THE BILINEAR MULTIPLIER PROBLEM FOR STRICTLY CONVEX COMPACT SETS

LOUKAS GRAFAKOS AND MARIA CARMEN REGUERA RODRÍGUEZ

Abstract. We study the question whether characteristic functions of strictly convex compact sets with smooth boundaries in $\mathbb{R}^{2n}$ are $L^p \times L^q \to L^r$ bounded bilinear Fourier multiplier operators on $\mathbb{R}^n \times \mathbb{R}^n$. When $n \geq 2$ we answer this question in the negative outside the local L^2 case, i.e., when $1/p + 1/q = 1/r$ and $2 \leq p, q, r' < \infty$ fails. Our proof is based on a suitable adaptation of the Kakeya type construction employed by Fefferman in the solution of the multiplier problem for the ball on $L^p(\mathbb{R}^2)$ for $p \neq 2$.

1. Introduction

It is well-known that the presence of curvature in the boundary of geometric regions affects negatively the L^p boundedness of the Fourier multiplier operators associated with the characteristic functions of these regions. This dramatic fact first made its appearance in the work of Fefferman [6] who showed that characteristic functions of balls in $\mathbf{R}^n$ are not bounded Fourier multiplier operators on $L^p(\mathbf{R}^n)$ when $p \neq 2$ and $n \geq 2$. Fefferman's proof was based on a variant of a construction of Besicovitch [1] employed in the solution of Kakeya's question concerning the smallest possible area of a set that contains line segments in all directions. On the latter, one may also consult the article of Cunningham [2].

The bilinear multiplier problem for the ball was studied by Diestel and Grafakos [4] who obtained that the characteristic function of the four-dimensional ball is not a bounded bilinear multiplier operator from $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^r(\mathbb{R}^2)$ outside the local L^2 case, i.e., when $1/p + 1/q = 1/r$ and $2 \leq p, q, r' < \infty$ fails. Here $r' = r/(r-1)$. This example can be lifted to higher dimensions, i.e. replace $\mathbb{R}^2$ by $\mathbb{R}^n$ for $n \geq 2$, by a bilinear version of de Leuuw's theorem [4]. Conversely, it was shown by Grafakos and Li [8] that the characteristic function of the unit disc in $\mathbb{R}^2$ is a bounded bilinear multiplier on $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ in the local L^2 case. The corresponding problem in $\mathbb{R}^{2n}$ for $n \geq 2$ remains unresolved. As of this writing it is also unknown whether the characteristic function of the unit disc in $\mathbb{R}^2$ is a bounded bilinear Fourier multiplier outside the local L^2 case.

In this article we address similar questions for strictly convex compact sets. This study is motivated by the following fact. In proving that the ball $|\xi|^2 + |\eta|^2 \leq 1$ in $\mathbf{R}^4$ is not a bilinear multiplier outside the local-L^2 case, one has to show the same result for the two duals of the ball multiplier, the special ellipsoids $|\xi + \eta|^2 + |\xi|^2 \leq 1$ and $|\xi + \eta|^2 + |\eta|^2 \leq 1$ in $\mathbf{R}^4$. These ellipsoids are strictly convex compact sets with smooth boundaries and the class of such sets is closed under bilinear duality. Therefore, this class provides a more appropriate general context for the study of this problem and it is quite natural to pursue the study in this framework.

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Throughout this paper, E denotes a strictly convex compact hypersurface in $\mathbb{R}^{2n}$ with smooth boundary. We have the following theorem concerning E.

Theorem 1. *Let $1 \le p, q, r' \le \infty$ be such that $1/p + 1/q = 1/r$ and at least one of them is strictly less than 2. Then the characteristic function of E is not a bounded bilinear Fourier multiplier from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$. In other words, the operator*

$$T_{\chi_E}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_E(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta$$

does not map $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ in this case.

In proving Theorem 1 we may assume that $n = 2$, as the two-dimensional counterexample can be “lifted” to higher dimensions via the multilinear version of de Leeuw’s theorem proved in [4]; for the linear case see de Leeuw [3].

2. The Kakeya Construction

We will use a Kakeya type construction to prove that the bilinear operator whose symbol is the characteristic function of the strictly convex set E is unbounded from $L^p(\mathbb{R}^2)\times L^q(\mathbb{R}^2)$ to $L^r(\mathbb{R}^2)$ when $\min(p,q,r') < 2$.

For a rectangle R in $\mathbb{R}^2$, we define R' to be the union of the two copies of R adjacent to R in the direction of its longest side. The proof of this lemma can be found in [7], page 738 and also in [13] page 435.

Lemma 1. *Let $\delta > 0$ be given. Then there exist a measurable subset U of $\mathbb{R}^2$ and a finite collection of rectangles R_j in $\mathbb{R}^2$ such that*

(1) *The R_j are pairwise disjoint.*
(2) *We have $1/2 \le |U| \le 3/2$.*
(3) *We have $|U| \le \delta \sum_j |R_j|$.*
(4) *For all j we have $|R'_j \cap U| \ge \frac{1}{12}|R_j|$.*

We are also going to use the following proposition, whose proof can be found in [4].

Proposition 1. *Let R be a rectangle in $\mathbb{R}^2$ and let v be a unit vector in $\mathbb{R}^2$ parallel to the longest side of R. Let R' be as above. Consider the half space $\mathcal{H}_v$ of $\mathbb{R}^4$ defined by*

$$\mathcal{H}_v = \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : \ (\xi+\eta)\cdot v \ge 0\}$$

Then for all $x \in \mathbb{R}^2$ the following estimate is valid:

$$\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi_{\mathcal{H}_v}(\xi,\eta)\widehat{\chi_R}(\xi)\widehat{\chi_R}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta \right| \ge \frac{1}{10}\chi_{R'}(x). \tag{2.1}$$

We denote by $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ the space of all bounded bilinear Fourier multipliers from $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^r(\mathbb{R}^2)$.

Consider the bilinear multiplier operator on $\mathbb{R}^2 \times \mathbb{R}^2$ whose symbol is the characteristic function of E, that is the operator

$$T_{\chi_E}(f,g)(x) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi_E(\xi_1,\xi_2,\xi_3,\xi_4)\widehat{f}(\xi_1,\xi_2)\widehat{g}(\xi_3,\xi_4)e^{2\pi i x\cdot(\xi_1+\xi_3,\xi_2+\xi_4)}d\xi_1 d\xi_2 d\xi_3 d\xi_4\,.$$

The following lemma is the most fundamental ingredient of the proof.

Lemma 2. *Let $v_1, v_2, ..., v_j, ...$ be a sequence of unit vectors in $\mathbb{R}^2$. Define the following sequence of half-spaces $\mathcal{H}_{v_j}$ in $\mathbb{R}^4$ as in Proposition 1.*

Assume that $T_{\chi_E} \in \mathcal{M}_{p,q,r}(\mathbb{R}^2)$ with norm bounded from above by $C = C(p,q,r)$. Then we have the following vector-valued inequality

$$\Big\| \Big(\sum_j |T_{\chi_{\mathcal{H}_{v_j}}}(f_j, g_j)|^2 \Big)^{1/2} \Big\|_r \leq C \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \Big\|_q . \tag{2.2}$$

for all functions f_j and g_j.

Proof. We assume that T_{χ_E} lies in $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ for some p, q, r. Without loss of generality we may assume that 0 lies in the interior of E.

For each unit vector $v \in \mathbb{R}^2$ find vector q_v in $\mathbb{R}^4$ such that $q_v + E$ is contained in $\mathcal{H}_v$ and touches $\mathcal{H}_v$ exactly at the origin. The strict convexity of E gives that the Gauss map is a diffeomorphism and thus that there is only one such point of contact. Moreover there is a closed ball B_v such that

$$B_v \subset q_v + E \subset \mathcal{H}_v$$

and all these sets intersect exactly at the origin; for a proof of these facts we refer to [12] and [5].

For $R > 0$, let RA denote the dilation of any set A about the origin, that is, $RA = \{Rx : x \in A\}$. Then the set $Rq_v + RE$ is contained in $\mathcal{H}_v$ and also touches $\mathcal{H}_v$ at the origin. Then we have

$$RB_v \subset Rq_v + RE \subset \mathcal{H}_v$$

and all these sets intersect exactly at the origin. For every fixed v, as $R \to \infty$ we have that RB_v fills up the whole half plane $\mathcal{H}_v$, and thus so does $Rq_v + RE$.

Consider the sequence of unit vectors v_k given in the statement of the lemma. We have constructed dilations and translations $Rq_v + RE$ of E such that $\chi_{Rq_v+RE} \to \chi_{\mathcal{H}_v}$ pointwise as $R \to \infty$. Thus

$$\lim_{R\to\infty} T_{\chi_{Rq_v+RE}}(f,g)(x) = T_{\chi_{\mathcal{H}_{v_k}}}(f,g)(x)$$

for all $x \in \mathbb{R}^2$ and good functions f and g. Consequently, using Fatou's lemma we can pass to the limit as $R \to \infty$ to obtain

$$\Big\| \Big(\sum_k |T_{\chi_{\mathcal{H}_{v_k}}}(f_k, g_k)|^2 \Big)^{1/2} \Big\|_r \leq \liminf_{R\to\infty} \Big\| \Big(\sum_k |T_{\chi_{Rq_{v_k}+RE}}(f_k, g_k)|^2 \Big)^{1/2} \Big\|_r \tag{2.3}$$

for good functions f_k, g_k. As bilinear multiplier norms are dilation invariant, it follows that for all $R > 0$ we have

$$\|\chi_{RE}\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^2)} = \|\chi_E\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^2)} = C.$$

Moreover, by the bilinear version of a theorem of Marcinkiewicz and Zygmund ([9], section 9), we have the following inequality for all $R > 0$ and all functions f_k, g_k

$$\Big\| \Big(\sum_k |T_{\chi_{RE}}(f_k, g_k)|^2 \Big)^{1/2} \Big\|_r \leq C \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_q .$$

Let $q_{v_k} = (q^1_{v_k}, q^2_{v_k})$. Putting these observations together we deduce

$$\begin{aligned}
&\Big\| \Big(\sum_k \big| T_{\chi_{\mathcal{H}_k}}(f_k, g_k) \big|^2 \Big)^{1/2} \Big\|_r \\
\le\ & \liminf_{R\to\infty} \Big\| \Big(\sum_k \big| T_{\chi_{Rq_{v_k}+RE}}(f_k, g_k) \big|^2 \Big)^{1/2} \Big\|_r \\
=\ & \liminf_{R\to\infty} \Big\| \Big(\sum_k \big| e^{2\pi i R (q^1_{v_k}+q^2_{v_k})\cdot(\cdot)} T_{\chi_{RE}}\big(e^{-2\pi i R q^1_{v_k}\cdot(\cdot)} f_k, e^{-2\pi i R q^2_{v_k}\cdot(\cdot)} g_k\big) \big|^2 \Big)^{1/2} \Big\|_r \\
\le\ & \liminf_{R\to\infty} \big\| \chi_{RE} \big\|_{\mathcal{M}_{p,q,r}} \Big\| \Big(\sum_k \big| e^{-2\pi i R q^1_{v_k}\cdot(\cdot)} f_k \big|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_k \big| e^{-2\pi i R q^2_{v_k}\cdot(\cdot)} g_k \big|^2 \Big)^{1/2} \Big\|_q \\
=\ & C \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_q ,
\end{aligned}$$

where the last equality follows from the dilation invariance of bilinear multiplier norms.

□

3. The main argument

We now prove the main result of this article, Theorem 1. We consider four cases.

Case (a): $p, q, r > 2$.

Reasoning by contradiction, let us suppose that χ_E is in $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ with norm C. Suppose that $\delta > 0$ is given. Let U and R_j be as in Lemma 1. Let v_j be the unit vector parallel to the longest side of R_j.We will estimate $\sum_j \int_U \big| T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x) \big|^2 dx$ from above and below to obtain the desired contradiction. On one hand we have

$$\begin{aligned}
&\sum_j \int_U \big| T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x) \big|^2 dx \\
\le\ & |U|^{\frac{r-2}{r}} \Big\| \Big(\sum_j \big| T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j}) \big|^2 \Big)^{1/2} \Big\|_r^2 \\
\le\ & C\, |U|^{\frac{r-2}{r}} \Big\| \Big(\sum_j \big| \chi_{R_j} \big|^2 \Big)^{1/2} \Big\|_p^2 \Big\| \Big(\sum_j \big| \chi_{R_j} \big|^2 \Big)^{1/2} \Big\|_q^2 \\
=\ & C\, |U|^{\frac{r-2}{r}} \Big(\sum_j |R_j| \Big)^{2/r} \\
\le\ & C\, \delta^{\frac{r-2}{r}} \sum_j |R_j|.
\end{aligned}$$

On the other hand, we get the following estimate

$$\begin{aligned}
\sum_j \int_U \big| T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x) \big|^2 dx \ &\ge\ \sum_j \int_U \big(\frac{1}{10} \chi_{R'_j}(x) \big)^2 dx \\
&=\ (\tfrac{1}{10})^2 \sum_j \big| U \cap R'_j \big|
\end{aligned}$$

$$\geq \quad \tfrac{1}{1200} \sum_j |R_j| \,.$$

Combining these two estimates, we obtain that

$$\tfrac{1}{1200} \sum_j |R_j| \leq C\, \delta^{\frac{r-2}{r}} \sum_j |R_j|$$

and therefore

$$\tfrac{1}{1200} \leq C\, \delta^{\frac{r-2}{r}}$$

for any $\delta > 0$. This is a contradiction since $r > 2$.

There are two more cases left in the Banach triangle $1 < p, q, r < \infty$.

Case (b) $p > 2$**,** $q < 2$**,** $r < 2$**,** and

Case (c) $p < 2$**,** $q > 2$**,** $r < 2$**.**

Both of these cases follow by duality and the fact that the dual operators of T_{χ_E} are also bilinear multiplier operators whose symbols are characteristic functions of sets with the same properties. Indeed, the multipliers of the two duals are the characteristic functions of the sets

$$\{\xi \in \mathbb{R}^4 : \; A_1\xi \in E\} = A_1^{-1}[E]$$

and

$$\{\xi \in \mathbb{R}^4 : \; A_2\xi \in E\} = A_2^{-1}[E]$$

where

$$A_1 = \begin{pmatrix} -I & -I \\ O & I \end{pmatrix} \qquad A_2 = \begin{pmatrix} I & O \\ -I & -I \end{pmatrix}.$$

Here I is the identity 2×2 matrix and O the zero 2×2 matrix. Since the matrices A_1 and A_2 are invertible, it follows that the sets $A_1^{-1}[E]$ and $A_1^{-1}[E]$ are images of E under linear transformations and they must also be compact and strictly convex. Thus in case (b) the pair (q, r) is replaced by (r', q') for which the counterexample of case (a) applies. Likewise in case (c).

We now show unboundedness outside the Banach case. We consider the remaining case.

Case (d): $1 \leq p, q < \infty$**,** $\frac{1}{2} < r \leq 1$**.**

Reasoning by contradiction, let us suppose $\chi_E \in \mathcal{M}_{p,q,r}(\mathbb{R}^2)$ for $\frac{1}{2} < r \leq 1$. As we are assuming that the set E is strictly convex compact hypersuface with smooth boundary, we can fill the half-space,

$$\mathcal{H} = \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \;\; : \;\; \xi_2 + \xi_4 > 0\}$$

using dilations and translations of E. Thus if $\chi_E \in \mathcal{M}_{p,q,r}$, then $\chi_{\mathcal{H}} \in \mathcal{M}_{p,q,r}(\mathbb{R}^2)$.

For $f, g \in S(\mathbb{R}^2)$, then we have

$$T_{\chi_{\mathcal{H}}}(f, g) = \frac{(I + i\mathbf{H}_{(0,1)})(fg)}{2},$$

where $\mathbf{H}_{\vec{\alpha}}$ denotes the two-dimensional directional Hilbert transform in the direction $\alpha \in \mathbb{R}^2$ and it is defined as

$$\mathbf{H}_{\vec{\alpha}}(f)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x - t\vec{\alpha}) \frac{dt}{t}.$$

The Fourier multiplier of the operator $T_{\chi_{\mathcal{H}}}$ is a characteristic function of a certain half-space passing through the origin in $\mathbb{R}^4$. Such operators can be viewed as 2-dimensional versions of billinear Fourier multipliers given by characteristic functions of half-planes passing through the origin in $\mathbb{R}^2$. The latter are related to the bilinear Hilbert transform and have been studied by Lacey and Thiele [10], [11].

It will suffice to show that the bilinear operator $(f,g) \to \mathbf{H}_{(0,1)}(fg)$ is not bounded from $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ to $L^r(\mathbb{R}^2)$ for $r \le 1$. We choose $f(x) = g(x) = \chi_{\{x:\, ||x|| \le 1\}}(x)$ and we set $x = (x_1, x_2)$. Then

$$\begin{aligned}\mathbf{H}_{(0,1)}(fg)(x) &= \mathbf{H}_{(0,1)}(f)(x) = \int_{\mathbb{R}} \chi_{\{x:\, ||x||\le 1\}}(x-(0,1)t)\frac{dt}{t} = \int_{-\sqrt{1-x_1^2}+x_2}^{\sqrt{1-x_1^2}+x_2} \frac{dt}{t} \\ &= \ln \frac{\sqrt{1-x_1^2}+x_2}{x_2-\sqrt{1-x_1^2}} = \ln\left(1+\frac{2\sqrt{1-x_1^2}}{x_2-\sqrt{1-x_1^2}}\right),\end{aligned}$$

but for $x_1 < \frac{1}{\sqrt{2}}$ and x_2 large enough, $\ln\left(1+\frac{2\sqrt{1-x_1^2}}{x_2-\sqrt{1-x_1^2}}\right)$ behaves as $\frac{2\sqrt{1-x_1^2}}{x_2-\sqrt{1-x_1^2}}$ which is not in $L^r(\mathbb{R}^2)$ for $r \le 1$. Hence $\chi_{\mathcal{H}} \notin \mathcal{M}_{p,q,r}(\mathbb{R}^2)$ and we reach the desired contradiction.

This argument proves that χ_E is not a bounded bilinear multiplier from $L^p(\mathbb{R}^2)\times L^q(\mathbb{R}^2)$ to $L^r(\mathbb{R}^2)$ for $1/2 < r \le 1$ whenever $1/p + 1/q = 1/r$. □.

Using a simple variant of the main argument, the main result can be proved for m-linear multiplier operators, $m \in \{1,2,3,\dots\}$, whose symbols are the characteristic functions of smooth compact strictly convex subsets of $\mathbb{R}^{mn}$ when $n \ge 2$. In doing so, one needs to first obtain easy extensions of Proposition 1 and Lemma 2 for any $m \in \mathbb{N}$.

Particular examples of this theorem appear in the case when E is an ellipsoid in $\mathbb{R}^{mn}$, that is, E is the image of the unit ball in $\mathbb{R}^{mn}$ under an invertible transformation.

Finally, we note that it is not necessary to assume that the curvature of the boundary of the convex set E is nonzero at every point. It suffices to assume that it is nonzero only in a small neighborhood of the boundary that has normal vectors of a certain form. Then one can construct a Kakeya set whose directions are contained in the set of normal directions of this piece of the boundary and the proof can be accomplished in the same way.

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: grafakosl@missouri.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA
E-mail address: mreguera@math.gatech.edu