SOME PROBLEMS IN HARMONIC ANALYSIS

WITH CONTRIBUTIONS BY

ALMUT BURCHARD, CIPRIAN DEMETER,
MICHAEL GOLDBERG, ALEXANDER KISELEV,
BEN KRAUSE, DETLEF MÜLLER,
ALEXANDER REZNIKOV, ANDREAS SEEGER,
CHRISTOPH THIELE, JAMES WRIGHT

In May 2016, we organized a conference in harmonic analysis in honor of Professor Michael Christ, on the campus of the University of Wisconsin in Madison. We are happy to present sixteen open problems, almost all of which were contributed by participants of a problem session held in the afternoon of May 19, 2016.

Loukas Grafakos
Diogo Oliveira e Silva
Malabika Pramanik
Andreas Seeger
Betsy Stovall

1. A Convolution Inequality on Spheres

Contributed by Almut Burchard

How can one extend the *Riesz–Sobolev inequality*

(1)
$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(x-y)h(y) \, \mathrm{d}x \, \mathrm{d}y \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^{\#}(x)g^{\#}(x-y)h^{\#}(y) \, \mathrm{d}x \, \mathrm{d}y$$

to spheres (and perhaps more general homogeneous spaces)? Here, f, g are nonnegative functions on $\mathbb{R}^d$ that decay at infinity, and $f^{\#}, g^{\#}$ are their symmetric decreasing rearrangements. The simplest case, where the middle function $g(x-y) = g^{\#}(x-y)$ is a decreasing function of the distance between x and y, extends directly from $\mathbb{R}^d$ to the standard spheres and hyperbolic spaces. The full inequality, where all three functions may vary, is equivalent by a convex duality argument to the statement that

(2)
$$\int \Phi(f * g) \le \int \Phi(f^{\#} * g^{\#})$$

for every increasing convex function Φ on $\mathbb{R}_+$. Eq. (1) is a refinement of the Brunn–Minkowski inequality and its proofs use the affine geometry of $\mathbb{R}^n$. A folklore result says that Eq. (1) fails for the natural convolution operator on the orthogonal group $O(d+1)$ (or the sphere S^d), given by

$$(f * g)(Q) = \int_{O(d+1)} f(y)g(Q^{-1}y),$$

where integration is with respect to the Haar measure, and $f^{\#}$ denotes symmetric decreasing rearrangement about the unit element (or the north pole on the sphere). (This result does not seem to exist in the literature. For $d = 2$, the inequality would imply a version of the isoperimetric inequality on $SO(3) = U(2)/\{\pm\}$ which is false for subsets whose complement is small.)

Consider restricting the convolution from the orthogonal group to the conjugacy class of reflections. For any vector $u \in S^d$ (viewed as the unit sphere in $\mathbb{R}^{d+1}$), denote by σ_u the Euclidean reflection on the hyperplane $u^{\perp} \subset \mathbb{R}^{d+1}$. Define a *convolution operator* for functions f, g on S^d by

$$(f * g)(u) = \int_{S^d} f(x)g(\sigma_u(x)) \, \mathrm{d}x \,.$$

Problem 1: *Given the above convolution on the sphere, does Eq. (2) hold for every convex increasing function Φ on $\mathbb{R}_+$? Here, $f^{\#}, g^{\#}$ are the symmetric decreasing rearrangements of f and g, i.e., $f^{\#}, g^{\#}$ are equimeasurable with f, g, and are non-increasing functions of the distance from the north pole.*

Clearly, $f * g$ is symmetric under the antipodal map $u \to -u$. Note also that $f^{\#} * g^{\#}$ is symmetric decreasing about the equator. By duality, an equivalent question is whether $\int (f * g)h$ increases if f, g are replaced by $f^{\#}, g^{\#}$, and h is rearranged to be symmetric decreasing about the equator.

The question is open in any dimension $d > 1$. On S^1, Eq. (1) reduces to a theorem of A. Baernstein [1], because the convolution operator agrees with the standard one except for a dilation.

The convolution operator appears in connection with two-point symmetrizations on the sphere [8], [16]. To provide some evidence in favor of the inequality, consider the case $\Phi(t) = t$. Fubini's theorem and the change of variables $y = \sigma_u(x)$ yield

$$\int_{S^d} (f * g)(u) \, \mathrm{d}u = \int_{S^d} \int_{S^d} f(x)g(y)|x - y|^{-(d-1)} \, \mathrm{d}x \, \mathrm{d}y \,,$$

which increases under symmetrization, since the Jacobian $|x - y|^{-(d-1)}$ is a symmetric decreasing function of the distance.

A. Burchard, Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4.

E-mail address: `almut@math.toronto.edu`

2. Questions about generalized additive energy

Contributed by Ciprian Demeter

For sets $A \subset \mathbb{R}^d$ consider the generalized additive energy

$$\mathbb{E}_n(A) = \#\{(a_1, \ldots, a_{2n}) \in A^{2n} : a_1 + \ldots + a_n = a_{n+1} + \ldots + a_{2n}\}.$$

Let $S^d \subset \mathbb{R}^{d+1}$ denote the d-dimensional unit sphere.

Problem 2: *Prove (or disprove) that $\mathbb{E}_2(A) \lesssim_\epsilon (\#A)^{2+\epsilon}$ if $A \subset S^2$.*

This is known for subsets of the two dimensional paraboloid $A \subset P^2$, see [5].

Problem 3: *Prove (or disprove) that $\mathbb{E}_3(A) \lesssim_\epsilon (\#A)^{3+\epsilon}$ if $A \subset S^1$ or $A \subset P^1$.*

For S^1, this would follow from the unit distance conjecture. The best known unconditional bound is $O(\#A^{7/2})$ via the Szemerédi–Trotter theorem (see [3]).

In the case where all points in A are $(\#A)^{-O(1)}$-separated all conclusions follow from decoupling theorems in [5].

C. Demeter, Department of Mathematics, Indiana University, Bloomington, IN 47405, USA.

E-mail address: `demeterc@indiana.edu`

3. Functions whose Fourier transform vanishes on the sphere

Contributed by Michael Goldberg

Problem 4: *Let $n \geq 2$. Does there exist a function $f \in L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)$ such that*

$$\widehat{f}\big|_{S^{n-1}} \equiv 0$$

and

$$\left| |1 - |\xi|^2 \right|^{-1/2} \widehat{f}(\xi) \notin L^2(\mathbb{R}^n)?$$

The Stein-Tomas theorem, and approximation by Schwartz functions, implies that restrictions of $\hat{f}(\xi)$ to rS^{n-1} are continuous in $L^2(S^{n-1})$ with respect to $r > 0$. Local integrability then dictates that vanishing of $\hat{f}$ on the unit sphere is necessary for $|1 - |\xi|^2|^{-1/2}\hat{f}$ to belong to $L^2(\mathbb{R}^n)$. We are curious whether it is a sufficient condition as well. The problem is phrased to suggest that perhaps it is not, and asks for a counterexample if one exists.

The more general question of finding sufficient conditions for $|1 - |\xi|^2|^{-\alpha}\hat{f} \in L^2(\mathbb{R}^n)$ is resolved on both sides of $\alpha = \frac{1}{2}$. For $0 \leq \alpha < \frac{1}{2}$ it suffices for $f \in L^{\frac{2n+2}{n+1+4\alpha}}(\mathbb{R}^n)$ [2]. For $\frac{1}{2} < \alpha < \frac{3}{2}$ it also suffices for $f \in L^{\frac{2n+2}{n+1+4\alpha}}(\mathbb{R}^n)$ so long as $\hat{f}|_{S^{n-1}} \equiv 0$ [24].

There is at least a superficial resemblance to the role of Hardy spaces in fractional integration. When $n = 1$ the resemblance is explicit. The sufficient condition for $|1 - |\xi|^2|^{-1/2}\hat{f}$ to belong to $L^2(\mathbb{R})$ is that $e^{ix}f(x), e^{-ix}f(x) \in H^1(\mathbb{R})$. This is a strictly stronger condition than $\hat{f}(\pm 1) = 0$, and one can construct solutions to the Problem in $L^1(\mathbb{R})$ based on functions such as

$$g_N(x) = \eta(x) - 2N^{-1}\cos(x)\eta(N^{-1}x)$$

and

$$h_N(x) = \eta(x - 2\pi N) - \eta(x + 2\pi N),$$

where η is a Schwartz function for which $\widehat{\eta}(\xi) = 1$ for all $|\xi| \leq 2$.

These examples don't seem to adapt well to higher dimensions, where the curvature of S^{n-1} turns into an impediment rather than an asset.

M. Goldberg, Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA.

E-mail address: `Michael.Goldberg@uc.edu`

4. REGULARITY OR BLOW UP FOR A 1D MODEL OF THE 3D EULER EQUATION

Contributed by Alexander Kiselev

The 3D Euler equation of fluid mechanics in vorticity form is given by

$$(3) \qquad \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \ \ u = \mathrm{curl}(-\Delta)^{-1}\omega, \ \ \omega(x,0) = \omega_0(x), \ \ \nabla \cdot \omega_0 = 0.$$

The equation can be set on $\mathbb{R}^3$ with decaying initial conditions, or on $\mathbb{T}^3$, or on a domain $\Omega \subset \mathbb{R}^3$, in which case boundary conditions are imposed on $\partial\Omega$. The question whether the solutions to 3D Euler equation remain globally smooth is a major open problem. One can consult [34] or [35], for example, for history and more details. For the 2D Euler equation, the term on the right hand side of (3) $-$ often called the vortex stretching term $-$ vanishes. Due to this simpler structure, the global regularity for 2D Euler equation has been known since the 1930s.

Here is a very natural 1D model of (3), proposed by De Gregorio [18, 19].

$$(4) \qquad \partial_t \omega + u\partial_x \omega = \omega\partial_x u, \ \ u_x = H\omega, \ \ \omega(x,0) = \omega_0(x).$$

Here $H\omega = \frac{1}{\pi}p.v. \int_{\mathbb{R}} \frac{\omega(y)}{\cdot -y}\,dy$ is the Hilbert transform of ω. For simplicity, let us think of (4) with periodic initial data. Note that the Biot–Savart law $u_x = H\omega$ is the direct and the only natural 1D analog of $u = \mathrm{curl}(-\Delta)^{-1}\omega$.

Before talking about (4), let us consider two simpler cases.

Case 1.
$$\partial_t \omega + u\partial_x \omega = 0, \ \ u_x = H\omega, \ \ \omega(x,0) = \omega_0(x).$$

This is the analog of the 2D Euler equation. Global regularity of solutions can be established similarly to that case.

Case 2.

$$(5) \qquad \partial_t \omega = \omega\partial_x u, \ \ u_x = H\omega, \ \ \omega(x,0) = \omega_0(x).$$

This equation has been considered by Constantin, Lax and Majda [13]. They proved that finite time blow up can happen for certain kinds of initial data. Moreover, the equation is explicitly solvable; by using the properties of the Hilbert transform, one can show that

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}.$$

The specific properties of the Hilbert transform that are used are $H^2 = -I$ and $H(\omega H\omega) = \frac{1}{2}((H\omega)^2 - \omega^2)$.

Here are two problems connected with this theme.

Problem 5: *What happens in the full De Gregorio model (4)? Can the nonlinear advection term $u\partial_x\omega$ somehow balance and cancel the effect of the term that produces blow up in the Constantin–Lax–Majda model? The mechanism for such cancellation is not clear, and so a reasonable first guess at the properties of the equation may be the possibility of finite time blow up. But in fact numerical simulations of (4) suggest global regularity [40, 46].*

Problem 6: *Consider an analog of the Constantin–Lax–Majda model on $\mathbb{R}^2$:*

$$\partial_t \omega = \omega \mathcal{C}\omega, \ \ \omega(x,0) = \omega_0(x),$$

where $\mathcal{C}$ is some Calderón-Zygmund operator in two dimensions. The magical properties of the Hilbert transform that allow (5) to be solved explicitly are gone. What can be said now? Is there still finite time blow up, and if so, how to understand its mechanism in a more robust way? This problem (or rather, a closely related one) is stated by Constantin and Sun [14] in connection with some complex fluids models.

A. Kiselev, Department of Mathematics, Rice University, Houston, TX 77005, USA.
E-mail address: `kiselev@rice.edu`

5. A MULTI-FREQUENCY CARLESON PROBLEM

Contributed by Ben Krause

Let $\{\theta_1, \ldots, \theta_K\} \subset \mathbb{R}$ be a collection of one-separated frequencies,

$$\min_{i \neq j} |\theta_i - \theta_j| \geq 1.$$

Let $P(t) := 1_{t<0}$ and $1_{[-c,c]} \leq \phi \leq 1_{[-2c,2c]}$, $1_{[-5c,5c]} \leq \varphi \leq 1_{[-10c,10c]}$ where ϕ, φ are Schwartz functions and $0 < c \ll 1$ is sufficiently small. Set

$$\widehat{S_\lambda f}(\xi) := P(\xi - \lambda)\phi(\xi - \lambda)\widehat{f}(\xi).$$

The following question arose in work on a discrete Carleson operator along the primes [12].

Problem 7: *What is the correct operator-norm growth in K of the "multi-frequency Carleson operator"*

$$\mathcal{C}f(x) := \sup_{|\lambda| \leq c} \left| \sum_{i=1}^{K} e(\theta_i x) S_\lambda(\varphi(\xi)\widehat{f}(\xi + \theta_i))^\vee(x) \right|?$$

Here, $e(t) := e^{2\pi i t}$. For applications, one may assume that $\{\theta_1, \ldots, \theta_K\}$ are rational frequencies each of whose denominators are comparable in size to K.

Remark. In [12, §3] it is shown that the L^2-operator norm of $\mathcal{C}$ grows like $\log^2 K$. This was accomplished by adapting a subtle argument due to Bourgain [4], using crucially the variational Carleson theorem due to Oberlin, Seeger, Tao, Thiele, and Wright [39].

Unfortunately, away from L^2, the argument of Bourgain breaks down, and the most efficient (cheap) way we found to estimate $\mathcal{C}$ in L^p, $p \neq 2$, was through Cauchy–Schwarz, vector-valued Carleson [28], and Rubio de Francia [41], keeping track of the loss of orthogonality near L^1 (see [31]). This lead to the following estimates

$$\|\mathcal{C}\|_{L^p \to L^p} \lesssim \begin{cases} K^{1/p} & \text{if } 1 < p \leq 2 \\ K^{1/2} & \text{if } 2 \leq p < \infty \end{cases}.$$

These estimates were strong enough to prove, for instance, the following discrete Carleson theorem along the primes:

$$\left\| \sup_{k \geq 0} \left| \sum_{p \in \pm \mathbb{P}} f(n-p) \frac{e(2^{-k}p)}{p} \log|p| \right| \right\|_{\ell^p(\mathbb{Z})} \lesssim \|f\|_{\ell^p(\mathbb{Z})}, \ 3/2 < p < 4,$$

where $\pm\mathbb{P} := \{\pm p : p \text{ is a prime}\}$. (The $\log|p|$ factor is inserted for normalization purposes.) The expected range $1 < p < \infty$ would follow from an estimate of the form

$$\|\mathcal{C}\|_{L^p \to L^p} \lesssim_\epsilon K^\epsilon$$

for every $\epsilon > 0$.

B. Krause, Department of Mathematics, The University of British Columbia, Vancouver, B.C., Canada V6T 1Z2.

E-mail address: benkrause@math.ubc.ca

6. Removal of Condition (R) for Fourier Restriction to Hypersurfaces in three dimensions

Contributed by Detlef Müller

Let S be a smooth hypersurface in $\mathbb{R}^3$ with Riemannian surface measure $d\sigma$. Assume that S is of *finite type*, i.e., that every tangent plane has finite order of contact with S, and consider the compactly supported measure $d\mu := \rho\, d\sigma$ on S, where $0 \leq \rho \in C_0^\infty(S)$. In my joint monograph with I. Ikromov [29], we have investigated the possibility of Stein–Tomas type restriction estimates of the form

$$(6) \qquad \left(\int_S |\widehat{f}|^2 \, d\mu \right)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

in an almost complete way. More precisely, we have determined the optimal range of p's for which (6) holds true, in terms of Newton polyhedra associated to S, provided S satisfies a certain condition (R). Every real-analytic surface S does satisfy (R), but we do not know if our description of this range remains valid for arbitrary finite type hypersurfaces without condition (R).

The following is a good toy model for testing this question.
Example. Let S be the graph of the function

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n, \qquad n, m \geq 2,$$

over a small neighborhood U of the origin in $\mathbb{R}^2$. The coordinates (x_1, x_2) are not adapted to the function ϕ in the sense of Varchenko, and the so-called Newton distance of ϕ in these coordinates (x_1, x_2) is given by

$$d := \frac{nm}{m+1}.$$

Adapted coordinates are $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by $\phi^a(y_1, y_2) = y_2^n$. As explained in our papers, from the (extended) Newton polyhedron of ϕ^a one can compute the so-called r-height h^r of ϕ, and our main theorem implies that (6) holds true if and only if $p' \geq p'_c = 2(h^r + 1)$. In this example, we find that the critical conjugate exponent p'_c is given by

$$p'_c = \begin{cases} 2d + 2, & \text{if } n \leq m+1, \\ 2n, & \text{if } n > m+1. \end{cases}$$

Problem 8: *Does this result remain valid if we perturb ϕ by a smooth function which is flat at the origin (provided we choose U sufficiently small), for instance for a perturbed function*

$$\phi_1(x_1, x_2) := (x_2 - x_1^m)^n + \varphi(x_1),$$

where all derivatives of φ vanish at the origin? For non-trivial φ, such a function ϕ_1 does not satisfy our condition (R), in contrast with $\phi_2(x_1, x_2) := (x_2 - x_1^m - \varphi(x_1))^n$.

For more information on this question the reader is referred to [29], [37], [38].

D. Müller, Mathematisches Seminar, Christian-Albrechts-Universität Kiel, D-24118 Kiel, Germany.
E-mail address: `mueller@math.uni-kiel.de`

7. Asymptotic behavior of the N-th polarization constant for "fat" Cantor sets

Contributed by Alexander Reznikov

Let $\mathcal{C} \subset [0,1]$ denote a "fat" Cantor set, where "fat" means $m_1(\mathcal{C}) > 0$, and m_1 denotes the Lebesgue measure on $[0,1]$. For the sake of this problem, one can take any particular Cantor set. Fix a number $s > 1$.

By ω_N we denote a set of points with possible repetitions, with cardinality $\#\omega_N = N$, counting multiplicities. For any such ω_N denote

$$P_s(\mathcal{C}; \omega_N) := \inf_{y \in \mathcal{C}} \sum_{x_j \in \omega_N} \frac{1}{|y - x_j|^s}.$$

Finally, denote

$$\mathcal{P}_s(\mathcal{C}; N) := \sup_{\omega_N \subset \mathcal{C}} P_s(\mathcal{C}; \omega_N).$$

It is well known, see [17], that there exist positive constants c_1 and c_2, such that

$$c_1 N^s \leqslant \mathcal{P}_s(\mathcal{C}; N) \leqslant c_2 N^s, \quad N \geqslant 2.$$

Problem 9: *Prove that*

$$\lim_{N \to \infty} \frac{\mathcal{P}_s(\mathcal{C}; N)}{N^s}$$

exists. Further, prove (or disprove)

$$\lim_{N \to \infty} \frac{\mathcal{P}_s(\mathcal{C}; N)}{N^s} = \frac{2(2^s - 1)\zeta(s)}{m_1(\mathcal{C})^s},$$

where $\zeta(s)$ is the classical Riemann zeta-function.

We remark that for the circle $\mathbb{T}$ instead of $\mathcal{C}$, it is known that the limit is equal to

$$\lim_{N \to \infty} \frac{\mathcal{P}_s(\mathbb{T}; N)}{N^s} = \frac{2(2^s - 1)\zeta(s)}{(2\pi)^s}.$$

We believe that the numerator should remain the same, and that the denominator is always the "length" of our set, raised to the power s.

A. Reznikov, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: `aleksandr.b.reznikov@vanderbilt.edu`

8. Three problems in Harmonic Analysis

Contributed by Andreas Seeger

8.1. **Convolutions with radial kernels.** Let σ_r be the surface measure on the sphere of radius r in $\mathbb{R}^d$, $d \geq 2$, and let η be a Schwartz function in $\mathcal{S}(\mathbb{R}^d)$ whose Fourier transform is zero on a neighborhood of the origin.

Problem 10: *For* $1 < p < 2$, $d > \frac{p}{2-p}$,

$$(7) \qquad \left\| \int_1^\infty (\eta * \sigma_r * g)(\cdot, r) \, \mathrm{d}r \right\|_{L^p(\mathbb{R}^d)} \leq C(p, d, \eta) \left(\int_1^\infty \int_{\mathbb{R}^d} |g(x, r)|^p \, \mathrm{d}x \, r^{d-1} \, \mathrm{d}r \right)^{\frac{1}{p}}.$$

If one applies the inequality to functions of the form $g(x, r) = f(x)k(r)$, one can derive effective characterizations of radial multipliers of $\mathcal{F}L^p(\mathbb{R}^d)$. In particular, if K is radial and $\widehat{K}$ is compactly supported away from the origin, then $\|\widehat{K}\|_{M^p} \approx \|K\|_{L^p}$. More generally, let ϕ be a nontrivial $C_c^\infty(\mathbb{R}^+)$ function. Then (7) implies

$$\|m(|\cdot|)\|_{M^p(\widehat{\mathbb{R}^d})} \approx \sup_{t>0} \left\| \mathcal{F}^{-1}[\phi(|\cdot|)m(t|\cdot|)] \right\|_{L^p(\mathbb{R}^d)}.$$

Thus, for such p, a convolution operator with radial kernel is bounded on $L^p(\mathbb{R}^d)$ if and only if it is bounded on the subspace $L_{\mathrm{rad}}^p(\mathbb{R}^d)$ consisting of radial L^p functions, cf. [23]. These statements are false if $2 \leq d \leq \frac{p}{2-p}$ (as then $\mathcal{F}L^p(\mathbb{R}^d)$ contains unbounded radial functions which are compactly supported away from the origin).

Inequality (7) is also known to imply an endpoint version of the so-called local smoothing conjecture (see [45] for the wave equation, and for an endpoint estimate for cone multipliers in $\mathbb{R}^{d+1}$, see [25], [26]). Consult [47], [48] for implications on the Bochner–Riesz, Fourier restriction, Kakeya–Nikodym, and other related conjectures.

Note that (7) is trivial for $p = 1$. Given $1 < p < 2$, it was shown in [25] that (7) holds in dimensions $d > \frac{p+2}{2-p}$. This gives a known nontrivial range of p in dimensions $d > 3$. See also [11] on improvements of (7) for special classes of functions. Weaker forms of the conjecture are known to hold in larger ranges, for example

$$(8) \qquad \left\| \int_1^\infty (\eta * \sigma_r * g)(\cdot, r) \, \mathrm{d}r \right\|_{L^p(\mathbb{R}^d)} \leq C_\varepsilon(p, d, \eta) \left(\int_1^\infty \int_{\mathbb{R}^d} |g(r, x)|^p \, \mathrm{d}x \, r^{d-1+\varepsilon} \, \mathrm{d}r \right)^{\frac{1}{p}}$$

holds for all $\varepsilon > 0$ if $d \geq \frac{3p-2}{2-p}$ (of course for fixed ε the range is better, depending on ε). This follows from the validity of the Wolff $\ell^{p'}$-decoupling inequalities ([52], [32]) in the optimal range $p' > \frac{2(d+1)}{d-1}$ which is a consequence of the more recent results by Bourgain and Demeter [5]. In particular (8) holds for all $\varepsilon > 0$ for the nontrivial ranges $p \in [1, \frac{6}{5}]$ in dimension two, and $p \in [1, \frac{4}{3}]$ in dimension three.

8.2. **Hyperbolic cross maximal function.** Let Φ be a C_c^∞ function on the real line and let $\Phi(0) \neq 0$. Define the operator T_t acting on functions in $\mathbb{R}^2$ by $\widehat{T_t f}(\xi) = \Phi(t^{-1}\xi_1^2 \xi_2^2)\widehat{f}(\xi)$. The operators T_t are bounded on $L^p(\mathbb{R}^2)$, with the same operator norm.

Problem 11: *Does the hyperbolic cross maximal inequality*

$$(9) \qquad \left\| \sup_{k \in \mathbb{Z}} |T_{2^k} f| \right\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$$

hold for some p?

If (9) held for some $p_0 \in (1, \infty)$, then (9) would hold for all $p \in (1, \infty)$, by standard Calderón–Zygmund techniques, see e.g. [15]. An affirmative answer would imply an affirmative answer to a question posed by Dappa and Trebels in [15], concerning the hyperbolic Riesz means H_s^λ defined by

$$\widehat{H_s^\lambda f}(\xi) = \Big(1 - \frac{\xi_1^2 \xi_2^2}{s^2}\Big)_+^\lambda \widehat{f}(\xi).$$

For sufficiently large λ, and all $f \in L^p(\mathbb{R}^2)$, does $H_s^\lambda f(x)$ converge to $f(x)$ almost everywhere, as $s \to \infty$?

8.3. **Rough singular integrals on L^1.** Let $d \geq 2$, $q > 1$, $\Omega \in L^q(S^{d-1})$, and assume $\int_{S^{d-1}} \Omega(x') \, d\sigma(x') = 0$. Let $0 < \varepsilon < 1$ and define the family of rough singular integral operators $\{T_\varepsilon\}$ by

$$T_\varepsilon f(x) = \int_{\varepsilon < |y| \leq 1} |y|^{-d} \Omega\Big(\frac{y}{|y|}\Big) f(x - y) \, dy.$$

It was shown in [42] that the operators T_ϵ are weak type $(1, 1)$ with constants independent of ε, see also [10], [27] for earlier results in low dimensions, and [49] for an extension to stratified groups.

Problem 12: *For $f \in L^1(\mathbb{R}^d)$, does $\lim_{\varepsilon \to 0} T_\varepsilon f(x)$ exist for almost every $x \in \mathbb{R}^d$?*
Equivalently, does $\sup_{0 < \varepsilon < 1} |T_\varepsilon f(x)|$ define an operator of weak type $(1, 1)$?

This problem has been around for a while and was proposed again in [43]. Andrei Lerner (personal communication, El Escorial 2016) pointed out renewed interest in it, in connection with problems on control of rough singular integral operators via sparse operators, and related questions about weighted norm inequalities, cf. [33].

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA.
E-mail address: `seeger@math.wisc.edu`

9. Triangular Hilbert transform

Contributed by Christoph Thiele

The triangular Hilbert transform is given as the principal value integral

$$\Lambda_3(f,g,h) = p.v. \iiint f(x,y)g(y,z)h(z,x)\frac{1}{x+y+z} \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where f, g, h are three test functions in the plane.

Problem 13: *Prove the a priori estimate*

$$|\Lambda_3(f,g,h)| \leq C\|f\|_{L^3}\|g\|_{L^3}\|h\|_{L^3},$$

with a universal constant C.

The problem will be solved if one proves any similar estimate with Lebesgue exponents 3 replaced by p, q, r on the right hand side, which then by homogeneity of the problem have to satisfy $1/p+1/q+1/r = 1$. Namely, one will have the symmetric estimate under permutations of the exponents, and interpolation will yield the central estimate with all exponents 3.

The triangular Hilbert transform is one of a family of simplex Hilbert transforms, one for each degree of multilinearity. The adjacent degrees are

$$\Lambda_2(f,g) = p.v. \iint f(x)g(y)\frac{1}{x+y} \, \mathrm{d}x\,\mathrm{d}y,$$

$$\Lambda_4(f,g,h,k)$$
$$= p.v. \iiiint f(x,y,z)g(y,z,u)h(z,u,x)k(u,x,y)\frac{1}{x+y+z+u} \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}u.$$

It turns out Λ_2 is equivalent to the classical Hilbert transform with the well known and relatively easy estimate

$$|\Lambda_2(f,g)| \leq C\|f\|_{L^2}\|g\|_{L^2},$$

while the estimate

$$|\Lambda_4(f,g,h,k)| \leq C\|f\|_{L^4}\|g\|_{L^4}\|h\|_{L^4}\|k\|_{L^4}$$

appears well out of reach of present technology. The case Λ_3 also requires some new ideas, but at least there is an encouraging variety of recent results in the vicinity of the problem which seem to suggest some helpful technology and possible routes of attack.

There is a dyadic model for the triangular Hilbert transform. A dyadic interval is an interval of the form $I = [2^k n, 2^k(n+1))$ with integers k, n where we assume n non-negative. If I_1, I_2, I_3 are three such dyadic intervals of the same length, we write $I_1 \oplus I_2 \oplus I_3 = 0$ if $n_1 \oplus n_2 \oplus n_3 = 0$, where addition is that of the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ identified with the natural numbers via binary expansion of the latter. The dyadic model is then

$$\sum_{k\in\mathbb{Z}} 2^{-k}\epsilon_k \sum_{|I|=|J|=|K|=2^k, I\oplus J\oplus K=0} \iiint f(x,y)g(y,z)h(z,x)h_I(x)h_J(y)h_K(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where ϵ_k are numbers in $[-1,1]$ and h_I is the Haar function associated to the dyadic interval I, normalized so that $\|h_I\|_{L^\infty} = 1$. If all ϵ_K are equal to 1, the form telescopes into the integral of the pointwise product fgh. Proving L^3 bounds as in the above problem for this

model operator is expected to be somewhat easier than the main problem, but be significant in terms of principle ideas for the attack on the main problem.

Of the many papers on the triangular Hilbert transform and related problems, we single out the recent ones [30] and [21] and refer for further contributions to the chain of references therein.

The paper [30] proves the desired bounds for the dyadic model of the triangular Hilbert transform under crucially simplifying structural assumptions on one of the three functions f, g, h, such as for example elementary tensor structure. Two types of structural assumptions are discussed there, leading to connections with objects of study in time-frequency analysis such as Carleson's operator controlling almost everywhere convergence of Fourier series and the bilinear Hilbert transform. It is expected that for the main problem time-frequency analysis has to be replaced by a more general theory that is not based on Fourier analysis but on more general expansions. Speculation goes as far as the circle of ideas around the recently solved Kadison–Singer problem [36] and its consequences for example on Riesz frames.

The paper [21] proves bounds on the truncated triangular Hilbert transform

$$\Lambda_{3,N}(f,g,h) = \iiint_{1<|x+y+z|<N} f(x,y)g(y,z)h(z,x)\frac{1}{x+y+z} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

with bounds on the growth of the constant that are in between trivial and desired, namely

$$|\Lambda_{3,N}(f,g,h)| \leq C(\log N)^{1/2}\|f\|_{L^3}\|g\|_{L^3}\|h\|_{L^3}.$$

The quest for such growth estimates goes back to [50] and [53], trivial being the power $(\log N)^1$ and desired being $(\log N)^0$. More generally, the paper [21] proves similar estimates for all simplex Hilbert transforms, with less improvement on the exponent as the degree of multilinearity increases. The technique in this paper is called "twisted technology", which also recently played a role in ergodic theory [22], proving quantitative bounds on convergence of ergodic means for two commuting transformations.

C. Thiele, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany.

E-mail address: thiele@math.uni-bonn.de

10. Problems on lacunary maximal functions

Contributed by James Wright

We consider a compactly supported finite Borel measure μ on $\mathbb{R}^d$ and define its dyadic dilates by $\langle \mu_k, f \rangle = \langle \mu, f(2^k \cdot) \rangle$. We present here three open problems about the corresponding lacunary maximal function given by

$$\mathcal{M}f(x) = \sup_{k \in \mathbb{Z}} |f * \mu_k(x)|.$$

The dilates 2^k can be replaced by more general lacunary dilates λ_k satisfying $\inf_k \lambda_{k+1}/\lambda_k > 1$. For more details and other open problems, see [44].

For the first two problems, suppose that μ satisfies the Fourier decay condition

$$(10) \qquad\qquad \left| \widehat{\mu}(\xi) \right| \; \leq \; C \, |\xi|^{-\varepsilon} \;\; \text{for some } \varepsilon > 0.$$

Problem 14: *Does $\mathcal{M}$ satisify a weak-type $(1,1)$ bound? That is, does $\mathcal{M} : L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$?*

To the best of our knowledge no counterexample and no example is known for the case when μ is a singular measure with the decay assumption (10) on $\widehat{\mu}$. We recall that the L^p-boundedness of $\mathcal{M}$ holds in the range $1 < p \leq \infty$, see *e.g.* [20].

Problem 15: *Does $\mathcal{M}$ map the Hardy space $H^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$?*

This is a more accessible problem and it is known to hold for μ being surface measure on the unit sphere S^{d-1}. This is due to M. Christ [9] and represents the first such result for a singular measure μ satisfying (10). More recently Christ's result has been extended to any μ satisfying a dimensional assumption on $\mathrm{supp}(\mu)$, coupled with an optimal L^p smoothing assumption near $p = 1$ for the corresponding averaging operator $f \to f * \mu$. See [44]. These assumptions are known to hold for surface measure on any hypersurface on which the Gaussian curvature does not vanish to infinite order. They also hold for arclength measure on a compact curve in $\mathbb{R}^3$ with nonvanishing curvature and torsion.

For the third and final problem, we examine lacunary maximal functions associated to measures μ which do not necessarily satisfy the Fourier decay condition (10).

Let Ω be a convex open domain in the plane with compact closure so that the origin is contained in Ω. We let σ be the arclength measure on the boundary $\partial\Omega$ and consider the lacunary maximal operator associated to $\partial\Omega$,

$$\mathcal{M}f(x) = \sup_{k \in \mathbb{Z}} \left| \int_{\partial\Omega} f(x - 2^k y) \, \mathrm{d}\sigma(y) \right|.$$

Problem 16: *For any fixed $1 < p < \infty$, give a necessary and sufficient geometric condition on $\partial\Omega$ which guarantees $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^2)$. Below we suggest a possible geometric condition.*

As mentioned above, if $|\widehat{\sigma}(\xi)| = O(|\xi|^{-\varepsilon})$ for some $\varepsilon > 0$, then $\mathcal{M}$ is L^p bounded for all p in the range $1 < p \leq \infty$. Now we are aiming for much weaker conditions.

The decay of $\widehat{\sigma}$ is strongly related to a geometric quantity. Given a unit vector θ, let $\ell^+(\theta)$ be the unique supporting line with θ an outer normal to $\partial\Omega$, i.e. the affine line perpendicular to θ which intersects $\partial\Omega$ so that Ω is a subset of the halfspace $\{x : x = y - t\theta : t > 0, y \in \ell^+(\theta)\}$. Similarly define $\ell^-(\theta)$ as the unique affine line perpendicular to θ which intersects $\partial\Omega$ and Ω is a subset of the halfspace $\{x : x = y + t\theta : t > 0, y \in \ell^-(\theta)\}$. For small $\delta > 0$, define the arcs (or 'caps')

$$\mathcal{C}^\pm(\theta, \delta) = \{y \in \partial\Omega : dist(y, \ell^\pm) \leq \delta\}.$$

By compactness considerations it is easy to see that there exists $\delta_0 > 0$ so that, for all $\theta \in S^1$ and all $\delta < \delta_0$, the arcs $\mathcal{C}^+(\theta, \delta)$ and $\mathcal{C}^-(\theta, \delta)$ are disjoint. Let $\Lambda(\theta, \delta)$ be the maximum of the length of these caps:

$$\Lambda(\theta, \delta) = \max_\pm \sigma(\mathcal{C}^\pm(\theta, \delta)).$$

The analytic significance of this quantity is that it gives a very good estimate for the size of Fourier transform $\widehat{\sigma}$, namely, for every $\theta \in S^1$ and $R \geq 1$,

$$|\widehat{\sigma}(R\theta)| \leq C_\Omega \Lambda(\theta, R^{-1}).$$

This is shown in [6] under the hypothesis for convex domains in the plane with smooth boundary, with no quantitative assumption on the second derivative. The general case follows by a simple approximation procedure.

In [44], the following theorem was established, giving a solution to Problem 16 when $p = 2$.

Theorem. *The operator $\mathcal{M}$ is bounded on $L^2(\mathbb{R}^2)$ if and only if*

$$\sup_{\theta \in S^1} \int_0^{\delta_0} \Lambda(\theta, \delta)^2 \frac{\mathrm{d}\delta}{\delta} < \infty.$$

By testing $\mathcal{M}$ on functions supported in thin strips, one obtains the necessary condition

$$\sup_{\|f\|_{L^p}=1} \|\mathcal{M}f\|_{Lp} \geq c \sup_{\theta \in S^1} \left(\int_0^{\delta_0} \Lambda(\theta, \delta)^p \frac{\mathrm{d}\delta}{\delta} \right)^{1/p}$$

for any p. We note that if $\partial\Omega$ has only one "flat" point near which $\partial\Omega$ can be parametrized as the graph of $C + \exp(-1/|t|^a)$ with $C \neq 0$, then this L^p condition holds if and only if $a < p$. Thus in this case L^2 boundedness of $\mathcal{M}$ holds if and only if $a < 2$.

Problem 16 (precise form): *For any $p \neq 2$, is the geometric condition*

$$\sup_{\theta \in S^1} \int_0^{\delta_0} \Lambda(\theta, \delta)^p \frac{\mathrm{d}\delta}{\delta} < \infty$$

sufficient for $\mathcal{M}$ to be bounded on $L^p(\mathbb{R}^2)$?

J. Wright, Maxwell Institute for Mathematical Sciences and the School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, Scotland.
E-mail address: `J.R.Wright@ed.ac.uk`

Bibliography

[1] A. Baernstein, *Convolution and rearrangement on circle.* Complex Variables and Elliptic Equations **12** (1989), 33–37.

[2] J.-G. Bak, D. McMichael, and D. Oberlin, L^p–L^q *estimates off the line of duality.* J. Austral. Math. Soc. Ser. A **58** (1995), 154–166.

[3] E. Bombieri and J. Bourgain, *A problem on sums of two squares.* Int. Math. Res. Not. IMRN 2015, **no. 11**, 3343–3407.

[4] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets.* With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein. Inst. Hautes Études Sci. Publ. Math. No. **69** (1989), 5–45.

[5] J. Bourgain and C. Demeter, *The proof of the ℓ^2 decoupling conjecture.* Ann. of Math. (2) **182** (2015), 351–389.

[6] J. Bruna, A. Nagel and S. Wainger, *Convex hypersurfaces and Fourier transforms.* Ann. of Math. (2) **127** (1988), 333–365.

[7] A. Burchard, *Cases of equality in the Riesz rearrangement inequality.* Ph.D. thesis, Georgia Institute of Technology, 1994.

[8] A. Burchard and M. Fortier, *Random polarizations.* Adv. Math. **334** (2013), 550–573.

[9] M. Christ, *Weak type (1,1) bounds for rough operators.* Annals of Math. **128** (1988), 19–42.

[10] M. Christ and J.-L. Rubio de Francia, *Weak type* $(1,1)$ *bounds for rough operators, II.* Invent. Math. **93** (1988), 225–237.

[11] L. Cladek, *Radial Fourier multipliers in $\mathbb{R}^3$ and $\mathbb{R}^4$.* Preprint, arXiv:1610.03201.

[12] L. Cladek, K. Henriot, B. Krause, I. Łaba and M. Pramanik, *A discrete Carleson theorem along the primes with a restricted supremum.* Preprint, arXiv:1604.08695.

[13] P. Constantin, P. D. Lax and A. Majda, *A simple one-dimensional model for the three-dimensional vorticity equation.* Comm. Pure Appl. Math. **38** (1985), 715–724.

[14] P. Constantin and W. Sun, *Remarks on Oldroyd-B and related complex fluid models.* Commun. Math. Sci. **10** (2012), 33–73.

[15] H. Dappa and W. Trebels, *On maximal functions generated by Fourier multipliers.* Ark. Mat. **23** (1985), 241–259.

[16] Q. Deng, Master's project. University of Toronto, 2013 (unpublished).

[17] T. Erdélyi and E. B. Saff, *Riesz polarization inequalities in higher dimensions.* J. Approx. Theory **171** (2013), 128–147.

[18] S. De Gregorio, *On a one-dimensional model for the three-dimensional vorticity equation.* J. Stat. Phys. **59** (1990), 1251–1263.

[19] S. De Gregorio, *A partial differential equation arising in a 1D model for the 3D vorticity equation.* Math. Methods Appl. Sci. **19** (1996), 1233–1255.

[20] J. Duoandikoetxea and J.-L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates.* Invent. Math. **84** (1986), 541–561.

[21] P. Durcik, V. Kovač and C. Thiele, *Power-type cancellation for the simplex Hilbert transform.* Preprint, arXiv1608.00156.

[22] P. Durcik, V. Kovač, C. Thiele and K. Škreb, *Norm-variation of ergodic averages with respect to two commuting transformations.* Preprint, arXiv1603.03317.

[23] G. Garrigós and A. Seeger, *Characterizations of Hankel multipliers.* Math. Ann. **342** (2008), 31–68.

[24] M. Goldberg, *The Helmholtz equation with L^p data and Bochner-Riesz multipliers*, Preprint, arXiv:1502.02066. To appear in Math. Res. Lett.

[25] Y. Heo, F. Nazarov and A. Seeger, *Radial Fourier multipliers in high dimensions.* Acta Math. **206** (2011), no. 1, 55–92.

[26] Y. Heo, F. Nazarov and A. Seeger, *On radial and conical Fourier multipliers.* J. Geom. Anal. **21** (2011), 96–117.

[27] S. Hofmann, *Weak* $(1,1)$ *boundedness of singular integrals with nonsmooth kernel.* Proc. Amer. Math. Soc. **103** (1988), 260–264.

[28] T. Hytönen and M. Lacey, *Pointwise convergence of vector-valued Fourier series.* Math. Ann. **357** (2013), 1329–1361.

[29] I. A. Ikromov and D. Müller, Fourier restriction for hypersurfaces in three dimensions and Newton polyhedra. Annals of Mathematics Studies, 260 pp., Princeton University Press, Princeton, NJ, 2016.

[30] V. Kovač, C. Thiele and P. Zorin-Kranich, *Dyadic triangular Hilbert transform of two general and one not too general function.* Forum of Mathematics, Sigma **3** (2015), e25, 27 pp.

[31] B. Krause, *Some optimizations for (maximal) multipliers in L^p.* Preprint, arXiv:1402.1804.

[32] I. Łaba and T. Wolff, *A local smoothing estimate in higher dimensions.* J. Anal. Math. **88** (2002), 149–171.

[33] A. Lerner, *On pointwise estimates involving sparse operators.* New York J. Math. **22** (2016), 341–349.

[34] A. Majda and A. Bertozzi, Vorticity and Incompressible Flow. Cambridge University Press, 2002.

[35] C. Marchioro and M. Pulvirenti, Mathematical Theory of Incompressible Nonviscous Fluids. Applied Mathematical Sciences Series (Springer-Verlag, New York), **96**, 1994.

[36] A. Marcus, D. Spielman and N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer preoblem.* Ann. of Math. (2) **182** (2015), 327–350

[37] D. Müller, L^p-L^2 *Fourier restriction for hypersurfaces in $\mathbb{R}^3$: Part I.* Preprint, arXiv:1208.6090v2.

[38] D. Müller, L^p-L^2 *Fourier restriction for hypersurfaces in $\mathbb{R}^3$: Part II.* Preprint, arXiv:1410.3298v1.

[39] R. Oberlin, A. Seeger, T. Tao, C. Thiele and J. Wright, *A variation norm Carleson theorem.* J. Eur. Math. Soc. (JEMS) **14** (2012), 421–464.

[40] H. Okamoto, T. Sakajo and M. Wunsch, *On a generalization of the Constantin–Lax–Majda equation.* Nonlinearity **21** (2008), 1–15.

[41] J. Rubio de Francia, *A Littlewood-Paley inequality for arbitrary intervals.* Rev. Mat. Iberoam. **1** (1985), 1–14.

[42] A. Seeger, *Singular integral operators with rough convolution kernels.* J. Amer. Math. Soc. **9** (1996), 95–105.

[43] A. Seeger, *A weak type bound for a singular integral.* Rev. Mat. Iberoamericana, **30** (2014), 961–978.

[44] A. Seeger and J. Wright, *Problems on averages and lacunary maximal functions.* Józef Marcinkiewicz Centenary Volume. Banach Center Publications **95** (2011), 235–250.

[45] C. D. Sogge, *Propagation of singularities and maximal functions in the plane.* Invent. Math. **104** (1991), 349–376.

[46] S. Steward and V. Sverak, in preparation.

[47] T. Tao, *The Bochner-Riesz conjecture implies the restriction conjecture.* Duke Math. J. **96** (1999), 363–376.

[48] T. Tao, *The weak-type endpoint Bochner-Riesz conjecture and related topics.* Indiana Univ. Math. J. **47** (1998), 1097–1124.

[49] T. Tao, *The weak-type $(1,1)$ of $L \log L$ homogeneous convolution operators.* Indiana Univ. Math. J. **48** (1999), 1547–1584.

[50] T. Tao, *Cancellation for the multilinear Hilbert transform.* Collect. Math. **67** (2016), 191–206.

[51] L. Wang and M. Madiman, *Beyond the entropy power inequality, via rearrangements.* IEEE Transactions on Information Theory **60** (2014), 5116– 5137.

[52] T. Wolff, *Local smoothing type estimates on L^p for large p.* Geom. Funct. Anal. **10** (2000), 1237–1288.

[53] P. Zorin-Kranich, *Cancellation for the simplex Hilbert transform.* To appear in Math. Res. Lett. Preprint, arXiv:1507.02436.

Editors:

L. Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: `grafakosl@missouri.edu`

D. Oliveira e Silva, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: `dosilva@math.uni-bonn.de`

M. Pramanik, Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, B.C.,, Canada V6T 1Z2
E-mail address: `malabika@math.ubc.ca`

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, 613 Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706, USA
E-mail address: `seeger@math.wisc.edu`

B. Stovall, Department of Mathematics, University of Wisconsin-Madison, 721 Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706, USA
E-mail address: `stovall@math.wisc.edu`