$L^2 \times L^2 \to L^1$ BOUNDEDNESS CRITERIA

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Abstract. We obtain a sharp $L^2 \times L^2 \to L^1$ boundedness criterion for a class of bilinear operators associated with a multiplier given by a signed sum of dyadic dilations of a given function, in terms of the L^q integrability of this function; precisely we show that boundedness holds if and only if $q < 4$. We discuss applications of this result concerning bilinear rough singular integrals and bilinear dyadic spherical maximal functions.

Our second result is an optimal $L^2 \times L^2 \to L^1$ boundedness criterion for bilinear operators associated with multipliers with L^∞ derivatives. This result provides the main tool in the proof of the first theorem and is also manifested in terms of the L^q integrability of the multiplier. The optimal range is $q < 4$ which, in the absence of Plancherel's identity on L^1, should be compared to $q = \infty$ in the classical $L^2 \to L^2$ boundedness for linear multiplier operators.

1. Introduction

A linear multiplier operator has the form

$$S_m(f)(x) = \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi$$

where m is a bounded function on $\mathbb{R}^n$ and f is a Schwartz function whose Fourier transform is defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$. Here $x\cdot y$ is the usual dot product on $\mathbb{R}^n$. An important question in harmonic analysis is to find sufficient conditions on m for S_m to admit a bounded extension from $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$. If this is the case, the function m is called an L^p Fourier multiplier. In view of Plancherel's theorem, m is an L^2 Fourier multiplier if and only if it is an L^∞ function.

In this work we investigate the $L^2 \times L^2 \to L^1$ boundedness of bilinear multiplier operators which is as central in this theory as the L^2 boundedness is in linear multiplier theory. In the linear case, S_m is

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bounded on L^2 exactly when m lies in L^∞. However, there does not exist such a straightforward characterization in this situation, due to the lack of Plancherel's theorem on L^1. This provides a strong motivation to search for sharp sufficient conditions for the $L^2 \times L^2 \to L^1$ boundedness of bilinear multiplier operators, i.e., operators that have the form

$$T_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi d\eta\,,$$

where f, g is a pair of Schwartz functions and m is a bounded function on $\mathbb{R}^{2n}$.

A classical sufficient condition for boundedness of T_m is the so-called Coifman-Meyer condition [4] on m, namely the requirement that

$$|\partial^\alpha m(\xi,\eta)| \le C_\alpha |(\xi,\eta)|^{-|\alpha|} \tag{1}$$

for sufficiently many α. This condition implies that T_m is bounded from $L^{p_1} \times L^{p_2}$ to L^p for $1 < p_1, p_2 < \infty$ when $1/p = 1/p_1 + 1/p_2$; see [4] for $p \ge 1$ and [19], [21] for $1/2 < p < 1$. In other words, this theorem says that linear Mikhlin multipliers on $\mathbb{R}^{2n}$ are bounded bilinear multipliers on $\mathbb{R}^n \times \mathbb{R}^n$. Analogous results for bilinear multipliers that satisfy Hörmander's [20] classical weakening of (1) for linear operators, expressed in terms of Sobolev spaces, was initiated by Tomita [25] and was subsequently further investigated by Grafakos, Fujita, Miyachi, Nguyen, Si, and Tomita among others; see [18], [9], [15], [22], [23], [17], [16]. Related to this we highlight that if the functions $m(2^k\cdot)\phi$ have s derivatives in $L^r(\mathbb{R}^{2n})$ $(1 < r < \infty)$ uniformly in $k \in \mathbb{Z}$, with ϕ being a suitable smooth bump supported in $1/2 < |(\xi,\eta)| < 2$, then T_m is bounded from $L^2 \times L^2$ to L^1 when $s > s_0 = \max(n/2, 2n/r)$ and s_0 cannot be replaced by any smaller number; see [14]. Thus more than $n/2$ derivatives of $m(2^k\cdot)\phi$ in $L^4(\mathbb{R}^{2n})$ uniformly in k are required of a generic multiplier m for T_m to map $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

There are, however, many other multipliers emerging naturally in the study of bilinear operators which do not fall under in the scope of the Coifman-Meyer condition. For one class of such multipliers, described below, we obtain a full characterization of $L^2 \times L^2 \to L^1$ boundedness.

Let us consider a function m on $\mathbb{R}^{2n}$ which satisfies, for some $\delta > 0$,

$$|m(\xi,\eta)| \le C' \min(|(\xi,\eta)|, |(\xi,\eta)|^{-\delta}), \tag{2}$$

and

$$|\partial^\alpha m(\xi,\eta)| \le C_\alpha \min(1, |(\xi,\eta)|^{-\delta}) \tag{3}$$

for all multiindices α. Unlike the case of Coifman-Meyer conditions (1), the rate of decay in (3) does not depend on the order of derivatives, and δ could be arbitrarily small, which means that conditions (2) and (3) are satisfied by a variety of functions which are not multipliers of Coifman-Meyer type.

Given a function m with properties (2) and (3), we set

$$m_k(\xi,\eta) = m(2^k(\xi,\eta))$$

for $k \in \mathbb{Z}$ and we define a multiplier

$$\sum_k r_k m_k, \tag{4}$$

where r_k is a given bounded sequence. We investigate boundedness properties of the bilinear operator

$$T := T_{\sum_{k\in\mathbb{Z}} r_k m_k} = \sum_{k\in\mathbb{Z}} r_k T_{m_k}. \tag{5}$$

A sufficient and essentially necessary condition in terms of the integrability of m for the boundedness of the operator T from $L^2 \times L^2$ to L^1 is given in the following theorem. In the sequel $\|T\|_{L^p\times L^q\to L^r}$ denotes the norm of T from $L^p \times L^q \to L^r$, $\lfloor b \rfloor$ denotes the integer part of a real number b, and $\mathcal{C}^M(\mathbb{R}^{2n})$ denotes the class of all functions on $\mathbb{R}^{2n}$ whose partial derivatives of order up to and including order M are continuous.

Theorem 1.1. *Let $1 \le q < 4$ and set $M_q' = \max\big(2n, \lfloor \frac{2n}{4-q} \rfloor + 1\big)$. Suppose that $m \in L^q \cap \mathcal{C}^{M_q'}$ is a function on $\mathbb{R}^{2n}$ satisfying (2) and (3) for all $|\alpha| \le M_q'$, and let $(r_k)_{k\in\mathbb{Z}}$ be a sequence such that $|r_k| \le 1$ for every $k \in \mathbb{Z}$. Then the bilinear operator T given in (5) has a bounded extension from $L^2 \times L^2 \to L^1$ that satisfies*

$$\|T\|_{L^2\times L^2\to L^1} \le const(C', C_\alpha, \delta, q, \|m\|_{L^q}), \tag{6}$$

where C', C_α and δ are the constants in (2) and (3).

Conversely, if $q \ge 1$ and inequality (6) is satisfied for every $m \in L^q$ fulfilling (2) and (3) and for every sequence $(r_k)_{k\in\mathbb{Z}}$ with $|r_k| \le 1$, then we must necessarily have $q < 4$.

Using standard duality and interpolation arguments, we can show that Theorem 1.1 in fact implies the following more general result.

Corollary 1.2. *For $1 \le q < 4$ and M_q' as in Theorem 1.1, let m be a function in $L^q(\mathbb{R}^{2n}) \cap \mathcal{C}^{M_q'}(\mathbb{R}^{2n})$ satisfying (2) and (3) for all $|\alpha| \le M_q'$. Assume that $(r_k)_{k\in\mathbb{Z}}$ is any bounded sequence. Then*

$$\|T\|_{L^{p_1}\times L^{p_2}\to L^p} < \infty$$

whenever $2 \le p_1, p_2 \le \infty$, $1 \le p \le 2$ and $1/p = 1/p_1 + 1/p_2$.

We note that operators T of type (5) include rough bilinear singular integrals, in particular those studied in [3] and [13]. Indeed, if $K(y,z) = \Omega(y,z)/|(y,z)|^{2n}$ is the kernel of a rough bilinear singular integral, with say Ω in $L^\infty(\mathbb{S}^{2n-1})$, and ψ is a smooth function supported in the unit annulus on $\mathbb{R}^{2n}$ satisfying $\sum_{j\in\mathbb{Z}} \psi(2^{-j}\cdot) = 1$, then the rough bilinear singular integral operator T_Ω is an operator of the form (5) with $m = \widehat{K\psi}$ and $r_k = 1$ for every k.

Theorem 1.1 and Corollary 1.2 can also be viewed as bilinear counterparts of the results in Duoandikoetxea and Rubio de Francia [6], in particular of Theorem B in that reference. In direct analogy to the linear case studied in [6], our results have immediate applications to the study of boundedness properties of rough bilinear singular integral operators studied in [13] and bilinear dyadic spherical maximal operators, whose continuous counterparts were studied in [8] and [1]. These applications are presented in Section 6.

In this work we also study multipliers m all of whose partial derivatives are merely in L^∞; we say that such functions lie in $\mathcal{L}^\infty$. To make things precise, we define the space

$$\mathcal{L}^\infty(\mathbb{R}^{2n}) = \big\{ m : \mathbb{R}^{2n} \to \mathbb{C} : \ \partial^\alpha m \text{ exist for all } \alpha \text{ and } \|\partial^\alpha m\|_{L^\infty} < \infty \big\}.$$

These functions play an important role in the study of multipliers of the form (4).

In the linear setting if $m \in L^\infty$, then the corresponding linear operator is bounded on L^2. One might guess that a bilinear operator T_m is bounded from $L^2 \times L^2$ to L^1 when m lies in $\mathcal{L}^\infty$. However Bényi and Torres [2] provided an important counterexample of a function $m \in \mathcal{L}^\infty$ for which the associated bilinear operator T_m is unbounded from $L^{p_1} \times L^{p_2}$ to L^p for any $1 \le p_1, p_2 < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$. The same authors also proved in [2] that if all derivatives $\partial^\alpha m$ have finite $L^1_\xi(L^2_\eta)$ and $L^1_\eta(L^2_\xi)$ norms, then T_m is bounded from $L^2 \times L^2$ to L^1. The counterexample of Bényi and Torres is also complemented by a subsequent positive result of Honzík and the first two authors [13, Corollary 8], who showed that the mere L^2 integrability of functions in $\mathcal{L}^\infty$ suffices to yield the $L^2 \times L^2 \to L^1$ boundedness of T_m. This result appeared in connection with the study of bilinear rough singular integrals and, on one hand it simplifies the mixed norm conditions in [2], on the other hand it eliminates any decay requirement on the derivatives of the multiplier m, making it more suitable than Hörmander type conditions in situations when only boundedness of derivatives of m is a priori known.

Next we investigate the magnitude of integrability of a given function m in $\mathcal{L}^\infty$ in order for the bilinear multiplier operator T_m to be bounded from $L^2 \times L^2 \to L^1$. We determine the optimal degree of integrability required to ensure the aforementioned boundedness. We have the following result.

Theorem 1.3. *Let* $1 \le q < 4$ *and set* $M_q = \left\lfloor \frac{2n}{4-q} \right\rfloor + 1$. *Let* $m(\xi,\eta)$ *be a function in* $L^q(\mathbb{R}^{2n}) \cap \mathcal{C}^{M_q}(\mathbb{R}^{2n})$ *satisfying*

$$\|\partial^\alpha m\|_{L^\infty} \le C_0 < \infty \quad \text{for all multiindices } \alpha \text{ with } |\alpha| \le M_q. \tag{7}$$

Then there is a constant C *depending on* n *and* q *such that the bilinear operator* T_m *with multiplier* m *satisfies*

$$\|T_m\|_{L^2 \times L^2 \to L^1} \le C\, C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{4}}. \tag{8}$$

Conversely, there is a function $m \in \bigcap_{q>4} L^q(\mathbb{R}^{2n}) \cap \mathcal{L}^\infty(\mathbb{R}^{2n})$ *such that the associated operator* T_m *does not map* $L^2 \times L^2$ *to* L^1.

We remark that the conditions that all derivatives $\partial^\alpha m$ have finite $L^1_\xi(L^2_\eta)$ and $L^1_\eta(L^2_\xi)$ norms imply that $\partial^\alpha m$ are in $L^{3/2}$ by Minkowski's inequality and interpolation, which yields that $m \in \mathcal{L}^\infty$ by Sobolev embedding. In particular $m \in L^{3/2} \cap \mathcal{L}^\infty$; hence our result covers that in [2]. As in Corollary 1.2, duality and interpolation combined with Theorem 1.3 in fact yields $L^{p_1} \times L^{p_2} \to L^p$ boundedness for the operator T_m in a substantially larger set of indices (p_1, p_2, p).

Corollary 1.4. *Let* $1 \le q < 4$ *and set* $M_q = \left\lfloor \frac{2n}{4-q} \right\rfloor + 1$. *Assume that* $m \in L^q(\mathbb{R}^{2n}) \cap \mathcal{C}^{M_q}(\mathbb{R}^{2n})$ *satisfies* (7). *Then*

$$\|T_m\|_{L^{p_1} \times L^{p_2} \to L^p} < \infty$$

for all indices (p_1, p_2, p) *satisfying* $2 \le p_1, p_2 \le \infty$, $1 \le p \le 2$, *and* $1/p = 1/p_1 + 1/p_2$.

The sufficiency directions of our proofs are based on the product-type wavelet method initiated by Honzík and the first two authors in [13] but incorporate several crucial combinatorial improvements, while the necessary directions use constructions inspired by those in [14].

2. Preliminary material

We recall some facts related to product-type wavelets. For a fixed $k \in \mathbb{N}$ there exist real-valued compactly supported functions ψ_F, ψ_M in $\mathcal{C}^k(\mathbb{R})$, which satisfy $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$, for all $0 \le \alpha \le k$ we have $\int_{\mathbb{R}} x^\alpha \psi_M(x)dx = 0$, and, if Ψ^G is defined by

$$\Psi^G(x) = \psi_{G_1}(x_1) \cdots \psi_{G_{2n}}(x_{2n})$$

for $G = (G_1, \dots, G_{2n})$ in the set

$$\mathcal{I} := \Big\{(G_1, \dots, G_{2n}) : \ G_i \in \{F, M\}\Big\},$$

then the family of functions

$$\bigcup_{\mu\in\mathbb{Z}^{2n}} \bigg[\Big\{\Psi^{(F,\dots,F)}(x-\mu)\Big\} \cup \bigcup_{\lambda=0}^{\infty} \Big\{2^{\lambda n}\Psi^G(2^\lambda x-\mu) : \ G \in \mathcal{I}\setminus\{(F,\dots,F)\}\Big\}\bigg]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $x = (x_1, \dots, x_{2n})$. The proof of this fact can be found in Triebel [26].

Let us denote by $\mathcal{J}$ the set of all pairs (λ, G) such that either $\lambda = 0$ and $G = (F, \dots, F)$, or λ is a nonnegative integer and $G \in \mathcal{I} \setminus \{(F, \dots, F)\}$. For $(\lambda, G) \in \mathcal{J}$ and $\mu \in \mathbb{Z}^{2n}$ we set

$$\Psi_\mu^{\lambda,G}(x) = 2^{\lambda n}\Psi^G(2^\lambda x - \mu), \qquad x \in \mathbb{R}^{2n}.$$

The following lemma is crucial in this work.

Lemma 2.1. *Let M be a positive integer. Assume that $m \in \mathcal{C}^{M+1}$ is a function on $\mathbb{R}^{2n}$ such that*

$$\sup_{|\alpha|\le M+1} \|\partial^\alpha m\|_{L^\infty} \le C_0 < \infty.$$

Then for $(\lambda, G) \in \mathcal{J}$ and $\mu \in \mathbb{Z}^{2n}$ we have

$$|\langle \Psi_\mu^{\lambda,G}, m\rangle| \le CC_0 2^{-(M+1+n)\lambda}, \tag{9}$$

provided that ψ_M has M vanishing moments.

This lemma is essentially Lemma 7 in [13] and its proof is omitted.

3. Proof of Theorem 1.3

Proof. Sufficiency. We utilize the wavelet decomposition of m described in Section 2. We recall that our wavelets are compactly supported and the function ψ_M has M vanishing moments, where M is a certain (large) number to be determined later.

For $(\lambda, G) \in \mathcal{J}$ and $\mu \in \mathbb{Z}^{2n}$ we set

$$b_\mu^{\lambda,G} = \langle \Psi_\mu^{\lambda,G}, m\rangle.$$

By [27, Theorem 1.64] and by the fact that the space L^q coincides with the Triebel-Lizorkin space $F^0_{q,2}$, we obtain

$$\|m\|_{L^q} \approx \Big\|\Big(\sum_{(\lambda,G)\in\mathcal{J}} \sum_{\mu\in\mathbb{Z}^{2n}} |b_\mu^{\lambda,G} 2^{\lambda n}\chi_{Q_{\lambda\mu}}|^2\Big)^{1/2}\Big\|_{L^q}, \tag{10}$$

where $Q_{\lambda\mu}$ is the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{1-\lambda}$.

Now, let us fix $(\lambda, G) \in \mathcal{J}$. To simplify notation, we will write b_μ instead of $b_\mu^{\lambda,G}$ in what follows. We also denote by $\tilde{Q}_{\lambda\mu}$ the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{-\lambda}$. Noting that these cubes are pairwise disjoint in μ (for the fixed value of λ), the equivalence (10) yields

(11)
$$\begin{aligned}\|m\|_{L^q} &\gtrsim 2^{\lambda n}\Big\|\Big(\sum_{\mu\in\mathbb{Z}^{2n}} |b_\mu|^2\chi_{Q_{\lambda\mu}}\Big)^{\frac12}\Big\|_{L^q} \geq 2^{\lambda n}\Big\|\Big(\sum_{\mu\in\mathbb{Z}^{2n}} |b_\mu|^2\chi_{\tilde{Q}_{\lambda\mu}}\Big)^{\frac12}\Big\|_{L^q}\\ &= 2^{\lambda n}\Big\|\sum_{\mu\in\mathbb{Z}^{2n}} |b_\mu|\chi_{\tilde{Q}_{\lambda\mu}}\Big\|_{L^q} = 2^{\lambda n(1-\frac{2}{q})}\Big(\sum_{\mu\in\mathbb{Z}^{2n}} |b_\mu|^q\Big)^{\frac1q}.\end{aligned}$$

Setting $b = (b_\mu)_{\mu\in\mathbb{Z}^{2n}}$, (11) becomes $\|b\|_{\ell^q} \leq C2^{\lambda n(\frac{2}{q}-1)}\|m\|_{L^q}$. Also, Lemma 2.1 implies that $\|b\|_{\ell^\infty} \leq CC_0 2^{-(M+n+1)\lambda}$, where M is the to be determined number of vanishing moments of ψ_M.

For a nonnegative integer r we define

$$U_r = \{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n = \mathbb{Z}^{2n} : 2^{-r-1}\|b\|_{\ell^\infty} < |b_{(k,l)}| \leq 2^{-r}\|b\|_{\ell^\infty}\}.$$

We can write U_r as a union of the following two disjoint sets:

$$\begin{aligned} U_r^1 &= \{(k,l) \in U_r : \mathrm{card}\{s : (k,s) \in U_r\} \geq 2^{\frac{rq}{2}}\|b\|_{\ell^q}^{\frac{q}{2}}\|b\|_{\ell^\infty}^{-\frac{q}{2}}\};\\ U_r^2 &= \{(k,l) \in U_r : \mathrm{card}\{s : (k,s) \in U_r\} < 2^{\frac{rq}{2}}\|b\|_{\ell^q}^{\frac{q}{2}}\|b\|_{\ell^\infty}^{-\frac{q}{2}}\}.\end{aligned}$$

Let us denote

$$E = \{k \in \mathbb{Z}^n : (k,l) \in U_r^1 \text{ for some } l \in \mathbb{Z}^n\}.$$

Then

$$\mathrm{card}E \cdot 2^{\frac{rq}{2}}\|b\|_{\ell^q}^{\frac{q}{2}}\|b\|_{\ell^\infty}^{-\frac{q}{2}} 2^{-q(r+1)}\|b\|_{\ell^\infty}^q \leq \sum_{(k,l)\in U_r^1} |b_{(k,l)}|^q \leq \|b\|_{\ell^q}^q,$$

and therefore

$$\mathrm{card}E \leq C2^{\frac{rq}{2}}\|b\|_{\ell^q}^{\frac{q}{2}}\|b\|_{\ell^\infty}^{-\frac{q}{2}} \leq C2^{\frac{rq}{2}}2^{\lambda n - \frac{\lambda n q}{2}}\|m\|_{L^q}^{\frac{q}{2}}\|b\|_{\ell^\infty}^{-\frac{q}{2}}.$$

Given $(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n$, it follows from the definition of $\Psi_{(k,l)}^{\lambda,G}$ that $\Psi_{(k,l)}^{\lambda,G}$ can be written in the tensor product form $\Psi_{(k,l)}^{\lambda,G} = \omega_{1,k}\omega_{2,l}$, where $\omega_{1,k}$ is a function of the variables $(x_1,\dots,x_n)$, $\omega_{2,l}$ is a function of $(x_{n+1},\dots,x_{2n})$ and $\|\omega_{1,k}\|_{L^\infty} \approx \|\omega_{2,l}\|_{L^\infty} = 2^{\frac{\lambda n}{2}}$. Define

$$m^{r,1} = \sum_{(k,l)\in U_r^1} b_{(k,l)}\Psi_{(k,l)}^{\lambda,G} = \sum_{(k,l)\in U_r^1} b_{(k,l)}\omega_{1,k}\omega_{2,l}.$$

Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Then

$$\begin{aligned}
\big\|T_{m^{r,1}}(f,g)\big\|_{L^1} &\le \Big\| \sum_{(k,l)\in U_r^1} b_{(k,l)} \mathcal{F}^{-1}(\omega_{1,k}\widehat{f}\,)\mathcal{F}^{-1}(\omega_{2,l}\widehat{g}\,)\Big\|_{L^1} \\
&\le \sum_{k\in E} \big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2} \Big\| \sum_{l:(k,l)\in U_r^1} b_{(k,l)}\omega_{2,l}\widehat{g}\,\Big\|_{L^2} \\
&\le\ C\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2} 2^{\frac{\lambda n}{2}} 2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2} \\
&\le\ C\Big(\sum_{k\in E}1\Big)^{1/2}\Big(\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\Big)^{\frac12} 2^{\frac{\lambda n}{2}} 2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2} \\
&\le\ C 2^{\frac{\lambda n}{2}+\lambda n(1-\frac{q}{4})} 2^{\frac{rq}{4}-r}\|m\|_{L^q}^{\frac{q}{4}}\|b\|_{\ell^\infty}^{1-\frac{q}{4}}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

Notice that in the estimates above we used the property that the supports of the functions $\omega_{1,k}$ and $\omega_{2,l}$ only have finite overlaps.

Now define

$$m^{r,2} = \sum_{(k,l)\in U_r^2} b_{(k,l)}\omega_{1,k}\omega_{2,l}.$$

Then

$$\begin{aligned}
\|T_{m^{r,2}}(f,g)\|_{L^1} &\le \Big\| \sum_{(k,l)\in U_r^2} b_{(k,l)} \mathcal{F}^{-1}(\omega_{1,k}\widehat{f}\,)\mathcal{F}^{-1}(\omega_{2,l}\widehat{g}\,)\Big\|_{L^1} \\
&\le \sum_{l:\,\exists k\,(k,l)\in U_r^2} \big\|\omega_{2,l}\widehat{g}\,\big\|_{L^2}\Big\|\sum_{k:(k,l)\in U_r^2} b_{(k,l)}\omega_{1,k}\widehat{f}\,\Big\|_{L^2} \\
&\le \Big(\sum_{l\in\mathbb{Z}^n}\big\|\omega_{2,l}\widehat{g}\,\big\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{l:\,\exists k\,(k,l)\in U_r^2}\Big\|\sum_{k:(k,l)\in U_r^2} b_{(k,l)}\omega_{1,k}\widehat{f}\,\Big\|_{L^2}^2\Big)^{\frac12} \\
&\le C2^{\frac{\lambda n}{2}}\|g\|_{L^2}\Big(\sum_{k:\,\exists l\,(k,l)\in U_r^2}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\sum_{l:(k,l)\in U_r^2}|b_{(k,l)}|^2\Big)^{\frac12} \\
&\le C2^{\frac{\lambda n}{2}}\|g\|_{L^2}2^{\frac{rq}{4}-r}\|b\|_{\ell^q}^{\frac{q}{4}}\|b\|_{\ell^\infty}^{1-\frac{q}{4}}\Big(\sum_{k\in\mathbb{Z}^n}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\Big)^{\frac12} \\
&\le C2^{\frac{\lambda n}{2}+\lambda n(1-\frac{q}{4})}2^{\frac{rq}{4}-r}\|m\|_{L^q}^{\frac{q}{4}}\|b\|_{\ell^\infty}^{1-\frac{q}{4}}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

Let

$$m^r = \sum_{(k,l)\in U_r} b_{(k,l)}\omega_{1,k}\omega_{2,l}.$$

Then the previous estimates yield

$$\|T_{m^r}\|_{L^2\times L^2\to L^1} \le C2^{\frac{\lambda n}{2}+\lambda n(1-\frac{q}{4})}2^{\frac{rq}{4}-r}\|m\|_{L^q}^{\frac{q}{4}}\|b\|_{\ell^\infty}^{1-\frac{q}{4}}.$$

Using that $\|b\|_{\ell^\infty} \leq CC_0 2^{-\lambda(M+1+n)}$, we obtain

$$\|T_{m^r}\|_{L^2\times L^2\to L^1} \leq CC_0^{1-\frac{q}{4}} 2^{-\lambda(\frac{4-q}{4}(M+1)-\frac{n}{2})} 2^{-r(1-\frac{q}{4})} \|m\|_{L^q}^{\frac{q}{4}}.$$

Thus, if we choose $M = \lfloor \frac{2n}{4-q} \rfloor$, we can sum over r and (λ, G) in order, which implies (8).

Necessity. We now pass to the converse direction. Let φ be a Schwartz function on $\mathbb{R}$ whose Fourier transform is supported in the interval $[-1/100, 1/100]$, and let $(b_j)_{j=1}^\infty$ and $(d_j)_{j=1}^\infty$ be two sequences of nonnegative numbers with only finitely many nonzero terms. Define functions f and g on $\mathbb{R}^n$ in terms of their Fourier transform by

$$\widehat{f}(\xi) = \sum_{j=1}^\infty b_j \widehat{\varphi}(\xi_1 - j) \prod_{r=2}^n \widehat{\varphi}(\xi_r - 1) \tag{12}$$

and

$$\widehat{g}(\eta) = \sum_{k=1}^\infty d_k \widehat{\varphi}(\eta_1 - k) \prod_{r=2}^n \widehat{\varphi}(\eta_r - 1), \tag{13}$$

with the convention that $\prod_{r=2}^n \widehat{\varphi}(\eta_r - 1) = 1$ if $n = 1$. Then both f and g are Schwartz functions whose L^2-norms are bounded by a constant multiple of $\big(\sum_{j=1}^\infty b_j^2\big)^{\frac{1}{2}}$ and $\big(\sum_{k=1}^\infty d_k^2\big)^{\frac{1}{2}}$, respectively.

Let $(a_j(t))_{j=1}^\infty$ denote the sequence of Rademacher functions; see, for instance, [11, Appendix C] for the definition and basic properties of Rademacher functions. For any $t \in [0,1]$, consider the function m_t given by

$$m_t(\xi,\eta) = \sum_{j=1}^\infty \sum_{k=1}^\infty a_{j+k}(t) c_{j+k} \psi(\xi_1 - j)\psi(\eta_1 - k) \prod_{r=2}^n \psi(\xi_r - 1)\psi(\eta_r - 1),$$

where $(c_\ell)_{l=2}^\infty$ is a bounded sequence of nonnegative numbers and ψ is a smooth function on $\mathbb{R}$ supported in the interval $[-1/10, 1/10]$ assuming value 1 in $[-1/20, 1/20]$. Then

$$\begin{aligned} &T_{m_t}(f,g)(x) \\ &= \sum_{j=1}^\infty \sum_{k=1}^\infty b_j d_k a_{j+k}(t) c_{j+k} (\varphi(x_1))^2 e^{2\pi i x_1 (j+k)} \prod_{r=2}^n e^{4\pi i x_r} (\varphi(x_r))^2 \\ &= \sum_{l=2}^\infty a_l(t) c_l e^{2\pi i x_1 l} (\varphi(x_1))^2 \sum_{j=1}^{l-1} b_j d_{l-j} \prod_{r=2}^n e^{4\pi i x_r} (\varphi(x_r))^2. \end{aligned}$$

Using Fubini's theorem and Khintchine's inequality (see, for instance, [11, Appendix C]), we obtain

(14)

$$\begin{aligned}
&\int_0^1 \|T_{m_t}(f,g)\|_{L^1}\,dt \\
&= \Big(\int_{\mathbb{R}} |\varphi(y)|^2\,dy\Big)^{n-1} \int_{\mathbb{R}}\int_0^1 \Big|\sum_{l=2}^{\infty} a_l(t) c_l e^{2\pi i x_1 l} (\varphi(x_1))^2 \sum_{j=1}^{l-1} b_j d_{l-j}\Big|\,dx_1\,dt \\
&\approx \Big(\int_{\mathbb{R}} |\varphi(y)|^2\,dy\Big)^{n-1} \int_{\mathbb{R}} \Bigg(\sum_{l=2}^{\infty}\Big(c_l|\varphi(x_1)|^2 \sum_{j=1}^{l-1} b_j d_{l-j}\Big)^2\Bigg)^{\frac12}\,dx_1 \\
&= \Bigg(\sum_{l=2}^{\infty} c_l^2 \Big(\sum_{j=1}^{l-1} b_j d_{l-j}\Big)^2\Bigg)^{\frac12} \Big(\int_{\mathbb{R}} |\varphi(y)|^2\,dy\Big)^{n}.
\end{aligned}$$

We now fix a positive integer $N \geq 2$ and set $b_j^N = d_j^N = 2^{-\frac{N}{2}}$ if $j = 2^N, \dots, 2^{N+1}-1$ and $b_j^N = d_j^N = 0$ otherwise. Then the sequences $(b_j^N)_{j=1}^\infty$ and $(d_j^N)_{j=1}^\infty$ belong to the unit ball in ℓ^2. We also observe that if $j \in \mathbb{N}$ and $l < 2^{N+1}$ then either $j < 2^N$ or $l-j < 2^N$, and if $l > 2^{N+2}-2$ then either $j > 2^{N+1}-1$ or $l-j > 2^{N+1}-1$. Consequently,

$$\sum_{j=1}^{l-1} b_j^N d_{l-j}^N = 0 \quad \text{if } l < 2^{N+1} \text{ or } l > 2^{N+2}-2. \tag{15}$$

On the other hand, if $5\cdot 2^{N-1} \leq l \leq 6 \cdot 2^{N-1}$ then

$$\sum_{j=1}^{l-1} b_j^N d_{l-j}^N \geq \sum_{j=\max\{2^N, l+1-2^{N+1}\}}^{\min\{2^{N+1}-1, l-2^N\}} 2^{-\frac{N}{2}} 2^{-\frac{N}{2}} \geq \sum_{j=2\cdot 2^{N-1}+1}^{3\cdot 2^{N-1}} 2^{-N} \geq \frac{1}{2}. \tag{16}$$

We define $c_l = (l-1)^{-\frac12}(\log e(l-1))^{\frac12}$. If f^N and g^N are given by (12) and (13), respectively, with b_j replaced by b_j^N and d_k replaced by d_k^N, then a combination of (14) and (16) yields

$$\begin{aligned}
&\int_0^1 \|T_{m_t}(f^N, g^N)\|_{L^1}\,dt \\
&\qquad \gtrsim \Bigg(\sum_{l=5\cdot 2^{N-1}}^{6\cdot 2^{N-1}} c_l^2\Bigg)^{\frac12} = \Bigg(\sum_{l=5\cdot 2^{N-1}}^{6\cdot 2^{N-1}} (l-1)^{-1}\log e(l-1)\Bigg)^{\frac12} \\
&\qquad \approx \big(\log^2 6\cdot 2^{N-1} - \log^2 5 \cdot 2^{N-1}\big)^{\frac12}
\end{aligned}$$

$$= \left(\log 6 \cdot 2^{N-1} + \log 5 \cdot 2^{N-1}\right)^{\frac{1}{2}} \left(\log 6 \cdot 2^{N-1} - \log 5 \cdot 2^{N-1}\right)^{\frac{1}{2}}$$
$$\approx N^{\frac{1}{2}}.$$

Consequently, for every $N \geq 2$ we can find $t_N \in [0,1]$ such that

$$\|T_{m_{t_N}}(f^N, g^N)\|_{L^1} \geq CN^{\frac{1}{2}}, \tag{17}$$

with C independent of N.

Let us now consider the function

$$m(\xi,\eta) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} s_{j+k}c_{j+k}\psi(\xi_1 - j)\psi(\eta_1 - k)\prod_{r=2}^{n}\psi(\xi_r - 1)\psi(\eta_r - 1),$$

where $(s_l)_{l=2}^{\infty}$ is a sequence taking values in $\{-1,1\}$ and satisfying $s_l = a_l(t_N)$ if $N \geq 2$ and $2^{N+1} \leq l \leq 2^{N+2} - 1$. By (15) we therefore obtain

$$\begin{aligned}
&T_m(f^N, g^N)(x) \\
&\quad = \sum_{l=2^{N+1}}^{2^{N+2}-1} s_l c_l e^{2\pi i x_1 l}(\varphi(x_1))^2 \sum_{j=1}^{l-1} b_j^N d_{l-j}^N \prod_{r=2}^{n} e^{4\pi i x_r}(\varphi(x_r))^2 \\
&\quad = \sum_{l=2^{N+1}}^{2^{N+2}-1} a_l(t_N) c_l e^{2\pi i x_1 l}(\varphi(x_1))^2 \sum_{j=1}^{l-1} b_j^N d_{l-j}^N \prod_{r=2}^{n} e^{4\pi i x_r}(\varphi(x_r))^2 \\
&\quad = T_{m_{t_N}}(f^N, g^N)(x),
\end{aligned}$$

which yields

$$\|T_m(f^N, g^N)\|_{L^1} \geq CN^{\frac{1}{2}},$$

and so T_m is unbounded from $L^2 \times L^2$ into L^1. We can also observe that m is a smooth function with all derivatives bounded and

$$\|m\|_{L^q} \approx \left(\sum_{l=2}^{\infty} c_l^q (l-1)\right)^{\frac{1}{q}} = \left(\sum_{l=2}^{\infty} (l-1)^{1-\frac{q}{2}} (\log e(l-1))^{\frac{q}{2}}\right)^{\frac{1}{q}},$$

which implies that $m \in \bigcap_{q>4} L^q$. □

Remark 1. A result related to Theorem 1.3 appeared in [14, Remark 2] asserting that the inequality

$$\|T_m\|_{L^2\times L^2 \to L^1} \leq C\|m\|_{L_s^q} \tag{18}$$

holds whenever $qs > 2n$ and $1 < q < 4$; here, L_s^q stands for the inhomogeneous Sobolev space with integrability index q and smoothness index s. We observe that the testing functions from the proof of Theorem 1.3 show that inequality (18) fails for $q > 4$.

Remark 2. The quantities on the right-hand side of inequalities (18) and (8) can be compared in certain situations via the classical Gagliardo-Nirenberg inequality [10, 24]. For instance, if n is an odd integer then

$$\|m\|_{L^{4}_{\frac{n+1}{2}}} \leq C \Big(\sup_{|\alpha|\leq n+1} \|\partial^\alpha m\|_{L^\infty}\Big)^{\frac{1}{2}} \|m\|_{L^2}^{\frac{1}{2}};$$

we point out, however, that the value $q = 4$ is not allowed in [14, Remark 2].

4. Proof of Theorem 1.1

Next we prove Theorem 1.1. The sufficiency part follows from Theorem 1.3 via an argument as in [13]; we provide a sketch of the proof for the sake of completeness.

Proof of Theorem 1.1. Sufficiency. Let ψ be a smooth function supported in the unit annulus such that $\sum_{j\in\mathbb{Z}} \psi(2^{-j}\cdot) = 1$. For every $j \in \mathbb{Z}$ we denote $\psi_j = \psi(2^{-j}\cdot)$ and $m_{j,0} = m\psi_j$. Further, for every $k \in \mathbb{Z}$ we set $m_{j,k}(\cdot) = m_{j,0}(2^k\cdot)$. If we define $\widetilde{T}_j$ to be the bilinear operator with multiplier $\sum_{k\in\mathbb{Z}} r_k m_{j,k}$, then we have $T = \sum_{j\in\mathbb{Z}} \widetilde{T}_j$.

The boundedness of $\sum_{j\leq 0} \widetilde{T}_j$ follows from the bilinear Coifman-Meyer theory, see [13, Proposition 3] for a detailed argument. We point out that the validity of the estimates (2) and (3) for (ξ,η) near the origin is essential here, and that this argument requires the restriction $M_q' \geq 2n$.

Next we sketch the proof when $j > 0$. The proof follows the argument in [13], with modifications based on Theorem 1.3. The goal is to obtain the $L^2 \times L^2 \to L^1$ boundedness of $\widetilde{T}_j$, with bounds forming a convergent series in j. We first decompose the multiplier $m_{j,0}$ into its diagonal and off-diagonal parts. Precisely, let $m_{j,0} = \sum_\omega a_\omega \omega$ be the wavelet decomposition of $m_{j,0}$, and let D_1 denote the collection of wavelets whose supports have a nonempty intersection with the set $\{(\xi,\eta) : 2^{-j}|\xi| \leq |\eta| \leq 2^j|\xi|\}$. Then we define the diagonal part $m^1_{j,0} = \sum_{\omega\in D_1} a_\omega \omega$, and we denote by $m^2_{j,0} = m_{j,0} - m^1_{j,0}$ the off-diagonal part of $m_{j,0}$. We remark that Theorem 1.3 yields an estimate for the norm of the bilinear multiplier operator corresponding to $m^1_{j,0}$ due to the fact that the proof is only based on estimates of the coefficients a_ω. Using this and a standard dilation argument we obtain the $L^2 \times L^2 \to L^1$ boundedness of the operator $\widetilde{T}^1_j$ associated with the bilinear multiplier $\sum_k r_k m^1_{j,0}(2^k\cdot)$ with bound

$$C\, j \|m_{j,0}\|_{L^q}^{\frac{q}{4}} C_0^{1-\frac{q}{4}} \leq C' \, \|m\|_{L^q}^{\frac{q}{4}} j 2^{-j\delta(1-\frac{q}{4})},$$

where

$$C_0 = \max_{|\alpha| \le \lfloor \frac{2n}{4-q} \rfloor + 1} \|\partial^\alpha m_{j,0}\|_{L^\infty} \le C''\, 2^{-j\delta},$$

by (3). We omit the details which can be obtained by a straightforward modification of the proof in [13, Section 4] with the help of Theorem 1.3.

Boundedness of the off-diagonal parts of the operator $\widetilde{T}_j$ can be proved by an argument as in [13, Section 5]. We note that this argument relies only on the decay of the L^∞-norms of the derivatives of $m_{j,0}$, which is guaranteed by condition (3); this argument requires the restriction $M_q' > n$.

Necessity. Throughout the proof we shall adopt the convention that whenever $x \in \mathbb{R}^n$ then x_r denotes the r-th coordinate of x, that is, $x = (x_1, \dots, x_n)$.

Let ψ be a nonzero Schwartz function on $\mathbb{R}$ whose Fourier transform is supported in $[-1/10, 1/10]$, and let $(a_l(t))_{l \in \mathbb{Z}^n}$ denote the sequence of Rademacher functions indexed by the elements of the set $\mathbb{Z}^n$. Given $N \in \mathbb{N}$, $N \ge 4$ we introduce the following sets:

$$\begin{aligned} I_N &= \{5 \cdot 2^{N-2} + 1, \dots, 6 \cdot 2^{N-2} - 1\}^n; \\ J_N &= \{5 \cdot 2^{N-1} + 2, \dots, 6 \cdot 2^{N-1} - 2\}^n; \\ L_N &= \{41 \cdot 2^{N-4} + 1, \dots, 43 \cdot 2^{N-4}\}^n. \end{aligned}$$

We observe that $L_N \subseteq J_N = I_N + I_N$. For $t \in [0,1]$ and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ we define

$$m_t(\xi, \eta) = \sum_{N=4}^{\infty} \sum_{j \in I_N} \sum_{k \in I_N} a_{j+k}(t) \prod_{r=1}^{n} c_{j_r + k_r} \widehat{\psi}(\xi_r - j_r) \widehat{\psi}(\eta_r - k_r),$$

where $c_l = l^{-1/2} (\log l)^{-1/n}$, $l \in \mathbb{N}$, $l \ge 42$. It is straightforward to observe that the family of functions m_t satisfies the estimates (2) and (3) with constants independent of t, and that

$$\begin{aligned} \|m_t\|_{L^q} &\lesssim \left(\sum_{N=4}^{\infty} \sum_{j \in I_N} \sum_{k \in I_N} \prod_{r=1}^{n} c_{j_r + k_r}^q \right)^{\frac{1}{q}} \lesssim \left(\sum_{N=4}^{\infty} \sum_{l \in J_N} \prod_{r=1}^{n} c_{l_r}^q 2^{nN} \right)^{\frac{1}{q}} \\ &= \left(\sum_{N=4}^{\infty} 2^{nN} \left(\sum_{p = 5 \cdot 2^{N-1} + 2}^{6 \cdot 2^{N-1} - 2} c_p^q \right)^n \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{N=4}^{\infty} 2^{nN(2 - \frac{q}{2})} N^{-q} \right)^{\frac{1}{q}} < \infty \end{aligned}$$

whenever $q \ge 4$.

Let $f = g$ be Schwartz functions on $\mathbb{R}$ such that $\widehat{f} = \widehat{g}$ is supported in the interval $(1,2)$ and $\widehat{f} = \widehat{g} = 1$ inside the interval $[5/4, 3/2]$, and let $F(x) = G(x) = \prod_{r=1}^n f(x_r) = \prod_{r=1}^n g(x_r)$. Let $K \in \mathbb{Z}$, $K \geq 4$. Then

$$\begin{aligned} &m_t(2^K\xi, 2^K\eta)\widehat{F}(\xi)\widehat{G}(\eta) \\ &= \sum_{N=4}^{\infty} \sum_{j\in I_N} \sum_{k\in I_N} a_{j+k}(t) \prod_{r=1}^{n} c_{j_r+k_r} \widehat{\psi}(2^K\xi_r - j_r)\widehat{\psi}(2^K\eta_r - k_r)\widehat{f}(\xi_r)\widehat{g}(\eta_r). \end{aligned}$$

Assume that $r \in \{1, \dots, n\}$ and $j_r \in \{5 \cdot 2^{N-2} + 1, \dots, 6 \cdot 2^{N-2} - 1\}$ (for some $N \geq 4$) are such that the function $\widehat{\psi}(2^K\xi_r - j_r)\widehat{f}(\xi_r)$ is not identically equal to 0. Using the support properties of $\widehat{\psi}$ and $\widehat{f}$ we deduce that there is $\xi_r \in (1,2)$ such that $|2^K\xi_r - j_r| \leq 1/10$. Thus,

$$2^K - \frac{1}{10} < 2^K\xi_r - \frac{1}{10} \leq j_r \leq 2^K\xi_r + \frac{1}{10} < 2^{K+1} + \frac{1}{10}.$$

Consequently, $2^K \leq j_r \leq 2^{K+1}$, which in turn implies that j_r belongs to the set $\{5 \cdot 2^{K-2} + 1, \dots, 6 \cdot 2^{K-2} - 1\}$.

Now, if j_r is as in the previous paragraph and ξ_r satisfies $|2^K\xi_r - j_r| \leq 1/10$ then

$$\frac{5}{4} \leq \frac{5}{4} + \frac{9}{10 \cdot 2^K} \leq \frac{j_r}{2^K} - \frac{1}{10 \cdot 2^K} \leq \xi_r \leq \frac{j_r}{2^K} + \frac{1}{10 \cdot 2^K} \leq \frac{6}{4} - \frac{9}{10 \cdot 2^K} \leq \frac{3}{2},$$

and so $\widehat{f}(\xi_r) = 1$.

The previous observations applied to both ξ and η yield

$$\begin{aligned} &m_t(2^K\xi, 2^K\eta)\widehat{F}(\xi)\widehat{G}(\eta) \\ &= \sum_{j\in I_K} \sum_{k\in I_K} a_{j+k}(t) \prod_{r=1}^{n} c_{j_r+k_r} \widehat{\psi}(2^K\xi_r - j_r)\widehat{\psi}(2^K\eta_r - k_r). \end{aligned} \tag{19}$$

Let S be a finite subset of $\{K \in \mathbb{Z} : K \geq 4\}$. We denote

$$T_t^S = \sum_{K\in S} T_{m_t(2^K \cdot)}.$$

Then, by (19), we obtain

$$\begin{aligned} &T_t^S(F,G)(x) \\ &= \sum_{K\in S} \sum_{j\in I_K} \sum_{k\in I_K} a_{j+k}(t) \prod_{r=1}^{n} c_{j_r+k_r} \frac{1}{2^{2K}} \left(\psi\left(\frac{x_r}{2^K}\right)\right)^2 e^{2\pi i x_r \cdot \frac{j_r+k_r}{2^K}} \\ &= \sum_{K\in S} \sum_{l\in J_K} a_l(t) \prod_{r=1}^{n} c_{l_r} \frac{1}{2^{2K}} \left(\psi\left(\frac{x_r}{2^K}\right)\right)^2 e^{2\pi i x_r \cdot \frac{l_r}{2^K}} \end{aligned}$$

$$\times \min\{l_r - 5\cdot 2^{K-1} - 1, 6\cdot 2^{K-1} - 1 - l_r\}.$$

We notice that the sets J_K are pairwise disjoint in K. By Fubini's theorem and Khintchine's inequality we write

$$\int_0^1 \|T_t^S(F,G)\|_{L^1}\,dt = \int_{\mathbb{R}^n}\int_0^1 |T_t^S(F,G)(x)|\,dt\,dx$$

$$\approx \int_{\mathbb{R}^n}\Bigg(\sum_{K\in S}\sum_{l\in J_K}\prod_{r=1}^n \frac{1}{2^{4K}}\left|\psi\left(\frac{x_r}{2^K}\right)\right|^4 c_{l_r}^2$$

$$\times \min\{l_r - 5\cdot 2^{K-1} - 1, 6\cdot 2^{K-1} - 1 - l_r\}^2\Bigg)^{\frac12} dx$$

$$\gtrsim \int_{\mathbb{R}^n}\left(\sum_{K\in S}\sum_{l\in L_K}\prod_{r=1}^n \frac{1}{2^{2K}}\left|\psi\left(\frac{x_r}{2^K}\right)\right|^4 c_{l_r}^2\right)^{\frac12} dx$$

$$= \int_{\mathbb{R}^n}\left(\sum_{K\in S}\prod_{r=1}^n \frac{1}{2^{2K}}\left|\psi\left(\frac{x_r}{2^K}\right)\right|^4 \sum_{l\in L_K}\prod_{r=1}^n c_{l_r}^2\right)^{\frac12} dx$$

$$= \int_{\mathbb{R}^n}\left(\sum_{K\in S}\left(\sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} c_p^2\right)^n \prod_{r=1}^n \frac{1}{2^{2K}}\left|\psi\left(\frac{x_r}{2^K}\right)\right|^4\right)^{\frac12} dx.$$

Since ψ is not identically equal to 0, there is $A>0$ such that

$$\int_{\{y\in\mathbb{R}:\ A\le y<2A\}} |\psi(y)|^2\,dy > 0.$$

Noticing that the sets $\{x\in\mathbb{R}:\ A\le \frac{x}{2^K} < 2A\}$ are pairwise disjoint in K, we can estimate

$$\int_0^1 \|T_t^S(F,G)\|_{L^1}\,dt$$

$$\gtrsim \int_{\mathbb{R}^n}\left(\sum_{K\in S}\left(\sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} c_p^2\right)^n \prod_{r=1}^n \frac{1}{2^{2K}}\left|\psi\left(\frac{x_r}{2^K}\right)\right|^4 \chi_{\{A\le \frac{x_r}{2^K}<2A\}}(x_r)\right)^{\frac12} dx$$

$$= \sum_{K\in S}\left(\sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} c_p^2\right)^{\frac n2} \prod_{r=1}^n \int_{\{x_r:\ A\le\frac{x_r}{2^K}<2A\}} \frac{1}{2^K}\left|\psi\left(\frac{x_r}{2^K}\right)\right|^2 dx_r$$

$$= \sum_{K\in S}\left(\sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} c_p^2\right)^{\frac n2} \left(\int_{\{y:\ A\le y<2A\}} |\psi(y)|^2\,dy\right)^n$$

$$\approx \sum_{K\in S}\left(\sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} c_p^2\right)^{\frac{n}{2}}.$$

We have

$$\sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} c_p^2 = \sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} p^{-1}(\log p)^{-\frac{2}{n}} \gtrsim K^{-\frac{2}{n}}.$$

Thus, taking $S=\{4,5,\dots,D\}$ for large D, we obtain

$$\int_0^1 \|T_t^S(F,G)\|_{L^1}\,dt \gtrsim \sum_{K=4}^{D}\left(\sum_{p=41\cdot 2^{K-4}+1}^{43\cdot 2^{K-4}} c_p^2\right)^{\frac{n}{2}} \gtrsim \sum_{K=4}^{D} K^{-1},$$

which tends to ∞ as $D\to\infty$. This contradicts (6). $\square$

5. Consequences and Corollaries

The following result, first proved in [13], provided the inspiration for the work in this article.

Proposition 5.1 ([13, Corollary 8]). *Suppose that $m(\xi,\eta)$ is a function in $L^2(\mathbb{R}^{2n})\cap\mathcal{C}^{4n+1}(\mathbb{R}^{2n})$ such that*

$$\sup_{|\alpha|\le 4n+1}\|\partial^\alpha m\|_{L^\infty}\le C_0<\infty\,.$$

Then there is a dimensional constant $C(n)$ such that the bilinear operator T_m with multiplier m satisfies

$$\|T_m\|_{L^2\times L^2\to L^1}\le C(n)\,C_0^{\frac{1}{5}}\|m\|_{L^2}^{\frac{4}{5}}. \tag{20}$$

Theorem 1.3 allows us to significantly sharpen the preceding result.

Corollary 5.2. *Suppose that $m(\xi,\eta)$ is a function in $L^2(\mathbb{R}^{2n})\cap\mathcal{C}^{n+1}(\mathbb{R}^{2n})$ such that*

$$\sup_{|\alpha|\le n+1}\|\partial^\alpha m\|_{L^\infty}\le C_0<\infty\,.$$

Then there is a dimensional constant $C(n)$ such that the bilinear operator T_m with multiplier m satisfies

$$\|T_m\|_{L^2\times L^2\to L^1}\le C(n)\,C_0^{\frac{1}{2}}\|m\|_{L^2}^{\frac{1}{2}}. \tag{21}$$

Moreover, the exponent $1/2$ is sharp in (21), in the sense that whenever K is a fixed positive integer and

$$\|T_m\|_{L^2\times L^2\to L^1}\le C\|m\|_{L^2}^{r} \tag{22}$$

holds for some $r > 0$ and for all $m \in \mathcal{C}^K(\mathbb{R}^{2n})$ satisfying

$$\sup_{|\alpha|\le K} \|\partial^\alpha m\|_{L^\infty} \le 1,$$

then $r \ge 1/2$.

Proof. Estimate (21) follows from (8) with $q = 2$. The sharpness can be seen by an argument similar to the necessity part of the proof of Theorem 1.3. Namely, it follows from the proof of Theorem 1.3 that (22) implies the validity of the inequality

$$\sum_{l=2}^{\infty} c_l^2 \left(\sum_{j=1}^{l-1} b_j d_{l-j}\right)^2 \lesssim \left(\sum_{l=2}^{\infty} c_l^2 (l-1)\right)^r \left(\sum_{j=1}^{\infty} b_j^2\right)\left(\sum_{k=1}^{\infty} d_k^2\right) \tag{23}$$

for all sequences $(b_j)_{j=1}^\infty$, $(d_k)_{k=1}^\infty$ and $(c_l)_{l=2}^\infty$ of nonnegative numbers such that $c_l \le \varepsilon$ for all l, where ε is a certain positive real number depending on n and K. Now, if we fix $N \in \mathbb{N}$ and choose

$$c_2 = \cdots = c_N = \varepsilon, \quad b_1 = \cdots = b_N = d_1 = \cdots = d_N = N^{-\frac{1}{2}}$$

and

$$c_i = b_i = d_i = 0 \quad \text{if} \quad i > N,$$

then (23) becomes

$$N \lesssim N^{2r}.$$

This implies that $r \ge 1/2$. $\square$

Remark 3. We would like to point out that the validity of the argument in the necessity part of the proof of Corollary 5.2 is not limited to the specific value of $q = 2$. Indeed, the proof can be easily modified to show the sharpness of the exponent $q/4$ in inequality (8) for any $q \in [1, 4)$.

The argument above can also be applied in the borderline case $q = 4$, showing that, for any fixed positive integer K, there is *no* positive constant ε for which

$$\|T_m\|_{L^2\times L^2\to L^1} \le C C_0^\varepsilon \|m\|_{L^4}^{1-\varepsilon},$$

where

$$C_0 = \sup_{|\alpha|\le K} \|\partial^\alpha m\|_{L^\infty}.$$

It is unknown to us whether T_m is bounded from $L^2 \times L^2 \to L^1$ for every $m \in L^4 \cap \mathcal{L}^\infty$; it seems, however, that the decay of the bound with C_0, rather than the mere fact that T_m is bounded, is what is relevant in many applications; see Theorem 1.1.

We complement Corollary 5.2 by showing that the requirement on the number $M_q = \left\lfloor \frac{2n}{4-q} \right\rfloor + 1$ of derivatives in Theorem 1.3 is optimal as well.

Proposition 5.3. *Let $1 \le q < 4$. Let $M = M(q)$ be an integer such that for all m in $L^q(\mathbb{R}^{2n}) \cap \mathcal{C}^M(\mathbb{R}^{2n})$ satisfying*

$$\|\partial^\alpha m\|_{L^\infty} \le C_0 < \infty \quad \text{for all multiindices } \alpha \text{ with } |\alpha| \le M \tag{24}$$

there is a constant C depending on n and q such that the bilinear operator T_m with multiplier m satisfies

$$\|T_m\|_{L^2\times L^2 \to L^1} \le C\, C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{4}}. \tag{25}$$

Then $M \ge \frac{2n}{4-q}$.

Proof. The proof is rather straightforward. We take

$$m(\xi,\eta) = \prod_{j=1}^n \psi(\xi_j)\psi(\eta_j)$$

and $f_\lambda(x) = g_\lambda(x) = 2^{-\lambda/2} \prod_{j=1}^n \varphi(2^{-\lambda}x_j)$, where ψ and φ are the functions from the necessity part of the proof of Theorem 1.3. Let $m_\lambda(\xi,\eta) = m(2^\lambda\xi, 2^\lambda\eta)$. Then $\|T_{m_\lambda}(f_\lambda, g_\lambda)\|_{L^1} \sim 1$, while $\|m_\lambda\|_{L^q} \sim 2^{-2n\lambda/q}$, and $C_0 \sim 2^{\lambda M}$. So (25) implies the inequality

$$1 \le C 2^{\lambda M(1-\frac{q}{4}) - \frac{n\lambda}{2}}.$$

This shows that $M \ge \frac{2n}{4-q}$ by letting $\lambda \to \infty$. $\square$

Next, we turn to the proofs of the claimed corollaries in Section 1.

Proof of Corollary 1.4. Consider the multipliers

$$m_1(\xi,\eta) = m(-(\xi+\eta),\eta), \qquad m_2(\xi,\eta) = m(\xi,-(\xi+\eta)) \tag{26}$$

of the two adjoints of T_m, $(T_m)^{*1}$ and $(T_m)^{*2}$. It is straightforward to verify that m_1 and m_2 belong to $L^q \cap \mathcal{C}^{M_q}$, with the L^q-norms of m_1 and m_2 being comparable to the L^q norm of m, and that

$$\sup_{|\alpha|\le M_q} \|\partial^\alpha m_i\|_{L^\infty} \le CC_0, \qquad i = 1,2.$$

Therefore, by Theorem 1.3 we have

$$\|T_{m_1}\|_{L^2\times L^2\to L^1},\, \|T_{m_2}\|_{L^2\times L^2\to L^1} \le C\, C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{4}},$$

which, by duality, implies that

$$\|T_m\|_{L^\infty\times L^2\to L^2},\, \|T_m\|_{L^2\times L^\infty\to L^2} \le C\, C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{4}}. \tag{27}$$

Interpolating between the estimates (27) and the estimate (8) from Theorem 1.3, we deduce via [12, Corollary 7.2.11] that

$$\|T_m\|_{L^{p_1}\times L^{p_2}\to L^p} \leq C\, C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{4}}$$

whenever $2 \leq p_1, p_2 \leq \infty$, $1 \leq p \leq 2$ and $1/p = 1/p_1 + 1/p_2$. □

Proof of Corollary 1.2. Let us first observe that the functions m_1 and m_2 defined in (26) satisfy estimates (2) and (3). We only verify (2) for m_1 here as the remaining inequalities can be proved similarly. Notice that $|\xi|+|\eta| \leq 2(|\xi+\eta|+|\eta|)$ and $|\xi+\eta|+|\eta| \leq 2(|\xi|+|\eta|)$, so (2) for m implies that

$$\begin{aligned} |m_1(\xi,\eta)| &= |m(-(\xi+\eta),\eta)| \\ &\leq C' \min\big(|(-(\xi+\eta),\eta)|, |(-(\xi+\eta),\eta)|^{-\delta}\big) \\ &\leq C \min\big(|(\xi,\eta)|, |(\xi,\eta)|^{-\delta}\big), \end{aligned}$$

which indeed proves (2) for m_1. Using this and arguing as in the proof of Corollary 1.4, we deduce the conclusion. □

6. Applications

6.1. **Rough bilinear singular integrals.** Let Ω be a function in $L^r(\mathbb{S}^{2n-1})$ for some $r > 1$ with vanishing integral. We denote $(y,z)' = (y,z)/|(y,z)| \in \mathbb{S}^{2n-1}$ and define the rough bilinear singular integral operator T_Ω by

$$T_\Omega(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^{2n}} \frac{\Omega((y,z)')}{|(y,z)|^{2n}} f(x-y)g(x-z)dydz.$$

This operator was introduced and first studied by Coifman and Meyer [3]. In [13] it was proved that T_Ω is bounded from $L^2 \times L^2$ to L^1 provided that $\Omega \in L^r(\mathbb{S}^{2n-1})$ for $r \geq 2$. As a consequence of Corollary 1.2 we can improve this result, answering partially question (b) raised in [13], as follows:

Theorem 6.1. *Let $r > 4/3$, and assume that $\Omega \in L^r(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega\, d\sigma = 0$. Then we have*

$$\|T_\Omega\|_{L^{p_1}\times L^{p_2}\to L^p} < \infty$$

whenever $2 \leq p_1, p_2 \leq \infty$, $1 \leq p \leq 2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Proof. We denote $K^0(x) = \frac{\Omega(x')}{|x|^{2n}}\psi(x)$, where ψ is a smooth function supported in the unit annulus of $\mathbb{R}^{2n}$ satisfying $\sum_{k\in\mathbb{Z}} \psi(2^{-k}\cdot) = 1$, and set $m = \widehat{K^0}$. It is well known that m satisfies conditions (2) and (3) (see, e.g., [5, Lemma 8.20]). Thanks to the embedding $L^{r_1}(\mathbb{S}^{2n-1}) \subseteq$

$L^{r_2}(\mathbb{S}^{2n-1})$ if $r_1 \geq r_2$, we may assume that $r \leq 2$. Then, by the Hausdorff-Young inequality, we obtain

$$\|m\|_{L^{r'}} \leq C\|K^0\|_{L^r} \leq C\|\Omega\|_{L^r}. \tag{28}$$

Since $r > 4/3$, we have $r' < 4$, and Corollary 1.2 applied with $r_k = 1$ for all k thus yields the boundedness of T_Ω from $L^2 \times L^2$ to L^1. $\square$

6.2. **Rough bilinear singular integrals of R. Fefferman type.** In the previous subsection we studied the rough bilinear singular integral with kernel

$$K(x) = \frac{\Omega(x')}{|x|^{2n}},$$

which is a smooth function in the radial direction. Fefferman [7] observed that, in the linear case, smoothness of the kernel in the radial direction is unnecessary and obtained boundedness of the singular integral operator with kernel of the form

$$K(x) = \rho(|x|)\frac{\Omega(x')}{|x|^{2n}}, \tag{29}$$

where ρ is any bounded function. Let us now consider the bilinear operator T_K associated with this kernel. Motivated by an extension of the above mentioned result due to Duoandikoetxea and Rubio de Francia [6], we slightly relax the boundedness assumption on ρ and assume that it satisfies the less restrictive condition

$$\int_0^R |\rho(r)|^2 dr \leq C_\rho R. \tag{30}$$

Our result is the following theorem.

Theorem 6.2. *Suppose that Ω lies in $L^r(\mathbb{S}^{2n-1})$ with $r > 4/3$ and has vanishing integral over $\mathbb{S}^{2n-1}$. Let ρ be a function on the real line satisfying* (30)*. Then the bilinear singular integral operator T_K with kernel K given by* (29) *satisfies*

$$\|T_K\|_{L^{p_1} \times L^{p_2} \to L^p} < \infty$$

whenever $2 \leq p_1, p_2 \leq \infty$, $1 \leq p \leq 2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Moreover, if $\Omega \in L^\infty$ and $\rho \in L^\infty$, then T_K is bounded from $L^{p_1} \times L^{p_2}$ to L^p for $1 < p_1, p_2 < \infty$, $1/2 < p < \infty$, and $1/p = 1/p_1 + 1/p_2$.

To prove Theorem 6.2 we need the following variant of Corollary 1.2.

Proposition 6.3. *Let $1 \leq q < 4$ and let $(M_k)_{k\in\mathbb{Z}}$ be a sequence of multipliers belonging to $\mathcal{C}^{M'_q}(\mathbb{R}^{2n})$, where $M'_q = \max\big(2n, \lfloor\frac{2n}{4-q}\rfloor + 1\big)$,* *and satisfying*

$$\sup_{k\in\mathbb{Z}} \|M_k(2^{-k}\cdot)\|_{L^q} < \infty. \tag{31}$$

Assume, moreover, that

$$|M_k(\xi,\eta)| \le C \min(|2^k(\xi,\eta)|, |2^k(\xi,\eta)|^{-\delta}), \tag{32}$$

and

$$|\partial^\alpha M_k(\xi,\eta)| \le C_\alpha 2^{k|\alpha|} \min(1, |2^k(\xi,\eta)|^{-\delta}) \tag{33}$$

for all multiindices α with $|\alpha| \le M_q'$ and some fixed $\delta > 0$. Let T_{M_k} be the bilinear operator associated with multiplier M_k, and define $T = \sum_{k\in\mathbb{Z}} T_{M_k}$. Then

$$\|T\|_{L^{p_1}\times L^{p_2}\to L^p} < \infty$$

whenever $2 \le p_1, p_2 \le \infty$, $1 \le p \le 2$ and $1/p = 1/p_1 + 1/p_2$.

Proposition 6.3 coincides with Corollary 1.2 if $M_k(\xi,\eta) = m(2^k(\xi,\eta))$. In the general case, the proof of Corollary 1.2 translates verbatim into the proof of Proposition 6.3; we leave the details to the interested reader.

We shall also need the following lemma which follows from the proof of [6, Corollary 4.1].

Lemma 6.4. *Let $k \in \mathbb{Z}$ and let K be as in Theorem 6.2. If $K^k = K\psi(2^{-k}\cdot)$, where ψ is a smooth function supported in the unit annulus of $\mathbb{R}^{2n}$, and δ is a positive real number satisfying $2\delta r' < 1$, then we have*

$$|\widehat{K^k}(\xi,\eta)| \le C\|\Omega\|_{L^r} \min(|2^k(\xi,\eta)|, |2^k(\xi,\eta)|^{-\delta})$$

and

$$|\partial^\alpha \widehat{K^k}(\xi,\eta)| \le C_\alpha \|\Omega\|_{L^r} 2^{k|\alpha|} \min(1, |2^k(\xi,\eta)|^{-\delta})$$

for all multiindices α.

Proof of Theorem 6.2. Let ψ be a smooth function supported in the unit annulus of $\mathbb{R}^{2n}$ such that $\sum_{k\in\mathbb{Z}} \psi(2^{-k}\cdot) = 1$. For $k \in \mathbb{Z}$ we denote $\psi_k = \psi(2^{-k}\cdot)$ and $M_k = \widehat{K\psi_k}$. Then $T_K = \sum_{k\in\mathbb{Z}} T_{M_k}$, and the first statement will thus follow if we verify the assumptions (32), (33) and (31) of Proposition 6.3.

The validity of conditions (32) and (33) follows from Lemma 6.4. Let us now show that condition (31) is fulfilled with $q = r' < 4$. We will assume throughout the proof that $q' = r \le 2$. This assumption can be made without loss of generality thanks to the embedding $L^{r_1}(\mathbb{S}^{2n-1}) \subseteq L^{r_2}(\mathbb{S}^{2n-1})$ when $r_1 \ge r_2$. For any fixed $k \in \mathbb{Z}$ we have

$$\|M_k(2^{-k}\cdot)\|_{L^q} = 2^{\frac{2kn}{q}} \|M_k\|_{L^q} = 2^{\frac{2kn}{q}} \|\widehat{K\psi_k}\|_{L^q} \lesssim 2^{\frac{2kn}{q}} \|K\psi_k\|_{L^{q'}},$$

where the last estimate follows from the Hausdorff-Young inequality. Now,

$$
\begin{aligned}
\|K\psi_k\|_{L^{q'}}^{q'} &= \int_{\mathbb{R}^{2n}} |\rho(|x|)\psi_k(x)|^{q'} \frac{|\Omega(x')|^{q'}}{|x|^{2nq'}}\, dx \\
&\lesssim \int_{2^{k-1}}^{2^{k+1}} |\rho(r)|^{q'} r^{2n(1-q')-1} \int_{\mathbb{S}^{2n-1}} |\Omega(\theta)|^{q'}\, d\theta\, dr \\
&\lesssim 2^{k(2n(1-q')-1)} \|\Omega\|_{L^{q'}}^{q'} \int_{2^{k-1}}^{2^{k+1}} |\rho(r)|^{q'}\, dr \\
&\lesssim 2^{k(2n(1-q')-\frac{q'}{2})} \|\Omega\|_{L^{q'}}^{q'} \left(\int_{2^{k-1}}^{2^{k+1}} |\rho(r)|^2\, dr \right)^{\frac{q'}{2}} \\
&\lesssim 2^{2kn(1-q')} \|\Omega\|_{L^{q'}}^{q'},
\end{aligned}
$$

by (30). Altogether,

$$
\|M_k(2^{-k}\cdot)\|_{L^q} \le C 2^{\frac{2kn}{q}} 2^{-\frac{2kn}{q}} \|\Omega\|_{L^{q'}} = C\|\Omega\|_{L^r},
$$

with C independent of k, as desired.

We now turn to the second statement. We denote by T_j the bilinear operator associated with the multiplier

$$
m_j(\xi,\eta) = \sum_{k\in\mathbb{Z}} M_k(\xi,\eta)\psi(2^{k-j}(\xi,\eta)).
$$

The $L^{p_1}\times L^{p_2}\to L^p$ boundedness of $\sum_{j\le 0} T_j$ follows from the bilinear Coifman-Meyer theory by an argument analogous to the one in [13, Proposition 3]. Let us thus assume that $j>0$ in what follows. Then the operator T_j is bounded from $L^2\times L^2$ to L^1 with bound $C2^{-j\delta}$, where $\delta\sim 1/r'$. If both Ω and ρ are bounded, following the argument in [13, Section 6] we deduce that T_j is a bilinear Calderón-Zygmund operator with constant $C_\varepsilon j^\varepsilon$ for any $\varepsilon\in(0,1)$. Interpolating between the two estimates as in [13, Lemma 12], we obtain $\|T_j\|_{L^{p_1}\times L^{p_2}\to L^p} \le C2^{-j\delta_1}$ with $\delta_1=\delta_1(p_1,p_2)>0$, where $1<p_1,p_2<\infty$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Summing over j, the boundedness of T follows. $\square$

6.3. **Bilinear dyadic spherical maximal operators.** In this subsection we study the bilinear dyadic spherical maximal operator given by

$$
A^d(f,g)(x) = \sup_{k\in\mathbb{Z}} |A_{2^k}(f,g)(x)|, \tag{34}
$$

where $A_t(f,g)(x) = \int_{\mathbb{S}^{2n-1}} f(x-ty)g(x-tz)d\sigma(y,z)$. The $L^2 \times L^2 \to L^1$ boundedness of this operator follows from Theorem 1.1 by a routine argument.

Theorem 6.5. *The bilinear dyadic spherical maximal operator is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ whenever $n \geq 2$ and $2 \leq p_1, p_2 \leq \infty$, $1 \leq p \leq 2$ and $1/p = 1/p_1 + 1/p_2$.*

Proof. Let
$$\varphi(y,z) = \psi(y)\psi(z),$$
where ψ is a radially decreasing Schwartz function on $\mathbb{R}^n$ such that
$$\left(\int_{\mathbb{R}^n} \psi(x)\,dx\right)^2 = \int_{\mathbb{R}^{2n}} \varphi(y,z)\,dy\,dz = |\mathbb{S}^{2n-1}|.$$
We define $\mu = d\sigma - \varphi$ and observe that, by [11, Theorem 2.1.10],
$$A^d(f,g)(x) \leq |\mathbb{S}^{2n-1}| M(f)(x) M(g)(x) + M_\mu(f,g)(x),$$
where M is the Hardy-Littlewood maximal operator and
$$M_\mu(f,g)(x) = \sup_{k \in \mathbb{Z}} |A_{\mu,k}(f,g)|(x)$$
with
$$\begin{aligned} A_{\mu,k}(f,g)(x) &= \int_{\mathbb{R}^{2n}} f(x - 2^k y) g(x - 2^k z) d\mu(y,z) \\ &= \int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\mu}(2^k(\xi,\eta)) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta. \end{aligned}$$
Let $m = \widehat{\mu}$. Since both $\widehat{d\sigma}$ and $\widehat{\varphi}$ are continuously differentiable and $\widehat{d\sigma}(0) = \widehat{\varphi}(0)$, Taylor's remainder theorem yields the estimate $|m(\xi,\eta)| \lesssim |(\xi,\eta)|$ in a neighborhood of the origin. Furthermore, it is well-known that m satisfies (3) with $\delta = \frac{2n-1}{2}$ (see [5, p. 178]), which implies, in particular, that $m \in L^q$ for every $q > \frac{4n}{2n-1}$. Thus, $m \in L^q$ for some $q < 4$ provided that $n \geq 2$.

Using Khintchine's inequality and Fubini's theorem, we can control $\|M_\mu(f,g)\|_{L^p}^p$ by
$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |A_{\mu,k}(f,g)|^2 \Big)^{1/2} \Big\|_{L^p}^p \approx \int_0^1 \Big\| \sum_{k \in \mathbb{Z}} r_k(t) A_{\mu,k}(f,g) \Big\|_{L^p}^p dt,$$
where $\{r_k\}$ is the sequence of Rademacher functions. Consequently, by Corollary 1.2, we obtain that
$$\Big\| \sum_{k \in \mathbb{Z}} r_k(t) A_{\mu,k}(f,g) \Big\|_{L^p} \leq C \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$
with C independent of t. This concludes the proof. □

References

[1] J. Barrionuevo, L. Grafakos, D. He, P. Honzík, L. Oliveira. *Bilinear spherical maximal function.* Math. Res. Lett., to appear. (arXiv:1704.03586)

[2] A. Bényi, R. Torres. *Almost orthogonality and a class of bounded bilinear pseudodifferential operators.* Math. Res. Lett. **11** (2004), 1–11.

[3] R. R. Coifman, Y. Meyer. *On commutators of singular integrals and bilinear singular integrals.* Trans. Amer. Math. Soc. **212** (1975), 315–331.

[4] R. R. Coifman, Y. Meyer. *Commutateurs d' intégrales singulières et opérateurs multilinéaires.* Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[5] J. Duoandikoetxea. *Fourier Analysis.* Vol. **29**, Graduate Studies in Mathematics, Amer. Math. Soc., Providence RI, 2000.

[6] J. Duoandikoetxea, J. L. Rubio de Francia. *Maximal and singular integral operators via Fourier transform estimates.* Invent. Math. **84** (1986), 541–561.

[7] R. Fefferman. *A note on singular integrals.* Proc. Amer. Math. Soc. **74** (1979), 266–270.

[8] D. Geba, A. Greenleaf, A. Iosevich, E. Palsson, E. Sawyer. *Restricted convolution inequalities, multilinear operators and applications.* Math. Res. Lett. **20** (2013), 675–694.

[9] M. Fujita, N. Tomita. *Weighted norm inequalities for multilinear Fourier multipliers.* Trans. Amer. Math. Soc. **364** (2012), 6335–6353.

[10] E. Gagliardo. *Ulteriori proprietà di alcune classi di funzioni in più variabili.* (Italian) Ricerche Mat. **8** (1959), 24–51.

[11] L. Grafakos. *Classical Fourier Analysis.* 3rd Ed., GTM **249**, Springer NY, 2014.

[12] L. Grafakos. *Modern Fourier Analysis.* 3rd Ed., GTM **250**, Springer NY, 2014.

[13] L. Grafakos, D. He, P. Honzík. *Rough bilinear singular integrals.* Adv. Math. **326** (2018), 54–78.

[14] L. Grafakos, D. He, P. Honzík. *The Hörmander multiplier theorem, II: The bilinear local L^2 case.* Math. Zeit. **289** (2018), 875–887.

[15] L. Grafakos, A. Miyachi, N. Tomita. *On multilinear Fourier multipliers of limited smoothness.* Canad. J. Math. **65** (2013), 299–330.

[16] L. Grafakos, A. Miyachi, H. V. Nguyen, N. Tomita. *Multilinear Fourier multipliers with minimal Sobolev regularity, II.* J. Math. Soc. Japan **69** (2017), 529–562.

[17] L. Grafakos, H. V. Nguyen. *Multilinear Fourier multipliers with minimal Sobolev regularity, I.* Colloq. Math. **144** (2016), 1–30.

[18] L. Grafakos, Z. Si. *The Hörmander multiplier theorem for multilinear operators.* J. Reine Angew. Math. **668** (2012), 133–147.

[19] L. Grafakos, R. H. Torres. *Multilinear Calderón-Zygmund Theory.* Adv. Math. **165** (1999), 124–164.

[20] L. Hörmander. *Estimates for translation invariant operators in L^p spaces.* Acta Math. **104** (1960), 93–140.

[21] C. Kenig, E. M. Stein. *Multilinear estimates and fractional integration.* Math. Res. Lett. **6** (1999), 1–15.

[22] A. Miyachi, N. Tomita. *Minimal smoothness conditions for bilinear Fourier multipliers.* Rev. Mat. Iber. **29** (2013), 495–530.

[23] A. Miyachi, N. Tomita. *Boundedness criterion for bilinear Fourier multiplier operators.* Tohoku Math. J. **66** (2014), 55–76.

[24] L. Nirenberg. *On elliptic partial differential equations.* Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162.

[25] N. Tomita. *A Hörmander type multiplier theorem for multilinear operators.* J. Funct. Anal. **259** (2010), 2028–2044.

[26] H. Triebel. *Bases in function spaces, sampling, discrepancy, numerical integration.* EMS Tracts in Mathematics, **11**, European Mathematical Society (EMS), Zürich, 2010.

[27] H. Triebel. *Theory of function spaces. III.* Monographs in Mathematics, **100**, Birkhäuser Verlag, Basel, 2006.

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