A WEAK-TYPE ESTIMATE FOR COMMUTATORS

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ABSTRACT. Let K be a smooth Calderón-Zygmund convolution kernel on $\mathbf{R}^2$ and suppose we are given a function $a \in L^\infty$. The two-dimensional commutator

$$Tf(x) = \int K(x-y)f(y) \fint_{[x,y]} a(z)\, dz\, dy$$

was shown to be bounded on $L^p(\mathbf{R}^2)$, $p > 1$ by Christ and Journé [1]. In this article, we show that this operator is also of weak type $(1,1)$.

1. INTRODUCTION

Suppose that we have a smooth Calderón-Zygmund convolution kernel K on $\mathbf{R}^2$ and a function $a \in L^\infty$. Christ and Journé [1] introduced the two-dimensional commutator operator

$$Tf(x) = \text{p.v.} \int K(x-y)f(y) \fint_{[x,y]} a(z)\, dz\, dy\,. \tag{1}$$

Here $\fint_{[x,y]} a(z)\, dz = \int_0^1 a\big((1-t)x + ty\big)\, dt$ denotes the average of the function a over the line segment $[x,y]$ that joins the points x and y in $\mathbf{R}^2$, and f is a Schwartz function on $\mathbf{R}^2$. The operator Tf is a singular integral operator with kernel

$$L(x,y) = K(x-y) \fint_{[x,y]} a(z) dz.$$

The smooth Calderón-Zygmund kernel K on $\mathbf{R}^2 \setminus \{0\}$ is assumed to satisfy the size condition

$$|K(x)| \le C|x|^{-2}, \tag{2}$$

the cancelation condition

$$\int_{R<|x|<2R} K(x)\, dx = 0$$

for all $R > 0$, and the smoothness condition

$$|\nabla K(x)| \le C|x|^{-3}\,. \tag{3}$$

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Obviously $L(x,y)$ inherits the standard size condition ($|L(x,y)| \le C|x-y|^{-2}$) from property (2) of the kernel K, but it does not inherit any smoothness in the usual pointwise sense due to the presence of the nonsmooth function $(x,y) \to \fint_{[x,y]} a(z)dz$.

The operator T was shown to be bounded on L^p by Christ and Journé [1] for $1 < p < \infty$. It is not possible to prove the weak $(1,1)$ estimate for T by a straightforward application of the Calderón-Zygmund method, since the kernel L is not a smooth function in general. Therefore, we adapt the method of Christ [2] (further developed by Christ and Rubio de Francia [3] and by Hofmann [4]) to the present setting to obtain the result below. This method was originally used for nonsmooth operators with homogeneous convolution kernels but seems to be relevant to the operator in question. The following theorem is our main result:

Theorem 1. *The operator T defined in* (1) *is of the weak type* $(1,1)$*, i.e. there is a constant $C>0$ such that*

$$\big|\{x\in \mathbf{R}^2:\ |T(f)(x)|>\alpha\}\big| \le C\,\|a\|_\infty \frac{\|f\|_1}{\alpha} \tag{4}$$

for all $\alpha > 0$.

2. Proof of the Theorem 1

We introduce a smooth function ϕ on $[0,\infty)$ such that $0\le\phi\le 1$, $\phi(x)=1$ for $x\in[0,1]$ and $\phi(x)=0$ for $x\in[2,\infty)$. We decompose the kernel K as $K=\sum_{k=-\infty}^{\infty} K_k$, where $K_k(x) = (\phi(2^{-k+1}|x|)-\phi(2^{-k}|x|))K(x)$ and define the corresponding operators T_k and kernels L_k associated with K_k. Clearly we have $T=\sum_{k=-\infty}^{\infty} T_k$ and $L=\sum_{k=-\infty}^{\infty} L_k$.

It was shown in [1] that the operator T is bounded on $L^2(\mathbf{R}^2)$ with bound at most a multiple of $\|a\|_\infty$. In order to prove the weak $(1,1)$ estimate (4), we are going to apply the Calderón-Zygmund decomposition $f=g+b$ at height $\alpha/\|a\|_\infty$. The functions g,b satisfy: $\|g\|_1\le\|f\|_1$, $\|g\|_\infty \le C\alpha/\|a\|_\infty$ and $b=\sum_j b_{Q_j}$, the b_{Q_j} are supported in dyadic cubes Q_j with disjoint interiors, $\int b_{Q_j}(x)dx=0$, $\|b_{Q_j}\|_1 \le C\alpha|Q_j|/\|a\|_\infty$, and

$$\sum_j |Q_j| \le \|a\|_\infty \frac{\|f\|_1}{\alpha}.$$

If $5Q_j$ denotes the cube with five times the side length of Q_j and the same center, obviously, the set $E=\bigcup_j 5Q_j$ satisfies

$$|E| \le C\,\|a\|_\infty \frac{\|f\|_1}{\alpha}\,. \tag{5}$$

As in [2], we denote by $B_k=\sum b_{Q_j}$, where the sum is taken over all indices j such that the sidelength of Q_j is 2^k. We also set

$$\mathcal{Q}_k = \{Q_j\ :\ \text{sidelength}\, Q_j\ \text{ is }\ 2^k\}.$$

We now state the key lemma:

Lemma 1. *There exists $\epsilon > 0$ such that for any nonnegative integer s we have*

$$\Big\| \sum_{j\in\mathbb{Z}} T_j B_{j-s} \Big\|_2^2 \le C_K 2^{-\epsilon s} \alpha \|b\|_1 \|a\|_\infty, \tag{6}$$

where C_K is a constant dependent on the properties (2) *and* (3) *of K.*

Once this lemma is established, the theorem is quickly proved as follows: we write

$$Tb(x) = \sum_{s\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} T_j B_{j-s}.$$

Naturally, if $x \notin E$ and $s < 0$ we have $T_j B_{j-s}(x) = 0$. Therefore, we have

$$|\{|Tb| > \alpha/2\}| \le |E| + \left| \left\{ \Big| \sum_{s\ge 0} \sum_{j\in\mathbb{Z}} T_j B_{j-s} \Big| > \alpha/2 \right\} \right|.$$

From Lemma 1 we obtain

$$\Big\| \sum_{s\ge 0} \sum_{j\in\mathbb{Z}} T_j B_{j-s} \Big\|_2^2 \le C\alpha \|b\|_1 \|a\|_\infty$$

and thus, it follows from Chebychev's inequality that

$$\left| \left\{ \Big| \sum_{s\ge 0} \sum_{j\in\mathbb{Z}} T_j B_{j-s} \Big| > \alpha \right\} \right| \le C\, \frac{\|b\|_1}{\alpha}\, \|a\|_\infty \le C\, \frac{\|f\|_1}{\alpha}\, \|a\|_\infty.$$

In view of (5), an analogous estimate holds for the set $|E|$, while the estimate

$$|\{|Tg| > \alpha/2\}| \le \frac{4}{\alpha^2} \|g\|_2^2 \, \|T\|^2 \le C' \frac{\alpha \|f\|_1 / \|a\|_\infty}{\alpha^2} C \|a\|_\infty^2 = C''\, \frac{\|f\|_1}{\alpha}\, \|a\|_\infty$$

is a consequence of the L^2 boundedness of T with norm $\|T\| \le C\|a\|_\infty$, see [1].

3. Geometric estimates

In this section we focus on the proof of Lemma 1. We write

$$\Big\| \sum_{j\in\mathbb{Z}} T_j B_{j-s} \Big\|_2^2 = \sum_{i,j\in\mathbb{Z}} < T_j B_{j-s}, T_i B_{i-s} > = \sum_{i,j\in\mathbb{Z}} < \tilde{T}_i T_j B_{j-s}, B_{i-s} > , \tag{7}$$

where $\tilde{T}_i$ is the adjoint operator of T_i. The integral operator $\tilde{T}_i T_j$ has kernel

$$\mathcal{K}_{i,j}(y,x) = \int K_i(z-x) K_j(z-y) \left(\fint_{[x,z]} a(w)dw \right) \left(\fint_{[y,z]} a(w)dw \right) dz.$$

We will be interested in Hölder estimates for these kernels. It is an easy observation (based on a dilation argument) that if we have a Hölder smoothness estimate

$$|\mathcal{K}_{0,i}(y,x) - \mathcal{K}_{0,i}(y,x')| \le C_K \|a\|_\infty^2 |x-x'|^\epsilon,$$

where C_K depends on the size and smoothness of K, then we also have

$$|\mathcal{K}_{i,j+i}(y,x) - \mathcal{K}_{i,j+i}(y,x')| \le C_K \|a\|_\infty^2 2^{-2j} 2^{-j\epsilon} |x-x'|^\epsilon$$

for any integer j.

To handle the diagonal terms of the double sum in (7) we will need the following lemma.

Lemma 2. *There is an* $\epsilon > 0$ *such that for any* $i \in \mathbb{Z}$

$$| < T_i B_{i-s}, T_i B_{i-s} > | \le C_K 2^{-\epsilon s} \|a\|_\infty \alpha \|B_{i-s}\|_1.$$

Proof. We discuss the case $i = 0$, noting that the other choices of i follow by scaling. We have

$$| < T_0 B_{-s}, T_0 B_{-s} > | = | < \tilde{T}_0 T_0 B_{-s}, B_{-s} > | \le \|\tilde{T}_0 T_0 B_{-s}\|_\infty \|B_{-s}\|_1. \tag{8}$$

$\tilde{T}_0 T_0$ is an integral operator with kernel $\mathcal{K}_{0,0}$. To study its Hölder smoothness, we write

$$\mathcal{K}_{0,0}(y,x) - \mathcal{K}_{0,0}(y,x') = I + II,$$

where

$$I = \int \big(K_0(z-x) - K_0(z-x')\big) K_0(z-y) \left(⨍_{[x,z]} a(w)dw \right) \left(⨍_{[y,z]} a(w)dw \right) dz$$

$$II = \int K_0(z-x) K_0(z-y) \left(⨍_{[x,z]} a(t)dt - ⨍_{[x',z]} a(w)dw \right) \left(⨍_{[y,z]} a(w)dw \right) dz.$$

Term I above is clearly bounded by $C|x-x'|$, since K_0 is a smooth function with support in some fixed compact set. To estimate term II, we switch to polar coordinates $z = y + r\theta$ to write

$$\begin{aligned} II = \int_{\theta \in \mathbf{S}^1} \Bigg[\int_{1/2}^{2} & K_0(y - x' + r\theta) K_0(r\theta) \\ & \times \left(⨍_{[x,y+r\theta]} a(w)dw - ⨍_{[x',y+r\theta]} a(w)dw \right) \left(⨍_{[y,y+r\theta]} a(w)dw \right) r\, dr \Bigg] d\theta. \end{aligned} \tag{9}$$

To estimate (9), we split the integral over $\mathbf{S}^1$ in (9) as a sum of integrals over the arc

$$\left| \theta \pm \frac{x-y}{|x-y|} \right| < t_0$$

and its complement, for some value of t_0 to be specified later. The part of the outer integral in (9) over this arc is then trivially estimated by $C t_0 \|a\|_\infty^2$. (We will later pick t_0 to be $10|x-x'|^{1/10}$, so this term will produce the correct estimate.)

Matters therefore reduce to estimating the part of the outer integral in (9) over the set

$$\left| \theta \pm \frac{x-y}{|x-y|} \right| \ge t_0.$$

We estimate this part by replacing the integral in θ by 2π times the supremum over θ in this set of the expression inside square brackets in (9). We may therefore fix θ, x, y. We denote by $\psi(r)$ the function

$$\begin{aligned} \psi(r) &= K_0(y - x' + r\theta) K_0(r\theta) \left(⨍_{[y,y+r\theta]} a(w)dw \right) r \\ &= K_0(y - x' + r\theta) K_0(r\theta) \int_0^r a(y + s\theta)\, ds \end{aligned}$$

and we note that $|\psi'(r)| \lesssim \|a\|_\infty$. By a translation and a rotation, without loss of generality, we assume that $\theta = (1,0)$ and $y = 0$. We have

$$\int_{1/2}^{2} K_0(y - x' + r\theta)K_0(r\theta)\left(\fint_{[x,y+r\theta]} a(w)dw - \fint_{[x',y+r\theta]} a(w)dw\right)\left(\fint_{[y,y+r\theta]} a(w)dw\right) rdr$$
$$= \int_{1/2}^{2} \psi(r) \int_0^1 a\big(x + s((r,0) - x)\big)ds\, dr - \int_{1/2}^{2} \psi(r) \int_0^1 a\big(x' + s((r,0) - x')\big)ds\, dr.$$

The support properties of K_0 make the integrals above vanishing, unless both x and x' lie in the ball of radius 4, i.e., one has

$$|x|, |x'| \le 4.$$

We now make a substitution, essentially going back to the original coordinates. For the first integral above we set

$$u = (x_1 - r)s + x_1 \qquad\qquad v = x_2(1 - s)$$

and for the second integral we set

$$u' = (x_1' - r)s + x_1' \qquad\qquad v' = x_2'(1 - s)$$

where $x = (x_1, x_2)$ and $x' = (x_1', x_2')$. We compute r, s and the Jacobian and write

$$D = \int_{1/2}^{2} \psi(r) \int_0^1 a\big(x + s((r,0) - x)\big)ds\, dr - \int_{1/2}^{2} \psi(r) \int_0^1 a\big(x' + s((r,0) - x')\big)ds\, dr$$
$$= \int_A \psi\left(x_1 - x_2\frac{u - x_1}{x_2 - v}\right) a(u,v)\frac{dudv}{|x_2 - v|} - \int_{A'} \psi\left(x_1' - x_2'\frac{u' - x_1'}{x_2' - v'}\right) a(u',v')\frac{du'dv'}{|x_2' - v'|},$$

where A is the triangle with vertices $(1/2, 0)$, $(2, 0)$, x and A' is the triangle with vertices $(1/2, 0)$, $(2, 0)$, x'.

Define functions ε_1 and ε_2 such that $-\varepsilon_1(t) = \varepsilon_2(t) = 1$ when $t \le 1/2$; $\varepsilon_1(t) = \varepsilon_2(t) = 1$ when $1/2 < t < 2$; and $\varepsilon_1(t) = -\varepsilon_2(t) = -1$ when $t \ge 2$. We observe that

$$\int_A \frac{dudv}{|x_2 - v|} =$$
$$\varepsilon_1(x_1) \int_{\min(\frac12, x_1)}^{\max(\frac12, x_1)} \int_0^{\frac{x_2}{x_1 - \frac12}(u - \frac12)} \frac{dv}{x_2 - v}\, du + \varepsilon_2(x_1) \int_{\min(2, x_1)}^{\max(2, x_1)} \int_0^{\frac{x_2}{x_1 - 2}(u - 2)} \frac{dv}{x_2 - v}\, du$$
$$\varepsilon_1(x_1) \left|x_1 - \frac12\right| \int_0^1 \log\frac1t\, dt + \varepsilon_2(x_1)\, |x_1 - 2| \int_0^1 \log\frac1t\, dt = \frac32 \int_0^1 \log\frac1t\, dt = \frac32,$$

where log is the logarithm with base e. This number is certainly independent of the position of x.

Now, we assume that $|x - x'| \le 1/1000$, $|x| \ge 10\,|x - x'|^{1/10}$ and we set

$$t_0 = 10\,|x - x'|^{1/10}.$$

Since $\left|(1,0) \pm \frac{x}{|x|}\right| \ge t_0$, using similar triangles we see that

$$|x_2| \ge |x|\,|x - x'|^{1/10} \ge 10\,|x - x'|^{1/5}; \tag{10}$$

also one has

$$|x_2'| \geq |x'|\,|x-x'|^{1/10} \geq 9\,|x-x'|^{1/5}\,. \tag{11}$$

Let $B(y,\delta)$ be the ball of radius δ centered at y. An easy calculation yields the estimate

$$\int_{A\cap B(x,|x-x'|^{1/4})} \frac{dudv}{|x_2-v|} \leq |x-x'|^{1/4}\,\mathcal{A}_x\,, \tag{12}$$

where $\mathcal{A}_x$ is the angle, measured in radians, formed by the vectors V_x (starting at x and ending at $(1/2,0)$) and W_x (starting at x and ending at $(2,0)$). Elementary geometric considerations yield:

$$\mathcal{A}_x \leq \pi \sin\Big(\frac{1}{2}\mathcal{A}_x\Big) \leq \frac{3\pi}{4}\max\Big(\frac{1}{|V_x|},\frac{1}{|W_x|}\Big) \leq \frac{C}{|x_2|}\,,$$

which combined with (10) and (12) yields the estimate

$$\int_{A\cap B(x,|x-x'|^{1/4})} \frac{dudv}{|x_2-v|} \leq C\,\frac{|x-x'|^{1/4}}{|x-x'|^{1/5}} = C\,|x-x'|^{1/20}\,. \tag{13}$$

By analogy we also have the estimate

$$\int_{A'\cap B(x,|x-x'|^{1/4})} \frac{dudv}{|x_2'-v|} \leq C\,|x-x'|^{1/20}\,. \tag{14}$$

Moreover, denoting by $A'\Delta A = (A\setminus A')\cup(A'\setminus A)$ the symmetric difference of A and A', we claim that

$$\int_{(A\Delta A')\setminus B(x,|x-x'|^{1/4})} \frac{dudv}{|x_2'-v|} \leq C\,|x-x'|^{1/20}. \tag{15}$$

We explain (15). First we note that in (15) the expressions $|x_2-v|$ and $|x_2'-v|$ are comparable if $(u,v)\notin B(x,|x-x'|^{1/4})$. Next, we prove (15) in the case where $x_2=x_2'$. By similar triangles we obtain that

$$|(A\Delta A')\cap\{(y_1,y_2):\ y_2=v\}| \leq \frac{2\,v\,|x-x'|}{x_2}$$

We may therefore estimate the integral in (15) by

$$\int_{(A\Delta A')\setminus B(x,|x-x'|^{1/4})} \frac{dudv}{|x_2-v|} \leq \int_0^{x_2-c\,|x-x'|^{1/4}} \frac{1}{x_2-v}\,\frac{2\,v\,|x-x'|}{x_2}\,dv \tag{16}$$

where c is the minimum of the slopes of the lines containing the vectors V_x, W_x, $V_{x'}$, and $W_{x'}$. In view of (10), we note that these slopes are greater than $C\,|x-x'|^{1/5}$. Therefore, we obtain the estimate

$$\frac{2\,|x-x'|}{x_2}\int_{C\,|x-x'|^{9/20}}^{x_2} \frac{x_2-v}{v}\,dv \leq C\,|x-x'|^{4/5}\,\log\frac{4/C}{|x-x'|} \leq C'\,|x-x'|^{1/20} \tag{17}$$

for the expression in (16).

Next, we have the cases $x_2>x_2'>0$ and $x_2'>x_2>0$. By symmetry we only look at the case $x_2>x_2'>0$. Here we extend one of the sides of the shorter triangle A' to make it have the same height as the taller one. Then we find a point x'' on the extended side such that $x_2'' = x_2$. Simple geometric considerations give that $|x'-x''| \le |x-x'| + |x-x'|^{4/5} \le 2\,|x-x'|^{4/5}$. Then we replace the triangle A' by the larger triangle A'' the vertex of which is x'' and base is the same as A' and we replace the symmetric difference $A\Delta A'$ by the larger one $A\Delta A''$ and the ball $B(x,|x-x'|^{1/4})$ by the smaller ball $B(x,(|x-x''|/2)^{5/16})$. Then we have

$$\int_{(A\Delta A')\setminus B(x,|x-x'|^{1/4})} \frac{dudv}{|x_2-v|} \le \int_{(A\Delta A'')\setminus B(x,(|x-x''|/2)^{5/16})} \frac{dudv}{|x_2-v|}$$

and matters reduce to the previous case where x'' plays the role of x'. One obtains an estimate similar to (17) in which the power $9/20 = 1/4 + 1/5$ is replaced by $41/80 = 5/16 + 1/5$. The same conclusion follows.

To estimate the remaining part of the difference D, we use the smoothness of $\psi(r)$. In particular, for $(u,v) \in (A \cap A') \setminus B(x,|x-x'|^{1/4})$ we have

$$\left| x_1 - x_2\frac{u-x_1}{x_2-v} - x_1' - x_2'\frac{u-x_1'}{x_2'-v} \right| \le \frac{|x-x'|}{|(x_2-v)(x_2'-v)|}.$$

By a slope argument similar to the above, $|x_2-v| \ge C|x-x'|^{1/4+1/5}$, and we obtain the estimate

$$\left| \psi\left(x_1 - x_2\frac{u-x_1}{x_2-v}\right) - \psi\left(x_1' - x_2'\frac{u-x_1'}{x_2'-v}\right) \right| \le C\|a\|_\infty |x-x'|^{1/20}.$$

Collecting the preceding estimates, we deduce that for

$$|x-y| \ge 10|x-x'|^{1/10} \qquad \text{and} \qquad |x-x'| \le 1/1000$$

we have the Hölder estimate

$$|\mathcal{K}_{0,0}(y,x) - \mathcal{K}_{0,0}(y,x')| \le C\|a\|_\infty^2 |x-x'|^{1/20}.$$

Therefore, for any y we obtain

$$\begin{aligned} |\tilde{T_0} T_0 B_{-s}(y)| &= \left| \int \mathcal{K}_{0,0}(y,x) B_{-s}(x)\,dx \right| \\ &\le C\,2^{-s/20}\|a\|_\infty^2 \left| \bigcup_{Q_j\in\mathcal{Q}_{-s}:\,Q_j\cap B(y,4)\neq\emptyset} Q_j \right| \\ &\le C\,2^{-s/20}\|a\|_\infty \alpha\,, \end{aligned}$$

since all the cubes Q_j that appear in the preceding union are disjoint and are contained in the disc of radius 8 centered at the point y, thus the measure of their union is at most a constant. This estimate combined with (8) yields the proof of Lemma 2 when $i = 0$, while the case of a general i follows by a dilation argument. □

The off-diagonal terms of the double sum in (7) are handled by the following result:

Lemma 3. *There is an $\epsilon > 0$ such that for any $i \in \mathbb{Z}$ and $s \in \mathbb{N}$, we have*

$$\sum_{j<i-3} |< T_j B_{j-s}, T_i B_{i-s} >| \le C_K 2^{-\epsilon s}\|a\|_\infty \alpha \|B_{i-s}\|_1. \tag{18}$$

Proof. The proof is very similar to that of the previous lemma. We seek to estimate the smoothness of the kernel $\mathcal{K}_{i,j}$ of $\tilde{T}_i T_j$, which is given by the formula

$$\mathcal{K}_{i,j}(y,x) = \int K_j(z-x)K_i(z-y)\left(\fint_{[x,z]} a(w)\,dw\right)\left(\fint_{[y,z]} a(w)\,dw\right)dz.$$

We then proceed as in the previous case, with the following differences: after we switch to the polar coordinates the analog of (9) is

$$\begin{aligned}(19)\qquad &\int_{\theta\in A}\int_{2^{i-1}}^{2^{i+1}} K_j(y-x'+r\theta)K_i(r\theta)\\ &\times\left(\fint_{[x,y+r\theta]} a(w)dw - \fint_{[x',y+r\theta]} a(w)dw\right)\left(\fint_{[y,y+r\theta]} a(w)dw\right) r\,dr\,d\theta,\end{aligned}$$

where A is an arc in S^1. Let us momentarily assume that $j=0$; then $i>3$ and A is an arc of length of about 2^{-i}. Indeed, let $\mathcal{U}$ be the smallest cone with vertex at the origin which contains the disc of radius 2 centered at $y-x'$. Then since $1/2 \le |y-x'+r\theta| \le 2$, it follows that $2^{i-2} \le |y-x'| \le 2^{i+2}$ and thus the angle of $\mathcal{U}$ is at most a constant multiple of 2^{-i}. Setting

$$\psi(r) = K_0(y-x'+r\theta)K_i(r\theta)\left(\fint_{[y,y+r\theta]} a(w)dw\right) r\,,$$

we have the estimates $\|\psi\|_\infty \le C2^{-i}\|a\|_\infty$ and $\|\psi'\|_\infty \le C2^{-i}\|a\|_\infty$. Next we choose $t_0 = 10\cdot 2^{-i}|x-x'|^{1/10}$ to play the same role as in the proof of the previous lemma, which corresponds to the fact that the kernel K_0 is supported in a ball of diameter 2 and either $|y-x|$ or $|y-x'|$ is about 2^i or $\mathcal{K}_{i,0}(y,x)=0$. Also, we do not need the restriction $|x-y|\ge 10|x-x'|^{1/10}$ anymore. In this case θ lies in an arc of approximate size 2^{-i} and r lies in the interval $[|y-x'|-2, |y-x'|+2]$ which has length 4. An easy calculation yields the Hölder estimate

$$|\mathcal{K}_{i,0}(y,x) - \mathcal{K}_{i,0}(y,x')| \le C2^{-2i}\|a\|_\infty^2|x-x'|^{1/20} \tag{20}$$

for $i>3$. By a scaling argument this means that for $j<i-3$ we have

$$|\mathcal{K}_{i,j}(y,x) - \mathcal{K}_{i,j}(y,x')| \le C\|a\|_\infty^2 2^{-j/20}2^{-2i}|x-x'|^{1/20}. \tag{21}$$

Having established this, we continue the proof of the lemma by writing

$$\sum_{j<i-3} |< T_jB_{j-s}, T_iB_{i-s} >| \le \sum_{j<i-3} \left\|\tilde{T}_iT_jB_{j-s}\right\|_\infty \|B_{i-s}\|_1.$$

Thus estimate (18) reduces to showing that for all i we have

$$\sum_{j<i-3} \left\|\tilde{T}_iT_jB_{j-s}\right\|_\infty \le C_K 2^{-\epsilon s}\|a\|_\infty\alpha\,. \tag{22}$$

We take a cube Q_k with side length 2^{j-s} and use the Hölder estimate (21) to obtain

$$\left|\int \mathcal{K}_{i,j}(y,x)b_{Q_k}(x)dx\right| \le C2^{(j-s)/20}2^{-j/20}2^{-2i}\|a\|_\infty^2\|b_{Q_k}\|_1 = C2^{-s/20}2^{-2i}\|a\|_\infty^2\|b_{Q_k}\|_1.$$

For a fixed y the function $\mathcal{K}_{i,j}(y,\cdot)$ is supported in the ball $B(y,4\cdot 2^i)$ and therefore we obtain

$$\left|\int \mathcal{K}_{i,j}(y,x)\sum_k b_{Q_k}(x)dx\right| \le C\|a\|_\infty^2 2^{-s/20}2^{-2i} \sum_{\substack{Q_k\in\mathcal{Q}_{j-s}\\ Q_k\cap B(y,4\cdot 2^i)\neq\emptyset}} \left\|b_{Q_k}\right\|_1 .$$

We now sum over $j < i-3$ and we use that the cubes Q_k are disjoint to deduce

$$\sum_{j<i-3}\sum_{\substack{Q_k\in\mathcal{Q}_{j-s}\\ Q_k\cap B(y,4\cdot 2^i)\neq\emptyset}} \left\|b_{Q_k}\right\|_1 \le \frac{C\,\alpha}{\|a\|_\infty}\sum_{j<i-3}\sum_{\substack{Q_k\in\mathcal{Q}_{j-s}\\ Q_k\subset B(y,5\cdot 2^i)\neq\emptyset}} |Q_k| \le \frac{20\,C\alpha}{\|a\|_\infty}2^{2i},$$

since all the cubes that appear in preceding double sum are disjoint and contained in a disc of radius $5\cdot 2^i$, hence the sum of their measures is at most $25\pi\, 2^{2i}$. Combining the previous estimates concludes the proof of Lemma 3. □

To finish the proof of the Lemma 1 we use Lemma 2 to estimate the terms $i=j$ of the double sum (7), Lemma 2 and the Cauchy-Schwarz inequality to handle the cases $0<|i-j|\le 3$, and Lemma 3 together with symmetry for the remaining cases $|i-j|>3$.

4. Conclusion

In this article we proved a weak-type estimate for the two-dimensional commutator (1). The obvious question is whether there is an analogous estimate for the higher dimensional version of the operator in (1). In the case of the singular operators with rough kernels such a higher-dimensional weak type $(1,1)$ estimate was proved by Seeger [5] using a Fourier transform approach and later by Tao [6] via a combinatorial technique. (The two-dimensional case was previously obtained by Hofmann [4], while the case of dimensions $n\le 7$ was claimed by Christ and Rubio de Francia.) At present, it is not clear how to adapt these approaches to the case of the corresponding commutators in dimensions $n\ge 3$.

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