb{R}^n)$ is a BBF space. By the proof of [108, Lemma 2.31], we conclude that $\Phi_{r_\Phi^-}$ is of lower type 1 and of upper type r_Φ^+/r_Φ^-, and

$$\left[L^\Phi(\mathbb{R}^n)\right]^{1/r_\Phi^-} = L^{\Phi_{r_\Phi^-}}(\mathbb{R}^n).$$

From this and [65, Lemma 1.2.4], it follows that $\mathcal{M}$ is weakly bounded on $[L^\Phi(\mathbb{R}^n)]^{1/r_\Phi^-}$. Thus, all the assumptions of Theorem 2.22 are satisfied for $X := L^\Phi(\mathbb{R}^n)$ with $1 \le r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1,\infty)$ satisfying $n(1/r_\Phi^- - 1/p) < 1$. Then, from this and Theorem 2.22 with $X := L^\Phi(\mathbb{R}^n)$, we deduce the desired conclusion, completing the proof of Theorem 5.16. □

The following theorem is a corollary of Theorem 4.12.

Theorem 5.17. *Let Φ be an Orlicz function with positive lower type r_Φ^- and positive upper type r_Φ^+. Assume that $1 < r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1, r_\Phi^-)$. Then $f \in W^{1,L^\Phi(\mathbb{R}^n)}(\mathbb{R}^n)$ if and only if $f \in L^\Phi(\mathbb{R}^n)$ and*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^\Phi(\mathbb{R}^n)} < \infty;$$

moreover,

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^\Phi(\mathbb{R}^n)}^p = \frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{L^\Phi(\mathbb{R}^n)}^p$$

with $K(p,n)$ as in (1.3)*.*

Proof. By the proof of Theorem 5.16 we conclude that

$$\left[L^\Phi(\mathbb{R}^n)\right]^{1/p} = L^{\Phi_p}(\mathbb{R}^n),$$

and $[L^\Phi(\mathbb{R}^n)]^{1/p}$ is a BBF space, where Φ_p is as in (5.9) with r_Φ^- replaced by p. Moreover, by the proof of [108, Lemma 4.5], [65, Theorem 1.2.1], and dual theorem of $L^\Phi(\mathbb{R}^n)$ (see, for instance, [88, Theorem 13]), we further conclude that, if $1 < r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1, r_\Phi^-)$, then $L^\Phi(\mathbb{R}^n)$ and $[L^\Phi(\mathbb{R}^n)]'$ have absolutely continuous norms, and $\mathcal{M}$ is bounded on $([L^\Phi(\mathbb{R}^n)]^{1/p})'$. Thus, the assumptions of Theorem 4.12 are satisfied for $X := L^\Phi(\mathbb{R}^n)$ with $1 < r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1, r_\Phi^-)$. The conclusion of Theorem 4.12 yields the claim of Theorem 5.17. □

From the proof of Theorem 5.17, we deduce that all the assumptions of Theorems 2.20, 3.4, and 4.8 with $X := L^{\Phi}(\mathbb{R}^n)$ are satisfied. Using this, we obtain the following corollary.

Corollary 5.18. *Let Φ be an Orlicz function with positive lower type r_{Φ}^- and positive upper type r_{Φ}^+. Assume that $1 < r_{\Phi}^- \le r_{\Phi}^+ < \infty$ and $p \in [1, r_{\Phi}^-)$. Then Theorems 2.20, 3.4, and 4.8 hold with X replaced by $L^{\Phi}(\mathbb{R}^n)$.*

Remark 5.19. We point out that, when $L^{\Phi}(\mathbb{R}^n) := L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$, Theorem 5.17 reduces to [4, Theorem 1.3] with $A(t) = t^p$ for any $t \in [0, \infty)$.

5.6 Orlicz-slice spaces

We recall the definition of Orlicz-slice spaces and briefly describe some related facts. Throughout this subsection, we assume that $\Phi : [0, \infty) \to [0, \infty)$ is an Orlicz function with positive lower type r_{Φ}^- and positive upper type r_{Φ}^+. For any given $t,\ r \in (0, \infty)$, the *Orlicz-slice space* $(E_{\Phi}^r)_t(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ with the finite quasi-norm

$$\|f\|_{(E_{\Phi}^r)_t(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[\frac{\|f \mathbf{1}_{B(x,t)}\|_{L^{\Phi}(\mathbb{R}^n)}}{\|\mathbf{1}_{B(x,t)}\|_{L^{\Phi}(\mathbb{R}^n)}} \right]^r dx \right\}^{\frac{1}{r}}.$$

The Orlicz-slice spaces were introduced in [108] as a generalization of both the slice spaces of Auscher and Mourgoglou [7, 8] and the Wiener amalgam spaces in [50, 52, 64]. According to [108, Lemma 2.28] and [107, Remark 7.41(i)], the Orlicz-slice space $(E_{\Phi}^r)_t(\mathbb{R}^n)$ is a ball Banach function space, but in general is not a Banach function space.

The following result is a corollary of Theorem 2.22.

Theorem 5.20. *Let $t \in (0, \infty)$, $r \in [1, \infty)$, and Φ be an Orlicz function with positive lower type r_{Φ}^- and positive upper type r_{Φ}^+. Assume that both $1 \le r_{\Phi}^- \le r_{\Phi}^+ < \infty$ and $p \in [1, \infty)$ satisfy $n(1/\min\{r_{\Phi}^-, r\} - 1/p) < 1$. Let $K(p, n)$ be as in (1.3). Then, for any $f \in C_{\mathrm{c}}^2(\mathbb{R}^n)$,*

$$\lim_{s \to 1^-} (1 - s) \left\| \left[\int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^p}{|\cdot - y|^{n+sp}}\, dy \right]^{\frac{1}{p}} \right\|_{(E_{\Phi}^r)_t(\mathbb{R}^n)}^p = \frac{K(p, n)}{p} \, \| |\nabla f| \|_{(E_{\Phi}^r)_t(\mathbb{R}^n)}^p.$$

Proof. First, from [108, Lemma 2.28] and [107, Remark 7.41(i)], it follows that the Orlicz-slice space $(E_{\Phi}^r)_t(\mathbb{R}^n)$ is a BBF space. Then, by [108, Lemma 2.31], we conclude that

$$\left[(E_{\Phi}^r)_t(\mathbb{R}^n)\right]^{1/\min\{r_{\Phi}^-, r\}} = \left(E_{\Phi_{\min\{r_{\Phi}^-, r\}}}^{r/\min\{r_{\Phi}^-, r\}} \right)_t (\mathbb{R}^n),$$

where $\Phi_{\min\{r_{\Phi}^-, r\}}$ is as in (5.9) with r_{Φ}^- replaced by $\min\{r_{\Phi}^-, r\}$, which is of lower type $r_{\Phi}^-/\min\{r_{\Phi}^-, r\}$ and of upper type $r_{\Phi}^+/\min\{r_{\Phi}^-, r\}$. From this and [107, Proposition 7.57], we deduce that $\mathcal{M}$ is weakly bounded on $[(E_{\Phi}^r)_t(\mathbb{R}^n)]^{1/\min\{r_{\Phi}^-, r\}}$. Thus the assumptions of Theorem 2.22 are satisfied for $X := (E_{\Phi}^r)_t(\mathbb{R}^n)$ with $1 \le r_{\Phi}^- \le r_{\Phi}^+ < \infty$ and $p \in [1, \infty)$ satisfying $n(1/\min\{r_{\Phi}^-, r\} - 1/p) < 1$. The conclusion of Theorem 2.22 yields the claimed assertion of Theorem 5.20. □

The following theorem is a corollary of Theorem 4.12.

Theorem 5.21. *Let $t \in (0,\infty)$, $r \in [1,\infty)$, and Φ be an Orlicz function with positive lower type r_Φ^- and positive upper type r_Φ^+. Assume $1 < r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1, \min\{r_\Phi^-, r\})$. Then f lies in $W^{1,(E_\Phi^r)_t(\mathbb{R}^n)}(\mathbb{R}^n)$ if and only if $f \in (E_\Phi^r)_t(\mathbb{R}^n)$ and*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{(E_\Phi^r)_t(\mathbb{R}^n)}^p < \infty;$$

moreover,

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{(E_\Phi^r)_t(\mathbb{R}^n)}^p = \frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{(E_\Phi^r)_t(\mathbb{R}^n)}^p$$

with $K(p,n)$ as in (1.3).

Proof. In view of the proof of Theorem 5.16 we have that

$$\left[(E_\Phi^r)_t(\mathbb{R}^n)\right]^{1/p} = (E_{\Phi_p}^{r/p})_t(\mathbb{R}^n),$$

and $[(E_\Phi^r)_t(\mathbb{R}^n)]^{1/p}$ is a BBF space, where Φ_p is as in (5.9) with r_Φ^- replaced by p. Furthermore, using [108, Theorem 2.26 and Lemmas 4.4 and 4.5] and the assumptions that $1 < r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1, \min\{r_\Phi^-, r\})$, we conclude that $(E_\Phi^r)_t(\mathbb{R}^n)$ and $[(E_\Phi^r)_t(\mathbb{R}^n)]'$ have absolutely continuous norms and that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $([(E_\Phi^r)_t(\mathbb{R}^n)]^{1/p})'$. Thus the assumptions of Theorem 4.12 are satisfied for $X := (E_\Phi^r)_t(\mathbb{R}^n)$ with $1 < r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1, \min\{r_\Phi^-, r\})$. The conclusion of this theorem completes the proof of Theorem 5.21. □

From the proof of Theorem 5.21, we deduce that the assumptions of Theorems 2.20, 3.36, 3.4, and 4.8 with $X := (E_\Phi^r)_t(\mathbb{R}^n)$ are satisfied. Summarizing, we have the following result.

Corollary 5.22. *Let $t \in (0,\infty)$, $r \in [1,\infty)$, and Φ be an Orlicz function with positive lower type r_Φ^- and positive upper type r_Φ^+. Assume $1 < r_\Phi^- \le r_\Phi^+ < \infty$ and $p \in [1, \min\{r_\Phi^-, r\})$. Then Theorems 2.20, 3.4, and 4.8 are valid for $X = (E_\Phi^r)_t(\mathbb{R}^n)$.*

5.7 Lorentz spaces

The *Lorentz space* $L^{q,r}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that, when $q, r \in (0,\infty)$,

$$\|f\|_{L^{q,r}(\mathbb{R}^n)} := \left\{\int_0^\infty \left[t^{\frac{1}{q}} f^*(t)\right]^r \frac{dt}{t}\right\}^{\frac{1}{r}} < \infty,$$

where f^* denotes the *decreasing rearrangement of* f, defined by setting, for any $t \in [0,\infty)$,

$$f^*(t) := \inf\{s \in (0,\infty) : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \le t\}.$$

We adopt the convention $\inf\emptyset = \infty$, thus having $f^*(t) = \infty$ whenever $d_f(s) > t$ for all $s \ge 0$.

Obviously, when $q, r \in (1,\infty)$, the Lorentz space $L^{q,r}(\mathbb{R}^n)$ is a Banach function space and hence a ball Banach function space; when $q, r \in (0,\infty)$, $L^{q,r}(\mathbb{R}^n)$ is a quasi-Banach function space and hence a ball quasi-Banach function space (see, for instance, [47, Theorem 1.4.11]).

The following result is a corollary of Theorem 2.15.

Theorem 5.23. *Let q, $r \in (1,\infty)$ and $p \in [1,\infty)$ satisfy $n(1/q-1/p)<1$. Let $K(p,n)$ be as in* (1.3). *Then, for any $f \in C_c^2(\mathbb{R}^n)$,*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^{q,r}(\mathbb{R}^n)}^p=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{L^{q,r}(\mathbb{R}^n)}^p.$$

Proof. From the conclusion in [93, p. 87], it follows that the Lorentz space $L^{q,r}(\mathbb{R}^n)$ is a BBF space. Using this fact and Theorem 2.15, it follows that in order to prove the required conclusion, it would suffice to show that $L^{q,r}(\mathbb{R}^n)$ is locally n/α-doubling with $n/\alpha \in (0,1+n/p)$. Indeed, by the definition of $L^{q,r}(\mathbb{R}^n)$, for any $B_0 := B(\mathbf{0},r)\in\mathbb{B}$, we find that

$$\|\mathbf{1}_{B_0}\|_{L^{q,r}(\mathbb{R}^n)}=\left\{\int_0^{|B(\mathbf{0},r)|}t^{\frac{r}{q}-1}\,dt\right\}^{\frac{1}{r}}=\left(\frac{r}{q}\right)^r|B_0|^{1/q}.$$

This implies that, for any $\lambda\in[1,\infty)$,

$$\|\mathbf{1}_{\lambda B_0}\|_{L^{q,r}(\mathbb{R}^n)}=\left(\frac{r}{q}\right)^r|\lambda B_0|^{1/q}=\lambda^{n/q}\left(\frac{r}{q}\right)^r|B_0|^{1/q}=\lambda^{n/q}\|\mathbf{1}_{B_0}\|_{L^{q,r}(\mathbb{R}^n)}.$$

Thus, $L^{q,r}(\mathbb{R}^n)$ is locally n/q-doubling. Observe that the assumption that $n(1/q-1/p)<1$ gives $n/q\in(0,1+n/p)$ and this completes the proof of Theorem 5.23. □

The following result is a consequence of Theorem 4.12.

Theorem 5.24. *Let q, $r\in(1,\infty)$ and $p\in[1,\min\{q,r\})$. Then $f\in W^{1,L^{q,r}(\mathbb{R}^n)}(\mathbb{R}^n)$ if and only if $f\in L^{q,r}(\mathbb{R}^n)$ and in this case we have*

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^{q,r}(\mathbb{R}^n)}^p<\infty;$$

moreover,

$$\lim_{s\to 1^-}(1-s)\left\|\left[\int_{\mathbb{R}^n}\frac{|f(\cdot)-f(y)|^p}{|\cdot-y|^{n+sp}}\,dy\right]^{\frac{1}{p}}\right\|_{L^{q,r}(\mathbb{R}^n)}^p=\frac{K(p,n)}{p}\,\|\,|\nabla f|\,\|_{L^{q,r}(\mathbb{R}^n)}^p$$

with $K(p,n)$ as in (1.3).

Proof. In view of the identity in [47, Remark 1.4.7],

$$[L^{q,r}(\mathbb{R}^n)]^{1/p}=L^{q/p,r/p}(\mathbb{R}^n)$$

and thus $[L^{q,r}(\mathbb{R}^n)]^{1/p}$ is a BBF space (see, for instance, [93, p. 87]). Then, from this and [47, Theorem 1.4.16 (vi)], it follows that

$$[L^{q,r}(\mathbb{R}^n)]'=L^{q',r'}(\mathbb{R}^n)\quad\text{and}\quad\left([L^{q,r}(\mathbb{R}^n)]^{1/p}\right)'=L^{(q/p)',(r/p)'}(\mathbb{R}^n).\tag{5.10}$$

Moreover, by [101, Remark 3.4(iii)], we conclude that $L^{q,r}(\mathbb{R}^n)$ and $[L^{q,r}(\mathbb{R}^n)]'$ both have absolutely continuous norms. Furthermore, the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded from $L^{t,s}(\mathbb{R}^n)$ to itself when $1 < t, s < \infty$; this can be obtained by interpolation from the endpoint cases where $(t, s) = (t, 1)$ and $(t, s) = (t, \infty)$, which can be found, for instance in [29]. This, combined with (5.10), further implies that $\mathcal{M}$ is bounded on $([L^{q,r}(\mathbb{R}^n)]^{1/p})'$. Thus, all the assumptions of Theorem 4.12 are satisfied for $X := L^{q,r}(\mathbb{R}^n)$ with $q, r \in (1, \infty)$ and $p \in [1, \min\{q, r\})$. From this and Theorem 4.12 with $X := L^{q,r}(\mathbb{R}^n)$, we deduce the claimed conclusions. This finishes the proof of Theorem 5.24. □

From the proof of Theorem 5.24, we deduce that the assumptions of Theorems 2.13, 3.4, 3.36, and 4.8 with $X := L^{q,r}(\mathbb{R}^n)$ are satisfied. Thus we obtain the following results.

Corollary 5.25. *Let $q, r \in (1, \infty)$ and $p \in [1, \min\{q, r\})$. Then Theorems 2.13, 3.4, 3.36, and 4.8 are valid for $X = L^{q,r}(\mathbb{R}^n)$.*

6 Final remarks

We prove the identity concerning the value of the constant in (1.3). Applying an orthogonal transformation we may assume that e is the unit vector $e_1 = (1, 0, \dots, 0)$ in $\mathbb{S}^{n-1}$. Using the identity in [47, Appendix D3] we write

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} |\xi \cdot e_1|^p \, d\sigma(\xi) &= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} |s|^p (1 - s^2)^{\frac{n-3}{2}} \, ds \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{0}^{1} (s^2)^{\frac{p-1}{2}} (1 - s^2)^{\frac{n-3}{2}} 2s \, ds \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{0}^{1} u^{\frac{p-1}{2}} (1 - u)^{\frac{n-3}{2}} \, du \\
&= \frac{2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)},
\end{aligned}$$

in view of the Beta function identity

$$\int_0^1 u^{a-1}(1-u)^{b-1} du = B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b), \ \forall\, a, b > 0.$$

Acknowledgements. The authors would like to thank the referee for their careful reading and for providing many motivating remarks which indeed improved the exposition of this article and led to Proposition 5.3. Indeed, the upper estimate (5.3) of Proposition 5.3 is attributed to the referee.

Data availability Our manuscript has no associated data.

Declarations

Conflict of interest The authors state that there is no conflict of interest.

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