INTERPLAY BETWEEN DISTRIBUTIONAL ESTIMATES AND BOUNDEDNESS IN HARMONIC ANALYSIS

DMITRIY BILYK AND LOUKAS GRAFAKOS

Abstract. We prove that certain boundedness properties of operators yield distributional estimates that have exponential decay at infinity. Such distributional estimates imply local exponential integrability and apply to many operators such as m-linear Calderón-Zygmund operators and their maximal counterparts.

1. Introduction and the main result

It is a classical result that the Hilbert transform H and its maximal counterpart H_* defined for functions f on the line by the identities

$$H(f)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t|\geq\varepsilon} f(x-t)\,\frac{dt}{t}\,, \qquad H_*(f)(x) = \sup_{\varepsilon>0} \frac{1}{\pi} \left| \int_{|t|\geq\varepsilon} f(x-t)\,\frac{dt}{t} \right|,$$

satisfy, for all measurable sets F of finite measure, the distributional estimates

$$|\{|H(\chi_F)| > \lambda\}| + |\{|H_*(\chi_F)| > \lambda\}| \leq C\,|F| \begin{cases} \lambda^{-1} & \text{when } \lambda < 1 \\ e^{-c\lambda} & \text{when } \lambda \geq 1 \end{cases} \tag{1}$$

for a pair of constants C, c. For a proof of this result we refer to Garsia [1] in which explicit properties of the kernel $1/t$ of H are exploited.

In this note we show that distributional estimates of the type (1) hold for a variety of linear (and sublinear) operators that may not have the rich structure of the Hilbert transform. In fact, we prove that any linear operator of restricted weak type $(1,1)$, whose adjoint is also of restricted weak type $(1,1)$, must satisfy the distributional estimate (1), provided it has a bounded kernel or can be written as a pointwise limit of linear operators with bounded kernels. Our results also apply to m-linear operators that are of restricted weak type $(1,\dots,1,1/m)$ and whose adjoints have the same property. Extensions of this result to certain maximal operators are also obtained.

We will be working with a multilinear operator T defined on the m-fold product of spaces of measurable functions on measure spaces (X_j,μ_j) that contain the simple functions. We assume that T takes values in the set of measurable functions on another measure space (X,μ). We denote by T^{*j} the adjoint with respect to the jth variable, where $j\in\{1,2,\dots,m\}$. The operator T^{*j} satisfies

$$\int g\,T(f_1,\dots,f_m)\,d\mu = \int f_j\,T^{*j}(\dots,f_{j-1},g,f_{j+1},\dots)\,d\mu_j$$

for all functions $f_1,\dots,f_m$, g in the corresponding domains; (an implicit assumption is that all integrals converge absolutely.) We also set $T^{*0}=T$.

2000 *Mathematics Subject Classification.* 46B70, 42B99.

Key words and phrases. multilinear operators, distributional estimates.

Both authors have been supported by the National Science Foundation under grant DMS 0099881.

We say that a multilinear operator T is of restricted weak type $(p_1, \dots, p_m, q)$ if there is a positive constant A such that for all measurable sets $F_1, \dots, F_m$ of finite measure we have

$$\|T(\chi_{F_1}, \dots, \chi_{F_m})\|_{L^{q,\infty}} \le A\, \mu_1(F_1)^{\frac{1}{p_1}} \dots \mu_m(F_m)^{\frac{1}{p_m}}\,. \tag{2}$$

Here $\|g\|_{L^{q,\infty}} = \sup_{\lambda>0} \lambda |\{|g| > \lambda\}|^{1/q}$ and the smallest constant A so that (2) is satisfied for all sets $F_1, \dots, F_m$ is called the restricted weak type $(p_1, \dots, p_m, q)$ constant of T.

We have the following result.

Theorem 1.1. *Suppose that for some $p_k \ge 1$ $(k = 1, \dots, m)$ T is of restricted weak type $(p_1, \dots, p_m, q)$, where q satisfies $\frac{1}{q} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$.*

Suppose that for all $j = 1, \dots, m$, T^{*j} is of restricted weak type $(p_{1,j}, \dots, p_{m,j}, q_j)$, where $\frac{1}{q_j} = \frac{1}{p_{1,j}} + \dots + \frac{1}{p_{m,j}}$, $p_{k,j} \ge 1$, and $p_{j,j} = 1$.

Suppose also that T maps $L^{\alpha p_1} \times \dots \times L^{\alpha p_m}$ to $L^{\alpha q}$ for some $\alpha \ge q^{-1}$.

Then there are constants C, c (depending on the previous indices, T, and m) such that for all measurable sets $F_1, \dots, F_m$ of finite measure we have

$$\mu(\{|T(\chi_{F_1}, \dots, \chi_{F_m})| > \lambda\}) \le C\left(\mu_1(F_1)^{\frac{1}{p_1}} \dots \mu_m(F_m)^{\frac{1}{p_m}}\right)^q \begin{cases} \lambda^{-q} & \text{when } \lambda < 1 \\ e^{-c\lambda} & \text{when } \lambda \ge 1. \end{cases} \tag{3}$$

Remark 1.2. *In many cases, the assumption that T maps $L^{\alpha p_1} \times \dots \times L^{\alpha p_m}$ into $L^{\alpha q}$ can be removed. For example, if T has a bounded kernel, this condition follows from the restricted weak type conditions on T^{*j} via the multilinear interpolation theorem in* [5]. *The same conclusion follows for operators that can be can be written as a limit of a sequence of operators with bounded kernels.*

Theorem 1.1 is motivated by properties of the bilinear Hilbert transform which satisfies the restricted weak type assumptions modulo some logarithmic factors. A similar conclusion for this operator is valid as well (cf. the subsequent work of the authors).

Setting all exponents p_k, $p_{k,j}$ equal to 1, we obtain the following important corollary:

Corollary 1.3. *Suppose that for $j = 0, 1, \dots m$, T^{*j} is of restricted weak type $(1, \dots, 1, 1/m)$. Suppose also that T maps $L^{mq} \times \dots \times L^{mq}$ to L^q for some $q \ge 1$.Then there are constants C, c such that for all measurable sets $F_1, \dots, F_m$ of finite measure we have*

$$\mu(\{|T(\chi_{F_1}, \dots, \chi_{F_m})| > \lambda\}) \le C\left(\mu_1(F_1) \dots \mu_m(F_m)\right)^{1/m} \begin{cases} \lambda^{-1/m} & \text{when } \lambda < 1 \\ e^{-c\lambda} & \text{when } \lambda \ge 1. \end{cases} \tag{4}$$

Remark 1.4. *As in the proof of Theorem 1.1, the assumption that T maps $L^{mq} \times \dots \times L^{mq}$ to L^q can be dropped if T is assumed to have a bounded kernel or can be written as a limit of a sequence of operators with bounded kernels, see* [5].

The distributional estimates (4) imply that T^{*j} is bounded from $L^{pm} \times \dots \times L^{pm}$ to L^p for all indices $1/m < p < \infty$. It follows by duality and standard multilinear interpolation [6], [2], that T is bounded from $L^{p_1} \times \dots \times L^{p_m}$ to L^p whenever $1 < p_1, \dots, p_m < \infty$ and $p^{-1} = p_1^{-1} + \dots + p_m^{-1}$. Therefore Corollary 1.3 recovers and strengthens the result in [5] in this case.

In particular, in the linear case (i.e., $m = 1$), Corollary 1.3 implies that estimate (4) holds for the Hilbert transform and other self adjoint (or skew adjoint) singular integrals that are of weak type $(1, 1)$.

In Section 3 we extend Corollary 1.3 for certain maximal singular integral operators.

May 17, 2004.

2. The proof of Theorem 1.1

For simplicity we denote the measure of any set S that appears in the sequel by $|S|$ with the understanding that this may be either $\mu_j(S)$ or $\mu(S)$ depending on the context. Let T be as in the statement of Theorem 1.1. We first prove the following lemma.

Lemma 2.1. *There is a constant C_1 such that for all measurable sets of finite measure $E, F_1, \dots, F_m$ there is a subset S of E such that $|S| \geq \frac{1}{2}|E|$ and*

$$\left| \int_S T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu \right| \leq C_1 \, |E|^{1-\frac{1}{q}} |F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}} .$$

Proof. Define

$$\Omega = \left\{ x : |T(\chi_{F_1}, \dots, \chi_{F_m})(x)| > 2^{\frac{1}{q}} A \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} \right\},$$

where A is the restricted weak $(p_1, \dots, p_m, q)$ constant of T. Then $|\Omega| < \frac{1}{2}|E|$. Now we set $S = E \setminus \Omega$. We have $|S| \geq \frac{1}{2}|E|$ and also

$$\left| \int_S T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu \right| \leq 2^{\frac{1}{q}} A \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} \, |E| = C_1 \, |E|^{1-\frac{1}{q}} |F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}} .$$

□

Lemma 2.2. *There is a constant C_2 such that for all measurable sets of finite measure $E, F_1, \dots, F_m$ that satisfy $|E|^{\frac{1}{q}} \leq |F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}$ we have*

$$\left| \int_E T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu \right| \leq C_2 \, |E| \left(1 + \log \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} \right) .$$

Proof. Let us denote $F_i^{(0)} = F_i$ for $i = 1, \dots, m$. We now proceed inductively. At the j^{th} step we choose the index k_j such that $|F_{k_j}^{(j)}| = \max(|F_1^{(j)}|, \dots, |F_m^{(j)}|)$. By Lemma 2.1 applied to T^{*k_j} for exponents $p_{1,k_j}, \dots, p_{m,k_j}, q_{k_j}$ with the roles of E and F_{k_j} interchanged, we can choose $S_{k_j}^{(j)} \subset F_{k_j}^{(j)}$ such that $|S_{k_j}^{(j)}| \geq \frac{1}{2}|F_{k_j}^{(j)}|$ and

$$\left| \int_{S_{k_j}^{(j)}} T^{*k_j}(\chi_{F_1}, \dots, \chi_E, \dots \chi_{F_m})\, d\mu_{k_j} \right| \leq C \, \frac{|E| \prod_{i \neq k_j} |F_i^{(j)}|^{\frac{1}{p_{i,k_j}}}}{|F_{k_j}^{(j)}|^{\frac{1}{q_{k_j}}-1}} \leq C \, |E|.$$

We now define $F_i^{(j+1)} = F_i^{(j)} \setminus S_i^{(j)}$ for all $i = 1, \dots, m$, where we set $S_i^{(j)} = \emptyset$ for $i \neq k_j$. We proceed by induction and we stop at the first integer n such that $|E|^{\frac{1}{q}} \geq |F_1^{(n)}|^{\frac{1}{p_1}} \dots |F_m^{(n)}|^{\frac{1}{p_m}}$. (Such an integer always exists since the quantity $|F_1^{(j)}|^{\frac{1}{p_1}} \dots |F_m^{(j)}|^{\frac{1}{p_m}}$ gets smaller by at least a factor of $(\frac{1}{2})^{\frac{1}{\max p_i}}$ when j is replaced by $j+1$.) Obviously, the number of steps n is at most $C(1 + \log \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}})$.

We now have the sequence of estimates

$$
\begin{aligned}
\Big|\int_E T(\chi_{F_1},\dots,\chi_{F_m})\,d\mu\Big| &= \Big|\int_E T(\dots,\chi_{S^{(0)}_{k_0}}+\chi_{F^{(1)}_{k_0}},\dots)\,d\mu\Big|\\
&\le \Big|\int_{S^{(0)}_{k_0}} T^{*k_0}(\dots,\chi_E,\dots)\,d\mu_{k_0}\Big| + \Big|\int_E T(\chi_{F^{(1)}_1},\dots,\chi_{F^{(1)}_m})\,d\mu\Big|\\
&\le C\,|E| + \Big|\int_E T(\chi_{F^{(1)}_1},\dots,\chi_{F^{(1)}_m})\,d\mu\Big|.
\end{aligned}
$$

Writing $\chi_{F^{(1)}_{k_1}}$ as $\chi_{S^{(1)}_{k_1}}+\chi_{F^{(2)}_{k_1}}$ and applying this argument $n-1$ more times we obtain that the previous expression is controlled by

$$
\begin{aligned}
& n\,C\,|E| + \Big|\int_E T(\chi_{F^{(n)}_1},\dots,\chi_{F^{(n)}_m})\,d\mu\Big|\\
\le\;& C\,|E|\Big(1+\log\frac{|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}}{|E|^{\frac1q}}\Big) + \|T\|_{(\alpha p_1,\dots,\alpha p_m,\alpha q)}\,|F^{(n)}_1|^{\frac{1}{\alpha p_1}}\dots|F^{(n)}_m|^{\frac{1}{\alpha p_m}}\,|E|^{\frac{1}{(\alpha q)'}}\\
\le\;& C_2\,|E|\Big(1+\log\frac{|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}}{|E|^{\frac1q}}\Big),
\end{aligned}
$$

where in the second line from the bottom we have used Hölder inequality and the fact that T is of strong type $(\alpha p_1,\dots,\alpha p_m,\alpha q)$.

$\square$

Combining Lemmata 2.1 and 2.2 we obtain the following:

Corollary 2.3. *There is a constant C_3 such that for all $E, F_1,\dots,F_m$ measurable sets of finite measure there is a subset $S = S_{E,F_1,\dots,F_m}$ of E with $|S|\ge\frac12|E|$ such that*

$$
\Big|\int_S T(\chi_{F_1},\dots,\chi_{F_m})\,d\mu\Big| \le C_3\,|E|\min\Big(1\,,\ \frac{|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}}{|E|^{\frac1q}}\Big)\Big(1+\log^+\frac{|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}}{|E|^{\frac1q}}\Big).
$$

We are now ready to prove the distributional estimate (3).

For a given $\lambda>0$, we set

$$
\begin{aligned}
E^1_\lambda &= \{\mathrm{Re}\,T(\chi_{F_1},\dots,\chi_{F_m})>\lambda\}\,,\\
E^2_\lambda &= \{\mathrm{Re}\,T(\chi_{F_1},\dots,\chi_{F_m})<-\lambda\}\,,\\
E^3_\lambda &= \{\mathrm{Im}\,T(\chi_{F_1},\dots,\chi_{F_m})>\lambda\}\,,\\
E^4_\lambda &= \{\mathrm{Im}\,T(\chi_{F_1},\dots,\chi_{F_m})<-\lambda\}\,.
\end{aligned}
$$

We shall prove the required estimate for a fixed E^j_λ. Suppose $|E^j_\lambda|^{\frac1q}\ge|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}$. Then by Corollary 2.3 there is a subset S^j_λ of E^j_λ of at least half its measure so that

$$
\frac{\lambda}{2}|E^j_\lambda|\le\lambda|S^j_\lambda|\le\Big|\int_{S^j_\lambda}T(\chi_{F_1},\dots,\chi_{F_m})\,d\mu\Big|\le C_3\,\frac{|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}}{|E^j_\lambda|^{\frac1q-1}},
$$

which implies

$$
|E^j_\lambda|\le(2C_3)^q\,\big(|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}\big)^q\lambda^{-q}\,.
$$

But this in turn implies that if $\lambda > 2C_3$, we must have $|E^j_\lambda|^{\frac{1}{q}} \le |F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}$. In this case, Corollary 2.3 gives

$$\frac{\lambda}{2}|E^j_\lambda| \le C_3\,|E^j_\lambda|\left(1+\log\frac{|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}}{|E^j_\lambda|^{\frac{1}{q}}}\right),$$

from which one easily deduces that $|E^j_\lambda| \le C\,e^{-c\lambda}\big(|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}\big)^q$. Summing over $j = 1,2,3,4$ we deduce the required conclusion with a constant four times as large.

3. Extensions to (multi)sublinear operators

Next we prove the following extension of Corollary 1.3 for operators that may be sublinear in each variable. Our setting here will be $\mathbf{R}^n$ (endowed with Lebesgue measure) and M will denote the Hardy-Littlewood maximal operator.

Theorem 3.1. *Suppose a positive sublinear operator T_* satisfies the following Cotlar-type inequality*

$$T_*(f_1,\dots,f_m) \le A\Big[M(T(f_1,\dots,f_m)) + \prod_{j=1}^m M(f_j)\Big]$$

for some operator T that satisfies estimate (3)*. Then there exist constants $C_*, c_* > 0$ such that for $\lambda > 1$*

$$|\{T_*(\chi_{F_1},\dots,\chi_{F_m}) > \lambda\}| \le C_*\,e^{-c_*\lambda}\big(|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}\big)^q.$$

Proof. Obviously, it is enough to show that $M(T(\chi_{F_1},\dots,\chi_{F_m}))$ satisfies the required distributional estimate, since $M(\chi_{F_j}) \le 1$. We denote $\Omega_\lambda = \{M(T(\chi_{F_1},\dots,\chi_{F_m})) > \lambda\}$ and $f = T(\chi_{F_1},\dots,\chi_{F_m})$ and we set $E_j = \{x : 2^{j-1}\lambda < |f(x)| \le 2^j\lambda\}$ for $j \ge 0$. By our assumption we have

$$|E_j| \le C\,e^{-c2^j\lambda}\big(|F_1|^{\frac{1}{p_1}}\dots|F_m|^{\frac{1}{p_m}}\big)^q.$$

We claim that there exists a constant $B > 0$, such that for all $x \in \Omega_\lambda$ there is an integer $k \ge 0$ and a ball I containing x with the property that

$$|I \cap E_k| \ge B2^{-2k}\lambda|I|. \tag{5}$$

Indeed, if it were not the case, then for any ball I containing x we would have

$$\begin{aligned}\frac{1}{|I|}\int_I |f(z)|dz &= \frac{1}{|I|}\int_{I\cap\{|f|\le\frac{\lambda}{2}\}}|f(z)|dz + \sum_{j=0}^\infty \frac{1}{|I|}\int_{I\cap E_j}|f(z)|dz\\ &\le \frac{\lambda}{2} + B\sum_{j=0}^\infty 2^{-2j}2^{j+1}\lambda\\ &= \lambda(\frac{1}{2}+4B) < \frac{3}{4}\lambda\end{aligned}$$

for $B < \frac{1}{16}$. But this would imply that $M(f)(x) < \lambda$ and that $x \notin \Omega_\lambda$, a contradiction.

For each $x \in \Omega_\lambda$ we denote by k_x the smallest k for which (5) holds and we set

$$\Omega^k_\lambda = \{x \in \Omega_\lambda : k_x = k\}.$$

It is easy to see that

$$\Omega^k_\lambda \subset \{M(\chi_{E_k}) \ge B2^{-2k}\lambda\}.$$

Thus, using weak type (1,1) property of M, we obtain

$$|\Omega_\lambda^k| \le B' 2^{2k} \lambda^{-1} |E_k| \le B'' 2^{2k - c' 2^k \lambda} \lambda^{-1} \big(|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}} \big)^q .$$

Now the required estimate for $\lambda \ge 1$ is obtained by summing the series

$$|\Omega_\lambda| = \sum_{j=0}^{\infty} |\Omega_\lambda^j| .$$

□

4. Applications to m-linear Calderón-Zygmund operators

Bounded m-linear operators from $L^{p_1}(\mathbf{R}^n) \times \dots \times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ (for some exponents $1 < p_1, \dots, p_m < \infty$ with $p^{-1} = p_1^{-1} + \dots + p_m^{-1}$) are called multilinear Calderón-Zygmund if they have the form

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, dy_1 \dots dy_m, \tag{6}$$

for some distributional kernel $K(x, y_1, \dots, y_m)$ that coincides with a function defined away from the diagonal $x = y_1 = y_2 = \dots = y_m$ that satisfies the size estimate

$$|K(y_0, y_1, \dots, y_m)| \le \frac{A}{\big(\sum_{k,\, l=0}^m |y_k - y_l| \big)^{mn}}, \tag{7}$$

and, for some $\epsilon > 0$, the regularity condition

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y_j', \dots, y_m)| \le \frac{A}{\big(\sum_{k,\, l=0}^m |y_k - y_l| \big)^{mn+\epsilon}}, \tag{8}$$

whenever $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2} \max_{0 \le k \le m} |y_j - y_k|$. In view of the result in [3], an m-linear Calderón-Zygmund operator T must be bounded from the product $L^1 \times \dots \times L^1$ to $L^{1/m,\infty}$. As the properties of the kernel K are symmetric in all variables, it follows that for any j between 1 and m we also have that $T^{*j} : L^1 \times \dots \times L^1 \to L^{1/m,\infty}$. Thus multilinear Calderón-Zygmund operators satisfy the hypotheses of Corollary 1.3. It follows that they must also satisfy the distributional estimates (4).

Next we show that the maximal multilinear Calderón-Zygmund operators also satisfy the distributional estimates (4). We define the maximal truncated operator as

$$T_*(f_1, \dots, f_m) = \sup_{\delta > 0} |T_\delta(f_1, \dots, f_m)|,$$

where we set

$$T_\delta(f_1, \dots, f_m)(x) = \int_{|x-y_1|^2 + \dots + |x-y_m|^2 < \delta^2} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, dy_1 \dots dy_m .$$

It was proved in [4] that T_* satisfies the pointwise estimate

$$T_*(f_1, \dots, f_m) \le C_\eta \Big[(M(|T(f_1, \dots, f_m)|^\eta))^{1/\eta} + \prod_{j=1}^m M(f_j) \Big]. \tag{9}$$

for some $C_\eta > 0$ whenever $0 < \eta < \infty$.

Using Theorem 3.1 we therefore deduce the following conclusion.

Proposition 4.1. *If T is a multilinear Calderón-Zygmund operator, then T_* satisfies the distributional estimate* (4).

May 17, 2004.

Proof. The estimate for $\lambda < 1$ follows from the weak type $(1, \dots, 1, \frac{1}{m})$ property of T_* (cf. [4]); the estimate for $\lambda > 1$ follows from Theorem 3.1 and (9) with $\eta = 1$. □

The next corollary is an immediate consequence of the results just obtained. Naturally the same conclusion applies to any operator that satisfies the hypotheses of Theorem 1.1 or Theorem 3.1 accordingly.

Corollary 4.2. *There is a constant $c_1 > 0$ so that for any multilinear Calderón-Zygmund operator T, for any ball B, and for any tuple of measurable sets $F_1, \dots, F_m$ of finite measure we have*

$$\int_B e^{c_1\, T(\chi_{F_1}, \dots, \chi_{F_m})}\, dx + \int_B e^{c_1\, T_*(\chi_{F_1}, \dots, \chi_{F_m})}\, dx < \infty\,.$$

5. Concluding remarks

One may wonder if the conclusion of Theorem 1.1 is still valid if it is assumed that T and its adjoints are bounded on some product of Lebesgue spaces $L^{p_1} \times \cdots \times L^{p_m}$ with all $p_j > 1$. We show that this is not the case even when $m = 1$.

Consider a linear operator T that maps L^q into $L^{q,\infty}$ for some $1 < q < 2$ and suppose that T^* has the same property. Then both T and T^* are L^p bounded for all $p \in (q, q')$. Suppose, furthermore, that T does not map L^r into itself for any $r \notin [q, q']$. Following the same procedure discussed in the proof of Theorem 1.1 we deduce that there is a constant C such that for all sets F of finite measure we have

$$|\{|T(\chi_F)| > \lambda\}| \le C\,|F| \begin{cases} \lambda^{-q} & \text{when } \lambda < 1 \\ \lambda^{-q'}(1 + \log \lambda)^{q'} & \text{when } \lambda \ge 1\,, \end{cases} \tag{10}$$

where $q' = q/(q-1)$. It is clear that the term $\lambda^{-q'}(1+\log\lambda)^{q'}$ cannot be replaced by a term of the form $e^{-c\lambda}$ as this would imply that T is bounded on L^p for all $p > q$ which we assume is not the case.

References

[1] A. Garsia, *Topics in Almost Everywhere Convergence*, Lectures in Advanced Mathematics, 4, Markham Publishing Co., Chicago, Ill. 1970.

[2] L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. **319** (2001), 151–180.

[3] L. Grafakos and R. Torres, *Multilinear Calderón-Zygmund theory*, Adv. in Math. **165** (2002), 124–164.

[4] L. Grafakos and R. Torres, *Maximal operator and weighted norm inequalities for multilinear singular integrals*, Indiana Univ. Math. J. **51** (2002), 1261–1276.

[5] L. Grafakos and T. Tao, *Multilinear interpolation between adjoint operators*, J. Funct. Anal. **199** (2003), 379–385.

[6] S. Janson, *On interpolation of multi-linear operators*, Function spaces and applications, Cwikel, Peetre, Sagher, and Wallin (Eds), Proceedings, Lund 1986, Springer LNM **1302** (1988).

Dmitriy Bilyk, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: bilykd@math.missouri.edu

Loukas Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: loukas@math.missouri.edu

May 17, 2004.