

LORENTZ ESTIMATES FOR MULTILINEAR CONVOLUTION OPERATORS

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ABSTRACT. We study O’Neil-type inequalities in Lorentz spaces for two extremal geometric configurations of translation-invariant multilinear convolution operators. For the one-parameter model, we complement the existing weak-kernel theory with estimates for kernels in the Lorentz space $L^{r,s}$, obtaining a stronger target Lorentz space when s is finite. For the full-dimensional model, we prove a Lorentz refinement of Oberlin’s multilinear Young inequality in the interior of its admissible range. For nonnegative full-dimensional kernels, we extend Hörmander’s necessary condition to the multilinear setting in Lorentz spaces and obtain an additional restriction on the secondary indices in the critical case. We also prove the failure of the full-dimensional weak L^1 endpoint, use additive combinatorics to construct one-parameter restricted weak-type counterexamples for $p < 1$, and obtain complementary positive results for kernels of special structure.

1. INTRODUCTION

This article is concerned with Lorentz norm estimates for multilinear convolution operators. Recall that for measurable functions f_1, f_2 on $\mathbb{R}^n$, the (linear) convolution $f_1 * f_2$ is a commutative operation defined pointwise by

$$(f_1 * f_2)(x) := \int_{\mathbb{R}^n} f_1(x - y) f_2(y) \, dy, \quad x \in \mathbb{R}^n,$$

provided that the integral converges absolutely for the given $x \in \mathbb{R}^n$.

The classical Young convolution inequality on $\mathbb{R}^n$ states that, for $1 \leq p_1, p_2, p \leq \infty$ satisfying the exponent relation

$$1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{1.1}$$

one has

$$\|f_1 * f_2\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \tag{1.2}$$

for all $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$. Here and throughout this paper, the L^p norm of a function f is denoted by $\|f\|_p = \left(\int_{\mathbb{R}^n} |f|^p \, dx \right)^{1/p}$.

Before stating the extension of estimate (1.2) to Lorentz spaces we recall the relevant definitions. Let f^* denote the decreasing rearrangement of a measurable function f on $\mathbb{R}^n$. For $0 < p < \infty$ and $0 < q \leq \infty$, we define the quantities

$$\|f\|_{p,q} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty. \end{cases}$$

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The Lorentz space $L^{p,q}(\mathbb{R}^n)$ consists of all measurable functions f for which $\|f\|_{p,q}$ is finite. In particular, $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, while $L^{p,\infty}(\mathbb{R}^n)$ is the usual weak Lebesgue space. We use the same notation for Lorentz quasi-norms on the other Euclidean spaces appearing in the paper. For standard properties of decreasing rearrangements and Lorentz spaces, see, for example, [8, Chapter 1].

O’Neil [20] proved that, if $1 < p_1, p_2, p < \infty$ satisfy Young’s exponent relation (1.1) and $0 < q_1, q_2, q \leq \infty$ satisfy

$$\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2},$$

then there exists a positive constant $C = C(p_1, p_2, q_1, q_2, q)$ such that

$$\|f_1 * f_2\|_{p,q} \leq C \|f_1\|_{p_1,q_1} \|f_2\|_{p_2,q_2}$$

for all $f_1 \in L^{p_1,q_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2,q_2}(\mathbb{R}^n)$. Further developments of convolution theory in Lorentz spaces include the work of Yap [23] and Blozinski [3, 4]; the latter also established the sharpness of the admissible range of indices. Nursultanov and Tikhonov [17] studied endpoint and weighted Lorentz estimates, while Nursultanov, Tikhonov, and Tleukhanova [18] obtained sharper estimates for convolution operator norms in Lebesgue spaces. These works form part of the linear background for the multilinear theory developed here.

In this work, we establish O’Neil-type Lorentz estimates for two classes of translation-invariant multilinear convolution operators. The first is the one-parameter model. Let $\alpha_1, \dots, \alpha_m$ be nonzero pairwise distinct real numbers. For measurable functions $f_1, \dots, f_m, g$ on $\mathbb{R}^n$, we define

$$T_g(f_1, \dots, f_m)(x) := \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(x - \alpha_i y) \right) g(y) \, dy, \quad (1.3)$$

assuming that the integral converges absolutely for the given $x \in \mathbb{R}^n$. Guliyev and Nazirova [14] established a weak-kernel analogue of Young’s inequality for this operator. More precisely, they proved that, if $1 < p_1, \dots, p_m, r < \infty$ and $1 \leq p < \infty$ satisfy

$$\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{r} - 1,$$

then there exists a positive constant $C = C(n, m, \alpha_1, \dots, \alpha_m, p_1, \dots, p_m, r)$ such that

$$\|T_g(f_1, \dots, f_m)\|_p \leq C \left(\prod_{i=1}^m \|f_i\|_{p_i} \right) \|g\|_{r,\infty},$$

for all $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, and all $g \in L^{r,\infty}(\mathbb{R}^n)$. Lorentz estimates for weak kernels were subsequently obtained by Guliyev, Ekinoglu, and Nazirova [13]. Our first main result complements this theory by capturing the contribution of a finite kernel Lorentz index to the target space; see Theorem 2.1.

The second is the full-dimensional model. If K is a measurable function on $(\mathbb{R}^n)^m$ and $f_1, \dots, f_m$ are measurable functions on $\mathbb{R}^n$, we define

$$T^K(f_1, \dots, f_m)(x) := \int_{(\mathbb{R}^n)^m} \left(\prod_{i=1}^m f_i(x - y_i) \right) K(y_1, \dots, y_m) \, dy_1 \cdots dy_m, \quad (1.4)$$

again under the assumption that the integral converges absolutely. Oberlin [19] established a multilinear version of Young’s convolution inequality for (1.4). More precisely,

Oberlin [19] showed that if $1 \leq p_1, \dots, p_m, r, p \leq \infty$ satisfy

$$m + \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i} + \frac{m}{r},$$

together with

$$\frac{1}{p} \leq \frac{1}{r} \quad \text{and} \quad 1 - \frac{1}{r} \leq \frac{1}{p_i}, \quad i = 1, \dots, m,$$

then there exists a positive constant $C = C(n, m, p_1, \dots, p_m, r, p)$ such that

$$\|T^K(f_1, \dots, f_m)\|_p \leq C \left(\prod_{i=1}^m \|f_i\|_{p_i} \right) \|K\|_r \quad (1.5)$$

for all $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, and all $K \in L^r((\mathbb{R}^n)^m)$.

Our second main result gives a Lorentz refinement of Oberlin's estimate (1.5) in the interior of this admissible range; see Theorem 2.2.

As the estimates considered here measure the size of K or g , we may as well regard these operators as $(m+1)$ -linear mappings in the input functions and the kernel. Moreover,

$$|T_g(f_1, \dots, f_m)| \leq T_{|g|}(|f_1|, \dots, |f_m|),$$

and

$$|T^K(f_1, \dots, f_m)| \leq T^{|K|}(|f_1|, \dots, |f_m|).$$

We therefore use without further mention the standard reduction to nonnegative functions and kernels. Note that for nonnegative functions the integrals in (1.3) and (1.4) are well defined with values in $[0, \infty]$.

Both models fit into a common measure-theoretic framework. If μ is a (nonnegative) regular Borel measure on $(\mathbb{R}^n)^m$, we consider the translation-invariant m -linear operator

$$T^\mu(f_1, \dots, f_m)(x) := \int_{(\mathbb{R}^n)^m} \left(\prod_{i=1}^m f_i(x - y_i) \right) d\mu(y_1, \dots, y_m),$$

defined for nonnegative measurable functions f_i . Operators of this form have been studied in several settings. The bilinear case with positive kernels was considered in [12], while restricted convolution inequalities and related multilinear operators associated with measures were studied in [7].

The full-dimensional operator (1.4) corresponds to the absolutely continuous measure μ_K defined by

$$d\mu_K(y_1, \dots, y_m) = K(y_1, \dots, y_m) dy_1 \cdots dy_m.$$

Thus

$$T^K = T^{\mu_K}.$$

For the one-parameter operator (1.3), define the n -dimensional subspace

$$\Gamma := \{(\alpha_1 y, \dots, \alpha_m y) : y \in \mathbb{R}^n\} \subseteq (\mathbb{R}^n)^m,$$

and let μ_g be the pushforward of the measure $g(y) dy$ under the map

$$y \mapsto (\alpha_1 y, \dots, \alpha_m y).$$

Then μ_g is supported on Γ , and

$$T_g = T^{\mu_g}.$$

Thus the full-dimensional model corresponds to absolutely continuous kernel measures, whereas the one-parameter model corresponds to kernel measures supported on

the lower-dimensional subspace Γ . The different dimensions of the two kernel domains lead to the different scaling relations appearing in the main theorems.

More generally, one may consider kernel measures obtained by pushing forward functions on $\mathbb{R}^{kn}$, where $1 \leq k \leq m$. This amounts to choosing linear maps

$$L_i : \mathbb{R}^{kn} \rightarrow \mathbb{R}^n, \quad i = 1, \dots, m,$$

and considering operators of the form

$$T_h^L(f_1, \dots, f_m)(x) := \int_{\mathbb{R}^{kn}} \prod_{i=1}^m f_i(x - L_i u) h(u) du,$$

where $L = (L_1, \dots, L_m)$. The corresponding kernel measure is supported on the image of the map

$$u \mapsto (L_1 u, \dots, L_m u)$$

from $\mathbb{R}^{kn}$ into $(\mathbb{R}^n)^m$. The full-dimensional operator corresponds to $k = m$, which is seen by writing $u = (y_i)_{i=1}^m$ and taking $L_i u = y_i$. The one-parameter operator corresponds to $k = 1$, with $u = y$ and $L_i u = \alpha_i y$. Unlike the two cases treated here, the intermediate range does not single out a canonical choice of these maps. We therefore leave its systematic study for future work.

2. STATEMENT OF MAIN RESULTS

Guliyev and Nazirova [14] obtained rearrangement inequalities and Lebesgue estimates for multilinear convolutions of the form (1.3), while Guliyev, Ekinoglu, and Nazirova [13] established Lorentz estimates for kernels in $L^{r,\infty}(\mathbb{R}^n)$. Within the common range of primary exponents, their result is recovered from the case $s = \infty$ below, together with the standard Lorentz embedding. For finite s , our theorem transfers the additional Lorentz summability of the kernel to the target, yielding a smaller secondary index and hence a stronger target Lorentz space. Our first main result concerns the one-parameter model and is as follows:

Theorem 2.1. *Let $\alpha_1, \dots, \alpha_m$ be nonzero pairwise distinct real numbers. Let $1 < p_1, \dots, p_m, r < \infty$ satisfy*

$$1 < \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{r} < 2, \quad (2.1)$$

and define $1 < p < \infty$ by

$$\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{r} - 1. \quad (2.2)$$

Let $0 < q_1, \dots, q_m, s \leq \infty$, and define $0 < q \leq \infty$ by

$$\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} + \frac{1}{s}. \quad (2.3)$$

Then there exists a positive constant

$$C = C(n, m, \alpha_1, \dots, \alpha_m, p_1, \dots, p_m, r, q_1, \dots, q_m, s)$$

such that

$$\|T_g(f_1, \dots, f_m)\|_{p,q} \leq C \left(\prod_{i=1}^m \|f_i\|_{p_i, q_i} \right) \|g\|_{r,s} \quad (2.4)$$

for all $f_i \in L^{p_i, q_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, and all $g \in L^{r,s}(\mathbb{R}^n)$.

The theorem of Guliyev, Ekinoglu, and Nazirova [13] also reaches the weak-kernel endpoint $p = 1$, under the corresponding condition on the input Lorentz indices, whereas Theorem 2.1 is restricted to $1 < p < \infty$. Thus the two results are complementary: their theorem treats this endpoint for kernels in $L^{r,\infty}(\mathbb{R}^n)$, while Theorem 2.1 captures the effect of a finite kernel Lorentz index in the open range.

Our second main result concerns the full-dimensional model. It gives a Lorentz refinement of Oberlin's multilinear Young inequality in the interior of its admissible range.

Theorem 2.2. *Let $1 < p_1, \dots, p_m, r, p < \infty$ satisfy*

$$m + \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i} + \frac{m}{r}. \quad (2.5)$$

Assume also that

$$\frac{1}{p} < \frac{1}{r} \quad \text{and} \quad 1 - \frac{1}{r} < \frac{1}{p_i}, \quad i = 1, \dots, m. \quad (2.6)$$

Let $0 < q_1, \dots, q_m, s \leq \infty$, and define $0 < q \leq \infty$ by

$$\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} + \frac{1}{s}. \quad (2.7)$$

Then there exists a positive constant

$$C = C(n, m, p_1, \dots, p_m, r, q_1, \dots, q_m, s)$$

such that

$$\|T^K(f_1, \dots, f_m)\|_{p,q} \leq C \left(\prod_{i=1}^m \|f_i\|_{p_i, q_i} \right) \|K\|_{r,s} \quad (2.8)$$

for all $f_i \in L^{p_i, q_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, and all $K \in L^{r,s}((\mathbb{R}^n)^m)$.

We note that, if q_0 is defined by

$$\frac{1}{q_0} = \sum_{i=1}^m \frac{1}{q_i} + \frac{1}{s},$$

then, by the standard inclusion

$$L^{p, q_0}(\mathbb{R}^n) \subseteq L^{p, q}(\mathbb{R}^n), \quad q_0 \leq q,$$

the conclusions of both theorems remain valid under the weaker condition

$$\frac{1}{q} \leq \sum_{i=1}^m \frac{1}{q_i} + \frac{1}{s}.$$

In this case, the constant may also depend on q .

The proofs of Theorems 2.1 and 2.2 are based on an interpolation argument, which proceeds in two steps. First, it employs restricted weak-type estimates at the vertices of a local box around the target exponent tuple. For the one-parameter model, these estimates follow from the weak-kernel theorem of Guliyev and Nazirova [14, Theorem 4.1]; for the full-dimensional model, they follow from Oberlin's multilinear Young inequality [19]. Viewing the kernel as an additional input, we then apply multilinear Marcinkiewicz interpolation to obtain the stated Lorentz bounds; see Proposition 3.3. For the underlying interpolation theory, see, among others, [22, 10, 5, 11].

We next turn to limitations of the full-dimensional theory. In the multilinear case $m \geq 2$, at the endpoint $p = r = 1$, the strong L^1 estimate for integrable kernels does not admit a weak- L^1 counterpart with both the kernel and the target in $L^{1,\infty}$; see Proposition 6.1. A more structural obstruction arises from translation invariance itself. Hörmander proved that a nonzero translation-invariant linear operator bounded from L^{p_1} to L^p must satisfy $p_1 \leq p$; see [15, Theorem 1.1]. A bilinear analogue appears in the work of Rodríguez-López [21], and the corresponding multilinear statement in Lebesgue spaces is also known; see [9].

Our third main result extends this obstruction to the multilinear Lorentz setting for nonnegative full-dimensional kernels. In the following result, we recover the necessary condition on the primary exponents and, in the critical case, obtain an additional necessary condition on the secondary Lorentz indices.

Theorem 2.3. *Let $0 < p_1, \dots, p_m, p < \infty$ and $0 < q_1, \dots, q_m, q \leq \infty$. Let K be a nonnegative measurable function on $(\mathbb{R}^n)^m$ that is nonzero on a set of positive measure. If there exists a positive constant C_K such that*

$$\|T^K(f_1, \dots, f_m)\|_{p,q} \leq C_K \prod_{i=1}^m \|f_i\|_{p_i, q_i} \quad (2.9)$$

for all $f_i \in L^{p_i, q_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, then we must have

$$\frac{1}{p} \leq \sum_{i=1}^m \frac{1}{p_i}. \quad (2.10)$$

Moreover, if equality holds in (2.10), then necessarily

$$\frac{1}{q} \leq \sum_{i=1}^m \frac{1}{q_i}. \quad (2.11)$$

Equivalently, for a nonnegative measurable kernel K , boundedness as in (2.9) forces $K = 0$ almost everywhere whenever either

$$\sum_{i=1}^m \frac{1}{p_i} < \frac{1}{p},$$

or

$$\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p} \quad \text{and} \quad \sum_{i=1}^m \frac{1}{q_i} < \frac{1}{q}.$$

Although conclusions of the type (2.10) were first obtained by Hörmander [15] for non-necessarily positive kernels, in this paper we focus on positive operators and for this reason we work with positive kernels. Our extension of Hörmander's theorem to Lorentz spaces presented here is based on an approach that is different from those in [15] and in [21].

We now return to the one-parameter model, where Theorem 2.1 concerns target exponents $p > 1$. Our fourth main result shows that a natural scale-invariant family of restricted weak-type estimates fails throughout the range $p < 1$, uniformly over measurable kernel sets and already in the one-dimensional bilinear case. The proof uses an additive-combinatorial construction in which an imbalance between sumset and difference-set cardinalities is amplified by passing to Cartesian powers and then transferred to measurable sets.

To state the result, we specialize (1.3) to $n = 1$, $m = 2$, $\alpha_1 = -1$, and $\alpha_2 = 1$, and retain the notation T_g :

$$T_g(f_1, f_2)(x) = \int_{\mathbb{R}} f_1(x + y) f_2(x - y) g(y) dy.$$

Theorem 2.4. *Let $\beta > 0$. There is no constant $C > 0$ such that*

$$\|T_{\chi_A}(\chi_{E_1}, \chi_{E_2})\|_{1/(1+\beta), \infty} \leq C |E_1| |E_2| |A|^\beta \quad (2.12)$$

for all measurable sets $E_1, E_2, A \subseteq \mathbb{R}$ of finite measure.

Thus, for every $\beta > 0$, the associated trilinear mapping is not of restricted weak type

$$\left(1, 1, \frac{1}{\beta}; \frac{1}{1+\beta}\right).$$

In particular, taking $\beta = 1$ shows that

$$\|T_{\chi_A}(\chi_{E_1}, \chi_{E_2})\|_{1/2, \infty} \leq C |E_1| |E_2| |A|$$

cannot hold with a constant independent of the measurable sets $E_1, E_2, A \subseteq \mathbb{R}$ of finite measure. Related questions for one-parameter trilinear operators were studied by Christ [6].

This uniform failure over arbitrary kernel sets does not preclude estimates for fixed kernels or for classes with additional structure. In the final part of the paper, we prove an $L^1 \times L^1 \rightarrow L^{1/2}$ estimate for cube kernels and obtain estimates, including some with target exponent $p < 1$, for radially decreasing kernels.

Outline. Section 3 establishes the restricted weak-type estimates used in the proofs of the two Lorentz bounds. The section concludes with the multilinear interpolation proposition that converts a local family of such estimates into a Lorentz bound.

Sections 4 and 5 prove Theorems 2.1 and 2.2, respectively. In each case, we construct a local box of admissible exponents and apply the interpolation result from Section 3.

Section 6 concerns limitations of the full-dimensional theory. It proves the failure of the weak L^1 endpoint and establishes the Hörmander-type necessary conditions in Theorem 2.3. Section 7 proves Theorem 2.4 by means of an additive-combinatorial construction. The final part of that section gives complementary positive results for kernels with additional structure, including a strong endpoint estimate for cube kernels and bounds with target exponent below 1 for radially decreasing kernels.

3. RESTRICTED WEAK-TYPE ESTIMATES AND MULTILINEAR INTERPOLATION

In this section we establish the restricted weak-type estimates used in the proofs of Theorem 2.1 and Theorem 2.2.

3.1. One-parameter estimates. The restricted weak-type estimates needed for the one-parameter model are consequences of the weak-kernel estimate of Guliyev and Nazirova [14, Theorem 4.1].

Lemma 3.1. *Let $\alpha_1, \dots, \alpha_m$ be nonzero pairwise distinct real numbers. Let $0 < \gamma_1, \dots, \gamma_m, \beta < 1$ satisfy*

$$1 < \sum_{i=1}^m \gamma_i + \beta \leq 2.$$

Define $1 \leq p_0 < \infty$ by

$$\frac{1}{p_0} = \sum_{i=1}^m \gamma_i + \beta - 1.$$

Then there exists a positive constant $C = C(n, m, \alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_m, \beta)$ such that

$$\|T_{\chi_A}(\chi_{E_1}, \dots, \chi_{E_m})\|_{p_0, \infty} \leq C \left(\prod_{i=1}^m |E_i|^{\gamma_i} \right) |A|^\beta \quad (3.1)$$

for all measurable sets $E_1, \dots, E_m, A \subseteq \mathbb{R}^n$ of finite measure.

Proof. Set

$$p_i := \frac{1}{\gamma_i}, \quad i = 1, \dots, m, \quad \text{and} \quad r := \frac{1}{\beta}.$$

Since $0 < \gamma_1, \dots, \gamma_m, \beta < 1$, we have

$$1 < p_1, \dots, p_m, r < \infty.$$

Moreover, by the definition of p_0 ,

$$\frac{1}{p_0} = \sum_{i=1}^m \gamma_i + \beta - 1 = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{r} - 1.$$

Since $1 \leq p_0 < \infty$, the weak-kernel estimate of Guliyev and Nazirova [14, Theorem 4.1] gives

$$\|T_{\chi_A}(\chi_{E_1}, \dots, \chi_{E_m})\|_{p_0} \leq C \left(\prod_{i=1}^m \|\chi_{E_i}\|_{p_i} \right) \|\chi_A\|_{r, \infty},$$

for all measurable sets $E_1, \dots, E_m, A \subseteq \mathbb{R}^n$ of finite measure. This implies (3.1). $\square$

3.2. Full-dimensional estimates. The corresponding restricted weak-type estimates for the full-dimensional model are based on Oberlin's multilinear Young inequality [19].

Lemma 3.2. *Let $0 < \gamma_1, \dots, \gamma_m, \beta < 1$ satisfy*

$$0 < \sum_{i=1}^m \gamma_i + m\beta - m < 1.$$

Define $1 < p_0 < \infty$ by

$$\frac{1}{p_0} = \sum_{i=1}^m \gamma_i + m\beta - m.$$

Assume also that

$$\frac{1}{p_0} < \beta, \quad \text{and} \quad 1 - \beta < \gamma_i, \quad i = 1, \dots, m.$$

Then there exists a positive constant $C = C(n, m, \gamma_1, \dots, \gamma_m, \beta)$ such that

$$\|T^{\chi_\Omega}(\chi_{E_1}, \dots, \chi_{E_m})\|_{p_0, \infty} \leq C \left(\prod_{i=1}^m |E_i|^{\gamma_i} \right) |\Omega|^\beta \quad (3.2)$$

for all measurable sets $E_1, \dots, E_m \subseteq \mathbb{R}^n$ and $\Omega \subseteq (\mathbb{R}^n)^m$ of finite measure.

Proof. Set

$$p_i := \frac{1}{\gamma_i}, \quad i = 1, \dots, m, \quad \text{and} \quad r := \frac{1}{\beta}.$$

Since $0 < \gamma_1, \dots, \gamma_m, \beta < 1$, we have

$$1 < p_1, \dots, p_m, r < \infty.$$

Moreover,

$$m + \frac{1}{p_0} = \sum_{i=1}^m \gamma_i + m\beta = \sum_{i=1}^m \frac{1}{p_i} + \frac{m}{r}.$$

The remaining assumptions give

$$\frac{1}{p_0} < \beta = \frac{1}{r},$$

and

$$1 - \frac{1}{r} = 1 - \beta < \gamma_i = \frac{1}{p_i}, \quad i = 1, \dots, m.$$

Thus Oberlin's multilinear Young inequality [19] applies and gives

$$\|T^{\chi_\Omega}(\chi_{E_1}, \dots, \chi_{E_m})\|_{p_0} \leq C \left(\prod_{i=1}^m \|\chi_{E_i}\|_{p_i} \right) \|\chi_\Omega\|_r,$$

for all measurable sets $E_1, \dots, E_m \subseteq \mathbb{R}^n$ and $\Omega \subseteq (\mathbb{R}^n)^m$ of finite measure, which implies (3.2). $\square$

3.3. Multilinear interpolation. The following proposition is a consequence of multilinear Marcinkiewicz interpolation; see [22, 10, 11, 9]. The parameter κ specifies the dimension κn of the kernel domain, with $\kappa = 1$ for the one-parameter model and $\kappa = m$ for the full-dimensional model.

Proposition 3.3. *Fix one of the following two cases: $\kappa = 1$ and $\mathcal{T}_h = T_h$, or $\kappa = m$ and $\mathcal{T}_h = T^h$. Let $0 < a_1, \dots, a_m, b < 1$ satisfy*

$$0 < \sum_{i=1}^m a_i + \kappa b - \kappa < 1,$$

and define $1 < p < \infty$ by

$$\frac{1}{p} = \sum_{i=1}^m a_i + \kappa b - \kappa.$$

Assume that there exist numbers a_i^-, a_i^+ with

$$0 < a_i^- < a_i < a_i^+ < 1, \quad i = 1, \dots, m,$$

and b_-, b_+ with

$$0 < b_- < b < b_+ < 1$$

such that, for every sign vector

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_0) \in \{-, +\}^{m+1},$$

the number p_ε defined by

$$\frac{1}{p_\varepsilon} = \sum_{i=1}^m a_i^{\varepsilon_i} + \kappa b_{\varepsilon_0} - \kappa$$

satisfies $1 < p_\varepsilon < \infty$, and the restricted weak-type estimate

$$\|\mathcal{T}_{\chi_A}(\chi_{E_1}, \dots, \chi_{E_m})\|_{p_\varepsilon, \infty} \leq C_0 \left(\prod_{i=1}^m |E_i|^{a_i^{\varepsilon_i}} \right) |A|^{b_{\varepsilon_0}} \quad (3.3)$$

holds for all measurable sets $E_1, \dots, E_m \subseteq \mathbb{R}^n$ and $A \subseteq (\mathbb{R}^n)^\kappa$ of finite measure, where C_0 is independent of ε and of the sets. Let $0 < q_1, \dots, q_m, s \leq \infty$ and define $0 < q \leq \infty$ by

$$\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} + \frac{1}{s}.$$

Then there exists a positive constant C such that

$$\|\mathcal{T}_h(f_1, \dots, f_m)\|_{p, q} \leq C \left(\prod_{i=1}^m \|f_i\|_{1/a_i, q_i} \right) \|h\|_{1/b, s} \quad (3.4)$$

for all $f_i \in L^{1/a_i, q_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, and all $h \in L^{1/b, s}((\mathbb{R}^n)^\kappa)$.

Proof. We verify that the exponent tuple in the conclusion lies in the interior of the convex hull of the tuples occurring in the restricted weak-type estimates.

Set

$$\theta_i := \frac{a_i - a_i^-}{a_i^+ - a_i^-}, \quad i = 1, \dots, m, \quad \text{and} \quad \theta_0 := \frac{b - b_-}{b_+ - b_-}.$$

Since

$$a_i^- < a_i < a_i^+, \quad i = 1, \dots, m,$$

and

$$b_- < b < b_+,$$

we have

$$0 < \theta_i < 1, \quad i = 0, 1, \dots, m.$$

Moreover,

$$a_i = (1 - \theta_i)a_i^- + \theta_i a_i^+, \quad i = 1, \dots, m,$$

and

$$b = (1 - \theta_0)b_- + \theta_0 b_+.$$

For each $i = 0, \dots, m$, define

$$w_i(-) := 1 - \theta_i, \quad w_i(+) := \theta_i.$$

For

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_0) \in \{-, +\}^{m+1},$$

set

$$\lambda_\varepsilon := \prod_{i=0}^m w_i(\varepsilon_i) > 0.$$

We have

$$\begin{aligned} \sum_{\varepsilon \in \{-, +\}^{m+1}} \lambda_\varepsilon &= \prod_{i=0}^m \left(\sum_{\sigma \in \{-, +\}} w_i(\sigma) \right) \\ &= \prod_{i=0}^m ((1 - \theta_i) + \theta_i) \\ &= 1. \end{aligned}$$

Moreover, for each $i = 1, \dots, m$,

$$\begin{aligned} \sum_{\varepsilon \in \{-, +\}^{m+1}} \lambda_\varepsilon a_i^{\varepsilon_i} &= \left(\sum_{\sigma \in \{-, +\}} w_i(\sigma) a_i^\sigma \right) \prod_{\substack{0 \leq j \leq m \\ j \neq i}} \left(\sum_{\sigma \in \{-, +\}} w_j(\sigma) \right) \\ &= (1 - \theta_i) a_i^- + \theta_i a_i^+ \\ &= a_i. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{\varepsilon \in \{-, +\}^{m+1}} \lambda_\varepsilon b_{\varepsilon_0} &= \left(\sum_{\sigma \in \{-, +\}} w_0(\sigma) b_\sigma \right) \prod_{j=1}^m \left(\sum_{\sigma \in \{-, +\}} w_j(\sigma) \right) \\ &= (1 - \theta_0) b_- + \theta_0 b_+ \\ &= b. \end{aligned}$$

Now, for every $\varepsilon \in \{-, +\}^{m+1}$ we define

$$\mathbf{v}_\varepsilon := \left(a_1^{\varepsilon_1}, \dots, a_m^{\varepsilon_m}, b_{\varepsilon_0}, \frac{1}{p_\varepsilon} \right),$$

and

$$\mathbf{v} := \left(a_1, \dots, a_m, b, \frac{1}{p} \right).$$

Since

$$\frac{1}{p_\varepsilon} = \sum_{i=1}^m a_i^{\varepsilon_i} + \kappa b_{\varepsilon_0} - \kappa,$$

we obtain

$$\begin{aligned} \sum_{\varepsilon \in \{-, +\}^{m+1}} \lambda_\varepsilon \frac{1}{p_\varepsilon} &= \sum_{\varepsilon \in \{-, +\}^{m+1}} \lambda_\varepsilon \left(\sum_{i=1}^m a_i^{\varepsilon_i} + \kappa b_{\varepsilon_0} - \kappa \right) \\ &= \sum_{i=1}^m a_i + \kappa b - \kappa \\ &= \frac{1}{p}. \end{aligned}$$

Consequently,

$$\mathbf{v} = \sum_{\varepsilon \in \{-, +\}^{m+1}} \lambda_\varepsilon \mathbf{v}_\varepsilon.$$

This proves that $\mathbf{v}$ is a convex combination of the vectors $\mathbf{v}_\varepsilon$, $\varepsilon \in \{-, +\}^{m+1}$. The result follows. $\square$

4. LORENTZ ESTIMATES FOR THE ONE-PARAMETER MODEL

In this section we prove Theorem 2.1. We first verify the local-box condition required by Proposition 3.3 with $\kappa = 1$, and then combine the resulting local box with the restricted weak-type estimates established in Section 3.

4.1. A local box of exponents.

Lemma 4.1. *Let $0 < a_1, \dots, a_m, b < 1$ and suppose that*

$$1 < \sum_{i=1}^m a_i + b < 2.$$

Then there exist numbers a_i^- and a_i^+ with

$$0 < a_i^- < a_i < a_i^+ < 1, \quad i = 1, \dots, m,$$

and also numbers b_-, b_+ with

$$0 < b_- < b < b_+ < 1,$$

such that, for every sign vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_0) \in \{-, +\}^{m+1}$, one has

$$1 < \sum_{i=1}^m a_i^{\varepsilon_i} + b_{\varepsilon_0} < 2. \quad (4.1)$$

Proof. Set

$$\Sigma := \sum_{i=1}^m a_i + b.$$

By assumption, $1 < \Sigma < 2$. Choose $\delta > 0$ so small that

$$\delta < \frac{1}{2} \min \left\{ a_1, \dots, a_m, b, 1 - a_1, \dots, 1 - a_m, 1 - b, \frac{\Sigma - 1}{m + 1}, \frac{2 - \Sigma}{m + 1} \right\}.$$

Then set

$$a_i^- = a_i - \delta, \quad a_i^+ = a_i + \delta, \quad i = 1, \dots, m,$$

and

$$b_- = b - \delta, \quad b_+ = b + \delta.$$

The inequalities in the statement follow from the choice of δ . Moreover, for every $\varepsilon \in \{-, +\}^{m+1}$,

$$\Sigma - (m + 1)\delta \leq \sum_{i=1}^m a_i^{\varepsilon_i} + b_{\varepsilon_0} \leq \Sigma + (m + 1)\delta.$$

Since $\Sigma - (m + 1)\delta > 1$ and $\Sigma + (m + 1)\delta < 2$, this proves (4.1). $\square$

4.2. Proof of Theorem 2.1. We apply Proposition 3.3 with

$$\mathcal{T}_h = T_h \quad \text{and} \quad \kappa = 1.$$

Set

$$a_i := \frac{1}{p_i}, \quad i = 1, \dots, m, \quad b := \frac{1}{r},$$

and take $\kappa = 1$. By (2.2),

$$\frac{1}{p} = \sum_{i=1}^m a_i + b - 1,$$

while (2.1) gives

$$1 < \sum_{i=1}^m a_i + b < 2.$$

Hence Lemma 4.1 provides numbers a_i^-, a_i^+ with

$$0 < a_i^- < a_i < a_i^+ < 1, \quad i = 1, \dots, m,$$

and numbers b_- , b_+ with

$$0 < b_- < b < b_+ < 1,$$

such that, for every

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_0) \in \{-, +\}^{m+1},$$

the number p_ε defined by

$$\frac{1}{p_\varepsilon} = \sum_{i=1}^m a_i^{\varepsilon_i} + b_{\varepsilon_0} - 1$$

satisfies

$$1 < p_\varepsilon < \infty.$$

Applying Lemma 3.1 with

$$\gamma_i = a_i^{\varepsilon_i}, \quad i = 1, \dots, m, \quad \beta = b_{\varepsilon_0},$$

we obtain

$$\|T_{\chi_A}(\chi_{E_1}, \dots, \chi_{E_m})\|_{p_\varepsilon, \infty} \leq C_\varepsilon \left(\prod_{i=1}^m |E_i|^{a_i^{\varepsilon_i}} \right) |A|^{b_{\varepsilon_0}}$$

for all measurable sets $E_1, \dots, E_m, A \subseteq \mathbb{R}^n$ of finite measure. Since there are only finitely many sign vectors ε , the constants C_ε may be replaced by a single constant independent of ε .

All the hypotheses of Proposition 3.3 are therefore satisfied with $\kappa = 1$. Using also (2.3), we conclude that

$$\|T_g(f_1, \dots, f_m)\|_{p, q} \leq C \left(\prod_{i=1}^m \|f_i\|_{1/a_i, q_i} \right) \|g\|_{1/b, s}.$$

Since

$$\frac{1}{a_i} = p_i, \quad i = 1, \dots, m, \quad \text{and} \quad \frac{1}{b} = r,$$

this is precisely (2.4). □

5. LORENTZ ESTIMATES FOR THE FULL-DIMENSIONAL MODEL

In this section we prove Theorem 2.2. We first verify the local-box condition required by Proposition 3.3 with $\kappa = m$, and then combine the resulting local box with the restricted weak-type estimates established in Section 3.

5.1. A local box of exponents.

Lemma 5.1. *Let $0 < a_1, \dots, a_m, b < 1$ and suppose that*

$$0 < \sum_{i=1}^m a_i + mb - m < 1, \quad \sum_{i=1}^m a_i + mb - m < b,$$

and

$$1 - b < a_i, \quad i = 1, \dots, m.$$

Then there exist numbers a_i^- , a_i^+ with

$$0 < a_i^- < a_i < a_i^+ < 1, \quad i = 1, \dots, m,$$

and numbers b_- , b_+ with

$$0 < b_- < b < b_+ < 1,$$

such that, for every sign vector

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_0) \in \{-, +\}^{m+1},$$

the quantity

$$\rho_\varepsilon := \sum_{i=1}^m a_i^{\varepsilon_i} + mb_{\varepsilon_0} - m$$

satisfies

$$0 < \rho_\varepsilon < 1, \quad \rho_\varepsilon < b_{\varepsilon_0}, \quad \text{and} \quad 1 - b_{\varepsilon_0} < a_i^{\varepsilon_i}, \quad i = 1, \dots, m.$$

Proof. Set

$$\rho := \sum_{i=1}^m a_i + mb - m.$$

By assumption,

$$0 < \rho < 1, \quad \rho < b, \quad a_i + b - 1 > 0, \quad i = 1, \dots, m.$$

Choose $\delta > 0$ so small that

$$\begin{aligned} \delta &< \min \left\{ \min_{1 \leq i \leq m} a_i, \min_{1 \leq i \leq m} (1 - a_i), b, 1 - b \right\}, \\ 2m\delta &< \min\{\rho, 1 - \rho\}, \\ (2m - 1)\delta &< b - \rho, \end{aligned}$$

and

$$2\delta < \min_{1 \leq i \leq m} (a_i + b - 1).$$

Define

$$a_i^- := a_i - \delta, \quad a_i^+ := a_i + \delta, \quad i = 1, \dots, m,$$

and

$$b_- := b - \delta, \quad b_+ := b + \delta.$$

The first condition on δ gives

$$0 < a_i^- < a_i < a_i^+ < 1, \quad i = 1, \dots, m,$$

and

$$0 < b_- < b < b_+ < 1.$$

Now fix

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_0) \in \{-, +\}^{m+1}.$$

Since each $a_i^{\varepsilon_i}$ differs from a_i by at most δ , and b_{ε_0} differs from b by at most δ , we have

$$|\rho_\varepsilon - \rho| = \left| \sum_{i=1}^m (a_i^{\varepsilon_i} - a_i) + m(b_{\varepsilon_0} - b) \right| \leq 2m\delta.$$

Using the second condition on δ we obtain

$$\rho_\varepsilon \geq \rho - 2m\delta > 0$$

and

$$\rho_\varepsilon \leq \rho + 2m\delta < 1.$$

Thus

$$0 < \rho_\varepsilon < 1.$$

Next we prove that $\rho_\varepsilon < b_{\varepsilon_0}$. Note that

$$b - \rho = b - \left(\sum_{i=1}^m a_i + mb - m \right) > 0.$$

Moreover,

$$\begin{aligned} |(b_{\varepsilon_0} - \rho_\varepsilon) - (b - \rho)| &= \left| -\sum_{i=1}^m (a_i^{\varepsilon_i} - a_i) - (m-1)(b_{\varepsilon_0} - b) \right| \\ &\leq m\delta + (m-1)\delta = (2m-1)\delta. \end{aligned}$$

Using the third condition on δ we get

$$b_{\varepsilon_0} - \rho_\varepsilon \geq b - \rho - (2m-1)\delta > 0.$$

Hence

$$\rho_\varepsilon < b_{\varepsilon_0}.$$

Finally, for each $i = 1, \dots, m$, we have

$$a_i^{\varepsilon_i} + b_{\varepsilon_0} - 1 = a_i + b - 1 + (a_i^{\varepsilon_i} - a_i) + (b_{\varepsilon_0} - b).$$

Therefore

$$a_i^{\varepsilon_i} + b_{\varepsilon_0} - 1 \geq a_i + b - 1 - 2\delta.$$

Using the fourth condition on δ we obtain

$$a_i^{\varepsilon_i} + b_{\varepsilon_0} - 1 > 0,$$

or equivalently,

$$1 - b_{\varepsilon_0} < a_i^{\varepsilon_i}, \quad i = 1, \dots, m.$$

This proves all the required inequalities. $\square$

5.2. Proof of Theorem 2.2. We apply Proposition 3.3 with

$$\mathcal{T}_h = T^h \quad \text{and} \quad \kappa = m.$$

Set

$$a_i := \frac{1}{p_i}, \quad i = 1, \dots, m, \quad b := \frac{1}{r},$$

and take $\kappa = m$. By (2.5),

$$\frac{1}{p} = \sum_{i=1}^m a_i + mb - m.$$

Since $1 < p < \infty$, we have

$$0 < \sum_{i=1}^m a_i + mb - m < 1.$$

Moreover, the assumptions (2.6) give

$$\sum_{i=1}^m a_i + mb - m < b$$

and

$$1 - b < a_i, \quad i = 1, \dots, m.$$

Hence Lemma 5.1 gives numbers a_i^-, a_i^+ with

$$0 < a_i^- < a_i < a_i^+ < 1, \quad i = 1, \dots, m,$$

and b_-, b_+ with

$$0 < b_- < b < b_+ < 1,$$

such that, for every

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_0) \in \{-, +\}^{m+1},$$

the quantity

$$\rho_\varepsilon := \sum_{i=1}^m a_i^{\varepsilon_i} + m b_{\varepsilon_0} - m$$

satisfies

$$0 < \rho_\varepsilon < 1, \quad \rho_\varepsilon < b_{\varepsilon_0},$$

and

$$1 - b_{\varepsilon_0} < a_i^{\varepsilon_i}, \quad i = 1, \dots, m.$$

Define p_ε by

$$\frac{1}{p_\varepsilon} := \rho_\varepsilon.$$

Then

$$1 < p_\varepsilon < \infty.$$

Applying Lemma 3.2 with

$$\gamma_i = a_i^{\varepsilon_i}, \quad i = 1, \dots, m, \quad \beta = b_{\varepsilon_0},$$

we obtain

$$\|T^{\chi_\Omega}(\chi_{E_1}, \dots, \chi_{E_m})\|_{p_\varepsilon, \infty} \leq C_\varepsilon \left(\prod_{i=1}^m |E_i|^{a_i^{\varepsilon_i}} \right) |\Omega|^{b_{\varepsilon_0}}$$

for all measurable sets $E_1, \dots, E_m \subseteq \mathbb{R}^n$ and $\Omega \subseteq (\mathbb{R}^n)^m$ of finite measure. Since there are only finitely many sign vectors, the constants C_ε may be replaced by a single constant independent of ε .

All the hypotheses of Proposition 3.3 are therefore satisfied with $\kappa = m$. Using also (2.7), we conclude that

$$\|T^K(f_1, \dots, f_m)\|_{p, q} \leq C \left(\prod_{i=1}^m \|f_i\|_{1/a_i, q_i} \right) \|K\|_{1/b, s}.$$

Since

$$\frac{1}{a_i} = p_i, \quad i = 1, \dots, m, \quad \frac{1}{b} = r,$$

this is precisely (2.8). □

6. SHARPNESS RESULTS FOR THE FULL-DIMENSIONAL MODEL

We complement the estimates of Section 5 with two results describing limitations of the full-dimensional theory. We first show that the strong endpoint estimate for integrable kernels does not extend to kernels in weak L^1 . We then derive necessary conditions for the boundedness of T^K when the kernel is nonnegative and nonzero, including an additional restriction on the Lorentz indices in the case

$$\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}.$$

6.1. Failure at the weak L^1 endpoint. If $K \in L^1((\mathbb{R}^n)^m)$, $1 \leq p_1, \dots, p_m \leq \infty$, and

$$\sum_{i=1}^m \frac{1}{p_i} = 1,$$

then Tonelli's theorem and Hölder's inequality give

$$\|T^K(f_1, \dots, f_m)\|_1 \leq \left(\prod_{i=1}^m \|f_i\|_{p_i} \right) \|K\|_1.$$

The next proposition shows that no uniform estimate of the same form holds when the L^1 norm of the kernel is replaced by its weak L^1 quasi-norm.

Proposition 6.1. *Let $m \geq 2$, and let $1 < p_1, \dots, p_m < \infty$ satisfy*

$$\sum_{i=1}^m \frac{1}{p_i} = 1.$$

There is no constant $C > 0$ such that

$$\|T^K(f_1, \dots, f_m)\|_{1,\infty} \leq C \left(\prod_{i=1}^m \|f_i\|_{p_i} \right) \|K\|_{1,\infty} \quad (6.1)$$

for all $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, and all $K \in L^{1,\infty}((\mathbb{R}^n)^m)$.

Proof. Let

$$D := mn,$$

and let ω_D denote the measure of the unit ball in $\mathbb{R}^D$. For $R > 16$, define

$$K_R(Y) := |Y|^{-D} \chi_{\{1 < |Y| < R/8\}}(Y), \quad Y \in \mathbb{R}^D.$$

We first check that the weak L^1 norm of K_R is bounded independently of R . If $\lambda \geq 1$, then $d_{K_R}(\lambda) = 0$, since $K_R(Y) \leq 1$ on its support. If $0 < \lambda < 1$, then

$$\{K_R > \lambda\} \subseteq \{|Y|^{-D} > \lambda\} = \{|Y| < \lambda^{-1/D}\}.$$

Therefore

$$|\{K_R > \lambda\}| \leq |\{|Y| < \lambda^{-1/D}\}| = \omega_D \lambda^{-1}.$$

Thus, for every $\lambda > 0$,

$$\lambda |\{K_R > \lambda\}| \leq \omega_D.$$

Hence

$$\|K_R\|_{1,\infty} = \sup_{\lambda > 0} \lambda |\{K_R > \lambda\}| \leq \omega_D, \quad (6.2)$$

uniformly in R .

Now set

$$Q_R := [0, R]^n, \quad f_i := \chi_{Q_R}, \quad i = 1, \dots, m.$$

Since

$$\sum_{i=1}^m \frac{1}{p_i} = 1,$$

we have

$$\prod_{i=1}^m \|f_i\|_{p_i} = \prod_{i=1}^m |Q_R|^{1/p_i} = \prod_{i=1}^m R^{n/p_i} = R^n. \quad (6.3)$$

Let

$$Q'_R := \left[\frac{R}{4}, \frac{3R}{4} \right]^n.$$

If $x \in Q'_R$ and $Y = (y_1, \dots, y_m)$ satisfies $1 < |Y| < R/8$, then

$$|y_i| \leq |Y| < \frac{R}{8}, \quad i = 1, \dots, m.$$

In particular, each coordinate of y_i has absolute value less than $R/8$. Since each coordinate of x lies between $R/4$ and $3R/4$, it follows that

$$x - y_i \in [0, R]^n = Q_R, \quad i = 1, \dots, m.$$

Therefore

$$\prod_{i=1}^m f_i(x - y_i) = 1$$

whenever $x \in Q'_R$ and $1 < |Y| < R/8$. Consequently,

$$T^{K_R}(f_1, \dots, f_m)(x) \geq \int_{\{1 < |Y| < R/8\}} |Y|^{-D} dY$$

for every $x \in Q'_R$.

Using polar coordinates in $\mathbb{R}^D$, we get

$$\begin{aligned} \int_{\{1 < |Y| < R/8\}} |Y|^{-D} dY &= D\omega_D \int_1^{R/8} \rho^{-D} \rho^{D-1} d\rho \\ &= D\omega_D \log(R/8). \end{aligned}$$

Thus

$$T^{K_R}(f_1, \dots, f_m)(x) \geq D\omega_D \log(R/8)$$

for every $x \in Q'_R$.

Taking

$$\lambda_R := \frac{1}{2} D\omega_D \log(R/8),$$

we have

$$Q'_R \subseteq \{T^{K_R}(f_1, \dots, f_m) > \lambda_R\}.$$

Therefore

$$\|T^{K_R}(f_1, \dots, f_m)\|_{1,\infty} \geq \lambda_R |Q'_R|.$$

Since

$$|Q'_R| = \left(\frac{R}{2}\right)^n,$$

we obtain

$$\|T^{K_R}(f_1, \dots, f_m)\|_{1,\infty} \geq c R^n \log(R/8), \quad (6.4)$$

for some constant c depending only on n and m .

If (6.1) were true with a constant C independent of R , then by (6.2) and (6.3) we would have

$$\|T^{K_R}(f_1, \dots, f_m)\|_{1,\infty} \leq C \omega_D R^n.$$

This contradicts (6.4) as $R \rightarrow \infty$ and finishes the proof. $\square$

6.2. Proof of Theorem 2.3. Let K be a nonnegative measurable function on $(\mathbb{R}^n)^m$ that is nonzero on a set of positive measure. We shall prove (2.10) and (2.11).

Choose $R, M > 0$ such that

$$K_0(Y) := \min\{K(Y), M\}\chi_{B(0,R)}(Y), \quad Y \in (\mathbb{R}^n)^m,$$

satisfies

$$c_0 := \int_{(\mathbb{R}^n)^m} K_0(Y) \, dY > 0,$$

where $B(0, R)$ denotes the ball in $(\mathbb{R}^n)^m$ of radius R centered at the origin.

By positivity,

$$0 \leq T^{K_0}(f_1, \dots, f_m) \leq T^K(f_1, \dots, f_m)$$

for nonnegative functions. Hence (2.9) also holds with K_0 in place of K and with the same constant C_K .

Set

$$Q := [-2R, 2R]^n, \quad Q' := [-R, R]^n.$$

If $x \in Q'$ and $Y = (y_1, \dots, y_m) \in B(0, R)$, then $x - y_i \in Q$ for every $i = 1, \dots, m$. Therefore

$$T^{K_0}(\chi_Q, \dots, \chi_Q)(x) = c_0, \quad x \in Q'. \quad (6.5)$$

For each $N \in \mathbb{N}$, choose points

$$x_1, \dots, x_N \in \mathbb{R}^n$$

so far apart that the sets

$$Q + x_j, \quad j = 1, \dots, N,$$

are pairwise disjoint. Since $Q' \subseteq Q$, the sets

$$Q' + x_j, \quad j = 1, \dots, N,$$

are also pairwise disjoint.

Now define

$$F^{(N)} := \sum_{j=1}^N \chi_{Q+x_j}.$$

By positivity and multilinearity,

$$T^{K_0}(F^{(N)}, \dots, F^{(N)}) \geq \sum_{j=1}^N T^{K_0}(\chi_{Q+x_j}, \dots, \chi_{Q+x_j}).$$

By translation invariance and (6.5),

$$T^{K_0}(\chi_{Q+x_j}, \dots, \chi_{Q+x_j}) \geq c_0 \chi_{Q'+x_j}, \quad j = 1, \dots, N.$$

Consequently,

$$T^{K_0}(F^{(N)}, \dots, F^{(N)}) \geq c_0 \sum_{j=1}^N \chi_{Q'+x_j}.$$

Taking the $L^{p,q}$ quasi-norm, we obtain

$$c_0 \left\| \sum_{j=1}^N \chi_{Q'+x_j} \right\|_{p,q} \leq \|T^{K_0}(F^{(N)}, \dots, F^{(N)})\|_{p,q}.$$

Since the sets $Q' + x_j$, $j = 1, \dots, N$, are pairwise disjoint,

$$\sum_{j=1}^N \chi_{Q'+x_j} = \chi_{\bigcup_{j=1}^N (Q'+x_j)},$$

and the union has measure $N|Q'|$. Therefore

$$\left\| \sum_{j=1}^N \chi_{Q'+x_j} \right\|_{p,q} = c_{p,q} |Q'|^{1/p} N^{1/p}.$$

It follows that

$$c_0 c_{p,q} |Q'|^{1/p} N^{1/p} \leq \|T^{K_0}(F^{(N)}, \dots, F^{(N)})\|_{p,q}.$$

On the other hand, (2.9) gives

$$\|T^{K_0}(F^{(N)}, \dots, F^{(N)})\|_{p,q} \leq C_K \prod_{i=1}^m \|F^{(N)}\|_{p_i, q_i}.$$

Since the sets $Q + x_j$, $j = 1, \dots, N$, are pairwise disjoint, for each $i = 1, \dots, m$ we have

$$\|F^{(N)}\|_{p_i, q_i} = c_{p_i, q_i} |Q|^{1/p_i} N^{1/p_i}.$$

Combining the preceding estimates yields

$$c_0 c_{p,q} |Q'|^{1/p} N^{1/p} \leq C_K \left(\prod_{i=1}^m c_{p_i, q_i} |Q|^{1/p_i} \right) N^{\sum_{i=1}^m 1/p_i}.$$

All factors other than the powers of N are independent of N . Absorbing them into a positive constant, we obtain

$$N^{1/p} \leq C N^{\sum_{i=1}^m 1/p_i}.$$

Since the constant is independent of N and the preceding inequality holds for every $N \in \mathbb{N}$, letting $N \rightarrow \infty$ yields (2.10).

Suppose now that

$$\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}.$$

For $i = 1, \dots, m$, define

$$G_i^{(N)} := \sum_{j=1}^N j^{-1/p_i} \chi_{Q+x_j}.$$

By positivity and multilinearity,

$$T^{K_0}(G_1^{(N)}, \dots, G_m^{(N)}) \geq \sum_{j=1}^N j^{-\sum_{i=1}^m 1/p_i} T^{K_0}(\chi_{Q+x_j}, \dots, \chi_{Q+x_j}).$$

Using translation invariance and (6.5), we obtain

$$T^{K_0}(G_1^{(N)}, \dots, G_m^{(N)}) \geq c_0 \sum_{j=1}^N j^{-\sum_{i=1}^m 1/p_i} \chi_{Q'+x_j}.$$

Since

$$\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p},$$

this becomes

$$T^{K_0}(G_1^{(N)}, \dots, G_m^{(N)}) \geq c_0 \sum_{j=1}^N j^{-1/p} \chi_{Q'+x_j}. \quad (6.6)$$

The sets $Q' + x_j$, $j = 1, \dots, N$, are pairwise disjoint and have the same measure. Hence the decreasing rearrangement of

$$\sum_{j=1}^N j^{-1/p} \chi_{Q'+x_j}$$

is equal to $j^{-1/p}$ on the interval

$$[(j-1)|Q'|, j|Q'|), \quad j = 1, \dots, N.$$

It follows from the definition of the Lorentz quasi-norm that there exists a positive constant $c = c(p, q)$, independent of N , such that

$$\left\| \sum_{j=1}^N j^{-1/p} \chi_{Q'+x_j} \right\|_{p,q} \geq c |Q'|^{1/p} (\log(N+1))^{1/q}.$$

Taking the $L^{p,q}$ quasi-norm in (6.6), we obtain

$$c_0 c |Q'|^{1/p} (\log(N+1))^{1/q} \leq \|T^{K_0}(G_1^{(N)}, \dots, G_m^{(N)})\|_{p,q}.$$

On the other hand, (2.9) gives

$$\|T^{K_0}(G_1^{(N)}, \dots, G_m^{(N)})\|_{p,q} \leq C_K \prod_{i=1}^m \|G_i^{(N)}\|_{p_i, q_i}.$$

For each $i = 1, \dots, m$, a similar computation with the disjoint cubes $Q + x_j$ gives

$$\|G_i^{(N)}\|_{p_i, q_i} \leq C_i |Q|^{1/p_i} (\log(N+1))^{1/q_i},$$

where C_i is independent of N . Combining the preceding estimates and absorbing all factors independent of N into the constant, we obtain

$$(\log(N+1))^{1/q} \leq C (\log(N+1))^{\sum_{i=1}^m 1/q_i}.$$

Since the constant is independent of N and the preceding inequality holds for every $N \in \mathbb{N}$, letting $N \rightarrow \infty$ yields (2.11) and completes the proof. $\square$

Taking

$$q_i = p_i, \quad i = 1, \dots, m, \quad q = p,$$

in Theorem 2.3 gives the corresponding Lebesgue result. In particular, if K is nonnegative,

$$\sum_{i=1}^m \frac{1}{p_i} < \frac{1}{p}$$

and

$$T^K : L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$$

is bounded, then $K = 0$ almost everywhere.

7. ONE-PARAMETER ESTIMATES WITH TARGET EXPONENT $p < 1$

In this section, we study the one-parameter model in the quasi-Banach range. We first prove Theorem 2.4 by means of an additive-combinatorial construction based on finite sets $P, Q \subseteq \mathbb{R}$ for which

$$|P + Q|^{1+\beta}$$

is large compared with

$$|P| |Q| |P - Q|^\beta.$$

Here and throughout this section, $|P|$ denotes the cardinality of a finite set P .

We conclude by recording complementary positive estimates that become available when additional structure is imposed on the kernel.

7.1. A counting lemma. We first isolate a simple calculation for finite sets that will be used below.

Lemma 7.1. *Let $N \geq 2$ and set*

$$P_N := \{2^0, 2^1, \dots, 2^{N-1}\}, \quad Q_N := -P_N.$$

Then

$$|P_N + Q_N| = N(N-1) + 1,$$

and

$$|P_N - Q_N| = \frac{N(N+1)}{2}.$$

Proof. Since $Q_N = -P_N$, we have

$$P_N + Q_N = P_N - P_N = \{2^i - 2^j : 0 \leq i, j \leq N-1\}.$$

The value 0 occurs when $i = j$. We claim that all strictly positive values

$$2^i - 2^j, \quad 0 \leq j < i \leq N-1,$$

are distinct. Suppose

$$2^i - 2^j = 2^k - 2^\ell, \quad i > j, \quad k > \ell.$$

Then

$$2^j(2^{i-j} - 1) = 2^\ell(2^{k-\ell} - 1).$$

Since $2^{i-j} - 1$ and $2^{k-\ell} - 1$ are odd, it follows that 2^j divides 2^ℓ and that 2^ℓ divides 2^j . Hence $j = \ell$, and then also $i = k$. There are exactly $N(N-1)/2$ pairs (i, j) with $0 \leq j < i \leq N-1$. Hence there are $N(N-1)/2$ strictly positive elements of $P_N + Q_N$. By symmetry, it has the same number of negative elements. Together with the single value 0, we obtain

$$|P_N + Q_N| = \frac{N(N-1)}{2} + \frac{N(N-1)}{2} + 1 = N(N-1) + 1.$$

Similarly,

$$P_N - Q_N = P_N + P_N = \{2^i + 2^j : 0 \leq i, j \leq N-1\}.$$

We claim that the values

$$2^i + 2^j, \quad 0 \leq i \leq j \leq N-1,$$

are all distinct. Suppose

$$2^i + 2^j = 2^k + 2^\ell, \quad i \leq j, \quad k \leq \ell.$$

If $j > \ell$, then

$$2^k + 2^\ell \leq 2^\ell + 2^\ell = 2^{\ell+1} \leq 2^j,$$

whereas

$$2^i + 2^j > 2^j,$$

a contradiction. Similarly, $\ell > j$ is impossible. Hence $j = \ell$, which implies $2^i = 2^k$, and therefore $i = k$. Thus the sums are distinct. There are exactly $N(N+1)/2$ pairs (i, j) with $0 \leq i \leq j \leq N-1$. Therefore

$$|P_N - Q_N| = \frac{N(N+1)}{2}.$$

This completes the proof. $\square$

We now derive the consequence of Lemma 7.1 that will be used in the proof of Theorem 2.4. For $\beta > 0$, set

$$R_N(\beta) := \frac{|P_N + Q_N|^{1+\beta}}{|P_N| |Q_N| |P_N - Q_N|^\beta}.$$

By Lemma 7.1, we obtain

$$R_N(\beta) = \frac{(N(N-1)+1)^{1+\beta}}{N^2 \left(\frac{N(N+1)}{2}\right)^\beta}.$$

Writing

$$N(N-1)+1 = N^2 \left(1 - \frac{1}{N} + \frac{1}{N^2}\right)$$

and

$$\frac{N(N+1)}{2} = \frac{N^2}{2} \left(1 + \frac{1}{N}\right),$$

we get

$$R_N(\beta) = 2^\beta \frac{\left(1 - \frac{1}{N} + \frac{1}{N^2}\right)^{1+\beta}}{\left(1 + \frac{1}{N}\right)^\beta}.$$

Therefore

$$R_N(\beta) \rightarrow 2^\beta \quad \text{as } N \rightarrow \infty.$$

Since $2^\beta > 1$, there exists $N = N(\beta) \geq 2$ such that

$$R_N(\beta) = \frac{|P_N + Q_N|^{1+\beta}}{|P_N| |Q_N| |P_N - Q_N|^\beta} > 1. \quad (7.1)$$

Indeed, if

$$\delta := \frac{2^\beta - 1}{2} > 0,$$

then for all sufficiently large N we have

$$R_N(\beta) > 2^\beta - \delta = \frac{2^\beta + 1}{2} > 1.$$

7.2. Proof of Theorem 2.4. Let $N = N(\beta)$ be chosen so that (7.1) holds, and set

$$P := P_N, \quad Q := Q_N,$$

where P_N and Q_N are the sets from Lemma 7.1. Thus

$$\frac{|P + Q|^{1+\beta}}{|P||Q||P - Q|^\beta} > 1. \quad (7.2)$$

For each $k \in \mathbb{N}$, set

$$\theta_j := \pi^j, \quad j = 0, \dots, k-1.$$

Since π is transcendental, the numbers

$$\theta_0, \theta_1, \dots, \theta_{k-1}$$

are linearly independent over $\mathbb{Q}$. Define

$$\Phi_k(x_0, \dots, x_{k-1}) := \sum_{j=0}^{k-1} x_j \theta_j.$$

Since the numbers θ_j are linearly independent over $\mathbb{Q}$, the map Φ_k is injective on every subset of $\mathbb{Z}^k$.

We now set

$$P^{(k)} := \Phi_k(P^k), \quad Q^{(k)} := \Phi_k(Q^k),$$

where P^k denotes the set of all k -tuples with entries in P , namely

$$P^k = \{(p_0, \dots, p_{k-1}) : p_j \in P, j = 0, \dots, k-1\},$$

and similarly for Q^k . Hence

$$|P^{(k)}| = |P^k| = |P|^k, \quad |Q^{(k)}| = |Q^k| = |Q|^k.$$

Moreover, by linearity of Φ_k ,

$$P^{(k)} + Q^{(k)} = \Phi_k((P + Q)^k),$$

and therefore

$$|P^{(k)} + Q^{(k)}| = |(P + Q)^k| = |P + Q|^k.$$

Similarly,

$$P^{(k)} - Q^{(k)} = \Phi_k((P - Q)^k),$$

and so

$$|P^{(k)} - Q^{(k)}| = |(P - Q)^k| = |P - Q|^k.$$

Next, fix $k \in \mathbb{N}$. Since the sets

$$P^{(k)}, \quad Q^{(k)}, \quad \frac{1}{2}(P^{(k)} - Q^{(k)}), \quad \frac{1}{2}(P^{(k)} + Q^{(k)})$$

are finite, we may choose $\eta > 0$ so small that the intervals centered at distinct points of each of these sets are pairwise disjoint. Write

$$I_\eta := (-\eta, \eta).$$

Thus, for instance,

$$P^{(k)} + I_\eta = \bigcup_{p \in P^{(k)}} (p - \eta, p + \eta),$$

and the intervals in this union are pairwise disjoint. Define

$$E_1 := P^{(k)} + I_\eta, \quad E_2 := Q^{(k)} + I_\eta,$$

and

$$A := \frac{1}{2}(P^{(k)} - Q^{(k)}) + I_\eta.$$

Then

$$|E_1| = 2\eta|P^{(k)}| = 2\eta|P|^k,$$

$$|E_2| = 2\eta|Q^{(k)}| = 2\eta|Q|^k,$$

and

$$|A| = 2\eta|P^{(k)} - Q^{(k)}| = 2\eta|P - Q|^k.$$

Now set

$$H(x) := \int_{\mathbb{R}} \chi_{E_1}(x+y) \chi_{E_2}(x-y) \chi_A(y) \, dy.$$

We obtain a lower bound for a level set of H . Let

$$s \in P^{(k)} + Q^{(k)}.$$

Choose one representation

$$s = p + q, \quad p \in P^{(k)}, \quad q \in Q^{(k)}.$$

If

$$x \in \frac{s}{2} + I_{\eta/4}$$

and

$$y \in \frac{p-q}{2} + I_{\eta/4},$$

then

$$x + y \in p + I_\eta \subseteq E_1, \quad x - y \in q + I_\eta \subseteq E_2,$$

and

$$y \in \frac{1}{2}(P^{(k)} - Q^{(k)}) + I_\eta = A.$$

Thus, for every

$$x \in \frac{s}{2} + I_{\eta/4},$$

we have

$$H(x) \geq \int_{\frac{p-q}{2} + I_{\eta/4}} 1 \, dy = |I_{\eta/4}| = \frac{\eta}{2}.$$

Consequently,

$$H(x) > \frac{\eta}{4}$$

on the set

$$S_k := \bigcup_{s \in P^{(k)} + Q^{(k)}} \left(\frac{s}{2} + I_{\eta/4} \right).$$

By the choice of η , the intervals in this union are disjoint. Therefore

$$|S_k| = \frac{\eta}{2}|P^{(k)} + Q^{(k)}| = \frac{\eta}{2}|P + Q|^k.$$

Since

$$S_k \subseteq \{H > \eta/4\},$$

the definition of the weak $L^{1/(1+\beta)}$ quasi-norm gives

$$\begin{aligned}\|H\|_{1/(1+\beta),\infty} &\geq \frac{\eta}{4}|S_k|^{1+\beta} \\ &= \frac{\eta}{4}\left(\frac{\eta}{2}|P+Q|^k\right)^{1+\beta} \\ &= \frac{\eta^{2+\beta}}{2^{3+\beta}}|P+Q|^{k(1+\beta)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|E_1||E_2||A|^\beta &= (2\eta|P|^k)(2\eta|Q|^k)(2\eta|P-Q|^k)^\beta \\ &= (2\eta)^{2+\beta}(|P||Q||P-Q|^\beta)^k.\end{aligned}$$

Consequently,

$$\frac{\|H\|_{1/(1+\beta),\infty}}{|E_1||E_2||A|^\beta} \geq \frac{1}{2^{5+2\beta}} \left(\frac{|P+Q|^{1+\beta}}{|P||Q||P-Q|^\beta} \right)^k.$$

By (7.2), the factor in parentheses is strictly larger than 1. Letting $k \rightarrow \infty$, the ratio tends to infinity. Hence no constant C can make (2.12) hold for all measurable sets $E_1, E_2, A \subseteq \mathbb{R}$ of finite measure. $\square$

7.3. Related positive results for structured kernels. Theorem 2.4 shows that a natural family of restricted weak-type estimates with target exponent $p < 1$ cannot hold uniformly over arbitrary measurable kernel sets, even in the one-dimensional bilinear case. This does not rule out estimates for fixed kernels or for classes of kernels with additional structure.

We now pass to the corresponding bilinear one-parameter operator on $\mathbb{R}^n$, obtained from (1.3) by taking $m = 2$, $\alpha_1 = -1$, and $\alpha_2 = 1$, and record two complementary positive results.

First, we consider cube kernels. By a *cube* we mean a half-open cube with sides parallel to the coordinate axes. A direct adaptation of the argument in [2, Lemma 4.1] gives the strong endpoint estimate

$$L^1 \times L^1 \longrightarrow L^{1/2}.$$

Proposition 7.2. *There exists a positive constant C_n , depending only on n , such that*

$$\|T_{\chi_Q}(f_1, f_2)\|_{1/2} \leq C_n |Q| \|f_1\|_1 \|f_2\|_1, \quad (7.3)$$

for every cube $Q \subseteq \mathbb{R}^n$ and all integrable functions f_1, f_2 on $\mathbb{R}^n$.

Next, we consider weak kernels with radial monotonicity. In this case, the weak L^r condition gives a pointwise estimate for the kernel, and this allows us to dominate T_g by the bilinear fractional integral

$$B_\alpha(f_1, f_2)(x) := \int_{\mathbb{R}^n} f_1(x+y) f_2(x-y) |y|^{\alpha-n} dy, \quad 0 < \alpha < n. \quad (7.4)$$

We say that a nonnegative measurable function g on $\mathbb{R}^n$ is *radially decreasing* if there exists a nonincreasing function $\psi : (0, \infty) \rightarrow [0, \infty)$ such that

$$g(y) = \psi(|y|) \quad \text{for a.e. } y \in \mathbb{R}^n.$$

Lemma 7.3. *Let $1 < r < \infty$, and let g be a radially decreasing function on $\mathbb{R}^n$. If $g \in L^{r,\infty}(\mathbb{R}^n)$, then there exists a positive constant $C = C(n, r)$ such that*

$$g(y) \leq C \|g\|_{r,\infty} |y|^{-n/r} \quad \text{for a.e. } y \in \mathbb{R}^n. \quad (7.5)$$

Proof. After modifying g on a set of measure zero, we may assume that $g(y) = \psi(|y|)$ for every $y \neq 0$, where ψ is nonincreasing. Fix $y \neq 0$ and note that if $0 < \lambda < g(y)$, then

$$g(z) \geq g(y) > \lambda \quad \text{whenever } |z| \leq |y|.$$

Therefore $B(0, |y|) \subseteq \{g > \lambda\}$, and hence

$$c_n |y|^n = |B(0, |y|)| \leq |\{g > \lambda\}|.$$

It follows that

$$\lambda (c_n |y|^n)^{1/r} \leq \lambda |\{g > \lambda\}|^{1/r} \leq \|g\|_{r,\infty},$$

for all $0 < \lambda < g(y)$. Letting $\lambda \rightarrow g(y)$ gives (7.5). $\square$

For $1 < r < \infty$, define

$$\alpha := n \left(1 - \frac{1}{r}\right). \quad (7.6)$$

Then $0 < \alpha < n$ and

$$|y|^{-n/r} = |y|^{\alpha-n}.$$

Let $\mathcal{P}_\alpha$ denote the interior of the pentagon in $[0, 1]^2$ with vertices

$$(1, 1), \quad (1, 0), \quad (\alpha/n, 0), \quad (0, \alpha/n), \quad (0, 1).$$

We use the known bounds for B_α ; see [16, 10, 1]. If

$$\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \in \mathcal{P}_\alpha \quad \text{and} \quad \frac{1}{p} + \frac{\alpha}{n} = \frac{1}{p_1} + \frac{1}{p_2},$$

then there exists a positive constant $C = C(\alpha, n, p_1, p_2)$ such that

$$\|B_\alpha(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2},$$

for all $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, 2$.

Combining Lemma 7.3 with these bounds for B_α gives the following immediate consequence.

Proposition 7.4. *Let $1 < r < \infty$, let α be given by (7.6), and let g be a nonnegative radially decreasing function in $L^{r,\infty}(\mathbb{R}^n)$. Suppose that*

$$\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \in \mathcal{P}_\alpha$$

and define $0 < p < \infty$ by

$$\frac{1}{p} + \frac{\alpha}{n} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then there exists a positive constant $C = C(n, r, p_1, p_2)$ such that

$$\|T_g(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \|g\|_{r,\infty} \quad (7.7)$$

for all $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$.

Since $\mathcal{P}_\alpha$ contains points for which

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} > 1,$$

Proposition 7.4 includes estimates with $p < 1$. Thus radial monotonicity gives an additional condition on the kernel under which bounds in the quasi-Banach range may hold.

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