FAILURE OF THE HÖRMANDER KERNEL CONDITION FOR MULTILINEAR CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. It is well known that the Hörmander smoothness condition $\sup_{y\neq 0}\int_{|x|\geq 2|y|}|K(x-y)-K(x)|dx<\infty$ implies weak type $(1,1)$ estimates for associated L^2-bounded Calderón-Zygmund operators. It has been an open question whether Hörmander's condition also suffices to guarantee weak type $(1,1,1/2)$ estimates for bilinear Calderón-Zygmund operators that are bounded at one point. In this paper we provide a negative answer to this question.

1. INTRODUCTION

Hörmander's [12] adaptation of the Calderón-Zygmund theorem says that an L^2-bounded convolution operator associated with a kernel K on $\mathbb{R}^d$ satisfying the smoothness condition

$$\|K\|_H = \sup_{y\neq 0}\int_{|x|\geq 2|y|}|K(x-y)-K(x)|dx<\infty \tag{1}$$

is also bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. By duality and interpolation this classical result implies that the operator also admits an L^p-bounded extension for all $p\in(1,\infty)$. Recent interest in multilinear extensions of the Calderón-Zygmund theory has led to the development of multilinear harmonic analysis; see [7, Chapter 7]. This area was introduced by Coifman and Meyer in their seminal work [3], [4], [5]. A fundamental result in this theory is that if an m-linear Calderón-Zygmund operator is bounded from $L^2\times\cdots\times L^2$ to $L^{2/m}$ and its kernel K satisfies an appropriate size condition and a standard Lipschitz smoothness condition on $\mathbb{R}^{md}$, then it is bounded from $L^1\times\cdots\times L^1$ to $L^{1/m,\infty}$; this result implies strong boundedness for the operator from product of Lebesgue spaces to another Lebesgue space L^p in the largest range of indices possible, and also implies weak type boundedness at the endpoints. Boundedness in the region where the target space

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is L^p with $p > 1$ was first proved by Coifman and Meyer [4], [5] and was extended to the case $p \leq 1$ by Kenig and Stein [13], and independently by Grafakos and Torres [11]. A natural question, inspired by the linear theory, is whether this result also holds if the kernel K, which is a function on $\mathbb{R}^{md} \setminus \{0\}$, satisfies only Hörmander's condition (1). This question has been around since 2002 and has attracted some attention. In this note, we provide a negative answer to this question. Our argument is mainly inspired by two ingredients related to bilinear rough singular integrals. The first is a reinforced and quantitative version of the counterexample in [6], while the second is the $L^2 \times L^2 \to L^1$ boundedness of bilinear rough singular integrals recently obtained in [8] and [9].

Our counterexample is a homogeneous kernel, i.e., a kernel that has the form:

$$K_\Omega(x_1,x_2) = \Omega((x_1,x_2)/|(x_1,x_2)|)|(x_1,x_2)|^{-2d}, \qquad (x_1,x_2) \in \mathbb{R}^{2d}$$

where Ω is integrable on the sphere $\mathbb{S}^{2d-1}$ with vanishing integral. The associated bilinear Calderón-Zygmund operator T_{K_Ω} is then defined as

$$T_{K_\Omega}(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^{2n}} K_\Omega(x-y_1,x-y_2) f(y_1) g(y_2) dy_1 dy_2.$$

We prove the following result:

Theorem 1. *Let $1 \leq q < \infty$. There exists an odd function Ω in $L^q(\mathbb{S}^{2d-1})$ such that the associated kernel K_Ω satisfies the Hörmander kernel condition* (1) *but the associated bilinear Calderón-Zygmund operator T_{K_Ω} does not map $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{R}^d)$ whenever $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{p} + \frac{2d-1}{q} > 2d$. In particular, this operator is not of weak type $(1,1,\frac{1}{2})$ when $1 \leq q < \frac{2d-1}{2d-2}$.*

If $\Omega \in L^q(\mathbb{S}^{2d-1})$ with $q \geq 2$ then T_{K_Ω} is always $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ bounded, see [8]; this result was later extended to $\frac{4}{3} < q \leq \infty$ in [9]. Thus Theorem 1 yields the following corollary:

Corollary 2. *Let $d \in \{1,2\}$. There exists an odd function Ω on $\mathbb{S}^{2d-1}$ such that K_Ω satisfies Hörmander's condition* (1) *and the associated operator T_{K_Ω} is bounded from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$, but is unbounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ whenever $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 \leq p_1, p_2 \leq \infty$ and $p < \frac{4}{2d+3}$. In particular, this operator is not of weak type $(1,1,\frac{1}{2})$.*

Remark 1. To obtain, via these techniques, an example of an $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ bounded bilinear Calderón-Zygmund operator whose kernel satisfies Hörmander's condition (1) but which does not satisfy a weak type $(1,1,\frac{1}{2})$ estimate in an arbitrary dimension d, we would need to know that

$$\|T_{K_\Omega}\|_{L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\to L^1(\mathbb{R}^d)} \le C\|\Omega\|_{L^q(\mathbb{S}^{2d-1})} \tag{2}$$

for all $q > 1$; but (2) remains open, as of this writing, for $1 < q \le \frac{4}{3}$.

Other versions of the Hörmander kernel condition in the multilinear setting are given in [15], [16] and [2]; these conditions are weaker than (1), so our example applies also in that case. Our result should be contrasted with the positive result in [17] concerning a stronger geometric version of condition (1).

Additionally, it was observed in [11] that if $\Omega \in L^1(\mathbb{R})$ is an odd function then the boundedness of T_{K_Ω} can be obtained as a consequence of the uniform boundedness of the bilinear Hilbert transforms, see [10], [14]. Thus, in particular, T_{K_Ω} is bounded on the hexagon described by the conditions

$$\Big|\frac{1}{p_1}-\frac{1}{p_2}\Big| < \frac{1}{2}, \qquad \Big|\frac{1}{p_1}-\frac{1}{p'}\Big| < \frac{1}{2}, \qquad \Big|\frac{1}{p_2}-\frac{1}{p'}\Big| < \frac{1}{2}, \tag{3}$$

where $p' = \frac{p}{p-1}$. We note that this hexagon contains points (p_1, p_2, p) with $p > 1$ arbitrarily close to 1. Another corollary of Theorem 1 is the following.

Corollary 3. *There exists an odd function Ω on $\mathbb{S}^1$ such that the kernel K_Ω satisfies the 2-dimensional Hörmander condition* (1) *and the associated operator T_{K_Ω} is bounded on the hexagon* (3) *but is unbounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^p(\mathbb{R})$ for $p < 1$.*

We prove the one-dimensional version of Theorem 1 in the next section. This is easy to read and contains the main idea. The proof in the d-dimensional case is given in Section 3; this contains an additional perturbation argument. We verify that K_Ω satisfies (1) in Section 4. In Section 5 we briefly discuss the multilinear situation.

2. Proof of Theorem 1 when $d = 1$

Define points on the circle $\mathbb{S}^1$

$$a_n = \left(\cos\left(\frac{\pi}{4}+\frac{\pi}{2^n}\right), \sin\left(\frac{\pi}{4}+\frac{\pi}{2^n}\right)\right)$$

and define circular arcs I_n^+ with endpoints a_n and a_{n+1} for $n = 10, 11, 12, \dots$. Let I_n^- be the reflection about the origin of I_n^+. We observe that the length ℓ_n of both I_n^+ and I_n^- is approximately 2^{-n}. Consider the function

$$\Omega = \sum_{n=10}^{\infty} h_n(\chi_{I_n^+} - \chi_{I_n^-})$$

where $h_n = 2^{n\delta}$ for some $\delta < 1/q$. Note that

$$\|\Omega\|_{L^q(\mathbb{S}^1)} \le c\Big(\sum_{n=10}^{\infty} h_n^q \ell_n\Big)^{\frac{1}{q}} \le c\Big(\sum_{n=10}^{\infty} 2^{n\delta q - n}\Big)^{\frac{1}{q}} < \infty$$

and that Ω is an odd function on $\mathbb{S}^1$.

For $0 < \varepsilon < \frac{1}{100}$ define $f_\varepsilon = (2\varepsilon)^{-\frac{1}{p_1}}\chi_{[-\varepsilon,\varepsilon]}$, $g_\varepsilon = (2\varepsilon)^{-\frac{1}{p_2}}\chi_{[-\varepsilon,\varepsilon]}$; these functions satisfy $\|f_\varepsilon\|_{L^{p_1}} = \|g_\varepsilon\|_{L^{p_2}} = 1$.

Let us fix an $x \in \mathbb{R}$ such that $\frac{11}{10} \le x \le \frac{12}{10}$. Then we have

(4)

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \ge (2\varepsilon)^{-\frac{1}{p_1}}(2\varepsilon)^{-\frac{1}{p_2}} \int_{|y_1|<\varepsilon}\int_{|y_2|<\varepsilon} \frac{\Omega\big(\frac{(x-y_1,x-y_2)}{|(x-y_1,x-y_2)|}\big)}{|(x-y_1,x-y_2)|^2}\, dy_1\, dy_2.$$

Let P_ε be all projections of points of the form $(x-y_1, x-y_2)$ onto the circle $\mathbf{S}^1$, where (y_1,y_2) is an arbitrary point in $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)$. As the point $(x-y_1,x-y_2)$ lies near the positive diagonal (that forms 45^o with the positive horizontal axis), this projection will only intersect circular caps I_n^+ and will never intersect caps I_n^-. In this case every term in the sum that defines Ω and appears in (4) is positive. We obtain

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \ge c\varepsilon^{-\frac{1}{p_1}}\varepsilon^{-\frac{1}{p_2}}\varepsilon \sum_{\substack{n\ge 10\\ I_n^+\subseteq P_\varepsilon}} \ell_n h_n$$

as $|(x-y_1,x-y_2)|^2 \approx 1$ and if $I_n^+ \subseteq P_\varepsilon$ then the set of those (y_1,y_2) satisfying $|y_1|<\varepsilon$, $|y_2|<\varepsilon$ and $(x-y_1,x-y_2)/|(x-y_1,x-y_2)| \in I_n^+$ has measure comparable to $\varepsilon\ell_n$, since x is so close to 1. As $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ we obtain for $\frac{11}{10} \le x \le \frac{12}{10}$ that

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \gtrapprox \varepsilon^{-\frac{1}{p}+1} \sum_{\substack{n:\\ 2^{-n}<c\varepsilon}} 2^{n\delta - n} \gtrapprox \varepsilon^{2-\frac{1}{p}-\delta},$$

which yields that $\|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)\|_{L^{p,\infty}} \gtrapprox \varepsilon^{2-\frac{1}{p}-\delta}$, and

$$\|T_{K_\Omega}\|_{L^{p_1}\times L^{p_2}\to L^{p,\infty}} \ge \frac{\|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)\|_{L^{p,\infty}}}{\|f_\varepsilon\|_{L^{p_1}}\|g_\varepsilon\|_{L^{p_2}}} \gtrapprox \varepsilon^{2-\frac{1}{p}-\delta}.$$

Choosing δ sufficiently close to $1/q$, we conclude that if $2-\frac{1}{p}-\frac{1}{q}<0$, then

$$\|T_{K_\Omega}\|_{L^{p_1}\times L^{p_2}\to L^{p,\infty}} = \infty.$$

To complete the proof of the main theorem we need to know that K_Ω satisfies Hörmander's condition (1). For this we prove the following lemma in which points in $\mathbb{R}^2$ will be denoted by capital letters.

Lemma 4. *Let $r>1$ and $\Omega_t = t^{-\frac{1}{r}}\chi_{I_t}$, where I_t is a circular arc of small length $t>0$ on the the circle $\mathbb{S}^1$. Then there is a constant $C_r<\infty$ such that*

$$\sup_{t>0}\sup_{Y\neq 0}\int_{|X|\geq 2|Y|}\left|K_{\Omega_t}(X-Y)-K_{\Omega_t}(X)\right|dX\leq C_r.$$

As the proof of Lemma 4 is contained in that of Lemma 5 proved later, we do not include it here. Lemma 4 gives that

$$\begin{aligned}\|K_\Omega\|_H &\leq \sum_{n=10}^{\infty} h_n \ell_n^{\frac{1}{r}}\Big(\Big\|\frac{1}{\ell_n^{\frac{1}{r}}}\chi_{I_n^+}\Big\|_H+\Big\|\frac{1}{\ell_n^{\frac{1}{r}}}\chi_{I_n^-}\Big\|_H\Big)\\ &\leq C\sum_{n=10}^{\infty} h_n\ell_n^{\frac{1}{r}} = C\sum_{n=10}^{\infty}2^{n\delta-n\frac{1}{r}}\end{aligned}$$

and this sum is convergent if we choose r such that $1<\delta<1/r$. This concludes the proof of Theorem 1 when $d=1$.

3. Proof of Theorem 1 when $d\geq 2$

We now extend the proof to higher dimensions. Fix a point

$$a=(\tfrac{1}{\sqrt{2d}},\dots,\tfrac{1}{\sqrt{2d}})\in\mathbb{S}^{2d-1}$$

and for $n=10,11,12,\dots$ define spherical annuli

$$A_n^+ = \mathbb{S}^{2d-1}\cap\Big(B(a,2^{-n})\setminus B(a,2^{-n-1})\Big).$$

Let A_n^- be the reflection about the origin of A_n^+. We observe that the measure v_n of both A_n^+ and A_n^- is approximately $2^{-n(2d-1)}$. Consider the function

$$\Omega=\sum_{n=10}^{\infty}h_n(\chi_{A_n^+}-\chi_{A_n^-})$$

where $h_n=2^{n\delta}$ for some $\delta<\frac{2d-1}{q}$. Note that

$$\|\Omega\|_{L^q(\mathbb{S}^{2d-1})}\leq c\Big(\sum_{n=10}^{\infty}h_n^q v_n\Big)^{\frac{1}{q}}\leq c\Big(\sum_{n=10}^{\infty}2^{n(\delta q-(2d-1))}\Big)^{\frac{1}{q}}<\infty$$

and that Ω is an odd function on $\mathbb{S}^{2d-1}$.

For $0<\varepsilon<\frac{1}{100d}$ define $f_\varepsilon=(2\varepsilon)^{-\frac{d}{p_1}}\chi_{[-\varepsilon,\varepsilon]^d}$, $g_\varepsilon=(2\varepsilon)^{-\frac{d}{p_2}}\chi_{[-\varepsilon,\varepsilon]^d}$; these functions satisfy $\|f_\varepsilon\|_{L^{p_1}}=\|g_\varepsilon\|_{L^{p_2}}=1$.

Let us fix an interval on the diagonal line in $\mathbb{R}^d$ defined by

$$I_d=\big\{x\in\mathbb{R}^d : x_1=x_2=\cdots=x_d\in[\tfrac{1}{\sqrt{d}}+\tfrac{1}{100d},\tfrac{1}{\sqrt{d}}+\tfrac{2}{100d}]\big\}. \tag{5}$$

Then for $x \in I_d$ we have

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \geq \tag{6}$$
$$(2\varepsilon)^{-\frac{d}{p_1}}(2\varepsilon)^{-\frac{d}{p_2}} \int_{[-\varepsilon,\varepsilon]^d}\int_{[-\varepsilon,\varepsilon]^d} \frac{\Omega\big(\frac{(x-y_1,x-y_2)}{|(x-y_1,x-y_2)|}\big)}{|(x-y_1,x-y_2)|^2}\, dy_1\, dy_2.$$

Let $P_{\varepsilon,x}$ be the set of all projections onto the sphere $\mathbb{S}^{2d-1}$ of points of the form $(x-y_1, x-y_2)$, where (y_1,y_2) is an arbitrary point in $[-\varepsilon,\varepsilon]^{2d}$. As the point $(x-y_1,x-y_2)$ lies near the positive diagonal, this projection will only intersect spherical annuli A_n^+ and will never intersect annuli A_n^-. In this case every term in the sum that defines Ω and appears in (6) is positive. We obtain

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \geq c\varepsilon^{-\frac{d}{p_1}}\varepsilon^{-\frac{d}{p_2}}\varepsilon \sum_{\substack{n\geq 10\\ A_n^+\subseteq P_{\varepsilon,x}}} v_n h_n$$

as $|(x-y_1,x-y_2)|^2 \approx 1$ and if $A_n^+ \subseteq P_{\varepsilon,x}$ then the set of those (y_1,y_2) satisfying $(y_1,y_2)\in[-\varepsilon,\varepsilon]^{2d}$ and $(x-y_1,x-y_2)/|(x-y_1,x-y_2)| \in A_n^+$ has measure comparable to εv_n, since x is so close to the unit sphere. Since $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$, we obtain

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \quad \gtrapprox \quad \varepsilon^{-\frac{d}{p}+1} \sum_{\substack{n:\\ 2^{-n}<c_d\varepsilon}} 2^{n\delta-n(2d-1)} \gtrapprox \varepsilon^{(2-\frac{1}{p})d-\delta},$$

whenever $x\in I_d$. In particular, in the last summation the term with $2^{-n_\varepsilon}\sim \frac{c_d}{10}\varepsilon$ would contribute essentially the same lower bound $\varepsilon^{(2-\frac{1}{p})d-\delta}$.

We now fix a point $x_0 \in I_d$. For any x such that $|x-x_0|\leq c_d'\varepsilon$ with c_d' a small positive constant, we define $P_{\varepsilon,x}$ as the projection of $(x,x)+[-\varepsilon,\varepsilon]^{2d}$ onto $\mathbb{S}^{2d-1}$. Recalling that P_{ε,x_0} contains $A_{n_\varepsilon}^+$ and that the distance between $A_{n_\varepsilon}^+$ and $\mathbb{S}^{2d-1}\setminus P_{\varepsilon,x_0}$ is greater than $\frac{c_d}{2}\varepsilon$, we obtain that $A_{n_\varepsilon}^+ \subset P_{\varepsilon,x}$ if c_d' is small enough, since the distance between the boundary of P_{ε,x_0} and the boundary of $P_{\varepsilon,x}$ is bounded by $c_d'\varepsilon$. In summary, for any point $x\in N_\varepsilon$, the $c_d'\varepsilon$-neighborhood of I_d with volume about ε^{d-1}, we have

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \gtrapprox \varepsilon^{-\frac{d}{p}+1} 2^{n_\varepsilon(\delta-2d+1)} \approx \varepsilon^{(2-\frac{1}{p})d-\delta}. \tag{7}$$

This yields

$$\|T_{K_\Omega}\|_{L^{p_1}\times L^{p_2}\to L^{p,\infty}} \geq \frac{\|T_{K_\Omega}(f_\varepsilon,g_\varepsilon)\|_{L^{p,\infty}(\mathbb{R}^d)}}{\|f_\varepsilon\|_{L^{p_1}}\|g_\varepsilon\|_{L^{p_2}}} \gtrapprox \varepsilon^{\frac{d-1}{p}+(2-\frac{1}{p})d-\delta}.$$

Choosing δ sufficiently close to $\frac{2d-1}{q}$, we conclude that if

$$2d-\tfrac{1}{p}-\tfrac{2d-1}{q}<0,$$

then

$$\|T_{K_\Omega}\|_{L^{p_1}(\mathbb{R}^d)\times L^{p_2}(\mathbb{R}^d)\to L^{p,\infty}(\mathbb{R}^d)} = \infty.$$

We have the following d-dimensional extension of Lemma 4.

Lemma 5. *Let $r > \frac{1}{2d-1}$ and $\Omega_t = t^{-\frac{1}{r}}\chi_{A_t}$, where A_t is a spherical cap of small radius t on the sphere $\mathbb{S}^{2d-1}$. Then there is a constant C that depends on d and r such that*

$$\sup_{t>0}\sup_{Y\neq 0}\int_{|X|\geq 2|Y|}\left|K_{\Omega_t}(X-Y)-K_{\Omega_t}(X)\right|dX \leq C. \tag{8}$$

We note that each spherical annulus A_n^+, A_n^- can be written as $B_n^+\setminus C_n^+$ or $B_n^-\setminus C_n^-$, where B_n^+, C_n^+ and B_n^-, C_n^- are spherical caps of radius approximately 2^{-n} centered at a and $-a$, respectively. Therefore, assuming Lemma 5, we obtain

$$\begin{aligned}\|K_\Omega\|_H &\leq \sum_{n=10}^{\infty} h_n 2^{-\frac{n}{r}}\left\|2^{\frac{n}{r}}\left(\chi_{B_n^+}-\chi_{C_n^+}-\chi_{B_n^-}+\chi_{C_n^-}\right)\right\|_H\\ &\leq C\sum_{n=10}^{\infty} h_n 2^{-\frac{n}{r}} = C\sum_{n=10}^{\infty} 2^{n\delta-\frac{n}{r}}\end{aligned}$$

and this sum is convergent if we choose $\delta < \frac{1}{r} < 2d-1$, which is possible since $\delta < \frac{2d-1}{q} \leq 2d-1$.

This finishes the proof of Theorem 1 for $d\geq 2$ assuming Lemma 5, which is proved in the next section.

4. Proof of Lemma 5

Let $X\in\mathbb{R}^{2d}$ and $X' = X/|X|$. It suffices to prove that

$$\int_{|X|\geq 2|Y|}\left|\Omega_t((X-Y)')-\Omega_t(X')\right|\frac{dX}{|X-Y|^{2d}} \leq C < \infty$$

as the part

$$\int_{|X|\geq 2|Y|}\left|\frac{\Omega_t(X')}{|X-Y|^{2d}}-\frac{\Omega_t(X')}{|X|^{2d}}\right|dX$$

is trivially bounded by $\|\Omega_t\|_{L^1(\mathbb{S}^{2d-1})}\leq C$ since $r > \frac{1}{2d-1}$.

But $|X-Y|\approx |X|$ and so we look at

$$\int_{2|Y|}^{\infty}\int_{\mathbb{S}^{2d-1}}\left|\Omega_t((s\theta-Y)')-\Omega_t(\theta)\right|d\theta\frac{ds}{s}. \tag{9}$$

The interior integral vanishes if both terms $\chi_{A_t}((s\theta-Y)')$ and $\chi_{A_t}(\theta)$ are 1 or 0. Thus we may consider the case when one term is one and the other is zero. In this case we estimate the expression on the left in (8) by

$$t^{-\frac{1}{r}}\int_{2|Y|}^{\infty}|\{\theta\in A_t,\big(\theta-\tfrac{Y}{s}\big)'\notin A_t\}|\frac{ds}{s}+t^{-\frac{1}{r}}\int_{2|Y|}^{\infty}|\{\theta\notin A_t,\big(\theta-\tfrac{Y}{s}\big)'\in A_t\}|\frac{ds}{s}.$$

Both A_t and the set of all $\theta\in\mathbb{S}^{2d-1}$ for which $\big(\theta-\frac{Y}{s}\big)'\in A_t$ have spherical measure at most ct^{2d-1}, where to show the latter we use the fact that $|\frac{Y}{s}|\le\frac{1}{2}$. Let us now assume that $\frac{|Y|}{s}\le\frac{t}{100}\ll 1$. In the first integral the set has spherical measure at most $c\frac{|Y|}{s}t^{2d-2}$, because it is comparable to $|A_t'\setminus A_t|$ with A_t' an appropriate rotation of A_t with displacement $\sim\frac{|Y|}{s}$. Similarly the set in the second integral has spherical measure at most $c\frac{|Y|}{s}t^{2d-2}$ as well. We therefore obtain the estimate for (9)

$$ct^{-\frac{1}{r}}\left[\int_{2|Y|}^{\frac{100|Y|}{t}}t^{2d-1}\frac{ds}{s}+\int_{\frac{100|Y|}{t}}^{\infty}\frac{|Y|}{s}t^{2d-2}\frac{ds}{s}\right]\le ct^{-\frac{1}{r}}[t^{2d-1}\log(t^{-1})]\le C<\infty,$$

since $2d-1-\frac{1}{r}>0$ and $t\le 1$. This proves (8).

5. The Multilinear case

The argument needed to prove a multilinear version of Theorem 1 is similar to the one performed above. We sketch it below for completeness.

Let Ω be an integrable function on the sphere $\mathbb{S}^{md-1}$ with vanishing integral. We define

$$K_\Omega(x_1,\dots,x_m)=\Omega((x_1,\dots,x_m)/|(x_1,\dots,x_m)|)|(x_1,\dots,x_m)|^{-md}$$

for $(x_1,\dots,x_m)\in\mathbb{R}^{md}$. The m-linear rough singular integral operator T_{K_Ω} is then defined by

$$T_{K_\Omega}(f_1,\dots,f_m)(x)=\text{p.v.}\int_{\mathbb{R}^{md}}K_\Omega(x-y_1,\dots,x-y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y},$$

where $d\vec{y}=dy_1\cdots dy_m$.

Let $1\le q<\infty$. We choose $a=(\frac{1}{\sqrt{md}},\dots,\frac{1}{\sqrt{md}})\in\mathbb{S}^{md-1}$, and define $\Omega=\sum_n h_n(\chi_{A_n^+}-\chi_{A_n^-})$ with $h_n=2^{n\delta}$ and $\delta<(md-1)/q$. Here, A_n^+ is a spherical annulus centered at point a whose radius is 2^{-n} and measure $\sim 2^{-(md-1)n}$, and A_n^- is its reflection with respect to the origin. We can easily check that $\Omega\in L^q(\mathbb{S}^{md-1})$.

Let $1\le p_1,\dots,p_m\le\infty$ and $p>0$ be such that $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. We take $f_j=(2\varepsilon)^{-d/p_j}\chi_{[-\varepsilon,\varepsilon]^d}$; then $\|f_j\|_{L^{p_j}(\mathbb{R}^d)}=1$ for $j=1,\dots,m$. Let I_d be as in (5) and let N_ε be a $c_d'\varepsilon$-neighborhood of I_d, then we can verify that

$$T_{K_\Omega}(f_1,\dots,f_m)(x) \geq c\varepsilon^{-\frac{d}{p}}\varepsilon \sum_{n:\ 2^{-n}\leq\varepsilon} |A_n^+|h_n \sim c\varepsilon^{-\frac{d}{p}+md-\delta}$$

for all $x \in N_\varepsilon$. Therefore

$$\|T_{K_\Omega}\|_{L^{p_1}\times L^{p_m}\to L^{p,\infty}} \gtrapprox \varepsilon^{md-\frac{1}{p}-\delta},$$

which tends to ∞ as $\varepsilon \to 0$ when $md < \frac{1}{p} + \frac{md-1}{q}$ if we choose δ close to $\frac{md-1}{q}$. It is straightforward to verify Lemma 6 in the multilinear setting under the condition $r > \frac{1}{md-1}$. In summary, we have showed the following.

Proposition 6. *For any $1 \leq q < \infty$ there is an odd function Ω in $L^q(\mathbb{S}^{md-1})$ such that the associated kernel K_Ω satisfies Hörmander's condition* (1) *but the Calderón-Zygmund operator T_{K_Ω} does not map $L^{p_1}(\mathbb{R}^d)\times\cdots\times L^{p_m}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ whenever $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$, $1\leq p_1,\dots,p_m\leq\infty$, and $\frac{1}{p}+\frac{md-1}{q} > md$. In particular, this operator is not of weak type $(1,\dots,1,\frac{1}{m})$ when $1 \leq q < \frac{md-1}{m(d-1)}$.*

Remark 2. It is known from [1] that the m-linear operator T_{K_Ω} is bounded from $L^2(\mathbb{R}^d)\times\cdots\times L^2(\mathbb{R}^d)$ to $L^{2/m}(\mathbb{R}^d)$ whenever $\Omega \in L^q(\mathbb{S}^{md-1})$ with $q > \frac{2m}{m+1}$. Thus, in the multilinear case, boundedness on the product of L^2 spaces and Hörmander's condition are not sufficient to yield the weak type $(1,1,\dots,1,1/m)$ endpoint when $d \leq 2$.

REFERENCES

[1] E. Buriánková, D. He, and P. Honzík, *Multilinear rough singular integrals*, in preparation.

[2] L. Chaffee, R. H. Torres, and X. Wu, *Multilinear weighted norm inequalities under integral type regularity conditions*, Harmonic analysis, partial differential equations and applications, 193–216, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2017

[3] R. R. Coifman; Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*. Transactions American Mathematical Society **212** (1975), 315–331.

[4] R. R. Coifman and Y. Meyer, *Commutateurs d' intégrales singulières et opérateurs multilinéaires*, Ann. de l' Inst. Fourier, Grenoble **28** (1978), 177–202.

[5] R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-différentiels*, Astérisque **57**, 1978.

[6] G. Diestel, L. Grafakos, P. Honzík, Z. Si, E. Terwilleger, *Method of rotations for bilinear singular integrals*, Commun. Math. Anal. 2011, Conference 3, 99–107.

[7] L. Grafakos, *Modern Fourier Analysis*, Third edition, Graduate Texts in Mathematics **250**, Springer, New York, 2014.

[8] L. Grafakos, D. He, and P. Honzík, *Rough bilinear singular integrals*, Adv. Math. **326** (2018), 54–78.

[9] L. Grafakos, D. He, and L. Slavíková, $L^2 \times L^2 \to L^1$ *boundedness criteria*, Math. Ann., to appear.

[10] L. Grafakos and X. Li, *Uniform bounds for the bilinear Hilbert transforms, I*, Ann. of Math. **159** (2004), 889–933.

[11] L. Grafakos and R. H. Torres, *Multilinear Calderón–Zygmund theory*, Adv. Math. **165** (2002), 124–164.

[12] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[13] C. E. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. Volume **6** (1999), 1–15.

[14] X. Li, *Uniform bounds for the bilinear Hilbert transform II.* Rev. Mat. Iberoam. **22** (2006), 1069–1126.

[15] J. M. Martell, C. Pérez, and R. Trujillo-González, *Lack of natural weighted estimates for some singular integral operators*, Trans. Amer. Math. Soc. **357** (2005), 385–396.

[16] K. Li, *Sparse domination theorem for multilinear singular integral operators with L^r-Hörmander condition*, Mich. Math. J. **67** (2018), 253–265.

[17] C. Pérez and R. H. Torres, *Minimal regularity conditions for the end-point estimate of bilinear Calderón-Zygmund operators*, Proc. Amer. Math. Soc. Ser. B **1**, (2014), 1–13.

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