SOME REMARKS ON MULTILINEAR MAPS AND INTERPOLATION

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Abstract. A multilinear version of the Boyd interpolation theorem is proved in the context of quasi-normed rearrangement-invariant spaces. A multilinear Marcinkiewicz interpolation theorem is obtained as a corollary. Several applications are given, including estimates for bilinear fractional integrals.

1. Introduction

In this article we give a version of the Boyd interpolation theorem for multilinear operators. We will be working with rearrangement invariant quasi-Banach spaces, which include all the well-known examples such as Orlicz spaces and Lorentz spaces.

We will consider the following situation. Consider $\mathbf{R}_+ = (0, \infty)$ with Lebesgue measure (which can of course be replaced by any infinite nonatomic measure space). We let $L_0(0, \infty)$ be the space of all real-valued measurable functions equipped with the topology of local convergence in measure. Let $\mathcal{E}$ be the space of all measurable functions which are bounded and supported on sets of finite measure. Now let $T : \mathcal{E}^n \to L_0(0, \infty)$ be a multilinear map (our results also apply to sublinear maps). We suppose that T is locally continuous i.e. continuous when restricted to $\prod_{k=1}^n L_\infty(E_k)$ for every choice of sets E_k of finite measure. We also suppose that T obeys a finite collection of weak type inequalities

$$\|T(\chi_{E_1}, \ldots, \chi_{E_n})\|_{L_{p,\infty}} \leq C \prod_{k=1}^n |E_k|^{\theta_k}$$

for every n-tuple of measurable sets $(E_1, \ldots, E_n)$. Here $L_{p,\infty}$ is the usual weak L_p space and $\theta_k > 0$ for every k. We then seek to characterize $(n+1)$-tuples of rearrangement-invariant spaces $(X_1, \ldots, X_n, Y)$ for which T extends to a bounded n-linear map from $X_1 \times \cdots \times X_n$ into Y. In general one needs two distinct hypotheses. The first consists of an assumption on the Boyd indices of the spaces $X_1, \ldots, X_n, Y$, as in the original Boyd interpolation theorem. The second hypothesis is that a certain n-linear test map associated with T is continuous.

Our main result (Theorem 4.1) gives a necessary (and often sufficient) condition on $(X_1, \ldots, X_n, Y)$ in the case when one has $n + 1$ such conditions which are sufficiently

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independent. Note that the original theorem of Boyd [2] corresponds to the case when $n = 1$ and there are two conditions of the type:

$$\|T(\chi_E)\|_{L_{p,\infty}} \leq C|E|^{1/p}.$$

We deduce Theorem 4.1 from a similar *homogeneous* Boyd-type theorem (Theorem 3.7) which is applicable for example to n-linear forms. As a corollary we obtain a multilinear version of the Marcinkiewicz interpolation theorem (Theorem 4.6).

Our work is related to work of Strichartz [17], Janson [5], and Christ [3]. Note that as in [5] and [17] (and in contrast to [3]) our multilinear assumptions consist only of a finite number of estimates. Our results also develop and extend earlier work of Sharpley (see [15], [16], and [1]).

In section 5 we give examples of multilinear interpolation. As one of our applications, we characterize the indices $(1/p, 1/q, 1/r)$, $0 < p, q, r \leq \infty$, for which the bilinear fractional integral operator

$$I_\alpha(f,g)(x) = \int_{\mathbf{R}^n} f(x+t)g(x-t)|t|^{\alpha-n}\,dt.$$

maps $L_p(\mathbf{R}^n) \times L_q(\mathbf{R}^n) \to L_r(\mathbf{R}^n)$. This characterization was also independently obtained by C. Kenig and E. M. Stein [10].

2. Preliminaries

In this section we set up the background required to state the multilinear Boyd interpolation theorem.

Let $L_0(0,\infty)$ be the space of all complex-valued measurable functions on $(0,\infty)$, with the topology of local convergence in measure. We define a quasi-Banach function space X on $(0,\infty)$ to be a subspace of L_0 equipped with a quasi-norm $\|\,\|_X$ such that:

- $\|f\|_X = 0$ if and only if $f = 0$ a.e.
- $\|\alpha f\|_X = |\alpha|\|f\|_X$, whenever $f \in X$ and $\alpha \in \mathbf{C}$.
- There exists a constant C so that if $f, g \in X$ then $\|f+g\|_X \leq C(\|f\|_X + \|g\|_X)$.
- X is complete (i.e. a quasi-Banach space) for $\|\,\|_X$.
- The injection $X \to L_0$ is continuous.
- If E is a set of finite measure then $\chi_E \in X$.
- If $f \in X$ and $g \in L_0$ with $|g| \leq |f|$ a.e. then $g \in X$ and $\|g\|_X \leq \|f\|_X$.
- If $0 \leq f_n \uparrow f$ a.e. and $f \in X$ then $\|f_n\|_X \uparrow \|f\|_X$.

By assumption X must contain the space $\mathcal{E}$ of all bounded measurable functions supported on sets of finite measure. We say that X is *minimal* if $\mathcal{E}$ is dense in X. We say that X is *maximal* if it has the property that if $0 \leq f_n \uparrow f$ a.e. with $\sup \|f_n\|_X < \infty$, then $f \in X$.

A quasi-Banach function space on $(0,\infty)$ which is either maximal or minimal (cf. [12]) is said to be a *rearrangement-invariant function space* or *r.i. space* if $\|f^*\|_X = \|f\|_X$ for all $f \in X$, where f^* is the decreasing rearrangement of $|f|$, i.e. $f^*(t) = \inf\{x : |\{|f| > x\}| \leq t\}$.

We say that X is r-convex if there is a constant C so that if $f_1,\dots,f_n\in X$ then

$$\|(\sum_{i=1}^n |f_i|^r)^{1/r}\|_X \le C(\sum_{i=1}^n \|f_i\|_X^r)^{1/r}.$$

For a discussion of r-convexity in the context of Banach lattices we refer to [12]; we refer to [7] for quasi-Banach lattices. Every Banach r.i. space is of course 1-convex, but there are examples of quasi-Banach r.i. spaces which fail to be r-convex for any $r>0$, see [6]. However it is very natural to assume r-convexity since all "practical spaces" are r-convex for some $r>0$.

Once an r.i. space X is defined on $(0,\infty)$ it may be transferred to any σ-finite measure space (Ω,μ) by defining $X(\Omega,\mu)$ to be the space of all measurable $f:\Omega\to\mathbf{C}$ such that $\|f\|_{X(\Omega)}=\|f^*\|_{X(0,\infty)}<\infty$. In general if Ω is a Polish space and μ is an infinite nonatomic Borel measure there is a measure-preserving bijection of Ω onto $(0,\infty)$. Thus there is no loss of generality in treating only the case of $\Omega=(0,\infty)$.

If X is an r.i. space then the dilation operators $D_a: X\to X$ given by

$$(D_af)(x)=f(x/a)$$

are well-defined and bounded. We define the Boyd indices by

$$p_X=\lim_{a\to\infty}\frac{\log a}{\log\|D_a\|}$$

and

$$q_X=\lim_{a\to 0}\frac{\log a}{\log\|D_a\|}.$$

Then $0<p_X\le q_X\le\infty$. We refer to [12] or [1] for relevant discussion. If $\epsilon>0$ then there is a constant $C=C(\epsilon,X)$ so that for all $f\in X$ we have

$$\|D_af\|_X\le C\max(a^{\frac{1}{p_X}+\epsilon},a^{\frac{1}{q_X}-\epsilon}). \tag{1}$$

It is sometimes useful to have the notion of a *carrier space* for an r.i. space X. Let $\widetilde{X}$ be a maximal quasi-Banach function space on $(0,\infty)$ with the property that the dilation operators D_a are bounded on $\widetilde{X}$ and $\|D_a\|_{\widetilde{X}}\le Ca^\kappa$ for some $\kappa>0$ and all $a\ge 1$. Then we can define an r.i. space X by requiring $f\in X$ if and only if $f^*\in\widetilde{X}$ and by setting $\|f\|_X=\|f^*\|_{\widetilde{X}}$. It is then easy to show that X is a maximal r.i. space and that $\alpha_X\le\kappa$. We will in this case refer to $\widetilde{X}$ as a carrier space for X. Notice, of course, that X is a carrier space for itself.

Examples of r.i. spaces are provided by the usual Lorentz spaces $L_{p,q}$ with (quasi)-norm

$$\|f\|_{L_{p,q}}=\begin{cases}\Big(\displaystyle\int_0^\infty [f^*(t)t^{1/p}]^q\frac{dt}{t}\Big)^{1/q} & \text{when } 0<q<\infty,\\ \sup_{t>0}f^*(t)t^{1/p} & \text{when } q=\infty\end{cases} \tag{2}$$

for $0<p,q\le\infty$. These spaces are 1-convex (i.e. normable) when $1<p<\infty$ and $1\le q\le\infty$ or if $p=q=1$. In general $L_{p,q}$ is q-convex if $q\le p$ and s-convex for any $s<p$ if $q>p$. The Boyd indices of $L_{p,q}$ both coincide with p. Note that all these spaces have natural carrier spaces which are weighted L_p-spaces.

The significance of the Boyd indices lies in the fact that they can be used to characterize all rearrangement-invariant Banach spaces X on which certain known operators are bounded. For instance the Hardy-Littlewood maximal operator is bounded on X (r.i. over $\mathbf{R}^n$) if and only if $q_X < \infty$, see [13], [18]. The Hilbert transform is bounded on X (r.i. over $\mathbf{R}$) if and only if $1 < p_X \le q_X < \infty$, see [2].

Let us now recall the Boyd interpolation theorem for $(0,\infty)$ (see [2] or [12], p.145):

Theorem 2.1. *Suppose $1 \le p < q < \infty$ and that $T : L_{p,1}+L_{q,1} \to L_0(0,\infty)$ is a linear map of weak types (p,p) and (q,q). Suppose X is an r.i. space with $p < p_X \le q_X < q$. Then T is a bounded map from X into itself.*

This result was extended to the case $0 < p < q < \infty$ in [8] (Theorem 1.3) with the additional assumption that X is r-convex for some $r > 0$.

The main purpose of this article is to obtain a multilinear version of Theorem 2.1. This is achieved in the next two sections. We first obtain a homogeneous multilinear version of Theorem 2.1 (Theorem 3.7), and from this we deduce an inhomogeneous version, Theorem 4.1.

3. The homogeneous multilinear Boyd theorem

Let $\mathcal{E}$ be the space of all measurable functions on $(0,\infty)$ which are bounded and have support of finite measure. We shall say that a map (usually n-linear) $T : \mathcal{E}^n \to Y$ in any topological vector space is locally continuous if its restriction to $\prod_{k=1}^n L_\infty(E_k)$ is continuous for every choice of sets E_k of finite measure.

Now suppose Θ is a finite subset of $(\overline{\mathbf{R}}_+)^n = [0,\infty)^n$ and Y is a quasi-Banach space. We say that an n-linear map $T : \mathcal{E}^n \to Y$ is $\Theta-$admissible if T is locally continuous and there is a constant M so that for every $\theta = (\theta_1, \dots, \theta_k) \in \Theta$ we have

$$\|T(\chi_{E_1}, \dots, \chi_{E_n})\|_Y \le M \prod_{k=1}^n |E_k|^{\theta_k}, \tag{3}$$

whenever $E_1, \dots, E_n$ have finite measure. The least such constant M is denoted by $\|T\|_\Theta$. In most of the work that follows, it will be convenient to take $\Theta \subset \mathbf{R}_+^n$ i.e. to require $\theta_k > 0$ for all θ, k.

Let us recall that a quasi-Banach space $(Y, \|\cdot\|)$ is called s-normed if there is a constant C such that for all $y_1, \dots, y_m \in Y$ we have

$$\|y_1 + \cdots + y_m\|^s \le C(\|y_1\|^s + \cdots + \|y_m\|^s).$$

Now let $\mathbf{X} = (X_1, \dots, X_n)$ be an n-tuple of r.i. spaces. We say that $\mathbf{X}$ has the *interpolation condition* (Θ, s), where $0 < s \le 1$, if for every s-normed quasi-Banach space Y and every Θ-admissible $T : \mathcal{E}^n \to Y$ there is a continuous extension $T : X_1 \times \cdots \times X_n \to Y$ with norm a constant multiple of $\|T\|_\Theta$. Note here that in the case $s = 1$ it is sufficient to take Y to be the scalar field $\mathbf{R}$ or $\mathbf{C}$ and hence we only consider n-linear forms.

We will need to establish some examples of Θ-admissible multilinear maps. We begin with a lemma.

Lemma 3.1. *Suppose* $0 < u < \infty$ *and* $0 < s < r < \infty$. *Then for any measurable set* $E \subset (0, \infty)$ *we have*

$$\left(su\int_E x^{su-1}dx\right)^{1/s} \leq \left(ru\int_E x^{ru-1}dx\right)^{1/r}.$$

In particular if $su < 1$ *then*

$$\left(su\int_E x^{su-1}dx\right)^{1/s} \leq |E|^u.$$

Proof. First note that for $t > 1$, we have $(t^{su}-1)^{1/s} \leq (t^{ru}-1)^{1/r}$ and also that $(t^{su}-1)^{1/s}(t^{ru}-1)^{-1/r}$ is increasing. This last fact follows from the observation that $t \to \frac{1}{r}\log(t^r-1) - \frac{1}{s}\log(t^s-1)$ is monotone decreasing and converges to zero at infinity. This implies that if E is an interval we have the desired inequality.

We now proceed to prove the result for E a disjoint union of m intervals using induction. Assume the required inequality is true for all unions of less than m disjoint intervals. Now if E is a finite union of m disjoint intervals $[v_j, w_j)$ for $1 \leq j \leq m$ where $v_1 < w_1 < \cdots < v_m < w_m$, we define $h > w_{m-1}$ by the condition that

$$h^{su} - w_{m-1}^{su} = (w_m^{su} - v_m^{su}) + (w_{m-1}^{su} - v_{m-1}^{su}).$$

If we had

$$h^{ru} - w_{m-1}^{ru} \leq (w_m^{ru} - v_m^{ru}) + (w_{m-1}^{ru} - v_{m-1}^{ru}), \tag{4}$$

then the inductive hypothesis applied to the $m-1$ intervals $[v_1, w_1), \ldots, [v_{m-2}, w_{m-2})$, and $[v_{m-1}, h)$ together with (4) would quickly give the desired conclusion. It suffices therefore to prove (4). This will follow from the fact that if α, β, γ, and δ are positive numbers satisfying $\alpha+\gamma = \beta+\delta$ and $\beta < \gamma < \delta$, then $\alpha^{r/s} + \gamma^{r/s} \leq \beta^{r/s} + \delta^{r/s}$ when $r > s$. Indeed, the assumptions above imply that $\beta < \alpha < \delta$ and clearly

$$\alpha^{r/s} + \gamma^{r/s} \leq \max_{\alpha\in(\beta,\delta)} \left(\alpha^{r/s} + (\beta+\delta-\alpha)^{r/s}\right) \leq \beta^{r/s} + \delta^{r/s}.$$

□

Let $\langle\cdot,\cdot\rangle$ denote the usual inner product on $\mathbf{R}^n$ and $\|\ \|$ the usual Euclidean norm. For each $\theta \in \mathbf{R}^n$ let θ_k denote its k^{th} coordinate. Suppose Θ is a finite subset of $(\overline{\mathbf{R}}_+)^n = [0,\infty)^n$. Define a sublinear map associated with Θ as follows

$$a(\xi) = a_\Theta(\xi) = \max_{\theta\in\Theta}\langle\xi,\theta\rangle. \tag{5}$$

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an n-tuple of r.i. spaces. We have the following theorem. See also Sharpley [15] for a somewhat similar result.

Theorem 3.2. *Let* $0 < s \leq 1$. *Consider the statements:*
(i) $\mathbf{X}$ *satisfies the interpolation condition* (Θ, s).
(ii) There exists a constant C *so that if* $f_1, \ldots, f_n \in \mathcal{E}$,

$$\left(\int_{\mathbf{R}^n}\prod_{k=1}^n (f_k^*(e^{\xi_k}))^s \exp(-sa(-\xi))d\xi\right)^{1/s} \leq C\prod_{k=1}^n \|f_k\|_{X_k}. \tag{6}$$

(iii) There exists a constant C so that if $f_1,\dots,f_n\in\mathcal{E}$,

$$\left(\int_{\mathbf{R}^n}\prod_{k=1}^n (F_k(e^{\xi_k}))^s\exp(-sa(-\xi))d\xi\right)^{1/s}\le C\prod_{k=1}^n\|f_k\|_{X_k}, \tag{7}$$

where

$$F_k(x)=\left(\frac{1}{x}\int_0^x (f_k^*(t))^s dt\right)^{1/s}.$$

(iv) There exists a constant C so that if $f_1,\dots,f_n\in\mathcal{E}$, then

$$\max_{\|\xi\|=1}\left(\int_0^\infty\prod_{k=1}^n (f_k^*(e^{t\xi_k}))^s\exp(-sta(-\xi))dt\right)^{1/s}\le C\prod_{k=1}^n\|f_k\|_{X_k}. \tag{8}$$

Then (ii) implies (i). Furthermore, if $\Theta\subset\mathbf{R}_+^n$ and s is small enough so that $s\theta_k<1$ for every $(\theta_1,\dots\theta_k)\in\Theta$ and every $1\le k\le n$, then (i), (ii), (iii), and (iv) are all equivalent.

Proof. First assume (ii) and that $T:\mathcal{E}^n\to Y$ is $\Theta-$admissible where Y is s-normed. Without loss of generality we assume $\|T\|_\Theta\le 1$. We first note that if f_k are supported in E_k and $\|f_k\|_{L_\infty}\le 1$ then we have an estimate:

$$\|T(f_1,\dots,f_n)\|_Y\le C\min_{\theta\in\Theta}\prod_{k=1}^n|E_k|^{\theta_k}, \tag{9}$$

where C depends only on s and n. To see this it suffices to get an estimate for positive functions f_k and then extend to signed and complex functions by additivity. But if f_k is positive we can write

$$f_k=\sum_{j=1}^\infty 2^{-j}\chi_{A_{jk}}$$

where $A_{jk}\subset E_k$. Expanding out we easily get estimate (9).

Now suppose $f_1,\dots,f_n\in\mathcal{E}$. We can write each f_k in the form

$$f_k=\sum_{m=-\infty}^\infty f_k\chi_{A_{km}}$$

where $|A_{km}|=2^m$ and $\|f_k\chi_{A_{km}}\|_{L_\infty}\le f^*(2^m)$. Now by (9) we have

$$\|T(f_1\chi_{A_{1m_1}},\dots,f_n\chi_{A_{nm_n}})\|_Y\le C\min_{\theta\in\Theta}2^{\sum_{k=1}^n\theta_k m_k}\prod_{k=1}^n f_k^*(2^{m_k}).$$

Now since $\|\ \|_Y$ is an $s-$norm after summing and making an obvious integral estimate we obtain

$$\|T(f_1,\dots,f_n)\|_Y^s\le C\int_0^\infty\cdots\int_0^\infty (f_1^*(x_1))^s\dots(f_n^*(x_n))^s\min_{\theta\in\Theta}\prod_{k=1}^n x_k^{s\theta_k-1}dx_1\dots dx_n.$$

The right-hand side is now estimated by $C\prod_{k=1}^n \|f_k\|_{X_k}$ using (6). We can now extend the definition of $T(f_1,\ldots,f_k)$ to $X_1\times\cdots\times X_n$ by noting that for $f_k\in X_k$ the sum

$$\sum_{m_1,\ldots,m_n} T(f_1\chi_{A_{1m_1}},\ldots,f_n\chi_{A_{nm_n}})$$

converges in Y. It is easy to check that this extends T unambiguously and continuously to $X_1\times\cdots\times X_n$. Thus (i) holds.

Now assume (i), $\Theta\subset\mathbf{R}_+^n$, and $s\theta_k<1$ for all $\theta\in\Theta$ and $1\le k\le n$. For each $\theta\in\Theta$ Lemma 3.1 gives that if $f_k=\chi_E\in\mathcal{E}$, then

$$\left(\int_0^\infty x^{s\theta_k-1}f_k(x)^s\,dx\right)^{1/s}\le (s\theta_k)^{-1/s}|E|^{\theta_k},$$

where as usually θ_k denotes the k^{th} coordinate of θ. It follows that if we define

$$T_\theta(f_1,\ldots,f_n)(x_1,\ldots,x_n)=\prod_{k=1}^n x_k^{\theta_k-\frac1s}f_k(x_k),$$

then $T_\theta:\mathcal{E}^n\to L_s((0,\infty)^n)$ is $\{\theta\}$-admissible and $\|T_\theta\|_{\{\theta\}}\le s^{-n/s}\prod_{k=1}^n\theta_k^{-1/s}$. If we define T by

$$T(f_1,\ldots,f_n)(x_1,\ldots,x_n)=\prod_{k=1}^n(\min_{\theta\in\Theta}x_k^{\theta_k-\frac1s})f_k(x_k)$$

then T is $\Theta-$admissible. It follows that we can find C so that (6) is valid and thus (ii) holds.

We now show that (ii) implies (iii). Observe that

$$(F_k(e^{\xi_k}))^s=\int_{-\infty}^{\xi_k}e^{\eta_k-\xi_k}(f_k^*(e^{\eta_k}))^s d\eta_k.$$

Hence

$$\begin{aligned}&\int_{\mathbf{R}^n}\prod_{k=1}^n(F_k(e^{\xi_k}))^s\exp(-sa(-\xi))d\xi\\ &=\int_{\mathbf{R}^n}\int_{\eta\le\xi}\prod_{k=1}^n(f_k^*(e^{\eta_k}))^s\exp(\langle\eta-\xi,\mathbf{1}\rangle-sa(-\xi))\,d\eta\,d\xi,\end{aligned}\tag{10}$$

where $\eta\le\xi$ means $\eta_k\le\xi_k$ for $1\le k\le n$ and $\mathbf{1}$ denotes the vector $(1,1,\ldots,1)$. For fixed η pick $\widetilde{\theta}\in\Theta$ so that $a(-\eta)=\langle\widetilde{\theta},-\eta\rangle$. Then

$$\int_{\xi\ge\eta}\exp(-\langle\xi,\mathbf{1}\rangle-sa(-\xi))d\xi\le\int_{\xi\ge\eta}\exp(-\langle\xi,\mathbf{1}-s\widetilde{\theta}\rangle)d\xi\le C\exp(-\langle\eta,\mathbf{1}\rangle-sa(-\eta))$$

since for some $\delta>0$ we have $1-s\theta_k>\delta>0$ for all $\theta\in\Theta$ and all k. Substituting back into (10) gives the required estimate.

Next we show (iii) implies (iv). To do this it suffices to note that the map $\xi\to\sum_{k=1}^n\log F_k(e^{\xi_k})$ is Lipschitz with a constant depending only on s and n unless some

f_k is zero. This means that if $\|\xi\| = 1$ we have an estimate:

$$\int_0^\infty \prod_{k=1}^n (F_k(e^{t\xi_k}))^s \exp(-sta(-\xi))dt \le C \int_K \prod_{k=1}^n (F_k(e^{\xi_k}))^s \exp(-sa(-\xi))d\xi.$$

where K is a cylinder of radius one with axis $\{t\xi : t \ge 0\}$. We can then deduce (8) from (7) since $f_k^* \le F_k$.

Finally (iv) implies (iii) (which implies (ii)) by just using polar coordinates.

□

We can extend this result somewhat to certain multilinear analogues of maximal operators. Denote by $L_0^+(0,\infty)$ the set of all nonnegative measurable functions on $(0,\infty)$. Let us say that a positively homogeneous (of degree 1) map $T : {\mathcal{E}_+}^n \to L_0^+(0,\infty)$ is *n-quasi-sublinear* with constant C so that for any k we have

$$\begin{aligned} &T(f_1,\dots,f_{k-1},(f_k+f_k'),f_{k+1},\dots,f_n) \\ &\le C(T(f_1,\dots,f_{k-1},f_k,f_{k+1},\dots,f_n)+T(f_1,\dots,f_{k-1},f_k',f_{k+1},\dots,f_n)), \end{aligned} \tag{11}$$

for all f_j, f_j'. Suppose T is n-quasi-sublinear. Then if we choose r so that $2^{1/r-1} = C$ we can use the proof of the Aoki-Rolewicz theorem ([9],[14]) to deduce the existence of a constant C' so that for any $1 \le k \le n$ and all m positive integers we have

$$T(f_1,\dots,f_{k-1},\sum_{j=1}^m g_j, f_{k+1},\dots,f_n) \le C'(\sum_{j=1}^m T(f_1,\dots,f_{k-1},g_j,f_{k+1},\dots,f_n)^r)^{1/r},$$

for all f_j and g_j. Based on this it is easy to show the following, by exactly the same argument as in Theorem 3.2:

Corollary 3.3. *Suppose $T : \mathcal{E}^n \to L_s(0,\infty)$ is n-quasi-sublinear with constant $C = 2^{1/r-1}$ where $0 < s \le r$, and that T is locally continuous. Let Θ be a finite subset of $\mathbf{R}_+^n$ and assume that there is a constant M so that*

$$\|T(\chi_{E_1},\dots,\chi_{E_n})\|_{L_s} \le M \inf_{\theta\in\Theta} \prod_{k=1}^n |E_k|^{\theta_k},$$

for all E_k of finite measure. If $\mathbf{X}$ is an n-tuple of r.i. spaces satisfying (6), then there is a constant C so that for $f_1,\dots,f_n \in \mathcal{E}$ we have

$$\|T(f_1,\dots,f_n)\|_{L_s} \le CM \prod_{k=1}^n \|f_k\|_{X_k}.$$

We now consider versions of the Boyd interpolation theorem in this setting.

Consider the convex hull co Θ; We define the open convex hull $\mathrm{co}_0\,\Theta$ to be the set of all $\sum_{\theta\in\Theta}\alpha_\theta\theta$ where $0 < \alpha_\theta < 1$ and $\sum_{\theta\in\Theta}\alpha_\theta = 1$. Then $\mathrm{co}_0\,\Theta$ is the interior of co Θ relative to the affine hyperplane it generates. We also define the *Boyd cube* $B_{\mathbf{X}}$ of $\mathbf{X}$ to be the set $\prod_{k=1}^n [1/q_{X_k}, 1/p_{X_k}]$, where p_{X_k}, q_{X_k} are the Boyd indices of X_k.

It will be convenient to introduce the following sublinear functional associated with $B_{\mathbf{X}}$

$$b(\xi) = b_{\mathbf{X}}(\xi) = \max_{\phi\in B_{\mathbf{X}}} \langle \xi, \phi \rangle. \tag{12}$$

Let us first note a simple consequence of Theorem 3.2.

Corollary 3.4. *Suppose that Θ is a finite subset of $(\overline{\mathbf{R}}_+)^n$ and that $\mathbf{X}$ satisfies the (Θ, s)-interpolation condition. Then $B_{\mathbf{X}} \cap co\, \Theta$ is nonempty.*

Proof. Suppose $B_{\mathbf{X}}, \mathrm{co}\, \Theta$ do not intersect. Then we can find $\eta \in \mathbf{R}^n$ so that

$$\max_{\theta\in\Theta}\langle \eta, \theta\rangle < \min_{\phi\in B_{\mathbf{X}}}\langle \eta, \phi\rangle.$$

Thus $a(\eta) = -b(-\eta) - 2\delta$ where $\delta > 0$. Now we refer to (1) to obtain for $f \in \mathcal{E}$,

$$\prod_{k=1}^n \|D_{e^{-t\eta_k}} f_k^*\|_{X_k} \le C \exp(t\delta + tb(-\eta)) \prod_{k=1}^n \|f_k^*\|_{X_k}$$

for $t \ge 0$. It follows from (6) that

$$\begin{aligned}\left(\int_{\mathbf{R}^n} \prod_{k=1}^n (f_k^*(e^{\xi_k + t\eta_k}))^s \exp(-sa(-\xi))d\xi\right)^{1/s} &\le C\exp(t\delta + tb(-\eta)) \prod_{k=1}^n \|f_k\|_{X_k} \\ &= C\exp(-ta(\eta) - t\delta) \prod_{k=1}^n \|f_k\|_{X_k}.\end{aligned}$$

Now $a(-\xi) + ta(\eta) \ge a(-\xi + t\eta)$. Thus we can reorganize to obtain

$$\left(\int_{\mathbf{R}^n} \prod_{k=1}^n (f_k^*(e^{\xi_k}))^s \exp(-sa(-\xi))d\xi\right)^{1/s} \le C\exp(-t\delta) \prod_{k=1}^n \|f_k\|_{X_k},$$

for every $t \ge 0$ which is absurd. □

Theorem 3.5. *Suppose Θ is a finite subset of $(\overline{\mathbf{R}}_+)^n$. Suppose $\mathbf{X}$ is an n-tuple of r.i. spaces such that $B_{\mathbf{X}} \cap co\, \Theta$ is a nonempty subset of $co_0\, \Theta$. Then $\mathbf{X}$ satisfies the (Θ, s) interpolation condition provided there is a constant C so that if $f_1, \ldots, f_n \in \mathcal{E}$,*

$$\left(\int_{H} \prod_{k=1}^n (f_k^*(e^{\xi_k}))^s \exp(-sa(-\xi))d\xi\right)^{1/s} \le C \prod_{k=1}^n \|f_k\|_{X_k}, \tag{13}$$

where H is the subspace of $\mathbf{R}^n$ of all ξ such that $\langle \xi, \theta\rangle$ is constant for all $\theta \in \Theta$.

If $\Theta \subset \mathbf{R}^n_+$ and $s\theta_k < 1$ for every $\theta \in \Theta$ and $1 \le k \le n$, then inequality (13) is also necessary for $\mathbf{X}$ to satisfy the (Θ, s)-interpolation condition.

Proof. Let $B_\epsilon = \{\xi : \ d(\xi, B_{\mathbf{X}}) \le \epsilon\}$. Our assumption on $B_{\mathbf{X}}$ and a compactness argument give the existence of $\epsilon > 0$ so that $B_{2\epsilon} \cap P \subset \mathrm{co}\, \Theta$, where P is the affine plane generated by Θ. We note that:

$$\max_{\phi\in B_{2\epsilon}\cap P}\langle \eta, \phi\rangle = \inf_{\xi\in H} a(\xi) + b(\eta - \xi) + 2\epsilon\|\eta - \xi\|. \tag{14}$$

To see this observe that the right hand side obviously dominates the left-hand side and is a sublinear functional. It is easy to check that if $\langle \eta, \phi\rangle$ is dominated by the right-hand side then we have $\phi \in B_{2\epsilon} \cap P$.

We will also need (1) which implies that if $f_1,\dots,f_n\in\mathcal{E}$, then

$$\prod_{k=1}^n \|D_{e^{\eta_k}}f_k\|_{X_k}\le C\exp(b(\eta)+\epsilon\|\eta\|)\prod_{k=1}^n\|f_k\|_{X_k}$$

for some constant C.

Now suppose $\eta\in H^\perp$. Then for a fixed $\zeta\in H$ and $f_1,\dots,f_n\in\mathcal{E}$ we have

$$\begin{aligned}
&\left(\int_H\prod_{k=1}^n (f_k^*(e^{\xi_k+\eta_k}))^s\exp(-sa(-\xi-\eta))d\xi\right)^{1/s}\\
=&\left(\int_H\prod_{k=1}^n (f_k^*(e^{\xi_k+\eta_k+\zeta_k}))^s\exp(-sa(-\xi-\zeta)-sa(-\eta))d\xi\right)^{1/s}\\
\le&\exp(-a(-\eta)+a(\zeta))\left(\int_H\prod_{k=1}^n (f_k^*(e^{\xi_k+\eta_k+\zeta_k}))^s\exp(-sa(-\xi))d\xi\right)^{1/s}\\
\le&\exp(-a(-\eta)+a(\zeta))\prod_{k=1}^n\|D_{e^{-\eta_k-\zeta_k}}f_k\|_{X_k}\\
\le& C\exp(-a(-\eta)+a(\zeta)+b(-\eta-\zeta)+\epsilon\|\eta+\zeta\|)\prod_{k=1}^n\|f_k\|_{X_k}.
\end{aligned}$$

At this point we use (14) and the fact that $B_{2\epsilon}\cap P\subset \mathrm{co}\,\Theta$ to show that

$$\inf_{\zeta\in H}\big(a(\zeta)+b(-\eta-\zeta)+\epsilon\|\eta+\zeta\|\big)\le a(-\eta)-\epsilon\|\eta\|.$$

Thus we conclude that

$$\left(\int_H\prod_{k=1}^n (f_k^*(e^{\xi_k+\eta_k}))^s\exp(-sa(-\xi-\eta))d\xi\right)^{1/s}\le C\exp(-\epsilon\|\eta\|)\prod_{k=1}^n\|f_k\|_{X_k}.$$

Raise to the s^{th} power and integrate over $\eta\in H^\perp$ to obtain (6). Hence $\mathbf{X}$ satisfies the (Θ,s)-interpolation condition.

The last statement follows from (8). □

We now specialize to the case when Θ is a relatively large subset of $(\overline{\mathbf{R}}_+)^n$. Let us define the dimension of Θ, denoted dim Θ, to be the dimension of the affine plane passing through all the points in Θ. We say that Θ is affinely independent if the conditions

$$\sum_{\theta\in\Theta}\lambda_\theta\theta=\mathbf{0}\qquad\text{and}\qquad\sum_{\theta\in\Theta}\lambda_\theta=0$$

imply $\lambda_\theta=0$, $\forall\,\theta\in\Theta$. Obviously if Θ is affinely independent we have $|\Theta|=1+\dim\Theta$.

Theorem 3.6. *Suppose dim $\Theta=n$ (e.g. if Θ is an affinely independent subset of $(\overline{\mathbf{R}}_+)^n$ and $|\Theta|=n+1$.) Suppose $\mathbf{X}$ is an n-tuple of r.i. spaces such that $B_{\mathbf{X}}\cap co\,\Theta$ is a nonempty subset of $co_0\,\Theta$. Then $\mathbf{X}$ satisfies the (Θ,s) interpolation condition.*

Proof. In this case Theorem 3.5 applies with $H=\{0\}$. □

A more important case is the following:

Theorem 3.7. *Suppose that dim* $\Theta = n-1$ *and that* Θ *spans* $\mathbf{R}^n$. *Let* $0 < s \leq 1$ *and suppose that* $\mathbf{X}$ *is an* n*-tuple of r.i. spaces such that* $B_{\mathbf{X}} \cap \mathrm{co}\,\Theta$ *is a nonempty subset of* $\mathrm{co}_0\,\Theta$. *Pick a unique* $\sigma = (\sigma_k)_{k=1}^n$ *so that* $\langle \sigma, \theta \rangle = 1$ *for all* $\theta \in \Theta$. *Consider the following statements:*
(i) $\mathbf{X}$ *satisfies the* (Θ, s) *interpolation condition.*
(ii) There is a constant C *such that if* $f_1, \ldots, f_n \in \mathcal{E}$ *we have*

$$\left(\int_0^\infty x^{s-1} \prod_{k=1}^n (f_k^*(x^{\sigma_k}))^s dx \right)^{1/s} \leq C \prod_{k=1}^n \|f_k\|_{X_k}. \tag{15}$$

Then (ii) implies (i). Moreover, if $\Theta \subset \mathbf{R}_+^n$ *and* $s\theta_k < 1$ *for every* $\theta \in \Theta$ *and* $1 \leq k \leq n$, *then (i) and (ii) are equivalent.*

Furthermore if $\Theta \subset \mathbf{R}_+^n$ *and if* $s(\sum_{k=1}^n \theta_k) \leq 1$ *for every* $\theta \in \Theta$, *(i) and (ii) are also equivalent to:*
(iii) There is a constant C *so that if* $f_1, \ldots, f_n \in \mathcal{E}$,

$$\left(\int_0^\infty x^{s-1} \prod_{\sigma_k \neq 0} |f_k(x^{\sigma_k})|^s dx \right)^{1/s} \leq C \prod_{\sigma_k \neq 0} \|f_k\|_{X_k}. \tag{16}$$

Remarks. The existence of σ follows from the fact that the plane generated by Θ does not contain the origin. Note that the indices k for which $\sigma_k = 0$ become *redundant* in the sense that that (15) can be rewritten as

$$\left(\int_0^\infty x^{s-1} \prod_{\sigma_k \neq 0} (f_k^*(x^{\sigma_k}))^s dx \right)^{1/s} \leq C \prod_{\sigma_k \neq 0} \|f_k\|_{X_k}.$$

Before we prove Theorem 3.7, let us illustrate the hypothesis on the Boyd indices, by considering the special but rather typical case when co Θ is the intersection of a cube $\prod_{k=1}^n [\alpha_k, \beta_k]$ with the plane $\sum_{k=1}^n \theta_k = r^{-1}$. In this case $\sigma_k = r$ for all k. It may then easily be seen that the hypotheses on the Boyd indices are satisfied if we have both

$$\sum_{k=1}^n \frac{1}{q_{X_k}} \leq \frac{1}{r} \leq \sum_{k=1}^n \frac{1}{p_{X_k}} \tag{17}$$

and

$$\alpha_k < \frac{1}{q_{X_k}} \leq \frac{1}{p_{X_k}} < \beta_k \tag{18}$$

for all $1 \leq k \leq n$. However if for some l we have

$$\alpha_l + \sum_{k \neq l} \frac{1}{q_{X_k}} > \frac{1}{r} \tag{19}$$

then the lower bound condition on $q_{X_l}^{-1}$ in (18) can be removed. Similarly if

$$\beta_l + \sum_{k\neq l} \frac{1}{p_{X_k}} < \frac{1}{r} \tag{20}$$

then the upper bound condition on $p_{X_l}^{-1}$ can be removed.

Proof. The fact that (ii) implies (i) is an application of Theorem 3.5. Indeed, in this case H is one-dimensional, say $H = \{t\sigma\}_{t\in\mathbf{R}}$. Then equation (13) becomes

$$\left(\int_{-\infty}^{+\infty} \prod_{k=1}^{n} (f_k^*(e^{t\sigma_k}))^s e^{-st} dt\right)^{1/s} \leq C \prod_{k=1}^{n} \|f_k\|_{X_k}$$

which reduces to (15) by substituting $x = e^t$. The converse statement follows from Theorem 3.5.

We now prove that (i) implies (iii) under the extra hypothesis $s(\sum_{k=1}^n \theta_k) \leq 1$ for every $\theta \in \Theta$. Define a map $T : \mathcal{E}^n \to L_s((0,\infty)\times(0,1))$ by setting

$$T(f_1,\dots,f_n)(x,y) = x^{1-1/s} \prod_{\sigma_k\neq 0} f_k(x^{\sigma_k}) \prod_{\sigma_l=0} f_l(y).$$

We will show that T is $\Theta-$admissible.

Suppose (E_k) are sets of finite measure. Let $F = \{x :\ x^{\sigma_k} \in E,\ \forall\, \sigma_k \neq 0\}$ and let $G = [0,1] \cap \cap_{\sigma_k=0} E_k$. Then we have

$$T(\chi_{E_1},\dots,\chi_{E_n})(x,y) = x^{1-1/s}\chi_F(x)\chi_G(y)$$

and therefore

$$\|T(\chi_{E_1},\dots,\chi_{E_n})\| = \left(\int_F x^{s-1}dx\right)^{1/s} |G|^{1/s}.$$

Now suppose $\theta\in\Theta$. Let $r = (\sum_{\sigma_k\neq 0}\theta_k)^{-1}$. Clearly $s \leq r$. We have by Lemma 3.1

$$\begin{aligned}
\left(\int_F x^{s-1}dx\right)^{1/s} &\leq r^{1/r}s^{-1/s}\left(\int_F x^{r-1}dx\right)^{1/r} \\
&\leq r^{1/r}s^{-1/s}\prod_{j\in J}\left(\int_F x^{\sigma_k-1}dx\right)^{\theta_k} \\
&\leq r^{1/r}s^{-1/s}\prod_{k\in J}|\sigma_k|^{-\theta_k}|E_k|^{\theta_k}.
\end{aligned}$$

On the other hand since $|G| \leq 1$ and $\sum_{\sigma_k\neq 0}\theta_k \leq s^{-1}$,

$$|G|^{1/s} \leq \prod_{\sigma_k\neq 0} |E_k|^{\theta_k}.$$

Thus T is $\Theta-$admissible and hence T extends to a bounded n-linear form on $X_1 \times \cdots \times X_n$. Letting $f_k = \chi_{[0,1]}$ if $\sigma_k = 0$ and restricting gives (16).

Now it is clear that (iii) implies (ii) and so the proof is complete. $\square$

Corollary 3.8. *Suppose under the hypotheses of Theorem 3.7 we also have that for some fixed r*

$$\sum_{k=1}^{n}\theta_k=\frac{1}{r}$$

for every $\theta\in\Theta$*. Then* $\mathbf{X}$ *has the* (Θ,s) *interpolation condition if*

$$\left(\int_0^\infty x^{s/r-1}\prod_{k=1}^{n}(f_k^*(x))^s dx\right)^{1/s}\le C\prod_{k=1}^{n}\|f_k\|_{X_k}. \tag{21}$$

In particular if $r=s$ *then* $\mathbf{X}$ *has the* (Θ,s) *interpolation condition if and only if* $X_1\cdots X_n\subset L_s$ *where* $X_1\cdots X_n=\{f_1\ldots f_n;\ f_k\in X_k\}$.

Proof. In this case $\sigma_k=r$ for all k and (21) is a obtained by a simple change of variables from (15). □

Finally let us note an unusual case which can arise:

Theorem 3.9. *Suppose* $\Theta\subset\mathbf{R}_+^n$*, dim* $\Theta=n-1$ *and* Θ *does not span* $\mathbf{R}^n$*. Suppose* $\mathbf{X}$ *is an n-tuple of r.i. spaces such that* $B_{\mathbf{X}}\cap co\,\Theta$ *is a nonempty subset of* $co_0\,\Theta$*. Let* $\sigma=(\sigma_k)_{k=1}^n$ *be chosen so that* $\langle\sigma,\theta\rangle=0$ *for all* $\theta\in\Theta$*. Assume* $s>0$ *is such that* $s\theta_k<1$ *for every* $\theta\in\Theta$ *and* $1\le k\le n$*. Then the following are equivalent:*
(i) $\mathbf{X}$ *satisfies the* (Θ,s) *interpolation condition.*
(ii) There is a constant C so that for $f_1,\ldots,f_n\in\mathcal{E}$ *we have*

$$\left(\int_0^\infty x^{-1}\prod_{k=1}^{n}(f_k^*(x^{\sigma_k}))^s dx\right)^{1/s}\le C\prod_{k=1}^{n}\|f_k\|_{X_k}. \tag{22}$$

We omit the proof which is similar to the that of Theorem 3.7.

4. The inhomogeneous multilinear Boyd theorem and applications

Suppose that Θ is a finite subset of $(\overline{\mathbf{R}}_+)^n$ and that $\theta\to r_\theta$ is a map from Θ to $\mathbf{R}_+$. We denote $\phi_\theta=(\theta,r_\theta)$ and $\Phi=\{\phi_\theta:\ \theta\in\Theta\}$. Clearly $\Phi\subset\mathbf{R}^{n+1}$. Now consider the case when we are given a map $T:\mathcal{E}^n\to L_0(0,\infty)$, which we assume to be locally continuous. We will say that T satisfies a weak-type (θ,r_θ) estimate if there exists $M>0$ so that if $E_1,\ldots,E_n$ are sets of finite measure then

$$\|T(\chi_{E_1},\ldots,\chi_{E_n})\|_{L_{r,\infty}}\le M\prod_{k=1}^{n}|E_k|^{\theta_k}. \tag{23}$$

We now give a version of the Boyd interpolation theorem for this setting which follows almost immediately from Theorem 3.5. For simplicity we shall only treat the most important case.

Theorem 4.1. *Suppose* Θ *is a subset of* $(\overline{\mathbf{R}}_+)^n$ *with* $|\Theta|=n+1$ *and dim* $\Theta=n$*. Suppose for each* θ *we have* $0<r_\theta\le\infty$*. Let* $\sigma\in\mathbf{R}^n$ *be the unique solution of the equation*

$$\langle\sigma,\theta\rangle=\frac{1}{r_\theta}+\tau$$

where τ is independent of θ. Let $\mathbf{X}$ be an n-tuple of r.i. spaces and suppose Y is a maximal r.i. space which is s-convex for some $s > 0$. Suppose the Boyd cube $B_{\mathbf{X}} \times [1/q_Y, 1/p_Y]$ intersects $\mathrm{co}\,\Phi$ in a non-empty subset of $\mathrm{co}_0\,\Phi$.

Then in order that every locally continuous n-linear $T : \mathcal{E}^n \to L_0(0,\infty)$, which satisfies the weak type (θ, r_θ) estimate (23) for $\theta \in \Theta$, extends to a bounded n-linear map $T : \prod_{k=1}^n X_k \to Y$ (with norm a multiple of M), it is sufficient that there is a constant C so that if $f_1, \dots, f_n \in \mathcal{E}$ then

$$\|x^\tau \prod_{k=1}^n f_k^*(x^{\sigma_k})\|_Y \le C \prod_{k=1}^n \|f_k\|_{X_k}. \tag{24}$$

If in addition we have

$$0 < \frac{1}{r_\theta} \le \sum_{k=1}^n \theta_k$$

for every $\theta \in \Theta$, then (24) is also necessary and is equivalent to the condition that there exists a constant C so that for $f_1, \dots, f_n \in \mathcal{E}$ we have

$$\|x^\tau \prod_{\sigma_k \neq 0} f_k(x^{\sigma_k})\|_Y \le C \prod_{\sigma_k \neq 0} \|f_k\|_{X_k}. \tag{25}$$

Remark. The same conclusions can be obtained if $|\Theta| > n+1$ provided there is a solution of the equation $\langle \sigma, \theta \rangle = r_\theta^{-1} + \tau$.

Proof. We first choose $s > 0$ small enough so that Y is s-convex, $s < r_\theta$ for all $\theta \in \Theta$, and $\tau + \frac{1}{s} > 0$. Next define X_{n+1} to be the space of all $f \in L_0(0,\infty)$ so that $fg \in L_s(0,\infty)$ for all $g \in Y$ with the quasi-norm

$$\|f\|_{X_{n+1}} = \sup_{\|g\|_Y \le 1} \|fg\|_{L_s}.$$

Since Y is both maximal and s-convex we obtain that $g \in Y$ if and only if $\sup\{\|fg\|_{L_s} : \|f\|_{X_{n+1}} \le 1\}$ is finite and furthermore there is a constant C so that $\|f\|_Y \le \sup\{\|fg\|_{L_s} : \|f\|_{X_{n+1}} \le 1\}$. (If Y is s-convex with constant one then $C = 1$; this is easily seen by noting that Y^s is a Banach r.i. space and X_{n+1}^s is simply the Köthe dual space; in general we can always renorm Y to have s-convexity constant one.) It easy to calculate the Boyd indices of X_{n+1}; these are given by

$$\frac{1}{p_{X_{n+1}}} = \frac{1}{s} - \frac{1}{q_Y}, \ \frac{1}{q_{X_{n+1}}} = \frac{1}{s} - \frac{1}{p_Y}.$$

We refer to [12] for similar calculations for dual spaces.

Now if T satisfies the weak-type (θ, r_θ) estimate (23) for every $\theta \in \Theta$, then we consider the map $T' : \mathcal{E}^{n+1} \to L_s(0,\infty)$ defined by

$$T'(f_1, \dots, f_{n+1}) = T(f_1, \dots, f_n) f_{n+1}.$$

If $E_1, \dots, E_{n+1}$ are sets of finite measure then

$$\|T'(\chi_{E_1}, \dots, \chi_{E_{n+1}})\|_{L_s} \le \left(\int_{E_{n+1}} |T(\chi_{E_1}, \dots, \chi_{E_n})|^s dx\right)^{1/s}$$
$$\le M\left(\tfrac{r_\theta}{r_\theta - s}\right)^{1/s} |E_{n+1}|^{1/s-1/r_\theta} \prod_{k=1}^n |E_k|^{\theta_k}$$

for every $\theta \in \Theta$. Thus if we let $\psi_\theta = (\theta, \frac{1}{s} - \frac{1}{r_\theta})$ and $\Psi = \{\psi_\theta : \ \theta \in \Theta\}$ then T' is (Ψ, s)-admissible. It is clear from our discussion of X_{n+1} that we only need to show that T' extends to a bounded n-linear map on $X_1 \times \cdots \times X_{n+1}$.

We now use Theorem 3.7. We first argue that Ψ is linearly independent in $\mathbf{R}^{n+1}$. Indeed from the definition of τ this is equivalent to the linear independence of the points $(\theta, \tau + \frac{1}{s})$ which follows from the affine independence of Θ. We will show that the $(n+1)$-tuple $(X_1, \dots, X_{n+1})$ has the (Ψ, s)-interpolation condition. We note that our hypotheses on the Boyd indices of $\mathbf{X}$ and Y imply that the hypotheses on the Boyd indices for Theorem 3.6 hold. Define $\sigma_k' = \sigma_k(\tau + \frac{1}{s})^{-1}$ for $k \le n$ and $\sigma_{n+1}' = (\tau + \frac{1}{s})^{-1}$. Then

$$\langle \sigma', \psi_\theta \rangle = (\tau + \tfrac{1}{s})^{-1}(\tau + \tfrac{1}{r_\theta}) + (\tau + \tfrac{1}{s})^{-1}(\tfrac{1}{s} - \tfrac{1}{r_\theta}) = 1.$$

Now if $f_1, \dots, f_{n+1} \in \mathcal{E}^{n+1}$, we have

$$\left(\int_0^\infty x^{s-1} \prod_{k=1}^{n+1} |f_k^*(x^{\sigma_k'})|^s dx\right)^{1/s} = (\tau + \tfrac{1}{s})^{1/s} \left(\int_0^\infty x^{s\tau} |f_{n+1}^*(x)|^s \prod_{k=1}^n |f_k^*(x^{\sigma_k})|^s dx\right)^{1/s}.$$

Now it is clear that if we assume (24) then we obtain (15) in Theorem 3.7 and so $(X_1, \dots, X_{n+1})$ satisfies the interpolation condition (Θ, s).

For the second part we construct the map $T : \mathcal{E}^n \to L_0((0,\infty) \times (0,1))$ by

$$T(f_1, \dots, f_n)(x, y) = x^\tau \prod_{\sigma_k \ne 0} f_k(x^{\sigma_k}) \prod_{\sigma_j = 0} f_j(y).$$

Let $u_\theta = (\sum_{\sigma_k \ne 0} \theta_k)^{-1}$ so that $r_\theta \le u_\theta$. Then by arguments similar to those for Theorem 3.7 we have that if $E_1, \dots, E_n$ are measurable sets of finite measure,

$$\|T(\chi_{E_1}, \dots, \chi_{E_n})\|_{L_{r_\theta}} \le r_\theta^{-1/r_\theta} u_\theta^{1/u_\theta} \prod_{k=1}^n |E_k|^{\theta_k}.$$

Our hypotheses then guarantee that T maps $X_1 \times \cdots \times X_n$ into Y i.e. we have (25) and hence also (24). □

Let us isolate a simple special case:

Corollary 4.2. *Suppose that in the preceding theorem we have*

$$\sum_{k=1}^n \theta_k = \frac{1}{r_\theta}$$

for every $\theta \in \Theta$. Then (25) is equivalent to the inclusion $X_1 \cdots X_n \subset Y$, where $X_1 \cdots X_n$ is the set of all products $f_1 \dots f_n$ with $f_k \in X_k$.

Proof. We need only to observe that in this case $\sigma_k = 1$ for every k and $\tau = 0$. □

We next point out that under certain hypotheses, we can replace (24) with an alternative criterion:

Corollary 4.3. *Suppose that in Theorem 4.1, $\widetilde{Y}$ is a carrier space for Y with the property that $\|D_a\|_{\widetilde{Y}} \le C_0 a^\rho$ for all $0 < a < 1$ where $\rho > 0$ and that $\widetilde{Y}$ is s-convex for some $s > 0$. Then the sufficent condition (24) can be replaced by:*

$$\|x^\tau \prod_{k=1}^{n} f_k^*(x^{\sigma_k})\|_{\widetilde{Y}} \le C \prod_{k=1}^{n} \|f_k\|_{X_k}, \tag{26}$$

for $f_1, \dots, f_n \in \mathcal{E}$.

Proof. We note that in the proof of Theorem 24 we can take s small enough so $\widetilde{Y}$ is s-convex. Suppose $f_1, \dots, f_n \in \mathcal{E}$. Let $\varphi(x) = x^\tau \prod_{k=1}^n f_k^*(x^{\sigma_k})$. By assumption $\|\varphi\|_{\widetilde{Y}} \le C \prod_{k=1}^n \|f_k\|_{X_k}$. Now let ψ be defined by

$$\psi(x) = \left(\int_x^\infty \varphi(y)^s \frac{dy}{y} \right)^{1/s}.$$

By the s-convexity of $\widetilde{Y}$ we obtain that

$$\|\psi\|_{\widetilde{Y}} \le M \left(\int_0^1 a^{s\rho - 1} da \right)^{1/s} \|\phi\|_{\widetilde{Y}}$$

so that we have an estimate

$$\|\psi\|_{\widetilde{Y}} \le C_1 \|\varphi\|_{\widetilde{Y}}.$$

However ψ is decreasing and so $\|\psi\|_Y \le C_1 \|\varphi\|_{\widetilde{Y}}$. Now if $f_{n+1} \in \mathcal{E}$ we have that

$$\left(\int_0^\infty (f_{n+1}^*(x))^s \varphi(x)^s dx \right)^{1/s} \le CC_1 \prod_{k=1}^{n+1} \|f_k\|_{X_k}$$

and the proof is completed in the same way. □

At this point we note that we can use Corollary 3.3 to extend this result to n-quasi-sublinear maps.

Corollary 4.4. *Assume that $X_1, \dots, X_n, Y$ satisfy (24). Suppose $T : \mathcal{E}^n \to L_0(0, \infty)$ is n-quasi-sublinear, locally continuous, and satisfies the weak-type (θ, r_θ)-inequality (23) for every $\theta \in \Theta$. Then we have the estimate*

$$\|T(f_1, \dots, f_n)\|_Y \le CM \prod_{k=1}^{n} \|f_k\|_{X_k}$$

for $f_1, \dots, f_n \in \mathcal{E}$.

We omit the details of the proof. The key point to note is that we should choose s in the argument for Theorem 4.1 above sufficiently small so that $2^{1/s-1} \ge C$ where is the constant in (11).

It is also worth noting that we can give a similar result to Theorem 4.1 in the case when Θ fails to be affinely independent. This case is somewhat degenerate. For example in the case $n = 1$ it applies to linear operators which satisfy weak type estimates (p, q_1) and (p, q_2) where $q_1 \neq q_2$.

Theorem 4.5. *Suppose Θ is an affinely dependent subset of $(\overline{\mathbf{R}}_+)^n$ with $|\Theta| = n+1$. Suppose for each θ we have $0 < r_\theta \leq \infty$ and that the set $\Phi = \{(\theta, r_\theta^{-1}) : \theta \in \Theta\}$ is linearly independent in $\mathbf{R}^{n+1}$. Choose $\sigma \in \mathbf{R}^n$ so that $\langle \sigma, \theta \rangle = 1$ for all $\theta \in \Theta$. Let $\mathbf{X}$ be an n-tuple of r.i. spaces and suppose Y is a maximal r.i. space. Let $r = \min_{\theta\in\Theta} r_\theta$ and suppose $0 < s \leq 1$ is such that $s < 1$ if $r = 1$ and $s \leq r$ otherwise. Suppose also the Boyd cube $B_{\mathbf{X}} \times [1/q_Y, 1/p_Y]$ intersects co Φ in a non-empty subset of $co_0\,\Phi$.*

Then, in order that every locally continuous n-linear $T : \mathcal{E}^n \to L_0(0,\infty)$, which satisfies the weak type (θ, r_θ) estimate (23) for $\theta \in \Theta$, extends to a bounded n-linear map $T : \prod_{k=1}^n X_k \to Y$ (with norm a multiple of M), it is sufficient that there exists a constant C so that

$$\left(\int_0^\infty x^{s-1}\prod_{k=1}^n (f_k^*(x^{\sigma_k}))^s dx\right)^{1/s} \leq C\prod_{k=1}^n \|f_k\|_{X_k}, \tag{27}$$

for $f_1, \ldots, f_n \in \mathcal{E}$.

Remark. The existence and uniqueness of σ is a consequence of our hypotheses, since Θ generates a plane of dimension $n-1$ which cannot be a linear subspace.

Proof. Our hypotheses are such that the space $L_{r,\infty}$ is s-normable. In this case the convex set Φ generates a plane containing the line in the direction parallel to the basis vector e_{n+1}.

We first prove the result when $Y = L_{t,\infty}$ for some t. By the above remark we have $t > r$. Let $X_{n+1} = L_{u,\infty}$ where $\frac{1}{t} + \frac{1}{u} = \frac{1}{r}$. Let $\psi_\theta = (\theta, \frac{1}{r} - \frac{1}{r_\theta}) \in \mathbf{R}^{n+1}$ and $\Psi = \{\psi_\theta : \theta \in \Theta\}$. Now it is clear that (27) implies that the $(n+1)$-tuple $(X_1, \ldots, X_{n+1})$ satisfies the conditions of Theorem 3.7 for the (Ψ, s)-interpolation condition. We apply this to the map $T' : \mathcal{E}^{n+1} \to L_{r,\infty}$ where $T'(f_1, \ldots, f_{n+1}) = T(f_1, \ldots, f_n)f_{n+1}$. A routine calculation gives

$$\|T'(\chi_{E_1}, \ldots, \chi_{E_{n+1}})\|_{L_{r,\infty}} \leq CM|E|^{1/r-1/r_\theta}\prod_{k=1}^n |E_k|^{\theta_k}.$$

Hence we have the estimate

$$\|T(f_1, \ldots, f_n)f_{n+1}\|_{L_{r,\infty}} \leq CM\|f_{n+1}\|_{L_{u,\infty}}\prod_{k=1}^n \|f_k\|_{X_k}.$$

This implies the estimate

$$\|T(f_1, \ldots, f_n)\|_{L_{t,\infty}} \leq CM\prod_{k=1}^n \|f_k\|_{X_k}$$

by simply considering $f_{n+1} = \chi_E$ for E a set of finite measure. We have now proved our claim.

Next we consider the general case. By our assumptions on Φ we may find $t < p_Y \leq q_Y < u$ so that both $(X_1, \ldots, X_n, L_{t,\infty})$ and $(X_1, \ldots, X_n, L_{u,\infty})$ satisfy the interior condition on the Boyd indices. Hence T maps $X_1 \times \cdots \times X_n$ boundedly into $L_{t,\infty} \cap L_{u,\infty}$ with norm a multiple of M. But it is easy to calculate from the Boyd indices that $L_{t,\infty} \cap L_{u,\infty} \subset Y$. $\square$

The theorems below extend the classical Marcinkiewicz interpolation theorem to the multilinear setting.

Theorem 4.6. *Let $0 < p_{jk} \le \infty$ for $1 \le j \le n+1$ and $1 \le k \le n$, and also let $0 < p_j \le \infty$ for $1 \le j \le n+1$. Suppose that a locally continuous n-linear map $T : \mathcal{E}^n \to L_0$ satisfies*

$$\|T(\chi_{E_1}, \dots, \chi_{E_n})\|_{L_{p_j,\infty}} \le M|E_1|^{1/p_{j1}} \dots |E_n|^{1/p_{jn}}$$

for all sets E_j of finite measure and all $1 \le j \le n+1$. Assume that the system below

$$\begin{pmatrix} 1/p_{11} & 1/p_{12} & \dots & 1/p_{1n} & 1 \\ 1/p_{21} & 1/p_{22} & \dots & 1/p_{2n} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/p_{n1} & 1/p_{n2} & \dots & 1/p_{nn} & 1 \\ 1/p_{(n+1)1} & 1/p_{(n+1)2} & \dots & 1/p_{(n+1)n} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \\ -\tau \end{pmatrix} = \begin{pmatrix} 1/p_1 \\ 1/p_2 \\ \vdots \\ 1/p_n \\ 1/p_{n+1} \end{pmatrix},$$

has a unique *solution $(\sigma_1, \dots, \sigma_n, -\tau) \in \mathbf{R}^{n+1}$ with* not all *$\sigma_j = 0$. Suppose that $(1/q_1, \dots, 1/q_n, 1/q)$ lies in the open convex hull of the points $(1/p_{j1}, \dots, 1/p_{jn}, 1/p_j)$ in $\mathbf{R}^{n+1}$ and let $0 < t_k, t \le \infty$ satisfy*

$$\sum_{\substack{1 \le k \le n \\ \sigma_k \ne 0}} \frac{1}{t_k} \ge \frac{1}{t}\,. \tag{28}$$

Then T extends to a bounded n-linear map $T : \prod_{k=1}^n L_{q_k,t_k} \to L_{q,t}$ with norm a multiple of M.

Remark. We remark that the existence of the unique solution in the linear system of Theorem 4.6 is equivalent to the condition that the $n+1$ points $\theta_j = (1/p_{jk})_{k=1}^n$ are affinely independent in $\mathbf{R}^n$. We also note that as in Corollary 4.4 the result above is valid for n-quasi-sublinear maps.

Proof. We clearly only need to consider the case of equality in (28). It is clear that the Boyd index assumption of Theorem 4.1 is satisfied. Clearly we have

$$\sum_{k=1}^n \frac{\sigma_k}{q_k} = \tau + \frac{1}{q}.$$

Hence if $f_1, \dots, f_n \in \mathcal{E}$ we have

$$x^{1/q+\tau} \prod_{k=1}^n f_k^*(x^{\sigma_k}) = \prod_{k=1}^n x^{\sigma_k/q_k} f_k^*(x^{\sigma_k}).$$

Let $F(x) = x^\tau \prod_{\sigma_k \ne 0} f_k^*(x^{\sigma_k})$. Then

$$\left(\int_0^\infty (x^{1/q}F(x))^t \frac{dx}{x}\right)^{1/t} \le C \prod_{\sigma_k \ne 0} \left(\int_0^\infty (x^{\sigma_k/q_k} f_k^*(x^{\sigma_k}))^{t_k} \frac{dx}{x}\right)^{1/t_k}.$$

Now if $\sigma_k \neq 0$

$$\left(\int_0^\infty (x^{\sigma_k/q_k} f_k^*(x^{\sigma_k}))^{t_k} \frac{dx}{x}\right)^{1/t_k} = |\sigma_k|^{-1} \|f\|_{L_{q_k,t_k}}.$$

Thus we have an estimate

$$\left(\int_0^\infty (x^{1/q} F(x))^t \frac{dx}{x}\right)^{1/t} \leq C \prod_{\sigma_k \neq 0} \|f_k\|_{L_{q_k,t_k}}.$$

In view of Corollary 4.3 this completes the proof. □

There is a version of the above result for the degenerate case corresponding to Theorem 4.5:

Theorem 4.7. *Let $0 < p_{jk} \leq \infty$ for $1 \leq j \leq n+1$ and $1 \leq k \leq n$, and let $0 < p_j \leq \infty$ for $1 \leq j \leq n+1$. Suppose that a locally continuous n-linear map $T : \mathcal{E}^n \to L_0$ satisfies*

$$\|T(\chi_{E_1}, \dots, \chi_{E_n})\|_{L_{p_j,\infty}} \leq M|E_1|^{1/p_{j1}} \dots |E_n|^{1/p_{jn}}$$

for all subsets E_k of finite measure and all $1 \leq j \leq n+1$. Assume that the $n+1$ points $\theta_j = (1/p_{jk})_{k=1}^n$ are affinely dependent in $\mathbf{R}^n$, but the points $(\theta_j, 1/p_j)$ are linearly independent in $\mathbf{R}^{n+1}$. Suppose that $(1/q_1, \dots, 1/q_n, 1/q)$ lies in the open convex hull of the points $(1/p_{j1}, \dots, 1/p_{jn}, 1/p_j)$ in $\mathbf{R}^{n+1}$. Let $r = \min_{1 \leq j \leq n+1} p_j$ and $0 < t_k, t \leq \infty$ satisfy

$$\sum_{\substack{1 \leq k \leq n \\ \sigma_k \neq 0}} \frac{1}{t_k} \begin{cases} > 1 & \text{if} \quad r = 1, \\ \geq \frac{1}{r} & \text{if} \quad r \neq 1, \end{cases} \tag{29}$$

where $\{\sigma_k\}_{k=1}^n$ are the unique solutions of the system

$$\sum_{k=1}^n \frac{\sigma_k}{p_{jk}} = 1, \qquad 1 \leq j \leq n+1.$$

Then T extends to a bounded n-linear map $T : \prod_{k=1}^n L_{q_n,t_n} \to L_{q,t}$ with norm a multiple of M.

Proof. This is deduced from Theorem 4.5. It is clear our hypotheses guarantee the appropriate conditions on the Boyd indices. Pick any $0 < s \leq 1$ so that $s \leq r$ if $r \neq 1$ and $s < 1$ otherwise with

$$\frac{1}{s} \geq \sum_{\sigma_k \neq 0} \frac{1}{t_k}.$$

It then suffices to verify (27) in Theorem 4.5. To do this we can clearly suppose that

$$\frac{1}{s} = \sum_{\sigma_k \neq 0} \frac{1}{t_k}.$$

Suppose $f_1, \dots, f_n \in \mathcal{E}$ and set

$$F(x) = x \prod_{\sigma_k \neq 0} f_k^*(x^{\sigma_k}).$$

Then

$$F(x) = \prod_{\sigma_k \neq 0} x^{\sigma_k/q_k} f_k^*(x^{\sigma_k})$$

and so

$$\left(\int_0^\infty F(x)^s \frac{dx}{x} \right)^{1/s} \leq \prod_{\sigma_k \neq 0} |\sigma_k|^{-1} \|f_k\|_{L_{q_k,t_k}}.$$

This establishes (27) and completes the proof. □

5. Examples and applications

In this section we discuss some examples of multilinear interpolation. For simplicity we restrict ourselves to bilinear and trilinear examples.

Example 5.1. (Young's inequality and O'Neil's inequality) On a locally compact abelian group consider the bilinear operator $(f, g) \to f * g$, where $*$ denotes convolution. Let H denote the closed triangle in $\mathbf{R}^3$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$. The well known Young's inequality says that

$$\|f * g\|_{L_r} \leq C \|f\|_{L_p} \|g\|_{L_q} \tag{30}$$

holds if the point $(1/p, 1/q, 1/r)$ lies in the closure of the triangle H.

The three trivial estimates $\|f * g\|_{L_1} \leq \|f\|_{L_1} \|g\|_{L_1}$, $\|f * g\|_{L_\infty} \leq \|f\|_{L_1} \|g\|_{L_\infty}$, and $\|f * g\|_{L_\infty} \leq \|f\|_{L_\infty} \|g\|_{L_1}$ give (30) on the interior of H. The estimates on the sides follow from bilinear complex interpolation.

Applying Theorem 4.6 in the situation above we obtain O' Neil's inequality. If the point $(1/p, 1/q, 1/r)$ lies in the interior of the triangle H and $0 < s_1, s_2 \leq \infty$ and $1/s = 1/s_1 + 1/s_2$, then

$$\|f * g\|_{L_{r,s}} \leq C \|f\|_{L_{p,s_1}} \|g\|_{L_{q,s_2}}. \tag{31}$$

The special case $s_1 = p$, $s = s_2 = \infty$ is of particular interest. Observe that if $(1/p, 1/q, 1/r)$ lies in the interior of H, then $1/p + 1/q = 1/r + 1$, from which it follows that $p < r$, which in turn implies that

$$\|f * g\|_{L_r} \leq C \|f * g\|_{L_{r,p}} \leq C \|f\|_{L_p} \|g\|_{L_{q,\infty}}.$$

The inequality above provides a sharpening of Young's inequality since the space L_q is replaced by $L^{q,\infty}$.

More generally we can use Theorem 4.1 to obtain the following result:

Theorem 5.2. *Suppose X, Y, Z are r.i. spaces whose Boyd indices satisfy the conditions*

$$1 < p_X, p_Y, p_Z, q_X, q_Y, q_Z < \infty,$$
$$\frac{1}{p_X} + \frac{1}{p_Y} \geq 1 + \frac{1}{q_Z},$$

and

$$\frac{1}{q_X} + \frac{1}{q_Y} \leq 1 + \frac{1}{p_Z}.$$

Assume that Z is maximal and $s-$convex for some $s > 0$. Then $(f, g) \to f * g$ maps $X \times Y$ to Z provided the map $(f, g) \to xf(x)g(x)$ maps $X(0, \infty) \times Y(0, \infty)$ to $Z(0, \infty)$.

Remark. Of course we can state this theorem with less stringent requirements on the Boyd indices, namely that the Boyd cube intersects H in a subset of its relative interior. As in the discussion in the remarks after Theorem 3.7 this can be illustrated. We can allow for example $p_X \leq 1$ provided $q_Y < p_Z$, and $q_X = \infty$ is permissible provided $p_Y^{-1} < 1 + q_Z^{-1}$. Similarly $p_Z \leq 1$ is permissible if $p_X^{-1} + p_Y^{-1} < 2$ and $q_Z = \infty$ is permissible if $q_X^{-1} + q_Y^{-1} > 1$.

Example 5.3. Fix three numbers $0 < \alpha, \beta, \gamma < n$ such that $\alpha + \beta > n$, $\beta + \gamma > n$ and $\gamma + \alpha > n$. Consider now the trilinear fractional integral form

$$I_{\alpha,\beta,\gamma}(f, g, h) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x)g(y)h(z)|x - y|^{-\alpha}|y - z|^{-\beta}|z - x|^{-\gamma} dxdydz.$$

We claim that the following inequality is valid

$$|I_{\alpha,\beta,\gamma}(f, g, h)| \leq C\|f\|_{L_p}\|g\|_{L_q}\|h\|_{L_r}$$

if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{\alpha + \beta + \gamma}{n} = 3, \quad 1 < p, q, r < \infty, \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \tag{32}$$

Note that (32) requires $\alpha + \beta + \gamma < 2n$.

Examples can be given to prove the necessity of the conditions on the indices above. Let us prove here the sufficiency. The assumptions $\alpha+\beta > n$, $\beta+\gamma > n$, and $\gamma+\alpha > n$ imply $\alpha + \beta + \gamma > 3n/2$ and hence it follows from (32) that $1/p + 1/q + 1/r < 3/2$. Therefore the plane given by the first equation in (32) cuts the unit cube $[0, 1]^3$ at the six points $A_1 = (1, (2n - \alpha - \beta - \gamma)/n, 0)$, $A_2 = ((2n - \alpha - \beta - \gamma)/n, 1, 0)$, $A_3 = (0, 1, (2n-\alpha-\beta-\gamma)/n)$ $A_4 = (0, (2n-\alpha-\beta-\gamma)/n, 1)$ $A_5 = ((2n-\alpha-\beta-\gamma)/n, 0, 1)$, and $A_6 = (1, 0, (2n-\alpha-\beta-\gamma)/n)$. These six points form the vertices of a hexagon. It suffices to prove Lorentz space estimates at these vertices for characteristic functions. For instance at the vertex A_1 the estimate we need to establish is

$$\int_{E_1} \int_{E_2} \int_{E_3} |x - y|^{-\alpha}|y - z|^{-\beta}|z - x|^{-\gamma} dxdydz \leq C|E_1||E_2|^{2-\frac{\alpha+\beta+\gamma}{n}}. \tag{33}$$

First integrate in z. We have

$$\int_{E_3} |y - z|^{-\beta}|z - x|^{-\gamma} dz \leq \int_{\mathbf{R}^n} |y - z|^{-\beta}|z - x|^{-\gamma} dz = C|x - y|^{n-\beta-\gamma}, \tag{34}$$

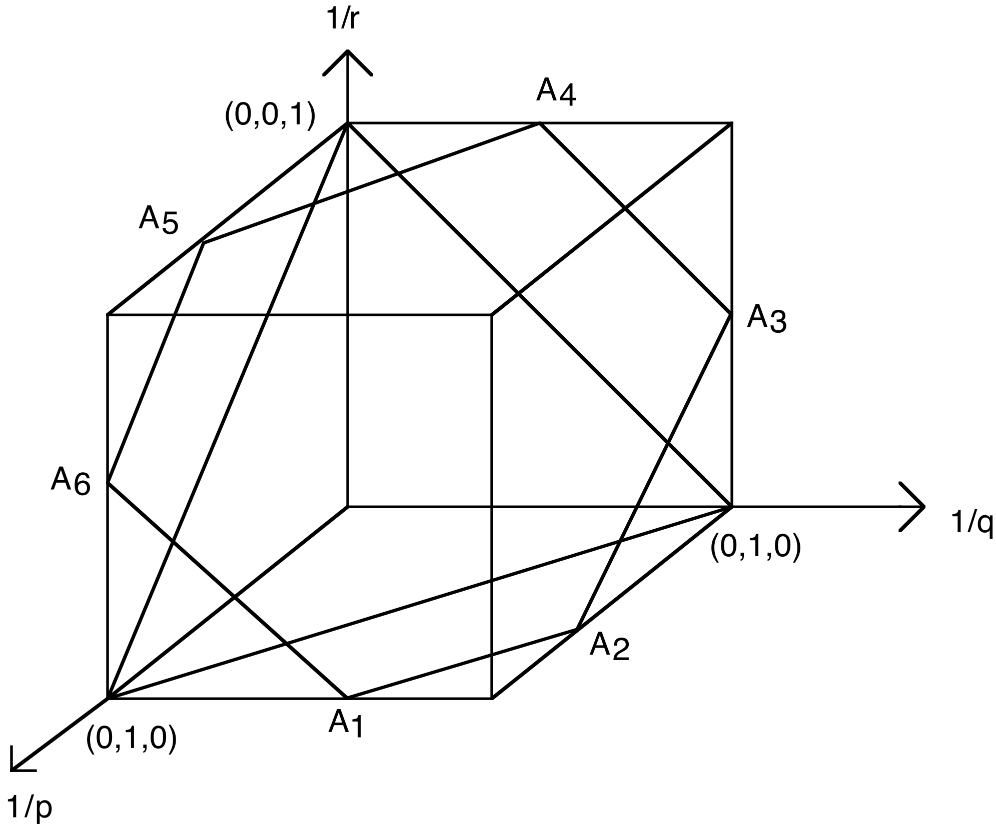

FIGURE 1. The set of all $(1/p, 1/q, 1/r)$ such that $|I_{\alpha,\beta,\gamma}(f, g, h)| \leq C\|f\|_{L_p}\|g\|_{L_q}\|h\|_{L_r}$.

for all $x \neq y$ since $\beta + \gamma > n$. The last equality above can be easily shown by a translation, a dilation, and a rotation. Using (34) we obtain

$$\begin{aligned}
&\int_{E_1}\int_{E_2}\int_{E_3} |x-y|^{-\alpha}|y-z|^{-\beta}|z-x|^{-\gamma}dxdydz \\
\leq &C\int_{E_1}\int_{E_2} |x-y|^{n-\alpha-\beta-\gamma}dydx \\
\leq &C\int_{E_1}\int_{|y|\leq c|E_2|^{1/n}} |y|^{n-\alpha-\beta-\gamma}dydx \\
\leq &C|E_1||E_2|^{(2n-\alpha-\beta-\gamma)/n},
\end{aligned}$$

which proves the required estimate (33). This example can be found in [3] when $n = 1$ and $\alpha = \beta = \gamma$.

In this example we have a trilinear form and it is appropriate to apply Corollary 3.8. Again simplifying our conditions on the Boyd indices gives:

Theorem 5.4. *Suppose X_1, X_2, X_3 are r.i. spaces on $\mathbf{R}^n$. Suppose the Boyd indices satisfy the conditions $1 < p_{X_i} \le q_{X_i} < \infty$ for $i = 1, 2, 3$ and*

$$\sum_{i=1}^{3} \frac{1}{q_{X_i}} \le 3 - \frac{\alpha+\beta+\gamma}{n} \le \sum_{i=1}^{3} \frac{1}{p_{X_i}}.$$

Then $I_{\alpha,\beta,\gamma}$ is bounded on $X_1 \times X_2 \times X_3$ provided the trilinear form $(f, g, h) \to x^{2-\frac{\alpha+\beta+\gamma}{n}} f(x)g(x)h(x)$ is bounded on $X_1(0,\infty) \times X_2(0,\infty) \times X_3(0,\infty)$.

Remark. Here as in the preceding example we can relax the conditions on the Boyd indices with the right extra hypotheses. For example if $p_{X_1} \le 1$ it is necessary that

$$\frac{1}{q_{X_2}} + \frac{1}{q_{X_3}} > 2 - \frac{\alpha+\beta+\gamma}{n}.$$

Example 5.5. Consider the operator

$$I(f,g)(x) = \int_{|t|\le 1} f(x+t)g(x-t)\,dt.$$

We will show I maps $L_p(\mathbf{R}^n) \times L_q(\mathbf{R}^n)$ into $L_r(\mathbf{R}^n)$ when $(1/p, 1/q, 1/r)$ lies in the closed convex hull of the points $(1,0,0)$, $(0,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$, and $(1,1,1/2)$.

By interpolation it suffices to establish boundedness estimates at these six points. Five of these estimates are trivial. We only prove that I maps $L_1 \times L_1 \to L_{1/2}$.

Suppose that we have established the estimate

$$\|I(f,g)\|_{L_{1/2}} \le C\|f\|_{L_1}\|g\|_{L_1} \tag{35}$$

for all f and g supported in two cubes of sidelength one. Then we prove (35) (with a larger constant) for all f and g integrable.

For each $k \in \mathbf{Z}^n$, let Q_k be the cube of sidelength one whose sides are parallel to the axes and whose lower left corner is $k \in \mathbf{Z}^n$. let $f_k = f\chi_{Q_k}$ and $g_m = g\chi_{Q_m}$. Then for each $k \in \mathbf{Z}^n$ there exist at most finitely many $m \in \mathbf{Z}^n$ such that $I(f_k, g_m)$ is nonzero. This is because the intersection of the sets $\{t : \ |t| \le 1\}$ and $\frac{1}{2}(Q_k - Q_m)$ has to be nonempty.

Now write

$$I(f,g) = \sum_{k\in\mathbf{Z}^n} \sum_{m\in\mathbf{Z}^n} I(f_k, g_m)$$

as a sum of a finite number of terms of the form

$$\sum_{k\in\mathbf{Z}^n} I(f_k, g_{k+d})$$

where $d \in \mathbf{Z}^n$ lies in a ball of radius at most a dimensional constant. Now

$$\|I(f,g)\|_{L_{1/2}} \le \left(\sum_{k\in\mathbf{Z}^n} \int_{\mathbf{R}^n} |I(f_k, g_{k+d})|^{1/2} dx \right)^2$$

$$\le C \left(\sum_{k\in\mathbf{Z}^n} \|f_k\|_{L_1}^{1/2} \|g_{k+d}\|_{L_1}^{1/2} \right)^2 \le C\|f\|_{L_1}\|g\|_{L_1},$$

by Cauchy-Schwarz, where the penultimate inequality above follows from the asumption that (35) holds for the functions f_k and g_m. Summing over d we obtain the required estimate $I : L_1 \times L_1 \to L_{1/2}$ with a larger constant.

We now prove (35) for f and g supported in cubes of sidelength one. (Think of $f = f_k$ and $g = g_{k+d}$.) Now observe that $I(f,g)$ is supported in a cube of sidelength two. Hölder's inequality gives

$$\|I(f,g)\|_{L_{1/2}} \le C\|I(f,g)\|_{L_1} \le C\int_{\mathbf{R}^n}\int_{|t|\le 1} |f(x+t)||g(x-t)|\,dtdx \le C\|f\|_{L_1}\|g\|_{L_1}.$$

Example 5.6. We now consider the bilinear fractional integral

$$I_\alpha(f,g)(x) = \int_{\mathbf{R}^n} f(x+t)g(x-t)|t|^{\alpha-n}dt,$$

where $0 < \alpha < n$. Homogeneity considerations imply that I_α can map $L_p(\mathbf{R}^n) \times L_q(\mathbf{R}^n) \to L_r(\mathbf{R}^n)$ only when

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{n}.$$

We will now show that I_α maps $L_p \times L_q \to L_r$ when the point $(1/p, 1/q, 1/r)$ lies in the open convex hull of the pentagon with vertices $(\frac{\alpha}{n}, 0, 0)$, $(1, 0, 1 - \frac{\alpha}{n})$, $(1, 1, \frac{2n-\alpha}{n})$, $(0, 1, 1 - \frac{\alpha}{n})$, and $(0, \frac{\alpha}{n}, 0)$. More precisely we will show that a weak-type estimate holds at each vertex of the pentagon below.

We first consider the vertex $(\frac{\alpha}{n}, 0, 0)$. Take $f = \chi_A$ and $g = \chi_B$, where A and B are measurable sets of finite measure. We have

$$\|I_\alpha(\chi_A, \chi_B)\|_{L_\infty} \le \sup_{x\in\mathbf{R}}\int_{-x+A} |t|^{\alpha-n}dt \le \int_{|t|\le c|A|} |t|^{\alpha-n}dt = C|A|^{\alpha/n}.$$

Likewise we obtain the required estimate at the vertex $(0, \frac{\alpha}{n}, 0)$.

The estimates at the vertices $(1, 0, 1 - \frac{\alpha}{n})$ and $(0, 1, 1 - \frac{\alpha}{n})$ follow from the estimates at the vertices $(\frac{\alpha}{n}, 0, 0)$ and $(0, \frac{\alpha}{n}, 0)$ respectively via duality. Alternatively, just observe that $I_\alpha(\chi_A, \chi_B) \le J_\alpha(\chi_A)$, where J_α is the usual fractional integral

$$(J_\alpha f)(x) = \int_{\mathbf{R}^n} f(x-y)|y|^{\alpha-n}dy,$$

and thus the estimate $\|I_\alpha(\chi_A, \chi_B)\|_{L_{n/(n-\alpha),\infty}} \le C|A|$ directly follows from the corresponding estimate for the linear operator.

Finally we are left with the estimate at the vertex $(1, 1, \frac{2n-\alpha}{n})$. For $j \in \mathbf{Z}$ we introduce operators

$$I_j(f,g)(x) = \int_{|t|\le 2^j} f(x+t)g(x-t)dt$$

and we note that for $f, g \ge 0$ we have

$$I_\alpha(f,g) \le C\sum_{j\in\mathbf{Z}} 2^{j(\alpha-n)} I_j(f,g).$$

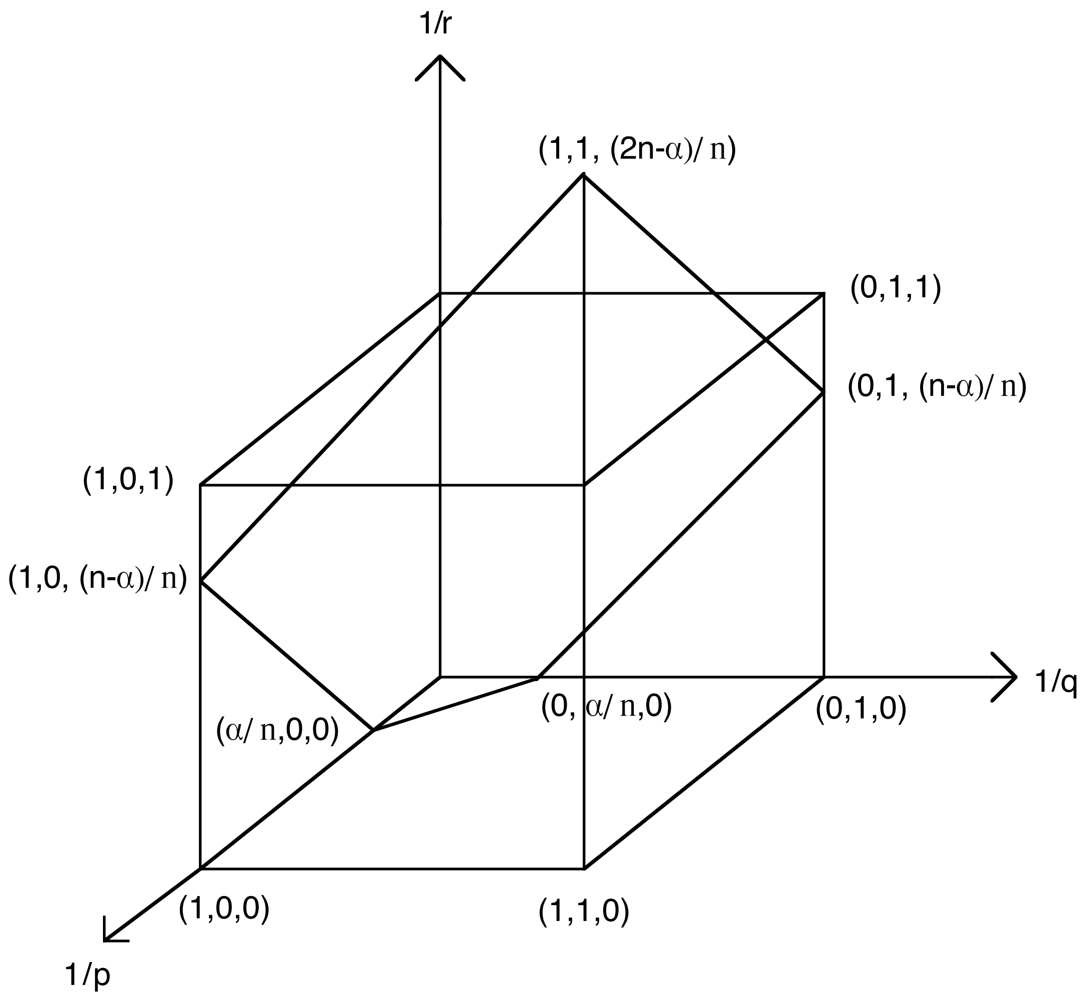

FIGURE 2. The set of all $(1/p, 1/q, 1/r)$ such that $I_\alpha : L_p \times L_q \to L_r$.

Next we observe that by a easy dilation argument I_j maps $L_1 \times L_1 \to L_{1/2}$ with norm bounded by a constant times 2^{jn}. This fact together with the observation

$$\int_E (I_j(f,g)(x))^{1/2}\,dx \le \left(\int_E I_j(f,g)(x)\,dx\right)^{1/2} |E|^{1/2} \le C\|f\|_{L_1}^{1/2}\|g\|_{L_1}^{1/2}|E|^{1/2},$$

implies that for any measurable set E with finite measure we have

$$\int_E (I_j(f,g)(x))^{1/2}\,dx \le \|f\|_{L_1}^{1/2}\|g\|_{L_1}^{1/2}\min(2^{jn}, |E|)^{1/2}. \tag{36}$$

Now pick $E = E_\lambda = \{x : |I_\alpha(f,g)(x)| > \lambda\}$. Then Chebychev's inequality and (36) give

$$\begin{aligned}\lambda^{1/2}|E_\lambda| &\leq \int_{E_\lambda} \Big|\sum_{j\in\mathbf{Z}} 2^{j(\alpha-n)} I_j(f,g)(x)\Big|^{1/2} dx \\ &\leq \sum_{j\in\mathbf{Z}} 2^{j(\alpha-n)/2} \int_{E_\lambda} |I_j(f,g)(x)|^{1/2} dx \\ &\leq \sum_{j\in\mathbf{Z}} 2^{j(\alpha-n)/2} \|f\|_{L_1}^{1/2} \|g\|_{L_1}^{1/2} \min(2^{jn}, |E_\lambda|)^{1/2} \\ &= C\|f\|_{L_1}^{1/2}\|g\|_{L_1}^{1/2}|E_\lambda|^{\alpha/2n}.\end{aligned}$$

This implies that

$$\lambda |E_\lambda|^{\frac{2n-\alpha}{n}} \leq C\|f\|_{L_1}\|g\|_{L_1}$$

which is the required weak type estimate at the vertex $(1, 1, \frac{2n-\alpha}{n})$. This example was studied in [4] when $r \geq 1$ and should be contrasted with the main result in [11]. The same result was independently obtained in [10]. To use the full strength of our results we apply Theorem 4.2 and the succeeding remark to obtain the following generalization for r.i. spaces.

Theorem 5.7. *Suppose X, Y, Z are r.i. spaces on $\mathbf{R}^n$ with Z maximal and s-convex for some $s > 0$. Suppose the Boyd indices of X, Y, Z satisfy the condition that the Boyd cube intersects the pentagon generated by $(\frac{\alpha}{n}, 0, 0), (1, 0, 1-\frac{\alpha}{n}), (1, 1, \frac{2n-\alpha}{n}), (0, 1, 1-\frac{\alpha}{n})$ and $(0, \frac{\alpha}{n}, 0)$ in a nonempty subset of the interior. Then in order that I_α maps $X \times Y$ to Z it is sufficient that $(f, g) \to x^\alpha f(x)g(x)$ maps $X(0,\infty) \times Y(0,\infty)$ to $Z(0,\infty)$.*

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