A BILINEAR FRACTIONAL INTEGRAL OPERATOR FOR EULER-RIESZ SYSTEMS

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Abstract. We establish a uniform estimate for a bilinear fractional integral operator via restricted weak-type endpoint estimates and Marcinkiewicz interpolation. This estimate is crucial in the integrability analysis of a tensor-valued bilinear fractional integral operator associated with Euler-Riesz systems modeling mean-field interactions induced by a singular kernel. The tensorial operator arises from a reformulation of the Euler-Riesz system that yields a gain in integrability for finite energy solutions through compensated integrability. Additionally, for smooth periodic solutions of the reformulated system, we derive a stability result.

1. Introduction

We consider the following Euler-Riesz system for $t \geq 0$ and $x \in \mathbb{R}^d$ (with $d \in \mathbb{N}$):

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma + \rho \nabla K_\alpha * \rho = 0, \end{cases} \tag{1.1}$$

where $\rho : [0, \infty) \times \mathbb{R}^d \to [0, \infty)$ denotes a density, $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ stands for the velocity and the exponent γ is greater than 1. The kernel K_α is given by

$$K_\alpha(x) = \tfrac{1}{d-\alpha} |x|^{\alpha - d} \tag{1.2}$$

with $0 < \alpha < d$; the term $\rho \nabla K_\alpha * \rho$ describes the nonlocal repelling interaction of particles. Smooth solutions (ρ, u) of (1.1) decaying sufficiently fast at infinity satisfy the conservation of energy and mass identities:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \tfrac{1}{2} \rho |u|^2 + \tfrac{1}{\gamma - 1} \rho^\gamma + \tfrac{1}{2} \rho (K_\alpha * \rho) \, \mathrm{d}x = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \rho \, \mathrm{d}x = 0 \, . \tag{1.3}$$

This, in particular, yields an a priori estimate for weak solutions, which implies the regularity $\rho \in L^\infty\big((0, \infty); L^1 \cap L^\gamma(\mathbb{R}^d)\big)$ for the density.

In this work, we exploit an intriguing connection between harmonic analysis and the theory of Euler-Riesz systems hinging on the study of a bilinear fractional integral operator. The approach is based on a reformulation of the interaction term in divergence form, as seen in (1.4) below, in conjunction with uniform bounds for an associated bilinear fractional integral operator that are established here. This reformulation is advantageous for three reasons: (i) On the one hand, all the terms of the equations are written

2020 *Mathematics Subject Classification.* 42B20, 42B37, 35Q35.

Key words and phrases. Marcinkiewicz interpolation, Euler-Riesz equations, compensated integrability.

in divergence form, allowing the derivatives to be absorbed by the test functions in a weak formulation. (ii) The harmonic analysis estimates lead to integrability properties of the nonlocal interaction term. (iii) Finally, for finite energy solutions, it provides a higher integrability estimate for the density in space-time, achieved by applying the compensated integrability theory for divergence-free positive symmetric tensors [29, 30, 31] to the setting of Euler-Riesz systems.

To illustrate, note that the only term of (1.1) that is not in divergence form is the interaction term $\rho \nabla K_\alpha * \rho$. Inspired by a calculation in [30] and exploiting the symmetry of the kernel K_α, one reaches the identity

$$\rho \nabla K_\alpha * \rho = \nabla \cdot S_\alpha(\rho) \tag{1.4}$$

where $S_\alpha(\rho)$ is a tensor defined by

$$S_\alpha(\rho)(t,x) = \tfrac{1}{2} \int_0^1 \int_{\mathbb{R}^d} \rho(t, x + (\theta - 1)y)\, \rho(t, x + \theta y)\, |y|^{\alpha - d - 2}\, y \otimes y \,\mathrm{d}y \,\mathrm{d}\theta. \tag{1.5}$$

Identity (1.4) is derived in Appendix A and yields a reformulation of (1.1) in which the equations are expressed as a divergence-free condition of a tensor that fits into the compensated integrability framework of [29]. This reformulation leads, in turn, to a higher integrability estimate for finite energy solutions thereby improving on the integrability provided by the energy identity; see Theorem 3.2.

To analyze $S_\alpha(\rho)$, we consider a bilinear fractional integral operator I_α^θ defined for nonnegative measurable functions f and g on $\mathbb{R}^d$ by

$$I_\alpha^\theta(f,g)(x) = \int_{\mathbb{R}^d} f(x + (\theta - 1)y)\, g(x + \theta y)\, |y|^{\alpha - d} \,\mathrm{d}y \tag{1.6}$$

with $0 < \alpha < d$ and $0 \le \theta \le 1$. The main result of this work provides a uniform bound in θ for I_α^θ with assumptions similar in style to the classical Hardy-Littlewood-Sobolev (HLS) inequality; see Theorem 2.1.

From the natural integrability of ρ induced by the energy identity, one may deduce using the classical HLS inequality that the term $\rho \nabla K_\alpha * \rho$ belongs to L^1 in space whenever $1 < \alpha < d$. By contrast, when employing the formulation (1.4) via the tensor $S_\alpha(\rho)$, one improves the range to $0 < \alpha < d$. This observation underlines the importance of the reformulation of (1.1) through identity (1.4).

The paper is organized as follows. In Section 2 we state the main theorem of this work and describe the associated results. In Section 3 we explain how the theory of compensated integrability leads to a higher integrability estimate for finite energy solutions of the Euler-Riesz system. Section 4 contains the proof of the main theorem and its corollary, Proposition 2.6, which yields an integrability result for the tensor given by (1.5). Finally, in Section 5, we establish a stability result for smooth periodic solutions of the reformulated Euler-Riesz system via the relative energy method.

2. Description of results

The main theorem of this work provides a uniform estimate for the bilinear fractional operator I_α^θ given by (1.6); see Theorem 2.1 below.

Fractional integral operators have been of great importance in harmonic analysis for several decades; however, in recent years, their bilinear analogues have also attracted research attention. In particular, an operator B_α, with $0<\alpha<d$, acting on nonnegative measurable functions of $\mathbb{R}^d$ as

$$B_\alpha(f,g)=\int_{\mathbb{R}^d} f(x-y)\,g(x+y)\,|y|^{\alpha-d}\,\mathrm{d}y$$

was first considered in [8] and later in [19, 12], in which optimal boundedness properties between Lebesgue spaces were established. It has subsequently been studied extensively by several authors; we refer to [6, 28, 25, 18, 16, 20, 17] for estimates concerning B_α (and related versions) on a variety of spaces. While the operator I_α^θ is quite similar to B_α when the dependence on the parameter θ is ignored, its study becomes more intricate when seeking estimates that are uniform in the auxiliary parameter θ.

These types of operators have sparked significant interest primarily due to the singular nature of their integrands, but also due to their proximity to Hilbert transforms. Notable examples include the linear fractional integral operator (also known as the Riesz potential) and the linear Hilbert transform. In our case, the bilinear operator I_α^θ is related to a bilinear Hilbert transform H^θ, given by

$$H^\theta(f,g)(x)=\text{p.v.}\int_{\mathbb{R}} f(x+(\theta-1)t)\,g(x+\theta t)\frac{\mathrm{d}t}{t}.$$

Uniform bounds in θ for this transform can be deduced by direct application of the results obtained in [13, 27]. Other boundedness results for similar bilinear Hilbert transforms can be found in [21, 22].

2.1. **Main result.**

Theorem 2.1. *Let $d\in\mathbb{N}$ be the dimension, $0<\alpha<d$, and p,q,r be integrability exponents satisfying*

$$1<p,q<\frac{d}{\alpha},\quad r\ge 1,\quad \textit{and}\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{\alpha}{d}.$$

Then there is a constant $C=C(\alpha,d,p,q)>0$ independent of θ such that for all $f\in L^p(\mathbb{R}^d)$ and $g\in L^q(\mathbb{R}^d)$ we have

$$\|I_\alpha^\theta(f,g)\|_{L^r(\mathbb{R}^d)}\le C\,\|f\|_{L^p(\mathbb{R}^d)}\,\|g\|_{L^q(\mathbb{R}^d)}. \tag{2.1}$$

Whenever $(1/p,1/q)$ lies in the interior of the square with vertices $(\alpha/d,\alpha/d)$, $(\alpha/d,1)$, $(1,\alpha/d)$, and $(1,1)$, then I_α^θ is bounded from $L^p(\mathbb{R}^d)\times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ *uniformly in* θ when $1/p+1/q=1/r+\alpha/d$. If one ignores the uniform bounds in θ, then I_α^θ is bounded from $L^p(\mathbb{R}^d)\times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ when the pair $(1/p,1/q)$ lies in the interior of the pentagon with vertices $(0,\alpha/d)$, $(0,1)$, $(1,1)$ $(1,0)$, and $(\alpha/d,0)$. See Figure 1.

The proof of Theorem 2.1 relies on a bilinear version of the Marcinkiewicz interpolation method, where from a finite set of restricted weak-type estimates, one deduces strong-type estimates; see Proposition 4.6. A more general version of this method, for multilinear operators, was established in [12, 14]. In particular, in [12], this method is deduced as a corollary of a Boyd interpolation theorem in a framework of quasi-normed rearrangement-invariant spaces.

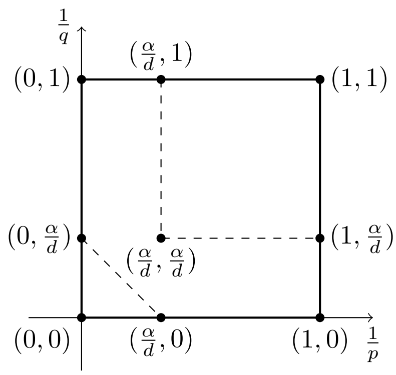

FIGURE 1. Region of boundedness

Finally, we would like to point out that the largest possible region in which uniform estimates hold for I_α^θ is, in fact, the open square with vertices $(\alpha/d,\alpha/d)$, $(\alpha/d,1)$, $(1,\alpha/d)$, and $(1,1)$. In fact, by interpolation, it suffices to verify that uniform bounds fail on the boundary of this square. To verify this assertion, let us assume that a uniform bound

$$\sup_{0<\theta<1}\|I_\alpha^\theta(f,g)\|_{L^r(\mathbb{R}^d)}\le C\,\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{\frac{d}{\alpha}}(\mathbb{R}^d)}$$

holds on the horizontal dotted line, that is, when $1/p+1/q=1/r+\alpha/d$ and $q=d/\alpha$, for some positive constant $C=C(\alpha,d)$. In this case, we must have $p=r$. By Fatou's lemma, it follows that

$$\|\liminf_{\theta\to 1} I_\alpha^\theta(f,g)\|_{L^p(\mathbb{R}^d)}\le C\,\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{\frac{d}{\alpha}}(\mathbb{R}^d)}$$

hence, for all Schwartz functions f and g we must have

$$\|fI_\alpha(g)\|_{L^p(\mathbb{R}^d)}\le C\,\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{\frac{d}{\alpha}}(\mathbb{R}^d)} \tag{2.2}$$

where I_α is the fractional integral operator

$$I_\alpha(g)(x)=\int_{\mathbb{R}^d}g(x+y)\,|y|^{\alpha-d}\,\mathrm{d}y=\int_{\mathbb{R}^d}g(x-y)\,|y|^{\alpha-d}\,\mathrm{d}y.$$

Now inserting $f(x)=f_{\epsilon,x_0}(x)=(1/\epsilon)^{\frac{d}{2}}e^{-\frac{\pi}{\epsilon}|x-x_0|^2}$ in (2.2) and letting $\epsilon\to 0$, we obtain

$$|I_\alpha(g)(x_0)|\le C\,\|g\|_{L^{\frac{d}{\alpha}}(\mathbb{R}^d)}$$

for all $x_0\in\mathbb{R}^d$. This would imply that I_α maps $L^{\frac{d}{\alpha}}(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, a fact known to be false; see [11, Example 5.1.4]. An analogous argument (letting $\theta\to 0$) indicates that a uniform bound also cannot hold on the vertical dotted line of Figure 1.

2.2. **Connections with the HLS inequality.**

Recall the HLS inequality [26]:

Proposition 2.2. *Let $p,q>1$ and $0<\alpha<d$ satisfy*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{\alpha}{d}.$$

If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{\alpha-d} g(y) \,\mathrm{d}x \,\mathrm{d}y \right| \leq C \, \|f\|_{L^p(\mathbb{R}^d)} \, \|g\|_{L^q(\mathbb{R}^d)} \tag{2.3}$$

for some $C = C(\alpha, d, p) > 0$.

Note that the assumptions on the integrability exponents p and q in Theorem 2.1 with $r = 1$ and Proposition 2.2 are exactly the same. Indeed, the assumptions of Proposition 2.2 imply that $p, q < d/\alpha$. To check this fact, suppose, without loss of generality, that $p > 1$ and $q \geq d/\alpha$. Then $1/p + 1/q < 1 + \alpha/d$, which is a contradiction.

Furthermore, the HLS inequality can be used to infer the L^1 boundedness of the operator I_α^θ, since for nonnegative measurable functions f and g, appropriate changes of variables yield

$$\|I_\alpha^\theta(f,g)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{\alpha-d} g(y) \,\mathrm{d}x \,\mathrm{d}y.$$

Thus, the uniform estimate (2.1) is particularly important if $r > 1$.

2.3. **A tensorial bilinear fractional integral operator.**

Consider a bilinear version of the tensor $S_\alpha(\rho)$, that is, define a tensorial bilinear fractional integral operator J_α for nonnegative measurable functions f and g on $\mathbb{R}^d$ by

$$J_\alpha(f,g)(x) = \int_0^1 \int_{\mathbb{R}^d} f(x + (\theta-1)y) \, g(x+\theta y) \, |y|^{\alpha-d-2} \, y \otimes y \,\mathrm{d}y \,\mathrm{d}\theta. \tag{2.4}$$

Note that $S_\alpha(\rho) = \frac{1}{2} J_\alpha(\rho,\rho)$. Additionally, identity (1.4) can be written in terms of J_α as

$$f \nabla K_\alpha * f = \nabla \cdot \left(\tfrac{1}{2} J_\alpha(f,f) \right). \tag{2.5}$$

As a consequence of Theorem 2.1, for the operator J_α we obtain the following result:

Proposition 2.3. *Let $0 < \alpha < d$ and p, q, r be integrability exponents satisfying*

$$1 < p, q < \frac{d}{\alpha}, \quad r \geq 1, \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{d}.$$

Then for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ we have

$$\|J_\alpha(f,g)\|_{L^r(\mathbb{R}^d)} \leq C \, \|f\|_{L^p(\mathbb{R}^d)} \, \|g\|_{L^q(\mathbb{R}^d)} \tag{2.6}$$

for some $C = C(\alpha, d, p, q) > 0$.

The previous proposition leads to an integrability result for $S_\alpha(\rho)$. Indeed, if

$$\rho \in L^1(\mathbb{R}^d) \cap L^\gamma(\mathbb{R}^d)$$

– as implied by the a priori bounds (1.3) – and $1 < 2dr/(d + \alpha r) \leq \gamma$, for some $r \geq 1$, then $S_\alpha(\rho) \in L^r(\mathbb{R}^d)$ and there exists a constant $C = C(\alpha, d, r) > 0$ such that

$$\|S_\alpha(\rho)\|_{L^r(\mathbb{R}^d)} \leq C \, \|\rho\|^2_{L^p(\mathbb{R}^d)} \tag{2.7}$$

where $p = 2dr/(d + \alpha r)$. Note that by interpolation, the right hand side of (2.7) is controlled by the norm $\|\rho\|_{L^1(\mathbb{R}^d)} + \|\rho\|_{L^\gamma(\mathbb{R}^d)}$.

2.4. **Reformulation of the Euler-Riesz system.**

Consider the Euler-Riesz system (1.1) supplemented with initial data ρ_0 and u_0. This system comprises a continuity equation for the conservation of mass and a second equation that ensures the conservation of momentum. These equations govern the dynamics of a compressible fluid with density ρ and linear velocity u, subject to pressure and interaction forces. The pressure function is given by $p(\rho) = \rho^\gamma$, with $\gamma > 1$ being the adiabatic exponent, and the interaction forces are modelled through the kernel K_α given by (1.2). For $d \geq 3$ and $\alpha = 2$, we recover the Euler-Poisson equations, as in that case, the interaction kernel K_2 is the Newtonian kernel. For existence theories on Euler-Riesz systems, we refer to [4, 5].

As observed in the introduction, smooth solutions of (1.1) satisfy a priori bounds of conservation of energy and mass. Given the adiabatic exponent, it is reasonable to consider solutions such that ρ belongs to $L^1 \cap L^\gamma$ in space. A natural question is whether this integrability can be improved by exploiting the structure of the equations. For finite energy solutions this can be accomplished by compensated integrability. Specifically, for a finite energy solution (ρ, u) of (1.1), one can prove that for each $T > 0$

$$\text{if} \quad \rho \in L^\infty\big(0,T; L^\gamma(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\big) \quad \text{then} \quad \rho \in L^{\gamma+\frac{1}{d}}\big((0,T)\times\mathbb{R}^d\big). \tag{2.8}$$

The first step towards (2.8) is to rewrite system (1.1) as a space-time divergence-free condition for an appropriate tensor. This is made possible through identity (1.4). This reformulates system (1.1) into a divergence-free positive symmetric tensor form, fitting in the compensated integrability theory of [29, 30, 31], thereby yielding the integrability improvement (2.8); see Section 3. We refer to [15] for an extension of this theory, and to [24] where a higher integrability estimate is obtained for one-dimensional finite energy solutions of an isentropic Euler system using a different methodology.

Next, we explore a possible weak formulation for the Euler-Riesz system (1.1). For the continuity equation take

$$\int_0^\infty \int_{\mathbb{R}^d} \rho \partial_t \varphi + \rho u \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0 \varphi_0 \,\mathrm{d}x = 0$$

where $\varphi \in C_c^1([0,\infty) \times \mathbb{R}^d)$ is a test function with $\varphi|_{t=0} = \varphi_0$. For the momentum equation we have two options, according to identity (1.4). Let $\xi \in C_c^1([0,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$ be a test function with $\xi|_{t=0} = \xi_0$. Using the left-hand side of (1.4), we get

$$\int_0^\infty \int_{\mathbb{R}^d} \rho u \cdot \partial_t \xi + (\rho u \otimes u + \rho^\gamma I_d) : \nabla \xi - \rho \nabla (K_\alpha * \rho) \cdot \xi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0 u_0 \cdot \xi_0 \,\mathrm{d}x = 0, \tag{2.9}$$

whereas, using the right-hand side of (1.4), we have

$$\int_0^\infty \int_{\mathbb{R}^d} \rho u \cdot \partial_t \xi + \big(\rho u \otimes u + \rho^\gamma I_d + S_\alpha(\rho)\big) : \nabla \xi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0 u_0 \cdot \xi_0 \,\mathrm{d}x = 0, \tag{2.10}$$

where I_d is the $d\times d$ identity matrix, and for two square matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A : B = \sum_{i,j} a_{ij} b_{ij}$.

Assume that $\rho \in L^1 \cap L^\gamma(\mathbb{R}^d)$. Using the HLS inequality we obtain:

(i) If $1 < \alpha < d$ and $\gamma \geq q = 2d/(d+\alpha-1)$, then $\rho\nabla K_\alpha * \rho \in L^1(\mathbb{R}^d)$ since

$$\|\rho\nabla K_\alpha * \rho\|_{L^1(\mathbb{R}^d)} \leq C(\alpha,d)\,\|\rho\|^2_{L^q(\mathbb{R}^d)}.$$

(ii) If $0 < \alpha < d$ and $\gamma \geq p = 2d/(d+\alpha)$, then $S_\alpha(\rho) \in L^1(\mathbb{R}^d)$ since

$$\|S_\alpha(\rho)\|_{L^1(\mathbb{R}^d)} \leq C(\alpha,d)\,\|\rho\|^2_{L^p(\mathbb{R}^d)}.$$

The second formulation is preferable as it is well-defined for a larger range of the parameters α and γ.

3. Compensated integrability

In this section we provide a proof for (2.8), which is one of the reasons for having considered the tensor $S_\alpha(\rho)$ and subsequently the bilinear fractional integral operator I^θ_α.

3.1. **A divergence-free positive symmetric tensor.**

First, we write system (1.1) as a space-time divergence-free condition for an appropriate tensor. Thanks to (1.4), system (1.1) can be reformulated into:

$$\nabla_{t,x} \cdot A_\alpha(\rho,u) = 0 \tag{3.1}$$

where the $(1+d)$-tensor $A_\alpha(\rho,u)$ is given by

$$A_\alpha(\rho,u) = \begin{bmatrix} \rho & (\rho u)^\top \\ \rho u & \rho u \otimes u + p(\rho) I_d + S_\alpha(\rho) \end{bmatrix}. \tag{3.2}$$

Next, we deduce some basic properties of the tensor $S_\alpha(\rho)$ given by (1.5) that are relevant for the subsequent analysis.

Proposition 3.1. *The tensor $S_\alpha(\rho)$ is symmetric, positive semi-definite and*

$$\det\big(p(\rho)I_d + S_\alpha(\rho)\big) \geq \begin{cases} p(\rho)^d, \\ \det S_\alpha(\rho). \end{cases}$$

Proof. It is clear that $S_\alpha(\rho)$ is symmetric since $y \otimes y$ is symmetric. Moreover, given a vector $v = v(x)$,

$$v^\top S_\alpha(\rho) v = \tfrac{1}{2}\int_{\mathbb{R}^d}\int_0^1 \rho(x+(\theta-1)y)\rho(x+\theta y)|y|^{\alpha-d-2}(y\cdot v)^2\,\mathrm{d}\theta\,\mathrm{d}y \geq 0$$

hence $S_\alpha(\rho)$ is positive semi-definite. Therefore, there exist nonnegative eigenvalues $\lambda_1,\dots,\lambda_d$, and respective eigenvectors $v_1,\dots,v_d$, that is, $S_\alpha(\rho)v_i = \lambda_i v_i$. Then $p(\rho)+\lambda_i$ is an eigenvalue of $p(\rho)I_d + S_\alpha(\rho)$, since $\big(p(\rho)I_d + S_\alpha(\rho)\big)v_i = \big(p(\rho)+\lambda_i\big)v_i$. Hence, given that $\lambda_i \geq 0$,

$$\det\big(p(\rho)I_d + S_\alpha(\rho)\big) = \prod_{i=1}^d \big(p(\rho)+\lambda_i\big) \geq \begin{cases} \displaystyle\prod_{i=1}^d p(\rho) = p(\rho)^d, \\ \displaystyle\prod_{i=1}^d \lambda_i = \det S_\alpha(\rho). \end{cases}$$

□

It follows that the tensor $A_\alpha(\rho,u)$ is symmetric and positive semi-definite. To check the latter, let $w=(w_0,\tilde w)$, with w_0 being a scalar and $\tilde w$ a d-dimensional vector, and note that

$$\begin{aligned} w^\top A w &= \begin{bmatrix} w_0 & \tilde w^\top \end{bmatrix} \begin{bmatrix} \rho & (\rho u)^\top \\ \rho u & \rho u\otimes u + p(\rho) I + S_\alpha(\rho) \end{bmatrix} \begin{bmatrix} w_0 \\ \tilde w \end{bmatrix} \\ &= \rho(w_0 + u\cdot \tilde w)^2 + p(\rho)|\tilde w|^2 + \tilde w^\top S_\alpha(\rho)\tilde w \\ &\geq 0 \end{aligned}$$

where in the last step we used the fact that $S_\alpha(\rho)$ is positive semi-definite.

Consequently $A_\alpha(\rho,u)$ is a divergence-free positive symmetric tensor.

3.2. **Higher integrability for finite energy solutions.**

Assume that (ρ,u) is a solution of (3.1) with finite mass and energy and such that $A_\alpha(\rho,u)$ belongs to $L^1\big((0,T)\times\mathbb{R}^d\big)\cap L^{1+\frac1d}_{loc}\big((0,T)\times\mathbb{R}^d\big)$ for each $T>0$. Note that by the conservation of mass and energy, it suffices to prescribe initial data (ρ_0,u_0) with finite mass and energy. We apply [29, Theorem 2.3] to the tensor $A_\alpha(\rho,u)$, along the same lines of the proof of [29, Theorem 3.1].

Set

$$\begin{gathered} \Sigma = (0,T)\times B_R, \quad B_R = \{x\in\mathbb{R}^d \mid |x|<R\}, \\ \partial\Sigma = \big(\{0\}\times B_R\big)\cup\big((0,T)\times\partial B_R\big)\cup\big(\{T\}\times B_R\big). \end{gathered}$$

The following estimate holds:

$$\int_0^T\int_{B_R}\big(\det A_\alpha(\rho,u)\big)^{\frac1d}\,\mathrm{d}x\,\mathrm{d}t \leq c_d\|A_\alpha(\rho,u)\nu\|_{L^1(\partial\Sigma)}^{1+\frac1d} \tag{3.3}$$

where ν is the outward normal vector to the boundary of Σ, given by

$$\nu = \begin{cases} (-1,0_d) & \text{on } \{0\}\times B_R, \\ z=(0,x/|x|) & \text{on } (0,T)\times B_R, \\ (1,0_d) & \text{on } \{T\}\times B_R. \end{cases}$$

Hence

$$\|A_\alpha(\rho,u)\nu\|_{L^1(\partial\Sigma)} = \int_{B_R}|(\rho,\rho u)|_{t=0} + |(\rho,\rho u)|_{t=T}\,\mathrm{d}x + \psi(R)$$

where

$$\psi(R) = \int_0^T\int_{\partial B_R}|A_\alpha(\rho,u)z|\,\mathrm{d}x\,\mathrm{d}t.$$

Since $A_\alpha(\rho,u)$ is integrable it follows that $\psi\in L^1(0,\infty)$. Indeed,

$$\begin{aligned} \int_0^\infty|\psi(R)|\,\mathrm{d}R &\leq \int_0^\infty\int_0^T\int_{\partial B_R}|A_\alpha(\rho,u)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}R \\ &= \int_0^\infty\frac{\mathrm{d}}{\mathrm{d}R}\int_0^T\int_{B_R}|A_\alpha(\rho,u)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}R \\ &= \|A_\alpha(\rho,u)\|_{L^1((0,T)\times\mathbb{R}^d)}. \end{aligned}$$

Therefore, there exists a sequence $R_n \to \infty$ such that $\psi(R_n) \to 0$. Considering this limit in (3.3), and using the conservation of mass and momentum, gives

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} \big(\det A_\alpha(\rho,u)\big)^{\frac{1}{d}} \,\mathrm{d}x\,\mathrm{d}t &\le c_d\Big(\int_{\mathbb{R}^d} |(\rho,\rho u)|_{t=0} + |(\rho,\rho u)|_{t=T}\,\mathrm{d}x\Big)^{1+\frac{1}{d}} \\
&= 2c_d\Big(\int_{\mathbb{R}^d} \sqrt{\rho_0^2+\rho_0^2|u_0|^2}\,\mathrm{d}x\Big)^{1+\frac{1}{d}} \\
&\le 2c_d\Big(\int_{\mathbb{R}^d} \rho_0 + \rho_0|u_0|\,\mathrm{d}x\Big)^{1+\frac{1}{d}}
\end{aligned}$$

where in the last inequality we used that $\sqrt{a^2+b^2} \le a+b$ for $a,b \ge 0$.

Now, using Proposition 3.1,

$$\begin{aligned}
\det A_\alpha(\rho,u) &= \rho \det\left(\rho u \otimes u + p(\rho) I + S_\alpha(\rho) - \rho u \frac{1}{\rho}\rho u^\top\right) \\
&= \rho \det\big(p(\rho) I_d + S_\alpha(\rho)\big) \\
&\ge \rho p(\rho)^d.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p(\rho)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^d} \big(\rho p(\rho)^d\big)^{\frac{1}{d}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^T \int_{\mathbb{R}^d} (\det A_\alpha(\rho,u))^{\frac{1}{d}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le 2c_d\left(\int_{\mathbb{R}^d} \tfrac{3}{2}\rho_0 + \tfrac{1}{2}\rho_0|u_0|^2\,\mathrm{d}x\right)^{1+\frac{1}{d}} \\
&\le 2c_d\left(\tfrac{3}{2}\int_{\mathbb{R}^d} \rho_0\,\mathrm{d}x + \int_{\mathbb{R}^d} \tfrac{1}{2}\rho_0|u_0|^2 + h(\rho_0) + \tfrac{1}{2}\rho_0 K * \rho_0\,\mathrm{d}x\right)^{1+\frac{1}{d}} \\
&< \infty
\end{aligned}$$

which establishes the desired higher integrability estimate given that $p(\rho) = \rho^\gamma$.

Letting $T \to \infty$ we obtain:

Theorem 3.2. *Solutions of the Euler-Riesz system* (1.1) *with* $0 < \alpha < d$ *and repelling potentials satisfy the a priori estimate*

$$\rho \in L^{\gamma+\frac{1}{d}}\big((0,\infty)\times\mathbb{R}^d\big).$$

Remark 3.3. *Note that the special case* $\alpha = 2$ *with* $d \ge 3$ *corresponds to the Euler-Poisson system*

$$\begin{aligned}
\partial_t\rho + \nabla\cdot(\rho u) &= 0, \\
\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) &= -\nabla\rho^\gamma - \rho\nabla\phi, \\
-\Delta\phi &= \rho,
\end{aligned} \tag{3.4}$$

commonly used in models for electrically charged fluids.

In this case, as the potential is given as the solution of Poisson's equation, one could also write

$$\rho\nabla\phi = \nabla\cdot\big(\tfrac{1}{2}|\nabla\phi|^2 I_d - \nabla\phi\otimes\nabla\phi\big)$$

however, the tensor being applied by the divergence on the right-hand side is not positive semi-definite, and therefore it does not fit into the theory of compensated integrability.

4. Bilinear harmonic analysis

The aim of this section is to prove Theorem 2.1. Since all the operators involved are positive, we assume that all the considered functions are nonnegative. Given that all the Lebesgue spaces in this section are over $\mathbb{R}^d$, we shorten the notation of $L^p(\mathbb{R}^d)$ to L^p.

4.1. **An auxiliary operator I^θ.**

For $0 \le \theta \le 1$, let I^θ be the bilinear operator defined for (nonnegative) measurable functions f and g on $\mathbb{R}^d$ by

$$I^\theta(f,g)(x) = \int_{|y|\le 1} f(x+(\theta-1)y)\, g(x+\theta y)\, \mathrm{d}y.$$

Lemma 4.1. *The operator I^θ maps $L^1 \times L^1$ to $L^{\frac{1}{2}}$ uniformly in θ. Precisely:*

$$\|I^\theta(f,g)\|_{L^{\frac{1}{2}}} \le C\, \|f\|_{L^1}\, \|g\|_{L^1} \tag{4.1}$$

with $C = 3^d 5^{2d}$.

Proof. We first prove (4.1) with $C = 3^d$ for integrable functions f and g supported in cubes with sides of length one parallel to the axes. Let $Q_0 = [0,1]^d$ and for each $k \in \mathbb{Z}^d$, let $Q_k = k + Q_0$ denote the cube with side length one whose sides are parallel to the axes and whose lower left corner is k. For $k = (k_1,\dots,k_d)$ and $l = (l_1,\dots,l_d)$ in $\mathbb{Z}^d$, assume that f is supported in Q_k and that g is supported in Q_l. Under these conditions, we claim that $I^\theta(f,g)$ is supported in a cube Q of side length 3. Indeed, for each $i = 1,\dots,d$, the inequalities

$$k_i \le x_i + (\theta-1)y_i \le k_i + 1, \qquad l_i \le x_i + \theta y_i \le l_i + 1,$$

together with $|y| \le 1$ and $0 \le \theta \le 1$ imply that

$$k_i - 1 \le x_i \le k_i + 2, \qquad l_i - 1 \le x_i \le l_i + 2,$$

which establishes the claim. Thus, for these f and g, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|I^\theta(f,g)\|_{L^{\frac{1}{2}}} &= \left(\int_{\mathbb{R}^d} \chi_Q |I^\theta(f,g)|^{\frac{1}{2}}\, \mathrm{d}x\right)^2 \\
&\le 3^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+(\theta-1)y) g(x+\theta y)\, \mathrm{d}y\, \mathrm{d}x \\
&\le 3^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(z-y) g(z)\, \mathrm{d}z\, \mathrm{d}y \\
&\le 3^d \|f\|_{L^1} \|g\|_{L^1}.
\end{aligned}$$

Now that we have established (4.1) for all integrable f and g supported in cubes with side length one, we proceed to the general case. For each k and m in $\mathbb{Z}^d$, set $f_k = \chi_{Q_k} f$, $g_m = \chi_{Q_m} g$.

Given $x \in \mathbb{R}^d$, we claim that if $I^\theta(f_k, g_m)(x) \neq 0$, then each m_i satisfies $k_i \leq m_i \leq k_i + 2$. Indeed, under the hypothesis $I^\theta(f_k, g_m)(x) \neq 0$, we have that $x + (\theta - 1)y \in Q_k$, $x + \theta y \in Q_m$, and so the conditions below hold

$$k_i \leq x_i + (\theta - 1)y_i \leq k_i + 1,$$

and

$$m_i \leq x_i + \theta y_i \leq m_i + 1,$$

for each $i = 1, \ldots, d$. Since $|y| \leq 1$, the conditions above imply that

$$k_i \leq x_i + \theta y_i - y_i \leq x_i + \theta y_i + 1 \leq m_i + 2,$$

and

$$m_i \leq x_i + \theta y_i = x_i + (\theta - 1)y_i + y_i \leq k_i + 2,$$

which establishes the claim. So, for any fixed $k \in \mathbb{Z}^d$, if $I^\theta(f_k, g_m)(x) \neq 0$, then $m = k + l$, where $l \in [-2, 2]^d \cap \mathbb{Z}^d = F$. Note that F contains at most 5^d elements.

We have:

$$|I^\theta(f, g)|^{\frac{1}{2}} \leq \sum_{k \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} |I^\theta(f_k, g_m)|^{\frac{1}{2}} = \sum_{l \in F} \sum_{k \in \mathbb{Z}^d} |I^\theta(f_k, g_{k+l})|^{\frac{1}{2}}$$

and so, using the fact that (4.1) with $C = 3^d$ holds for the functions f_k and g_{k+l}, it follows that

$$\begin{aligned} \|I^\theta(f, g)\|_{L^{\frac{1}{2}}} &\leq \left(\sum_{l \in F} \sum_{k \in \mathbb{Z}^d} \|I^\theta(f_k, g_{k+l})\|_{L^{1/2}}^{\frac{1}{2}} \right)^2 \\ &\leq 3^d \left(\sum_{l \in F} \sum_{k \in \mathbb{Z}^d} \|f_k\|_{L^1}^{\frac{1}{2}} \|g_{k+l}\|_{L^1}^{\frac{1}{2}} \right)^2 . \end{aligned}$$

Finally, applying the Cauchy-Schwarz inequality to the last term above yields

$$\begin{aligned} \|I^\theta(f, g)\|_{L^{\frac{1}{2}}} &\leq 3^d \left(\sum_{l \in F} \Big(\sum_{k \in \mathbb{Z}^d} \|f_k\|_{L^1} \Big)^{\frac{1}{2}} \Big(\sum_{k \in \mathbb{Z}^d} \|g_{k+l}\|_{L^1} \Big)^{\frac{1}{2}} \right)^2 \\ &\leq 3^d \left(\sum_{l \in F} \|f\|_{L^1}^{\frac{1}{2}} \|g\|_{L^1}^{\frac{1}{2}} \right)^2 \\ &\leq 3^d 5^{2d} \|f\|_{L^1} \|g\|_{L^1} \end{aligned}$$

which concludes the proof. □

4.2. **A dilated version of I^θ.**

In this section we consider a dilated version of I^θ, denoted by I^θ_j, for $j \in \mathbb{Z}$. This is defined as follows:

$$I^\theta_j(f,g)(x) = \int_{|y|\le 2^j} f(x+(\theta-1)y)\, g(x+\theta y)\,\mathrm{d}y.$$

Lemma 4.2. *The operator I^θ_j maps $L^1 \times L^1$ to L^1 uniformly in θ and in j. Precisely:*

$$\|I^\theta_j(f,g)\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}. \tag{4.2}$$

Proof. Let $f, g \in L^1$. By Fubini's theorem and the change of variables $x+(\theta-1)y = z$ it holds that

$$\|I^\theta_j(f,g)\|_{L^1} = \int_{|y|\le 2^j}\int_{\mathbb{R}^d} f(z)g(z+y)\,\mathrm{d}z\,\mathrm{d}y.$$

Using Fubini's theorem once more, together with the change of variables $z+y=w$, it follows that

$$\|I^\theta_j(f,g)\|_{L^1} = \int_{\mathbb{R}^d} f(z)\int_{|w-z|\le 2^j} g(w)\,\mathrm{d}w\mathrm{d}z \le \|f\|_{L^1}\|g\|_{L^1}$$

as desired. □

Lemma 4.3. *The operator I^θ_j maps $L^1 \times L^1$ to $L^{\frac12}$ uniformly in θ. Precisely:*

$$\|I^\theta_j(f,g)\|_{L^{\frac12}} \le 2^{dj}3^d5^{2d}\|f\|_{L^1}\|g\|_{L^1}. \tag{4.3}$$

Proof. This is a consequence of (4.1) via a dilation argument which we include for convenience.

We have:

$$\begin{aligned}
\|I^\theta_j(f,g)\|_{L^{\frac12}} &= \left(\int_{\mathbb{R}^d} |I^\theta_j(f,g)(2^jx)|^{\frac12} 2^{dj}\,\mathrm{d}x\right)^2\\
&= 2^{2dj}\left(\int_{\mathbb{R}^d}\left(\int_{|y|\le 2^j} f(2^jx+(\theta-1)y)g(2^jx+\theta y)\,\mathrm{d}y\right)^{\frac12}\mathrm{d}x\right)^2\\
&= 2^{2dj}\left(\int_{\mathbb{R}^d}\left(\int_{|y|\le 1} f(2^j(x+(\theta-1)y))g(2^j(x+\theta y))2^{dj}\,\mathrm{d}y\right)^{\frac12}\mathrm{d}x\right)^2\\
&= 2^{3dj}\|I^\theta(f_j,g_j)\|_{L^{\frac12}}
\end{aligned}$$

where $f_j(x) = f(2^jx)$ and $g_j(x) = g(2^jx)$.

Using (4.1), it follows that

$$\begin{aligned}
\|I^\theta_j(f,g)\|_{L^{\frac12}} &\le 2^{3dj}3^d5^{2d}\int_{\mathbb{R}^d} f_j(x)\,\mathrm{d}x\int_{\mathbb{R}^d} g_j(x)\,\mathrm{d}x\\
&= 2^{3dj}3^d5^{2d}\int_{\mathbb{R}^d} f(x)2^{-dj}\,\mathrm{d}x\int_{\mathbb{R}^d} g(x)2^{-dj}\,\mathrm{d}x\\
&= 2^{dj}3^d5^{2d}\|f\|_{L^1}\|g\|_{L^1},
\end{aligned}$$

as desired. □

Lemma 4.4. *There exists $c > 0$, depending only on d, such that for all measurable sets $E, A, B \subseteq \mathbb{R}^d$ it holds:*

$$\left(\int_E |I_j^\theta(\chi_A, \chi_B)|^{\frac{1}{2}} \,\mathrm{d}x\right)^2 \le \begin{cases} c|A||B|\min\{2^{dj}, |E|\}, & (4.4)\\ c|A||E|\min\{2^{dj}, |B|\}, & (4.5)\\ c|B||E|\min\{2^{dj}, |A|\}, & (4.6)\end{cases}$$

and

$$\int_E |I_j^\theta(\chi_A, \chi_B)| \,\mathrm{d}x \le c\min\{2^{dj}|E|, |A||B|\}. \tag{4.7}$$

Proof. First we prove estimate (4.4). Using (4.3) with $f = \chi_A$ and $g = \chi_B$ we have that

$$\begin{aligned}\left(\int_E |I_j^\theta(\chi_A, \chi_B)|^{\frac{1}{2}} \,\mathrm{d}x\right)^2 &\le \left(\int_{\mathbb{R}^d} |I_j^\theta(\chi_A, \chi_B)|^{\frac{1}{2}} \,\mathrm{d}x\right)^2\\ &\le 3^d 5^{2d} 2^{dj}|A||B|.\end{aligned}$$

On the other hand, by (4.2) and the Cauchy-Schwarz inequality it holds:

$$\begin{aligned}\left(\int_E |I_j^\theta(\chi_A, \chi_B)|^{\frac{1}{2}} \,\mathrm{d}x\right)^2 &= \left(\int_{\mathbb{R}^2} \chi_E |I_j^\theta(\chi_A, \chi_B)|^{\frac{1}{2}} \,\mathrm{d}x\right)^2\\ &\le |E||A||B|\\ &\le 3^d 5^{2d}|E||A||B|.\end{aligned}$$

Estimate (4.4) follows from combining the two estimates above.

Next, we turn our attention to estimates (4.5) and (4.6). We only give a proof of the former due to their symmetrical nature. First, we use the Cauchy-Schwarz inequality as above to obtain

$$\left(\int_E |I_j^\theta(\chi_A, \chi_B)|^{\frac{1}{2}} \,\mathrm{d}x\right)^2 \le |E| \int_{\mathbb{R}^d} I_j^\theta(\chi_A, \chi_B) \,\mathrm{d}x.$$

There are two ways to estimate the integral on the right-hand side of the previous inequality. One way is by $|A||B|$ (using (4.2)), and the other is as follows (using that $\chi_B \le 1$):

$$\begin{aligned}\int_{\mathbb{R}^d} I_j^\theta(\chi_A, \chi_B) \,\mathrm{d}x &\le \int_{\mathbb{R}^d} \int_{|y|\le 2^j} \chi_A(x + (\theta - 1)y) \,\mathrm{d}y \,\mathrm{d}x\\ &= \int_{|y|\le 2^j} \int_{\mathbb{R}^d} \chi_A(x + (\theta - 1)y) \,\mathrm{d}x \,\mathrm{d}y\\ &\le \nu_d 2^{dj}|A|\end{aligned}$$

where ν_d denotes the measure of the unit ball in $\mathbb{R}^d$. This proves (4.5).

In order to prove (4.7), we observe that

$$I_j^\theta(\chi_A, \chi_B) \le \nu_d 2^{dj}$$

from which it follows that

$$\int_E |I_j^\theta(\chi_A, \chi_B)| \,\mathrm{d}x \le \nu_d 2^{dj}|E|.$$

The previous inequality together with

$$\int_E |I_j^\theta(\chi_A, \chi_B)|\,\mathrm{d}x \le \int_{\mathbb{R}^d} |I_j^\theta(\chi_A, \chi_B)|\,\mathrm{d}x \le |A||B|$$

yields the desired estimate. □

4.3. **Bilinear Marcinkiewicz interpolation.**

Recall the definition of weak Lebesgue spaces. For $0 < r < \infty$, the weak L^r space, denoted by $L^{r,\infty}$, is the space of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{L^{r,\infty}} := \sup_{\lambda>0} \lambda \big|\{x \in \mathbb{R}^d :\ |f(x)| > \lambda\}\big|^{\frac{1}{r}} < \infty. \tag{4.8}$$

The map $\|\cdot\|_{L^{r,\infty}}$ is a quasi-norm, and the following holds [9]:

$$\|f\|_{L^{r,\infty}} \le \sup_{0<|E|<\infty} |E|^{-\frac{1}{s}+\frac{1}{r}} \left(\int_E |f|^s\,\mathrm{d}x\right)^{\frac{1}{s}} \tag{4.9}$$

where $0 < s < r$ and the supremum is taken over measurable sets $E \subseteq \mathbb{R}^d$ with finite measure.

Definition 4.5. Let $0 < p, q, r < \infty$. A bilinear operator U acting on measurable functions is said to be of restricted weak type (p, q, r) (with constant $c > 0$) if

$$\|U(\chi_A, \chi_B)\|_{L^{r,\infty}} \le c\,|A|^{\frac{1}{p}}|B|^{\frac{1}{q}} \tag{4.10}$$

for all measurable sets A and B with finite measure.

The next proposition, a version of the multilinear Marcinkiewicz interpolation, is the main step towards establishing Theorem 2.1. It yields strong-type bounds for bilinear operators, assuming only a finite set of restricted weak-type estimates. For a proof of the general case, see [14] or [10, Theorem 7.2.2 and Corollary 7.2.4].

Proposition 4.6. *Let $0 < p_i, q_i, r_i < \infty$ for $i = 1, 2, 3$. Suppose that the points*

$$\Big(\frac{1}{p_1}, \frac{1}{q_1}\Big), \quad \Big(\frac{1}{p_2}, \frac{1}{q_2}\Big), \quad \Big(\frac{1}{p_3}, \frac{1}{q_3}\Big),$$

do not lie on the same line in $\mathbb{R}^2$. For $0 < \theta_1, \theta_2, \theta_3 < 1$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$ consider the points $0 < p, q, r < \infty$ such that

$$\Big(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\Big) = \theta_1\Big(\frac{1}{p_1}, \frac{1}{q_1}, \frac{1}{r_1}\Big) + \theta_2\Big(\frac{1}{p_2}, \frac{1}{q_2}, \frac{1}{r_2}\Big) + \theta_3\Big(\frac{1}{p_3}, \frac{1}{q_3}, \frac{1}{r_3}\Big)$$

and

$$\frac{1}{r} \le \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

Let U be a bilinear operator that is of restricted weak-type (p_i, q_i, r_i) (with constant $c_i > 0$) for all $i = 1, 2, 3$. Then there is a constant $C > 0$ depending only on p_i, q_i, r_i, and θ_i $(i = 1, 2, 3)$ such that

$$\big\|U(f, g)\big\|_{L^r} \le C\, c_1^{\theta_1} c_2^{\theta_2} c_3^{\theta_3} \|f\|_{L^p} \|g\|_{L^q}$$

for all functions $f \in L^p$ and $g \in L^q$.

We note that the conclusion of Proposition 4.6 is also valid in the interior of the convex hull of four (or more) points at which initial restricted weak-type estimates are known. The reason is that any polygon can be written as a union of triangles.

4.4. **Proof of Theorem 2.1.**

Turning our attention to the bilinear fractional integral operator I_α^θ defined by (1.6), we note that by a dilation argument, if it maps $L^p \times L^q$ to L^r, then necessarily

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{d}. \tag{4.11}$$

Moreover, Theorem 2.1 affirms that I_α^θ is bounded uniformly in θ from $L^p \times L^q$ to L^r when (p,q) lies in the open square with vertices $(1,1)$, $(1, \frac{d}{\alpha})$, $(\frac{d}{\alpha}, 1)$, $(\frac{d}{\alpha}, \frac{d}{\alpha})$ and (4.11) holds. In this case, we have:

(i) If $(p,q) = (1,1)$, then $r = \frac{d}{2d-\alpha}$,
(ii) If $(p,q) = (1, \frac{d}{\alpha})$, then $r = 1$,
(iii) If $(p,q) = (\frac{d}{\alpha}, 1)$, then $r = 1$,
(iv) If $(p,q) = (\frac{d}{\alpha}, \frac{d}{\alpha})$, then $r = \frac{d}{\alpha}$.

Set

$$(p_1, q_1, r_1) = \left(1, 1, \frac{d}{2d-\alpha}\right), \quad (p_2, q_2, r_2) = \left(1, \frac{d}{\alpha}, 1\right)$$

$$(p_3, q_3, r_3) = \left(\frac{d}{\alpha}, 1, 1\right), \quad (p_4, q_4, r_4) = \left(\frac{d}{\alpha}, \frac{d}{\alpha}, \frac{d}{\alpha}\right).$$

To establish Theorem 2.1 it suffices to prove that I_α^θ is of restricted weak type (p_i, q_i, r_i) (with constant c_i that is independent of θ), for $i = 1, 2, 3, 4$. Then, the result follows by bilinear Marcinkiewicz interpolation, Proposition 4.6.

That is, we need to prove the following estimates:

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{\frac{d}{2d-\alpha},\infty}} \le c_1 |A||B|, \tag{4.12}$$

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{1,\infty}} \le c_2 |A||B|^{\frac{\alpha}{d}}, \tag{4.13}$$

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{1,\infty}} \le c_3 |A|^{\frac{\alpha}{d}}|B|, \tag{4.14}$$

$$\|I_\alpha^\theta(\chi_A, \chi_B)\|_{L^{\frac{d}{\alpha},\infty}} \le c_4 |A|^{\frac{\alpha}{d}}|B|^{\frac{\alpha}{d}}, \tag{4.15}$$

uniformly in θ, for all measurable sets A and B with finite measure.

In the proof of (4.12)-(4.15) we utilize the following lemma.

Lemma 4.7. *Let $d \in \mathbb{N}$, $0 < \alpha < d$ and $a, b > 0$. There exists $c = c(d, \alpha) > 0$ such that*

$$\left(\sum_{j \in \mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} (\min\{2^{dj}, a\})^{\frac{1}{2}} \right)^2 \le c\, a^{\frac{\alpha}{d}}, \tag{4.16}$$

$$\sum_{j \in \mathbb{Z}} 2^{(\alpha-d)j} \min\{2^{dj} a, b\} \le c\, a \left(\frac{b}{a}\right)^{\frac{\alpha}{d}}. \tag{4.17}$$

Proof. We only prove (4.16) as the other one is similar. Let $m = \max\{j \in \mathbb{Z} \mid 2^{dj} < a\}$. Then

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}}(\min\{2^{dj},a\})^{\frac{1}{2}} &= \sum_{j=-\infty}^{m} 2^{\frac{\alpha j}{2}} + \sum_{j=m+1}^{\infty} 2^{\frac{(\alpha-d)j}{2}} a^{\frac{1}{2}} \\ &= \sum_{k=0}^{\infty} 2^{-\frac{\alpha(k-m)}{2}} + \sum_{i=0}^{\infty} 2^{\frac{(\alpha-d)}{2}(i+m+1)} a^{\frac{1}{2}} \\ &= \Big(\sum_{k=0}^{\infty} 2^{-\frac{\alpha k}{2}}\Big) 2^{\frac{\alpha m}{2}} + \Big(\sum_{i=0}^{\infty} 2^{\frac{(\alpha-d)i}{2}}\Big) 2^{\frac{(\alpha-d)}{2}(m+1)} a^{\frac{1}{2}}.\end{aligned}$$

The desired inequality is achieved by noting that $2^{\frac{\alpha m}{2}} < a^{\frac{\alpha}{2d}}$ and $2^{\frac{(\alpha-d)}{2}(m+1)} \le a^{\frac{\alpha-d}{2d}}$. □

Proof of Theorem 2.1.

First, note that $\mathbb{R}^d$ can be expressed as the union of annuli:

$$\mathbb{R}^d = \bigcup_{j\in\mathbb{Z}} \left(B(2^j) \setminus B(2^{j-1})\right),$$

where $B(R)$ denotes the open ball in $\mathbb{R}^d$ centered at the origin with radius R.

Therefore:

$$\begin{aligned}I_\alpha^\theta(f,g)(x) &\le \sum_{j\in\mathbb{Z}} \int_{2^{j-1}\le|y|\le 2^j} f(x+(\theta-1)y)g(x+\theta y)|y|^{\alpha-d}\,\mathrm{d}y \\ &\le \sum_{j\in\mathbb{Z}} 2^{d-\alpha} \int_{2^{j-1}\le|y|\le 2^j} f(x+(\theta-1)y)g(x+\theta y)2^{(\alpha-d)j}\,\mathrm{d}y \\ &\le 2^{d-\alpha} \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I_j^\theta(f,g)(x).\end{aligned}$$

Let A, B be measurable sets of $\mathbb{R}^d$ with finite measure. In what follows, the positive constant C might change from line to line, but it will always be independent of θ.

To prove the restricted estimate (4.12) we use (4.4), (4.9) with $s = 1/2$, and (4.16) as follows:

$$\begin{aligned}\|I_\alpha^\theta(\chi_A,\chi_B)\|_{L^{\frac{d}{2d-\alpha},\infty}} &\le C \sup_{0<|E|<\infty} |E|^{-2+\frac{2d-\alpha}{d}} \left(\int_E \Big|\sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I_j^\theta(\chi_A,\chi_B)\Big|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \\ &\le C \sup_{0<|E|<\infty} |E|^{-\frac{\alpha}{d}} \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} \int_E I_j^\theta(\chi_A,\chi_B)^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \\ &\le C\,|A||B| \sup_{0<|E|<\infty} |E|^{-\frac{\alpha}{d}} \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} (\min\{2^{dj},|E|\})^{\frac{1}{2}}\right)^2 \\ &\le C\,|A||B|.\end{aligned}$$

Next we prove (4.13). Here we use (4.5), (4.9) with $s = 1/2$, and (4.16) as follows:

$$\begin{aligned}
\|I_\alpha^\theta(\chi_A,\chi_B)\|_{L^{1,\infty}} &\leq C \sup_{0<|E|<\infty} |E|^{-2+1} \left(\int_E \Big| \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I_j^\theta(\chi_A,\chi_B) \Big|^{\frac{1}{2}} \mathrm{d}x \right)^2 \\
&\leq C \sup_{0<|E|<\infty} |E|^{-1} \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} \int_E I_j^\theta(\chi_A,\chi_B)^{\frac{1}{2}} \mathrm{d}x \right)^2 \\
&\leq C \sup_{0<|E|<\infty} |E|^{-1}|A||E| \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} (\min\{2^{dj}, |B|\})^{\frac{1}{2}} \right)^2 \\
&\leq C\, |A||B|^{\frac{\alpha}{d}}.
\end{aligned}$$

The estimate (4.14) is based on (4.6) and is deduced similarly as the one above.

Finally, we turn to (4.15). Here we use (4.7), (4.9) with $s = 1$, and (4.17) as follows:

$$\begin{aligned}
\|I_\alpha^\theta(\chi_A,\chi_B)\|_{L^{\frac{d}{\alpha},\infty}} &\leq C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}} \int_E \Big| \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} I_j^\theta(\chi_A,\chi_B) \Big| \mathrm{d}x \\
&\leq C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}} \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} \int_E I_j^\theta(\chi_A,\chi_B)\, \mathrm{d}x \\
&\leq C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}} \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} \min\{2^{dj}|E|, |A||B|\} \\
&\leq C \sup_{0<|E|<\infty} |E|^{-1+\frac{\alpha}{d}} |E| \left(\frac{|A||B|}{|E|} \right)^{\frac{\alpha}{d}} \\
&= C\, |A|^{\frac{\alpha}{d}} |B|^{\frac{\alpha}{d}}.
\end{aligned}$$

This completes the proof of the theorem. □

4.5. **Proof of Proposition 2.3.**

First, we observe that the tensor $J_\alpha(f,g)$ is pointwise bounded by $\int_0^1 I_\alpha^\theta(f,g)\,\mathrm{d}\theta$. Indeed, for any $x \in \mathbb{R}^d$ and $v(x) \in \mathbb{R}^d \setminus \{0\}$ it holds that

$$\begin{aligned}
|J_\alpha(f,g)(x)v(x)| &= \left| \int_0^1 \int_{\mathbb{R}^d} f(x+(\theta-1)y)\, g(x+\theta y) |y|^{\alpha-d-2} (y\cdot v(x)) y \,\mathrm{d}y\,\mathrm{d}\theta \right| \\
&\leq \int_0^1 \int_{\mathbb{R}^d} f(x+(\theta-1)y)\, g(x+\theta y) |y|^{\alpha-d-2} |y\cdot v(x)||y| \,\mathrm{d}y\,\mathrm{d}\theta \\
&\leq \int_0^1 I_\alpha^\theta(f,g)(x)\,\mathrm{d}\theta |v(x)|
\end{aligned}$$

and hence

$$|J_\alpha(f,g)(x)| = \sup_{v(x)\neq 0} \frac{|J_\alpha(f,g)(x)v(x)|}{|v(x)|} \leq \int_0^1 I_\alpha^\theta(f,g)(x)\,\mathrm{d}\theta.$$

Therefore, using Jensen's inequality we deduce that

$$\|J_\alpha(f,g)\|_{L^r}^r \leq \int_{\mathbb{R}^d} \left| \int_0^1 I_\alpha^\theta(f,g)\,\mathrm{d}\theta \right|^r \mathrm{d}x \leq \int_{\mathbb{R}^d} \int_0^1 |I_\alpha^\theta(f,g)|^r \,\mathrm{d}\theta\,\mathrm{d}x.$$

Now, we use Fubini's theorem to obtain

$$\|J_\alpha(f,g)\|_{L^r}^r \leq \int_0^1 \int_{\mathbb{R}^d} |I_\alpha^\theta(f,g)|^r \,\mathrm{d}x\,\mathrm{d}\theta = \int_0^1 \|I_\alpha^\theta(f,g)\|_{L^r}^r \,\mathrm{d}\theta$$

from which the desired result follows upon applying Theorem 2.1.

5. Stability for Euler-Riesz systems

In this section, we establish a stability result for smooth solutions of an Euler-Riesz system with periodic boundary conditions, written according to identity (1.4). Two smooth solutions are compared using the relative energy functional. Using the abstract formalism developed in [7], we derive an identity that describes the time evolution of the relative energy. The right-hand side of the relative energy identity is controlled with the help of the HLS inequality and then Gronwall's lemma provides a stability result.

Stability results of this type have been obtained for similar systems of equations, where one of the considered solutions is assumed to be merely a weak or even measure-valued solution, yielding a weak-strong uniqueness or measure-valued versus strong uniqueness principle (see [1, 2, 3, 23] and references therein). The result obtained here can be phrased in the language of weak-strong stability, but we avoid doing that and we refer to [2] for details of such a formulation.

Let $T > 0$ and denote by $\mathbb{T}^d$ the d-dimensional open cube $(-1/2, 1/2)^d$. Consider the following Euler-Riesz system in $(0,T) \times \mathbb{T}^d$, expressed using the abstract functional framework developed in [7]:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot \big(\rho u \otimes u\big) + \rho \nabla \dfrac{\delta \mathcal{E}}{\delta \rho}(\rho) = 0, \\ \rho|_{t=0} = \rho_0, \ \ u|_{t=0} = u_0, \end{cases} \tag{5.1}$$

with the potential energy functional $\mathcal{E}$ defined as

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^d} h(\rho) + \kappa \tfrac{1}{2} \rho (K_\alpha * \rho) \, \mathrm{d}x$$

where $h(\rho) = \frac{1}{\gamma - 1} \rho^\gamma$ is the internal energy function, K_α is the kernel given by (1.2), and the constant κ represents the interaction strength and for this section it is allowed to take positive and negative values. The size of $|\kappa|$ will be restricted to ensure that the relative energy is nonnegative.

The density ρ and velocity u are assumed to be periodic in space with unit period.

5.1. **Relative energy identity.**

The functional derivative $\delta \mathcal{E} / \delta \rho$ is given by

$$\frac{\delta \mathcal{E}}{\delta \rho}(\rho) = h'(\rho) + \kappa K_\alpha * \rho \tag{5.2}$$

which can be computed through the formula

$$\left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho), \varphi \right\rangle = \int_{\mathbb{T}^d} \frac{\delta \mathcal{E}}{\delta \rho}(\rho) \varphi \, \mathrm{d}x :- \lim_{\delta \to 0} \frac{\mathcal{E}(\rho + \delta \varphi) - \mathcal{E}(\rho)}{\delta}$$

where φ is an arbitrary test function.

Furthermore, using (1.4), we have

$$\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho}(\rho) = \nabla \cdot R_\alpha(\rho) \tag{5.3}$$

where $R_\alpha(\rho) = p(\rho)I_d + \kappa S_\alpha(\rho)$, with $p(\rho) = \rho^\gamma$ being the pressure function. Note that the calculations in the Appendix A that lead to identity (1.4) are valid if one replaces $\mathbb{R}^d$ by $\mathbb{T}^d$ due to the symmetrical assumption on the torus.

The relative potential energy functional $\mathcal{E}(\cdot|\cdot)$ is defined as follows:

$$\begin{aligned}\mathcal{E}(\rho|\bar\rho) &= \mathcal{E}(\rho) - \mathcal{E}(\bar\rho) - \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \rho - \bar\rho \right\rangle \\ &= \int_{\mathbb{T}^d} h(\rho|\bar\rho) + \kappa \tfrac{1}{2}(\rho - \bar\rho)\big(K_\alpha * (\rho - \bar\rho)\big)\,\mathrm{d}x\end{aligned}$$

where $h(\rho|\bar\rho) = h(\rho) - h(\bar\rho) - h'(\bar\rho)(\rho - \bar\rho)$.

Next, we present the evolution of $\mathcal{E}(\rho|\bar\rho)$ over time, assuming that ρ and $\bar\rho$ evolve according to system (5.1). For the full details of the calculations involved, refer to [7]. The following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\rho|\bar\rho) = -\int_{\mathbb{T}^d} \nabla \bar u : R_\alpha(\rho|\bar\rho)\,\mathrm{d}x - \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho) - \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \nabla\cdot\big(\rho(u-\bar u)\big)\right\rangle \tag{5.4}$$

where

$$R_\alpha(\rho|\bar\rho) = p(\rho|\bar\rho)I_d + \kappa S_\alpha(\rho|\bar\rho).$$

Now, the linear velocities satisfy

$$\begin{cases}\partial_t u + (u\cdot\nabla)u + \nabla \dfrac{\delta\mathcal{E}}{\delta\rho}(\rho) = 0,\\[2ex] \partial_t \bar u + (\bar u\cdot\nabla)\bar u + \nabla \dfrac{\delta\mathcal{E}}{\delta\rho}(\bar\rho) = 0,\end{cases}$$

from which it can be deduced that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} \tfrac12 \rho|u-\bar u|^2\,\mathrm{d}x &= -\int_{\mathbb{T}^d} \nabla\bar u : \rho(u-\bar u)\otimes(u-\bar u)\,\mathrm{d}x \\ &\quad + \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho) - \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \nabla\cdot\big(\rho(u-\bar u)\big)\right\rangle.\end{aligned} \tag{5.5}$$

Combining (5.4) with (5.5) yields the relative total energy identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}(\rho|\bar\rho) + \int_{\mathbb{T}^d}\tfrac12\rho|u-\bar u|^2\,\mathrm{d}x\right) = -\int_{\mathbb{T}^d}\nabla\bar u : \big(\rho(u-\bar u)\otimes(u-\bar u) + R_\alpha(\rho|\bar\rho)\big)\,\mathrm{d}x. \tag{5.6}$$

5.2. **Stability of smooth solutions.**

Let (ρ, u) and $(\bar\rho, \bar u)$ be two smooth solutions of (5.1), and suppose additionally that for $(\bar\rho, \bar u)$ the density $\bar\rho$ is bounded away from vacuum, that is, there exist $\bar\delta > 0$ and $\bar M < \infty$ such that

$$\bar\delta \le \bar\rho(t,x) \le \bar M \quad \text{for } (t,x) \in [0,T)\times\mathbb{T}^d$$

and also

$$\nabla \bar u \in L^\infty\big(0,T; L^\infty(\mathbb{T}^d)\big).$$

The solutions (ρ, u) and $(\bar\rho, \bar u)$ satisfy the relative energy identity (5.6). Let $\Psi : [0,T) \to \mathbb{R}$ denote the relative energy between this pair of solutions,

$$\begin{aligned}\Psi(t) &= \int_{\mathbb{T}^d}\tfrac12\rho|u-\bar u|^2\,\mathrm{d}x + \mathcal{E}(\rho|\bar\rho)\\ &= \int_{\mathbb{T}^d}\tfrac12\rho|u-\bar u|^2 + h(\rho|\bar\rho) + \kappa\tfrac12(\rho-\bar\rho)\big(K_\alpha*(\rho-\bar\rho)\big)\,\mathrm{d}x.\end{aligned}$$

The objective is to prove a stability estimate connecting the behavior at time T to the initial behavior at time zero. The identity (5.6) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = &-\int_{\mathbb{T}^d} \nabla\bar{u} : \rho(u-\bar{u})\otimes(u-\bar{u})\,\mathrm{d}x - \int_{\mathbb{T}^d}(\nabla\cdot\bar{u})p(\rho|\bar{\rho})\,\mathrm{d}x \\ &- \kappa\int_{\mathbb{T}^d}\nabla\bar{u} : S_\alpha(\rho|\bar{\rho})\,\mathrm{d}x.\end{aligned} \tag{5.7}$$

To use Ψ as a yardstick for comparing the two solutions, we need to show that Ψ is nonnegative. This is based on two key ingredients. First, the HLS inequality (2.3) gives

$$\|(\rho-\bar{\rho})\big(K_\alpha * (\rho-\bar{\rho})\big)\|_{L^1(\mathbb{T}^d)} \le C_0\,\|\rho-\bar{\rho}\|^2_{L^p(\mathbb{T}^d)}$$

for some positive constant $C_0 = C_0(\alpha, d)$, where $p = 2d/(d+\alpha)$. This is improved by using interpolation and properties of the function $h(\rho|\bar{\rho})$ to show (see [23, Lemma 3.6] for the Newtonian potential and [2, Proposition 4.2] for the general case):

Lemma 5.1. *Consider the function $h(\rho) = \frac{1}{\gamma-1}\rho^\gamma$ with $\gamma \ge 2-\alpha/d$ and $0<\alpha<d$. Let $\rho \in L^\gamma(\mathbb{T}^d)$ be nonnegative, and let $\bar{\rho}\in L^\infty(\mathbb{T}^d)$ be bounded away from vacuum. Then, there exists a positive constant C_* such that*

$$\|(\rho-\bar{\rho})K_\alpha * (\rho-\bar{\rho})\|_{L^1(\mathbb{T}^d)} \le C_*\int_{\mathbb{T}^d} h(\rho|\bar{\rho})\,\mathrm{d}x. \tag{5.8}$$

Choosing κ so that $0<|\kappa|<\frac{2}{C_*}$ and setting $\lambda := 1-\frac{|\kappa|C_*}{2} > 0$, we obtain

$$\lambda\int_{\mathbb{T}^d} h(\rho|\bar{\rho})\,\mathrm{d}x \le \int_{\mathbb{T}^d} h(\rho|\bar{\rho}) + \kappa\tfrac{1}{2}(\rho-\bar{\rho})K_\alpha * (\rho-\bar{\rho})\,\mathrm{d}x$$

from which the nonnegativity of Ψ follows.

Next, we bound the terms on the right-hand side of identity (5.7) in terms of Ψ. The first term is bounded by the relative kinetic energy, and hence by Ψ. The bound for the second term is also clear as $p(\rho|\bar{\rho}) = (\gamma-1)h(\rho|\bar{\rho})$. Regarding the last term, we first observe that due to the quadratic nature of $S_\alpha(\rho)$ one has

$$S_\alpha(\rho|\bar{\rho}) = S_\alpha(\rho-\bar{\rho}).$$

Moreover, for any fixed time $t\in[0,T)$, the L^1-norm of $S_\alpha(\rho-\bar{\rho})$ is bounded by

$$\mathcal{I} := \tfrac{1}{2}\int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} |(\rho-\bar{\rho})(x+(\theta-1)y)||(\rho-\bar{\rho})(x+\theta y)||y|^{\alpha-d}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta$$

where the dependency on time is omitted for simplicity. We then estimate

$$\begin{aligned}\mathcal{I} &\le \tfrac{1}{2}\int_0^1\int_{\mathbb{T}^d}\int_{|z|<1} |(\rho-\bar{\rho})(z)||(\rho-\bar{\rho})(z+y)||y|^{\alpha-d}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\theta \\ &\le \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |(\rho-\bar{\rho})(z)|\chi_{(-2,2)^d}(z)|(\rho-\bar{\rho})(w)|\chi_{(-2,2)^d}(w)|z-w|^{\alpha-d}\,\mathrm{d}z\,\mathrm{d}w \\ &\le C(\alpha,d)\,\|(\rho-\bar{\rho})\chi_{(-2,2)^d}\|^2_{L^p(\mathbb{R}^d)}\end{aligned}$$

where $p = 2d/(d+\alpha)$, by the HLS inequality. Finally, the periodicity in space of $\rho-\bar{\rho}$ implies that

$$\|(\rho-\bar{\rho})\chi_{(-2,2)^d}\|_{L^p(\mathbb{R}^d)} = 4^d\|\rho-\bar{\rho}\|_{L^p(\mathbb{T}^d)}.$$

Hence, similarly to Lemma 5.1,

$$\begin{aligned}\int_{\mathbb{T}^d} \nabla \bar{u} : S_\alpha(\rho|\bar{\rho})\,\mathrm{d}x &\le \kappa \|\nabla \bar{u}\|_\infty \|S_\alpha(\rho-\bar{\rho})\|_{L^1(\mathbb{T}^d)} \\ &\le C\int_{\mathbb{T}^d} h(\rho|\bar{\rho})\,\mathrm{d}x \le C\Psi.\end{aligned}$$

In summary, we have obtained the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi \le C\Psi.$$

By Gronwall's lemma, for each $t \in [0,T)$, it follows that $\Psi(t) \le e^{CT}\Psi(0)$ which, together with the strict convexity of the internal energy function h, yields the desired stability result.

A weak-strong uniqueness theorem is proved in [2, Theorem 3.1] following the general approach outlined above. The method of proof differs in the treatment of the nonlocal term, achieved here via the use of the representation formula (1.4). This provides an improvement in the range of parameters α achieving the full range $0 < \alpha < d$. By contrast, the range of γ is still restricted by $\gamma \ge 2 - \alpha/d$.

Appendix A.

Here we give a formal proof that for our symmetric kernel $K_\alpha : \mathbb{R}^d \to \mathbb{R}$, with $K_\alpha(x) = \mathcal{K}_\alpha(|x|)$ it holds that

$$f\nabla K_\alpha * f = \nabla \cdot S_\alpha(f) \tag{A.1}$$

for any sufficiently smooth $f : \mathbb{R}^d \to \mathbb{R}$, where the tensor $S_\alpha(f)$ is given by

$$S_\alpha(f)(x) = -\frac{1}{2}\int_{\mathbb{R}^d}\int_0^1 \mathcal{K}_\alpha'(|y|)\,\frac{1}{|y|}\,f(x+(\theta-1)y)\,f(x+\theta y)\,y\otimes y\,\mathrm{d}\theta\,\mathrm{d}y. \tag{A.2}$$

Note that for $\mathcal{K}_\alpha(|y|) = \frac{1}{d-\alpha}|y|^{\alpha-d}$ we have $\mathcal{K}_\alpha'(|y|) = -|y|^{\alpha-d-1}$ and thus the corresponding tensor is positive semi-definite for nonnegative f.

To prove (A.1), we first deduce that

$$f\nabla K_\alpha * f(x) = -\frac{1}{2}\int_{\mathbb{R}^d} \mathcal{K}_\alpha'(y)\frac{y}{|y|}\nabla_x \cdot \int_0^1 y f(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta\,\mathrm{d}y. \tag{A.3}$$

Using the symmetry of the convolution one has:

$$\begin{aligned}(f\nabla K_\alpha * f)(x) &= f(x)\int_{\mathbb{R}^d}\mathcal{K}_\alpha'(|y|)\frac{y}{|y|}f(x-y)\,\mathrm{d}y \\ &= -f(x)\int_{\mathbb{R}^d}\mathcal{K}_\alpha'(|y|)\frac{y}{|y|}f(x+y)\,\mathrm{d}y\end{aligned}$$

where in the second equality we used the change of variables $y \to -y$.

Hence

$$f\nabla K_\alpha * f(x) = \frac{1}{2}\int_{\mathbb{R}^d}\mathcal{K}_\alpha'(|y|)\frac{y}{|y|}f(x)\big(f(x-y) - f(x+y)\big)\,\mathrm{d}y.$$

Now it is claimed that

$$f(x)\big(f(x-y) - f(x+y)\big) = -\nabla_x \cdot \int_0^1 y f(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta,$$

from which identity (A.3) follows. Indeed,

$$
\begin{aligned}
-f(x)\big(f(x-y)-f(x+y)\big) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\Big(f(x+(\theta-1)y)f(x+\theta y)\Big)\,\mathrm{d}\theta \\
&= \int_0^1 \big(\nabla f(x+(\theta-1)y)\cdot y\big)f(x+\theta y)+\big(\nabla f(x+\theta y)\cdot y\big)f(x+(\theta-1)y)\,\mathrm{d}\theta \\
&= \int_0^1 \nabla\big(f(x+(\theta-1)y)f(x+\theta y)\big)\,\mathrm{d}\theta\cdot y \\
&= \nabla_x\cdot\int_0^1 yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta
\end{aligned}
$$

as desired.

Consequently, component-wise one has:

$$
\begin{aligned}
\big(\nabla\cdot S_\alpha(f)(x)\big)_i &= -\frac{1}{2}\nabla_x\cdot\int_{\mathbb{R}^d}\int_0^1 \mathcal{K}'_\alpha(|y|)\frac{y_i}{|y|}yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta\,\mathrm{d}y \\
&= -\frac{1}{2}\int_{\mathbb{R}^d}\mathcal{K}'_\alpha(|y|)\frac{y_i}{|y|}\nabla_x\cdot\int_0^1 yf(x+(\theta-1)y)f(x+\theta y)\,\mathrm{d}\theta\,\mathrm{d}y \\
&= \big(f\nabla K_\alpha * f(x)\big)_i
\end{aligned}
$$

which establishes identity (A.1).

It is noted that for a d-dimensional cube $[-a,a]^d$ centered at the origin, with $a>0$, and periodic functions with period equal to $2a$, the formulas (A.1) and (A.2) are still valid with the integrations performed over $[-a,a]^d$.

Acknowledgments

We thank Prof. Dennis Serre for explaining the application of compensated integrability to Euler-Poisson systems that served as a starting point to this work. NJA would like to thank Dr. André Guerra for bringing his attention to bilinear integral operators. This research was partially funded by the Austrian Science Fund (FWF), project number 10.55776/F65. The research of AET was supported by KAUST baseline funds.

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