SHARP ESTIMATES FOR FOURIER MULTIPLIERS: LINEAR AND MULTILINEAR THEORY

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Abstract. We present some recent results in the theory of Fourier multipliers. These concern sharp versions of the classical multiplier theorems of Hörmander and of Marcinkiewicz and their bilinear analogues. We also discuss optimal, and in some cases necessary and sufficient, criteria for certain bilinear Fourier multiplier operators to be bounded from $L^2 \times L^2 \to L^1$.

1. The Hörmander multiplier theorem - an introduction

Fourier multipliers are formed by composing three operators: the Fourier transform, multiplication, and the inverse Fourier transform. The *Fourier transform* of a Schwartz function f on $\mathbb{R}^n$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx$$

and its *inverse Fourier transform* is defined as

$$f^\vee(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x\cdot\xi} dx.$$

Using these definitions, a general *Fourier multiplier operator* has the form

$$T_\sigma(f)(x) = \big(\widehat{f}\,\sigma\big)^\vee(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\sigma(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where σ is a bounded function on $\mathbb{R}^n$; the function σ has to be in L^∞ if T_σ is going to be bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some $p \in [1,\infty]$. An old and important problem in harmonic analysis is to find optimal sufficient conditions on σ so that the operator T_σ, initially defined on Schwartz functions, admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself. If this is the case for a given p, then σ is called *an L^p Fourier multiplier*. An easy application of Plancherel's identity $\|f\|_{L^2} = \|\widehat{f}\,\|_{L^2}$ yields that σ is an L^2 Fourier multiplier if and only if σ is a bounded function. Also, duality gives that σ is an L^p Fourier multiplier if and only if it is an $L^{p'}$ Fourier multiplier for any $p \in (1,\infty)$. Then

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it follows by interpolation that σ is also an L^q Fourier multiplier for any q between p and $p' = \frac{p}{p-1}$.

The first nontrivial multiplier result was provided by Mikhlin [39] who proved that if the condition

$$|\partial^\alpha \sigma(\xi)| \le C_\alpha |\xi|^{-|\alpha|}, \qquad \xi \neq 0, \tag{1.1}$$

holds for all multi-indices α with size $|\alpha| \le [n/2]+1$, then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. Here $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where α_j are nonnegative integers denoting the number of differentiations in the jth coordinate. This theorem is well suited for functions that are homogeneous of degree zero and are sufficiently differentiable on the unit sphere. The key point is that the αth derivative of a homogeneous of degree zero function is homogeneous of degree $-|\alpha|$ and the constant C_α in (1.1) is the L^∞ norm of $\partial^\alpha \sigma$ on $\mathbb{S}^{n-1}$ in this case.

Example: The functions

$$\begin{aligned}
\sigma_1(\xi_1, \xi_2, \xi_3) =& \frac{\xi_1 \xi_2 \xi_3}{(\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{3}{2}}} \\
\sigma_2(\xi_1, \xi_2, \xi_3) =& \frac{\xi_2 \xi_3}{i\xi_1^2 + (\xi_2^2 + \xi_3^2)} \\
\sigma_3(\xi_1, \xi_2, \xi_3) =& \frac{\xi_1 \xi_2}{1 + i(\xi_1^2 + \xi_2^2 + \xi_3^2)}
\end{aligned}$$

defined on $\mathbb{R}^3$ satisfy condition (1.1). To verify this assertion we use that σ_1, σ_2 are homogeneous of degree zero and smooth on $\mathbb{S}^2$; this implies that their α-th derivatives are homogeneous of degree $-|\alpha|$. For σ_3 we introduce the homogeneous of degree zero function $F(t, \xi_1, \xi_2, \xi_3) = \xi_1 \xi_2 \big(t^2 + i(\xi_1^2 + \xi_2^2 + \xi_3^2)\big)^{-1}$ on $\mathbb{R}^4$ and we note that its α-th derivative in the variables ξ_1, ξ_2, ξ_3 is homogeneous of degree $-|\alpha|$.

An extension of Mikhlin's result was obtained by Hörmander [37] who showed that the conclusion of Mikhlin's theorem still holds if condition (1.1) is replaced by

$$\sup_{k \in \mathbb{Z}} 2^{-kn+2k|\alpha|} \int_{2^k < |\xi| < 2^{k+1}} |\partial^\alpha \sigma(\xi)|^2 d\xi < \infty. \tag{1.2}$$

To compare conditions (1.1) and (1.2), for every multiindex α we introduce the function

$$M_\alpha(\xi) = |\partial^\alpha \sigma(\xi)| |\xi|^{|\alpha|}.$$

Then condition (1.1) requires M_α to be bounded on $\mathbb{R}^n \setminus \{0\}$ for all α with $|\alpha| \le [n/2]+1$, while (1.2) relaxes this assumption to the weaker requirement that averages of M_α with respect to the L^2 norms over dyadic annuli of the form $\{2^k < |\xi| < 2^{k+1}\}$ are uniformly

bounded. Additionally, it is useful to observe that Hörmander's condition (1.2) can be rewritten in the form

$$\sup_{k\in\mathbb{Z}} \|\partial^\alpha[\sigma(2^k\cdot)]\|_{L^2(A)} < \infty, \tag{1.3}$$

where $A = \{\xi \in \mathbb{R}^n : 1 < |\xi| < 2\}$ denotes the unit annulus in $\mathbb{R}^n$.

In order to obtain a sharp variant of the Hörmander multiplier theorem, we need to introduce derivatives of fractional order. Let Δ be the Laplacian on $\mathbb{R}^n$. We denote by $(I-\Delta)^{s/2}$ the operator given by multiplication by $(1+4\pi^2|\xi|^2)^{s/2}$ on the Fourier transform. Informally speaking, when s is an even natural number, $(I-\Delta)^{s/2}$ corresponds to taking all derivatives of a function up to and including order s. For other $s > 0$, $(I-\Delta)^{s/2}$ corresponds to taking all derivatives of a function up to and including the (potentially) fractional number s. Even when s is an odd integer, $(I-\Delta)^{s/2}$ contains all derivatives of a function up to and including order s, at least when measured in the L^p sense, i.e., for $1 < p < \infty$ and $s \in \mathbb{Z}^+$ we have

$$\left\|(I-\Delta)^{s/2} f\right\|_{L^p} \approx \sum_{|\alpha|\le s} \left\|\partial^\alpha f\right\|_{L^p}. \tag{1.4}$$

The quantity $\|(I-\Delta)^{s/2} f\|_{L^p}$ is denoted by $\|f\|_{L^p_s}$ and is referred to as the L^p Sobolev norm of order s of a function f; if $s \in \mathbb{Z}^+$, this is equivalent with the sum of the L^p norms of all partial derivatives of f up to and including order s, as indicated in (1.4).

A variant of Hörmander's result involving fractional derivatives can be formulated as follows: let $s > 0$ and let Ψ be a Schwartz function whose Fourier transform is supported in the annulus $\{\xi : 1/2 < |\xi| < 2\}$ and which satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. If for some $1 \le r \le 2$ and $s > n/r$, σ satisfies

$$\sup_{k\in\mathbb{Z}} \left\|(I-\Delta)^{s/2}\left[\widehat{\Psi}\sigma(2^k\cdot)\right]\right\|_{L^r(\mathbb{R}^n)} < \infty, \tag{1.5}$$

then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. We would like to point out that in the special case when s is a positive integer and $r = 2$, the present version of the Hörmander multiplier theorem is equivalent to the original one; this can be verified by making use of (1.3) and of the equivalence (1.4).

One may wonder if condition (1.5) still implies that σ is an L^p Fourier multiplier for some $p \in (1,\infty)$ if $s \le \frac{n}{2}$. This is indeed true and, roughly speaking, *the closer p is to 2, the fewer derivatives are needed in condition* (1.5). More precisely, Calderón and Torchinsky [3, Theorem 4.7] showed, via an interpolation argument, that T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself whenever condition (1.5) holds for all p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{s}{n}$ and $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{1}{r}$. It was observed in [22] that the assumption $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{1}{r}$ can be

replaced by a weaker one, namely, by $\frac{1}{r} < \frac{s}{n}$. We observe that the latter condition is also necessary as it is dictated by the embedding of $L^r_s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ (we recall that all L^p Fourier multipliers need to be bounded functions). Recently, Grafakos and Slavíková have eliminated the index r from the statement of this theorem, by showing that T_σ is L^p bounded if $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{s}{n}$ and the $L^{\frac{n}{s},1}(\mathbb{R}^n)$ quasinorm of $(I-\Delta)^{s/2}\big[\widehat{\Psi}\sigma(2^k\cdot)\big]$ is bounded by a finite constant uniformly over all $k \in \mathbb{Z}$. Here $L^{\frac{n}{s},1}$ is the Lorentz space, defined in Section 2.

In addition, it is known that if T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself for every σ satisfying (1.5), then $\left|\frac{1}{p}-\frac{1}{2}\right| \le \frac{s}{n}$. This can be shown using the following classical example of Hirschman [36, comments after Theorem 3c], Wainger [53, Part II], and Miyachi [40, Theorem 3]).

Example: Let $a > 0$, $a \neq 1$, $b > 0$, and assume that ϕ is a smooth function which vanishes in a neighborhood of 0 and is equal to 1 for large ξ in $\mathbb{R}^n$. Let $\sigma_{a,b}$ be the bounded function defined as

$$\sigma_{a,b}(\xi) = \phi(\xi)|\xi|^{-b}e^{i|\xi|^a}.$$

Then $\sigma_{a,b}$ satisfies condition (1.5) with $s = b/a$ and $r > n/s$ and $T_{\sigma_{a,b}}$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $|\frac{1}{p}-\frac{1}{2}| \le \frac{b/a}{n}$.

Additionally, Slavíková [46] recently constructed an example to show that L^p boundedness does not hold on the line $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{s}{n}$. This means that conditions $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{s}{n}$ and $rs > n$ are optimal for assumption (1.5). Prior to this, positive endpoint results on L^p and on H^1 involving Besov spaces were given by Seeger [43], [44], [45].

2. A sharp version of the Hörmander multiplier theorem

In this section, we discuss the aforementioned improvement to the Hörmander multiplier theorem in which the Lebesgue space $L^r(\mathbb{R}^n)$, $r > \frac{n}{s}$, in condition (1.5) is replaced by the locally larger Lorentz space $L^{\frac{n}{s},1}(\mathbb{R}^n)$. This space is defined in terms of the nonincreasing rearrangement of the function f, namely, the unique nonincreasing left-continuous function on $(0,\infty)$ equimeasurable with f, defined as follows:

$$f^*(t) = \inf\big\{r \ge 0 : \ |\{y \in \mathbb{R}^n : \ |f(y)| > r\}| < t\big\}.$$

We recall the definitions of Lorentz spaces. For any measurable function f on $\mathbb{R}^n$, we define

$$\|f\|_{L^{p,1}(\mathbb{R}^n)} = \int_0^\infty f^*(t)t^{\frac{1}{p}-1}\,dt$$

and

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{t>0} f^*(t) t^{\frac{1}{p}},$$

where $1 < p < \infty$. It can be shown that

$$\|f\|_{L^{p,1}(\mathbb{R}^n)} = p \int_0^\infty |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} \, d\lambda$$

and

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}}.$$

The space $L^{p',\infty}(\mathbb{R}^n)$, where $p' = \frac{p}{p-1}$, is a sort of a measure-theoretic dual of the space $L^{p,1}(\mathbb{R}^n)$, in view of the following version of Hölder's inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_{L^{p,1}(\mathbb{R}^n)} \|g\|_{L^{p',\infty}(\mathbb{R}^n)}. \tag{2.1}$$

We now discuss the proof of the following theorem.

Theorem 2.1. [31] *Let Ψ be a Schwartz function on $\mathbb{R}^n$ whose Fourier transform is supported in the annulus $1/2 < |\xi| < 2$ and satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$, $\xi \neq 0$. Let $p \in (1,\infty)$, $n \in \mathbb{N}$, and let $s \in (0,n)$ satisfy*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}. \tag{2.2}$$

Then for all functions f in the Schwartz class of $\mathbb{R}^n$ we have the a priori estimate

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}\left[\widehat{\Psi}\sigma(2^j\cdot)\right]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.3}$$

The Lorentz space $L^{\frac{n}{s},1}(\mathbb{R}^n)$ appears naturally in this context, since at least if derivatives of integer order are considered, this space is locally the largest rearrangement-invariant function space X such that if all partial derivatives of f of order up to and including order s lie in X, then f is bounded, see [49, 7].

The strategy to proving Theorem 2.1 is as follows: We show that inequality (2.3) holds for any $p \in (1,\infty)$, provided that $s \in (n/2, n)$, in Proposition 2.3 stated below. We then interpolate between this estimate with p near 1 and the trivial L^2 bound which essentially holds with zero derivatives.

In what follows, $B(x,r)$ denotes the ball centered at point x and having the radius r. If a ball of radius r is centered at the origin, we shall denote it simply by B_r.

We consider the centered Hardy-Littlewood maximal operator M defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy$$

for a measurable function f. An important property of this operator is that for any $p \in (1, \infty]$ there is a constant $C_{p,n}$ such that

$$\big\|M(f)\big\|_{L^p(\mathbb{R}^n)} \leq C_{p,n}\big\|f\big\|_{L^p(\mathbb{R}^n)}. \tag{2.4}$$

Moreover, when $p = 1$ there is an analogous inequality with $L^{1,\infty}$ in place of L^1 on the left in (2.4). There is also a vector-valued version of the preceding inequality due to Fefferman and Stein [16] saying:

$$\Big\|\Big(\sum_k M(f_k)^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n}\Big\|\Big(\sum_k |f_k|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}, \tag{2.5}$$

where $1 < p, q < \infty$.

We now consider a related maximal operator defined for $q \geq 1$ and $q < \infty$. We define a maximal operator M_{L^q} by

$$M_{L^q}(f)(x) = \sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^q\,dy\right)^{\frac{1}{q}}.$$

Observe that

$$M_{L^q}(f) = (M(|f|^q))^{\frac{1}{q}},$$

where M stands for the classical Hardy-Littlewood maximal operator, which coincides with M_{L^1}.

A crucial step in proving Proposition 2.3 is the next lemma, which in some sense sharpens the following estimate (see [18, Theorem 2.1.10])

$$\sup_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\frac{|f(x+2^{-j}y)|}{(1+|y|)^s}\,dy \leq C_{n,s}\,M(f)(x), \qquad s > n, \tag{2.6}$$

valid for measurable functions f; here $C_{n,s}$ is a constant depending only on the dimension and on $s > n$.

Lemma 2.2. *Assume that $n \in \mathbb{N}$, $s \in (0, n)$ and $q > \frac{n}{s}$. Then there is a positive constant C depending on n, s and q such that for any $j \in \mathbb{Z}$ and any measurable function f on $\mathbb{R}^n$,*

$$\left\|\frac{f(x+2^{-j}y)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n,dy)} \leq C M_{L^q}(f)(x), \quad x \in \mathbb{R}^n. \tag{2.7}$$

Proof. Setting $g(y) = f(x + 2^{-j}y)$, we obtain

$$\left\|\frac{f(x+2^{-j}y)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n,dy)} = \left\|\frac{g(y)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n,dy)} \tag{2.8}$$

and

$$
\begin{aligned}
M_{L^q}(f)(x) &= \sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|^q\,dy\right)^{\frac{1}{q}} \\
&= \sup_{r>0}\left(\frac{1}{2^{jn}|B(x,r)|}\int_{B(0,2^jr)}|f(x+2^{-j}z)|^q\,dz\right)^{\frac{1}{q}} \\
&= \sup_{r'>0}\left(\frac{1}{|B(0,r')|}\int_{B(0,r')}|g(y)|^q\,dy\right)^{\frac{1}{q}} \\
&= M_{L^q}(g)(0).
\end{aligned}
\tag{2.9}
$$

This says that we may assume, without loss of generality, that $j=0$ and $x=0$. Hence, it suffices to show that for any measurable function g on $\mathbb{R}^n$,

$$
\left\|\frac{g(y)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n,dy)} \le CM_{L^q}(g)(0).
\tag{2.10}
$$

If $M_{L^q}(g)(0)=\infty$, then inequality (2.10) holds trivially, so we can assume in what follows that $M_{L^q}(g)(0)<\infty$. Since the case $M_{L^q}(g)(0)=0$ is trivial as well (as g needs to vanish a.e. in this case), dividing the function g by the positive constant $M_{L^q}(g)(0)$, we can in fact assume that $M_{L^q}(g)(0)=1$.

Fix any $a>0$ and $k\in\mathbb{N}_0$. Then

$$
\begin{aligned}
|\{y\in B_{2^{k+1}}\setminus B_{2^k} : |g(y)|>a\}| &\le \frac{1}{a^q}\int_{B_{2^{k+1}}\setminus B_{2^k}}|g(y)|^q\,dy \\
&\le \frac{|B_{2^{k+1}}|}{a^q}\cdot\frac{1}{|B_{2^{k+1}}|}\int_{B_{2^{k+1}}}|g(y)|^q\,dy \\
&\le \frac{\omega_n 2^{(k+1)n}}{a^q},
\end{aligned}
$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Combining this with the trivial estimate

$$
|\{y\in B_{2^{k+1}}\setminus B_{2^k} : |g(y)|>a\}| \le \omega_n 2^{(k+1)n},
$$

we deduce that

$$
\begin{aligned}
&\left|\left\{y\in\mathbb{R}^n : \frac{|g(y)|}{(1+|y|)^s}>a\right\}\right| \\
&= \left|\left\{y\in B_1 : \frac{|g(y)|}{(1+|y|)^s}>a\right\}\right| + \sum_{k=0}^{\infty}\left|\left\{y\in B_{2^{k+1}}\setminus B_{2^k} : \frac{|g(y)|}{(1+|y|)^s}>a\right\}\right| \\
&\le |\{y\in B_1 : |g(y)|>a\}| + \sum_{k=0}^{\infty}\left|\left\{y\in B_{2^{k+1}}\setminus B_{2^k} : |g(y)|>2^{ks}a\right\}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq |\{y \in B_1 : |g(y)| > a\}| + \sum_{k=0}^{\infty} \omega_n 2^{(k+1)n} \min\left\{\frac{1}{2^{ksq} a^q}, 1\right\} \\
&\leq |\{y \in B_1 : |g(y)| > a\}| + \sum_{k \in \mathbb{N}_0 : 2^k < \frac{1}{a^{1/s}}} \omega_n 2^n \cdot 2^{kn} + \sum_{k \in \mathbb{N}_0 : 2^k \geq \frac{1}{a^{1/s}}} \frac{\omega_n 2^n}{a^q} \cdot 2^{k(n-sq)} \\
&\leq |\{y \in B_1 : |g(y)| > a\}| + \frac{C}{a^{\frac{n}{s}}}.
\end{aligned}
$$

Notice that in the last inequality we have used the fact that $n - sq < 0$. Hence,

$$
\begin{aligned}
\left\| \frac{g(y)}{(1+|y|)^s} \right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n, dy)} &= \sup_{a>0} a \left| \left\{ y \in \mathbb{R}^n : \frac{|g(y)|}{(1+|y|)^s} > a \right\} \right|^{\frac{s}{n}} \\
&\leq \sup_{a>0} a \, |\{y \in B_1 : |g(y)| > a\}|^{\frac{s}{n}} + C \\
&= \|g\|_{L^{\frac{n}{s},\infty}(B_1)} + C \\
&\leq C' \|g\|_{L^q(B_1)} + C \\
&\leq C' \omega_n^{\frac{1}{q}} M_{L^q}(g)(0) + C \\
&\leq C'',
\end{aligned}
$$

where $C' > 0$ is the constant from the embedding $L^q(B_1) \hookrightarrow L^{\frac{n}{s},\infty}(B_1)$. As $M_{L^q}(g)(0) = 1$, this proves (2.10), and thus (2.7). □

Assume that Ψ is the function from the statement of Theorem 2.1. For any $j \in \mathbb{Z}$ we define the Littlewood-Paley operator Δ_j^Ψ as

$$
\Delta_j^\Psi(f)(x) = \int_{\mathbb{R}^n} f(x-y) 2^{jn} \Psi(2^j y) dy.
$$

The associated square function is given by

$$
f \mapsto \Big(\sum_{j \in \mathbb{Z}} |\Delta_j^\Psi f|^2 \Big)^{\frac{1}{2}},
$$

and the Littlewood-Paley theorem asserts that if $1 < p < \infty$ then

$$
\|f\|_{L^p(\mathbb{R}^n)} \approx \Big\| \Big(\sum_{j \in \mathbb{Z}} |\Delta_j^\Psi f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^n)} \tag{2.11}
$$

for any $f \in L^p(\mathbb{R}^n)$.

Let us also recall some properties of the Fourier transform that will be needed in the sequel. It is well known that the Fourier transform is an isometry on L^2, and a bounded operator from L^1 into L^∞. An interpolation argument then yields that

$$
\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \tag{2.12}
$$

whenever $1 \le p \le 2$. This result is called the Hausdorff-Young inequality, and can be extended to the setting of Lorentz spaces as well. In particular, one has

$$\|\widehat{f}\|_{L^{p',1}(\mathbb{R}^n)} \le C\|f\|_{L^{p,1}(\mathbb{R}^n)} \tag{2.13}$$

for any $1 < p < 2$.

Proposition 2.3. *Let $p \in (1,\infty)$, $n \in \mathbb{N}$, $s \in (\frac{n}{2}, n)$. Let Ψ be as in Theorem 2.1. Then*

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.14}$$

Proof. Let

$$K = \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} < \infty\,.$$

Define a function Θ in terms of

$$\widehat{\Theta}(\xi) = \widehat{\Psi}(\xi/2) + \widehat{\Psi}(\xi) + \widehat{\Psi}(2\xi),$$

and observe that $\widehat{\Theta}$ is equal to 1 on the support of the function $\widehat{\Psi}$.

Let us denote by Δ_j^Ψ and Δ_j^Θ the Littlewood-Paley operators associated with Ψ and Θ, respectively. If f is a Schwartz function on $\mathbb{R}^n$, then standard manipulations yield

$$\begin{aligned}
\Delta_j^\Psi T_\sigma(f)(x) &= \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{\Psi}(2^{-j}\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi \\
&= \int_{\mathbb{R}^n} (\Delta_j^\Theta f)\widehat{\ }(\xi)\widehat{\Psi}(2^{-j}\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi \\
&= 2^{jn}\int_{\mathbb{R}^n} (\Delta_j^\Theta f)\widehat{\ }(2^j\xi')\widehat{\Psi}(\xi')\sigma(2^j\xi')e^{2\pi i x\cdot 2^j\xi'}d\xi' \\
&= \int_{\mathbb{R}^n} (\Delta_j^\Theta f)(x+2^{-j}y)\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\widehat{\ }(y)\,dy \\
&= \int_{\mathbb{R}^n} \frac{(\Delta_j^\Theta f)(x+2^{-j}y)}{(1+|y|)^s}(1+|y|)^s\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\widehat{\ }(y)\,dy.
\end{aligned}$$

By the Hölder inequality in Lorentz spaces (2.1), we therefore obtain

$$|\Delta_j^\Psi T_\sigma(f)(x)| \le \left\|\frac{(\Delta_j^\Theta f)(x+2^{-j}y)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n,dy)} \left\|(1+|y|)^s\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\widehat{\ }(y)\right\|_{L^{(\frac{n}{s})',1}(\mathbb{R}^n,dy)}.$$

Since $\frac{n}{s} < 2$, we can find a real number q such that $\frac{n}{s} < q < 2$. Lemma 2.2 now yields that

$$\left\|\frac{(\Delta_j^\Theta f)(x+2^{-j}y)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n,dy)} \le CM_{L^q}(\Delta_j^\Theta f)(x).$$

Using inequality (2.13) with $p = \frac{n}{s}$, we deduce that

$$\left\|(1+|y|)^s\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\widehat{\ }(y)\right\|_{L^{(\frac{n}{s})',1}(\mathbb{R}^n)} \le C\left\|(1+|y|^2)^{\frac{s}{2}}\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\widehat{\ }(y)\right\|_{L^{(\frac{n}{s})',1}(\mathbb{R}^n)}$$

$$\leq C\big\|(I-\Delta)^{\frac{s}{2}}\big[\widehat{\Psi}\sigma(2^j\cdot)\big]\big\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}$$
$$\leq CK.$$

Altogether, we obtain the estimate

$$|\Delta_j^{\Psi}T_\sigma(f)|(x)\leq CKM_{L^q}(\Delta_j^{\Theta}f)(x).$$

Assume that $p\geq 2$. Then applying the Littlewood-Paley theorem (2.11) and the Fefferman-Stein inequality (2.5) (since $\frac{p}{q}\geq\frac{2}{q}>1$) we obtain the following sequence of inequalities:

$$\begin{aligned}
\big\|T_\sigma(f)\big\|_{L^p(\mathbb{R}^n)} &\leq C\Big\|\Big(\sum_{j\in\mathbb{Z}}|\Delta_j^{\Psi}T_\sigma(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leq CK\Big\|\Big(\sum_{j\in\mathbb{Z}}|M_{L^q}(\Delta_j^{\Theta}f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)}\\
&= CK\Big\|\Big(\sum_{j\in\mathbb{Z}}(M(|\Delta_j^{\Theta}f|^q)^{\frac{2}{q}}\Big)^{\frac{q}{2}}\Big\|^{\frac{1}{q}}_{L^{\frac{p}{q}}(\mathbb{R}^n)}\\
&\leq CK\Big\|\Big(\sum_{j\in\mathbb{Z}}|\Delta_j^{\Theta}f|^{q\cdot\frac{2}{q}}\Big)^{\frac{q}{2}}\Big\|^{\frac{1}{q}}_{L^{\frac{p}{q}}(\mathbb{R}^n)}\\
&= CK\Big\|\Big(\sum_{j\in\mathbb{Z}}|\Delta_j^{\Theta}f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leq CK\|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

If $p\in(1,2)$ then the result follows by duality. □

To complete the proof, we need to properly interpolate between L^2 and L^{p_1}, for p_1 near 1. This is achieved via the following result, whose proof is not included here but the reader is referred to [31] for a proof.

Proposition 2.4. *Suppose that* $1<p_1<\infty$ *and* $0<s_1<n$. *If*

$$\|T_\sigma(f)\|_{L^{p_1}(\mathbb{R}^n)}\leq C\sup_{j\in\mathbb{Z}}\|(I-\Delta)^{\frac{s_1}{2}}[\widehat{\Psi}\sigma(2^j\cdot)]\|_{L^{\frac{n}{s_1},1}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)},\tag{2.15}$$

then

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)}\leq C\sup_{j\in\mathbb{Z}}\|(I-\Delta)^{\frac{s}{2}}[\widehat{\Psi}\sigma(2^j\cdot)]\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}$$

for any $1<p<\infty$ *and* $0<s<s_1$ *satisfying*

$$\frac{1}{s}\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{s_1}\left|\frac{1}{p_1}-\frac{1}{2}\right|.\tag{2.16}$$

Assuming Proposition 2.4, and using the estimate from Proposition 2.3 as the assumption (2.15), we complete the proof of Theorem 2.1 as follows:

Proof of Theorem 2.1. If $s \in (\frac{n}{2}, n)$, then inequality (2.3) follows from Proposition 2.3. If $s \leq \frac{n}{2}$, then we denote

$$\alpha = \frac{1}{s}\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Since $\alpha \in (0, \frac{1}{n})$, we can find $p_1 \in (1, \infty)$ and $s_1 \in (\frac{n}{2}, n)$ such that

$$\alpha < \frac{1}{s_1}\left|\frac{1}{p_1} - \frac{1}{2}\right|.$$

A combination of Propositions 2.3 and 2.4 thus yields the desired assertion (2.3). □

3. An example

Unlike the Mikhlin multiplier theorem, the Hörmander multiplier theorem and its extension due to Calderón and Torchinsky (see Section 1 for more details) apply to multipliers whose derivatives have infinitely many singularities, such as the multiplier

$$\sigma(x) = \sum_{k\in\mathbb{Z}} \phi(2^{-k}x)|2^{-k}x - a_k|^{\beta}, \tag{3.1}$$

where $\beta > 0$, ϕ is a smooth function supported in the set $\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\}$ and, for every $k \in \mathbb{Z}$, $a_k \in \mathbb{R}^n$ belongs to the same set.

As an application of Theorem 2.1 we show that the function σ in (3.1) continues to be an L^p Fourier multiplier for any $p \in (1, \infty)$ if $|2^{-k}x - a_k|$ is replaced by $\left(\log \frac{e4^n}{|2^{-k}x - a_k|^n}\right)^{-1}$. This conclusion cannot be reached via the multiplier theorems of Hörmander or Calderón and Torchinsky as the s-order derivative of the function $(\log \frac{e4^n}{|x|^n})^{-\beta}$, with $\beta > 0$, does not belong locally to any Lebesgue space $L^r(\mathbb{R}^n)$ with $r > n/s$ (but it does belong locally to the Lorentz space $L^{\frac{n}{s},1}(\mathbb{R}^n)$, as we will see below).

Example 3.1. *Assume that $n \in \mathbb{N}$, $n \geq 2$, and $\beta < 0$. Let ϕ be a smooth function supported in the set $A = \{x \in \mathbb{R}^n : 1/2 < |x| < 2\}$ and let $a_k \in A$, $k \in \mathbb{Z}$. Then the function*

$$\sigma(x) = \sum_{k\in\mathbb{Z}} \phi(2^{-k}x)\left(\log \frac{e4^n}{|2^{-k}x - a_k|^n}\right)^{\beta} \tag{3.2}$$

is an L^p Fourier multiplier for any $p \in (1, \infty)$.

To verify the statement of Example 3.1, we fix a positive integer s and observe that for any $j \in \mathbb{Z}$,

$$\begin{aligned}\|(I-\Delta)^{\frac{s}{2}}[\widehat{\Psi}\sigma(2^j\cdot)]\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \leq & \left\|(I-\Delta)^{\frac{s}{2}}\left[\widehat{\Psi}(x)\phi(x)\left(\log\frac{e4^n}{|x-a_j|^n}\right)^{\beta}\right]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \\ & + \left\|(I-\Delta)^{\frac{s}{2}}\left[\widehat{\Psi}(x)\phi(2x)\left(\log\frac{e4^n}{|2x-a_{j-1}|^n}\right)^{\beta}\right]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \\ & + \left\|(I-\Delta)^{\frac{s}{2}}\left[\widehat{\Psi}(x)\phi\big(x/2\big)\left(\log\frac{e4^n}{|\frac{x}{2}-a_{j+1}|^n}\right)^{\beta}\right]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}.\end{aligned}$$

In what follows, let us deal with the first term only, since the remaining two terms can be estimated in a similar way.

Fix $j \in \mathbb{Z}$ and denote

$$f_j(x) = \widehat{\Psi}(x)\phi(x)\left(\log\frac{e4^n}{|x-a_j|^n}\right)^{\beta}.$$

For any multiindex α satisfying $|\alpha| \geq 1$ we have

$$|\partial^\alpha f_j(x)| \leq C\chi_A(x)\left(\log\frac{e4^n}{|x-a_j|^n}\right)^{\beta-1}|x-a_j|^{-|\alpha|}.$$

Since $|A| \leq 2^n\omega_n$, where ω_n stands for the volume of the unit ball in $\mathbb{R}^n$, the previous estimate implies

$$(\partial^\alpha f_j)^*(t) \leq C\chi_{(0,2^n\omega_n)}(t)\left(\log\frac{e4^n\omega_n}{t}\right)^{\beta-1} t^{-\frac{|\alpha|}{n}},$$

where the constant C is independent of j. Therefore, if s is a positive integer and α is a multiindex with $1 \leq |\alpha| \leq s$, then

$$(\partial^\alpha f_j)^*(t) \leq C\chi_{(0,2^n\omega_n)}(t)\left(\log\frac{e4^n\omega_n}{t}\right)^{\beta-1} t^{-\frac{s}{n}}.$$

Consequently,

$$\sup_{1\leq|\alpha|\leq s}\|\partial^\alpha f_j\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \leq C\int_0^{2^n\omega_n}\left(\log\frac{e4^n\omega_n}{t}\right)^{\beta-1} t^{-1}\,dt < \infty. \tag{3.3}$$

Since each $|f_j|$ is bounded by a constant independent of j and compactly supported in the set A, we also have

$$\|f_j\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \leq C < \infty.$$

It remains to observe that the quantity $\|(I-\Delta)^{\frac{s}{2}}f_j\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}$ is equivalent to

$$\sum_{|\alpha|\leq s}\|\partial^\alpha f_j\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}.$$

This can be proved in exactly the same way as the corresponding result for the Lebesgue spaces (1.4), see, e.g., [48, Theorem 3, Chapter 5]. Therefore, we deduce that

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}[\widehat{\Psi}\sigma(2^j\cdot)]\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} < \infty$$

for any positive integer s. Theorem 2.1 now yields that σ is an L^p Fourier multiplier for any $p \in (1,\infty)$.

4. The Marcinkiewicz multiplier theorem

We recall the classical version of the Marcinkiewicz multiplier theorem:

Theorem 4.1. *Suppose that $\sigma(\xi_1,\dots,\xi_n)$ is a function on $\mathbb{R}^n$ such that*

$$\left|\partial_1^{\beta_1}\cdots\partial_n^{\beta_n}\sigma(\xi_1,\dots,\xi_n)\right| \le A_{\beta_1,\dots,\beta_n}|\xi_1|^{-\beta_1}\cdots|\xi_n|^{-\beta_n} \tag{4.1}$$

for all $\beta_j \in \{0,1\}$, $j=1,\dots,n$. Then σ is an L^p Fourier multiplier for all $1<p<\infty$ with bound

$$\|T_\sigma\|_{L^p\to L^p} \le C_{n,p} \sup_{\beta_j\in\{0,1\}} A_{\beta_1,\dots,\beta_n}.$$

Example: The following functions satisfy conditions (4.1) for all $\beta_j \in \mathbb{Z}^+\cup\{0\}$:

$$m_1(\xi) = \frac{\xi_1}{\xi_1 + i(\xi_2^2+\cdots+\xi_n^2)},$$

$$m_2(\xi) = \frac{|\xi_1|^{\alpha_1}\cdots|\xi_n|^{\alpha_n}}{(\xi_1^2+\xi_2^2+\cdots+\xi_n^2)^{\alpha/2}},$$

where $\alpha_1+\alpha_2+\cdots+\alpha_n=\alpha$, $\alpha_j>0$,

$$m_3(\xi) = \frac{\xi_2\xi_3^2}{i\xi_1+\xi_2^2+\xi_3^4}.$$

The functions m_1 and m_2 are defined on $\mathbb{R}^n\setminus\{0\}$ and m_3 on $\mathbb{R}^3\setminus\{0\}$.

These examples and many other examples that satisfy conditions (4.1) are invariant under a set of dilations in the following sense: suppose that there exist $k_1,\dots,k_n\in\mathbb{R}^+$ and $s\in\mathbb{R}$ such that the smooth function m on $\mathbb{R}^n\setminus\{0\}$ satisfies

$$m(\lambda^{k_1}\xi_1,\dots,\lambda^{k_n}\xi_n) = \lambda^{is}m(\xi_1,\dots,\xi_n)$$

for all $\xi_1,\dots,\xi_n\in\mathbb{R}$ and $\lambda>0$. Then m satisfies condition (4.1). Indeed, differentiation gives

$$\lambda^{\alpha_1k_1+\cdots+\alpha_nk_n}\partial^\alpha m(\lambda^{k_1}\xi_1,\dots,\lambda^{k_n}\xi_n) = \lambda^{is}\partial^\alpha m(\xi_1,\dots,\xi_n)$$

for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. Now for every $\xi \in \mathbb{R}^n \setminus \{0\}$ pick the unique $\lambda_\xi > 0$ such that $(\lambda_\xi^{k_1}\xi_1, \dots, \lambda_\xi^{k_n}\xi_n) \in \mathbb{S}^{n-1}$. Then $\lambda_\xi^{k_j\alpha_j} \le |\xi_j|^{-\alpha_j}$, and it follows that

$$|\partial^\alpha m(\xi_1, \dots, \xi_n)| \le \left[\sup_{\mathbb{S}^{n-1}} |\partial^\alpha m|\right] \lambda_\xi^{\alpha_1 k_1 + \dots + \alpha_n k_n} \le C_\alpha |\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n}.$$

Example: Let $\kappa \in \mathbb{R}$. Consider the function

$$m(\xi, \eta) = \left(\frac{1 + |\xi + \eta|^2}{(1 + |\xi|^2)(1 + |\eta|^2)}\right)^\kappa$$

defined on $\mathbb{R}^{2n}$. Define the function

$$M(\xi, t, \eta, s) = \left(\frac{t^2 s^2 + |s\xi + t\eta|^2}{(t^2 + |\xi|^2)(s^2 + |\eta|^2)}\right)^\kappa$$

on $\mathbb{R}^{2n+2}$. Notice that

$$M(\lambda\xi, \lambda t, \mu\eta, \mu s) = M(\xi, t, \eta, s)$$

for any $\lambda, \mu > 0$. This says that for fixed (η, s), the function $(\xi, t) \mapsto M(\xi, t, \eta, s)$ is homogeneous of degree zero, hence $\partial_\xi^\alpha M(\xi, t, \eta, s)$ is homogeneous of degree $-|\alpha|$ in (ξ, t). By the same argument the function $(\eta, s) \mapsto \partial_\eta^\beta \partial_\xi^\alpha M(\xi, t, \eta, s)$ is homogeneous of degree $-|\beta|$ in (η, s). From these observations we conclude that

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| = \left|\partial_\xi^\alpha \partial_\eta^\beta M(\xi, 1, \eta, 1)\right| \le \frac{C_{\alpha,\beta,s}}{(1 + |\xi|)^{|\alpha|}(1 + |\eta|)^{|\beta|}}$$

Here α and β are multiindices of n entries. It follows that

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| \le C_{\alpha,\beta,s} |\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n} |\eta_1|^{-\beta_1} \cdots |\eta_n|^{-\beta_n},$$

that is, condition (4.1) holds for all $\beta_j \in \mathbb{Z}^+ \cup \{0\}$.

Let us now study a product-type Sobolev space version of the Marcinkiewicz multiplier theorem. We define $(I - \partial_\ell^2)^{\frac{\gamma_\ell}{2}} f$ as the linear operator $((1 + 4\pi^2|\eta_\ell|^2)^{\frac{\gamma_\ell}{2}} \widehat{f}(\eta))^\vee$ associated with the multiplier $(1 + 4\pi^2|\eta_\ell|^2)^{\frac{\gamma_\ell}{2}}$. We present here a proof of the Marcinkiewicz multiplier theorem in which only $|1/p - 1/2| + \varepsilon$ derivatives per variable are required to guarantee L^p boundedness of T_σ, instead of a full derivative as in (4.1).

Theorem 4.2. *Let $n \in \mathbb{N}$, $n \ge 2$. Suppose that $1 \le r < \infty$ and ψ is a Schwartz function on the line whose Fourier transform is supported in $[-2, -1/2] \cup [1/2, 2]$ and which satisfies $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for all $\xi \ne 0$. Let $\gamma_\ell > 1/r$, $\ell = 1, \dots, n$. If a function σ on $\mathbb{R}^n$ satisfies*

$$\sup_{j_1,\dots,j_n\in\mathbb{Z}} \left\|(I - \partial_1^2)^{\frac{\gamma_1}{2}} \cdots (I - \partial_n^2)^{\frac{\gamma_n}{2}} \left(\widehat{\psi}(\xi_1) \cdots \widehat{\psi}(\xi_n) \sigma(2^{j_1}\xi_1, \dots, 2^{j_n}\xi_n)\right)\right\|_{L^r} < \infty, \tag{4.2}$$

then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$ with

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \min(\gamma_1, \dots, \gamma_n). \tag{4.3}$$

Moreover, (4.3) *is optimal in the sense that if T_σ is L^p-bounded for every σ satisfying* (4.2)*, then the strict inequality in* (4.3) *must necessarily hold.*

Carbery [4] first formulated a version of Theorem 4.2 in which the multiplier lies in a product-type L^2-based Sobolev space. Carbery and Seeger [5, Remark after Prop. 6.1] obtained Theorem 4.2 in the case when $\gamma_1 = \dots = \gamma_n > \big|\frac{1}{p} - \frac{1}{2}\big| = \frac{1}{r}$. The positive direction of their result also appeared in [6, Condition (1.4)] but this time the range of p is $\big|\frac{1}{p} - \frac{1}{2}\big| < \frac{1}{r}$. The present variant of Theorem 4.2 appeared in [32].

Let us now focus on the proof of Theorem 4.2. We use ψ to denote the bump from Theorem 4.2; further, θ will stand for the function on the line satisfying

$$\widehat{\theta}(\eta) = \widehat{\psi}(\eta/2) + \widehat{\psi}(\eta) + \widehat{\psi}(2\eta).$$

One can observe that $\widehat{\theta}$ is supported in $\{\frac{1}{4} \le |\xi| \le 4\}$ and $\widehat{\theta} = 1$ on the support of $\widehat{\psi}$.

To simplify the notation, if $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we shall write

$$2^J\xi = \big(2^{j_1}\xi_1, \dots, 2^{j_n}\xi_n\big)$$

and

$$\widehat{\psi}(\xi) = \prod_{\ell=1}^n \widehat{\psi}(\xi_\ell), \quad \widehat{\theta}(\xi) = \prod_{\ell=1}^n \widehat{\theta}(\xi_\ell).$$

Let $k \in 1, \dots, n$. For $j \in \mathbb{Z}$ we define the Littlewood-Paley operators associated to the bumps ψ and θ by

$$\Delta_j^{\psi,k}(f)(x) = \int_{\mathbb{R}} f(x_1, \dots, x_{k-1}, x_k - y, x_{k+1}, \dots, x_n) 2^j \psi(2^j y) dy$$

and

$$\Delta_j^{\theta,k}(f)(x) = \int_{\mathbb{R}} f(x_1, \dots, x_{k-1}, x_k - y, x_{k+1}, \dots, x_n) 2^j \theta(2^j y) dy.$$

We begin with the following lemma:

Lemma 4.3. *Let $1 \le r < \infty$, let $1 \le \rho < 2$ satisfy $1 \le \rho \le r$ and let $\gamma_1, \dots, \gamma_n$ be real numbers such that $\gamma_\ell \rho > 1$, $\ell = 1, \dots, n$. Then, for any function f on $\mathbb{R}^n$ and for all integers $j_1, \dots j_n$, we have*

$$|\Delta_{j_1}^{\psi,1} \cdots \Delta_{j_n}^{\psi,n} T_\sigma(f)| \le C\, K \Big[M^{(1)} \cdots M^{(n)}(|\Delta_{j_1}^{\theta,1} \cdots \Delta_{j_n}^{\theta,n} f|^\rho)\Big]^{\frac{1}{\rho}}, \tag{4.4}$$

where $M^{(\ell)}$ denotes the one-dimensional Hardy-Littlewood maximal operator in the ℓ-th coordinate and

$$K = \sup_{j_1,\dots,j_n\in\mathbb{Z}} \Big\| (I-\partial_1^2)^{\frac{\gamma_1}{2}} \cdots (I-\partial_n^2)^{\frac{\gamma_n}{2}} \Big[\sigma\big(2^{j_1}\xi_1,\dots,2^{j_n}\xi_n\big)\widehat{\psi}(\xi_1)\cdots\widehat{\psi}(\xi_n)\Big]\Big\|_{L^r}.$$

Proof. Throughout the proof we shall use the notation introduced above and, whenever $J=(j_1,\dots,j_n)$, we write

$$\Delta_J^{\psi} f = \Delta_{j_1}^{\psi,1}\cdots\Delta_{j_n}^{\psi,n} f, \quad \Delta_J^{\theta} f = \Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f.$$

Since $\widehat{\theta}$ is equal to 1 on the support of $\widehat{\psi}$, we have

$$\begin{aligned}
&\Delta_J^{\psi} T_\sigma(f)(x_1,\dots,x_n)\\
&= \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{\psi}(2^{-J}\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi\\
&= \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{\theta}(2^{-J}\xi)\widehat{\psi}(2^{-J}\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi\\
&= \int_{\mathbb{R}^n} (\Delta_J^{\theta} f)\widehat{\ }(\xi)\widehat{\psi}(2^{-J}\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi\\
&= \int_{\mathbb{R}^n} 2^{j_1+\cdots+j_n}(\Delta_J^{\theta} f)\widehat{\ }(2^J\xi')\widehat{\psi}(\xi')\sigma(2^J\xi')e^{2\pi i(2^J x\cdot\xi')}d\xi'\\
&= 2^{j_1+\cdots+j_n}\int_{\mathbb{R}^n} (\Delta_J^{\theta} f)(y)\left[\widehat{\psi}(\xi')\sigma(2^J\xi')\right]\widehat{\ }(2^J(x-y))dy\\
&= \int_{\mathbb{R}^n} \frac{2^{j_1+\cdots+j_n}(\Delta_J^{\theta} f)(y)}{\prod_{\ell=1}^n(1+2^{j_\ell}|x_\ell-y_\ell|)^{\gamma_\ell}}\cdot\prod_{\ell=1}^n(1+2^{j_\ell}|x_\ell-y_\ell|)^{\gamma_\ell}\left[\widehat{\psi}(\xi)\sigma(2^J\xi)\right]\widehat{\ }(2^J(x-y))\,dy.
\end{aligned}$$

Hölder's inequality thus yields that $|\Delta_J T_\sigma(f)(x)|$ is bounded by

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} 2^{j_1+\cdots+j_n}\frac{|(\Delta_J^{\theta} f)(y)|^\rho}{\prod_{\ell=1}^n(1+2^{j_\ell}|x_\ell-y_\ell|)^{\gamma_\ell\rho}}\,dy\right)^{\frac{1}{\rho}}\\
&\cdot\left(\int_{\mathbb{R}^n} 2^{j_1+\cdots+j_n}\left|\prod_{\ell=1}^n(1+2^{j_\ell}|x_\ell-y_\ell|)^{\gamma_\ell}\cdot\left[\widehat{\psi}(\xi)\sigma(2^J\xi)\right]\widehat{\ }(2^J(x-y))\right|^{\rho'}dy\right)^{\frac{1}{\rho'}},
\end{aligned}$$

where, when $\rho=1$, the second term in the product is to be interpreted as

$$\left\|\prod_{\ell=1}^n(1+2^{j_\ell}|x_\ell-y_\ell|)^{\gamma_\ell}\cdot\left[\widehat{\psi}(\xi)\sigma(2^J\xi)\right]\widehat{\ }(2^J(x-y))\right\|_{L^\infty}.$$

Since $\gamma_\ell\rho>1$ for all $\ell=1,\dots,n$, n consecutive applications of (2.6) yield the estimate

$$\left(\int_{\mathbb{R}^n} 2^{j_1+\cdots+j_n}\frac{|(\Delta_J^{\theta} f)(y)|^\rho}{\prod_{\ell=1}^n(1+2^{j_\ell}|x_\ell-y_\ell|)^{\gamma_\ell\rho}}\,dy\right)^{\frac{1}{\rho}} \le C\Big[M^{(1)}\cdots M^{(n)}\big(|\Delta_J^{\theta} f|^\rho\big)(x)\Big]^{\frac{1}{\rho}}.$$

We now write

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} 2^{j_1+\cdots+j_n}\Big|\prod_{\ell=1}^n(1+2^{j_\ell}|x_\ell-y_\ell|)^{\gamma_\ell}\Big[\widehat{\psi}(\xi)\sigma(2^J\xi)\Big]\widehat{\ }(2^J(x-y))\Big|^{\rho'}dy\Big)^{\frac{1}{\rho'}}\\
&\le\Big(\int_{\mathbb{R}^n}\Big|\prod_{\ell=1}^n(1+|y_\ell|^2)^{\frac{\gamma_\ell}{2}}\Big[\widehat{\psi}(\xi)\sigma(2^J\xi)\Big]\widehat{\ }(y)\Big|^{\rho'}dy\Big)^{\frac{1}{\rho'}}\\
&\le\Big\|(I-\partial_1^2)^{\frac{\gamma_1}{2}}\cdots(I-\partial_n^2)^{\frac{\gamma_n}{2}}\Big[\sigma\big(2^J\xi\big)\widehat{\psi}(\xi)\Big]\Big\|_{L^\rho}\\
&\le C\Big\|(I-\partial_1^2)^{\frac{\gamma_1}{2}}\cdots(I-\partial_n^2)^{\frac{\gamma_n}{2}}\Big[\sigma\big(2^J\xi\big)\widehat{\psi}(\xi)\Big]\Big\|_{L^r} \qquad (4.5)\\
&\le CK.
\end{aligned}$$

Notice that the second inequality is the Hausdorff-Young inequality (2.12) while (4.5) is a consequence of the Kato-Ponce inequality [29] (if $\rho < r$). A combination of the preceding estimates yields (4.4). □

Proposition 4.4. *Let $1 \le r < \infty$ and let $\gamma_\ell > \max\{1/2, 1/r\}$, $\ell = 1, \dots, n$. If a function σ on $\mathbb{R}^n$ satisfies (4.2), then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$.*

Proof. Suppose first that $p > 2$. Since $\gamma_\ell > \max\{1/2, 1/r\}$, $\ell = 1, \dots, n$, we can find $\rho \in [1, 2)$ such that $\rho \le r$ and $\rho\gamma_\ell > 1$, $\ell = 1, \dots, n$. Then

$$\begin{aligned}
\big\|T_\sigma(f)\big\|_{L^p(\mathbb{R}^n)} &\le C_p(n)\Big\|\Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}}|\Delta_{j_1}^{\psi,1}\cdots\Delta_{j_n}^{\psi,n}T_\sigma(f)|^2\Big)^{\frac12}\Big\|_{L^p}\\
&\le C_p(n)K\Big\|\Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}}\Big[M^{(1)}\cdots M^{(n)}(|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n}f|^\rho)\Big]^{\frac{2}{\rho}}\Big)^{\frac12}\Big\|_{L^p}\\
&\le C_p(n)K\Big\|\Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}}|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n}f|^2\Big)^{\frac12}\Big\|_{L^p}\\
&\le C_p(n)K\|f\|_{L^p}.
\end{aligned}$$

Notice that the second inequality follows from Lemma 4.3 and the third inequality is obtained by applying the Fefferman-Stein inequality (2.5) on the Lebesgue space $L^{\frac{p}{2}}$ in each of the variables $y_1, \dots, y_n$. Observe that the Fefferman-Stein inequality makes use of the assumptions $2/\rho > 1$ and $p/2 > 1$. The first and last inequality follow from Proposition 4.5 below.

The case $1 < p < 2$ follows by a duality argument, while the case $p = 2$ is a consequence of Plancherel's theorem and of a Sobolev embedding into L^∞. □

Proposition 4.5. *If ψ satisfies $\sum_{j\in\mathbb{Z}}\widehat{\psi}(2^{-j}\xi)=1$ for all $\xi\neq 0$, then we have*

$$\Big\|\Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}}|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n}f|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{R}^n)}\approx\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.6}$$

where $\widehat{\theta}(\xi)=\widehat{\psi}(2\xi)+\widehat{\psi}(\xi)+\widehat{\psi}(2^{-1}\xi)$.

Before we discuss the proofs we recall the Rademacher functions, which we plan to use. The Rademacher functions are defined on $[0,1]$ as follows: $r_0(t)=1$; $r_1(t)=1$ for $0\le t\le 1/2$ and $r_1(t)=-1$ for $1/2<t\le 1$; $r_2(t)=1$ for $0\le t\le 1/4$, $r_2(t)=-1$ for $1/4<t\le 1/2$, $r_2(t)=1$ for $1/2<t\le 3/4$, and $r_2(t)=-1$ for $3/4<t\le 1$; and so on. According to this definition, we have that $r_j(t)=\operatorname{sgn}(\sin(2^j\pi t))$ for $j=0,1,2,\dots$. They are mutually independent random variables on $[0,1]$ that satisfy Khintchine's inequalities:

For any $0<p<\infty$ and for any complex-valued square summable sequences $\{z_j\}$ we have

$$B_p\Big(\sum_j|z_j|^2\Big)^{\frac12}\le\Big\|\sum_j z_jr_j\Big\|_{L^p([0,1])}\le A_p\Big(\sum_j|z_j|^2\Big)^{\frac12} \tag{4.7}$$

for some constants $0<A_p,B_p<\infty$ that depend only on p.

These inequalities also extend to several variables. Set

$$F_n(t_1,\dots,t_n)=\sum_{j_1}\cdots\sum_{j_n}c_{j_1,\dots,j_n}r_{j_1}(t_1)\cdots r_{j_n}(t_n),$$

for $t_j\in[0,1]$, where $c_{j_1,\dots,j_n}$ is a sequence of complex numbers.

For any $0<p<\infty$ and for any complex-valued square summable sequence of n variables $\{c_{j_1,\dots,j_n}\}_{j_1,\dots,j_n}$, we have the following inequalities for F_n:

$$B_p^n\Big(\sum_{j_1}\cdots\sum_{j_n}|c_{j_1,\dots,j_n}|^2\Big)^{\frac12}\le\big\|F_n\big\|_{L^p([0,1]^n)}\le A_p^n\Big(\sum_{j_1}\cdots\sum_{j_n}|c_{j_1,\dots,j_n}|^2\Big)^{\frac12},$$

where A_p,B_p are the constants in (4.7).

Using the Rademacher functions we can now prove Proposition 4.5.

Proof. We begin by noting that (4.6) holds when $n=1$, as $\sum_{j\in\mathbb{Z}}\widehat{\theta}(2^{-j}\xi)=3$ when $\xi\neq 0$; see [18, Theorem 6.1.6, Corollary 6.1.7] So it suffices to prove (4.6) in higher dimensions. Using the preceding inequalities we write:

$$\begin{aligned}&\Big\|\Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}}|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n}f|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{R}^n)}^p\\&\quad=\int_{\mathbb{R}^n}\Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}}|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n}f|^2\Big)^{\frac p2}dx\end{aligned}$$

$$\approx \int_{\mathbb{R}^n}\int_{[0,1]^n}\Big|\sum_{j_1,\dots,j_n\in\mathbb{Z}} r_{j_1}(t_1)\cdots r_{j_n}(t_n)\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n}f(x)\Big|^p\,dt_1\cdots dt_n\,dx$$

$$\int_{\mathbb{R}^n}\int_{[0,1]^n}\Big|\Big(\sum_{j_1\in\mathbb{Z}} r_{j_1}(t_1)\Delta_{j_1}^{\theta,1}\Big)\cdots\Big(\sum_{j_n\in\mathbb{Z}} r_{j_n}(t_n)\Delta_{j_n}^{\theta,n}\Big)f(x)\Big|^p\,dt_1\cdots dt_n\,dx$$

$$\approx \int_{\mathbb{R}^{n-1}}\int_{[0,1]^{n-1}}\int_{\mathbb{R}}\Big(\sum_{j_1\in\mathbb{Z}}\Big|\Delta_{j_1}^{\theta,1}\Big\{\prod_{i=2}^{n-1}\Big(\sum_{j_i\in\mathbb{Z}} r_{j_i}(t_i)\Delta_{j_i}^{\theta,i}\Big)f\Big\}(x_1,x')\Big|^2\Big)^{\frac{p}{2}}dx_1\,dt_2\cdots dt_n dx'$$

where $x'=(x_2,\dots,x_n)$. We now apply the Littlewood-Paley theorem (2.11) in the first variable x_1 to eliminate the square function in j_1 and replace the inner integral by the p-th power of the function in the curly brackets. We then continue the same reasoning to the remaining variables $x_2,\dots,x_n$ to conclude the proof of (4.6). □

5. The interpolation argument needed in the proof of Theorem 4.2

When $p=2$ no derivatives are required of σ for T_σ to be bounded. To mitigate the effect of the requirement of the derivatives of σ for T_σ to be bounded on L^p for $p\neq 2$, we apply an interpolation argument between $p=2$ and p near 1.

We shall use the notation introduced at the beginning of the previous section, and we shall denote

$$\Gamma\big(\{s_\ell\}_{\ell=1}^n\big)=\Gamma(s_1,\dots,s_n)=(I-\partial_1^2)^{\frac{s_1}{2}}\cdots(I-\partial_n^2)^{\frac{s_n}{2}}.$$

The following result will be the key interpolation estimate:

Theorem 5.1. *Fix $1<p_0,p_1,r_0,r_1<\infty$, $0<s_1^0,\dots,s_n^0,s_1^1,\dots,s_n^1<\infty$. Suppose that $r_0s_\ell^0>1$ and $r_1s_\ell^1>1$ for all $\ell=1,\dots,n$. Let ψ be as before. Assume that for $k\in\{0,1\}$ we have*

$$\|T_\sigma(f)\|_{L^{p_k}}\le K_k\sup_{j_1,\dots,j_n\in\mathbb{Z}}\Big\|\Gamma(s_1^k,\dots,s_n^k)\Big[\sigma(2^J\xi)\prod_{\ell=1}^n\widehat{\psi}(\xi_\ell)\Big]\Big\|_{L^{r_k}}\|f\|_{L^{p_k}}\tag{5.1}$$

for all $f\in\mathscr{C}_0^\infty(\mathbb{R}^n)$. For $0<\theta<1$ and $\ell=1,\dots,n$ define

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{r}=\frac{1-\theta}{r_0}+\frac{\theta}{r_1},\quad s_\ell=(1-\theta)s_\ell^0+\theta s_\ell^1.$$

Then there is a constant C_* such that for all $f\in\mathscr{C}_0^\infty(\mathbb{R}^n)$ we have

$$\|T_\sigma(f)\|_{L^p}\le C_*K_0^{1-\theta}K_1^\theta\sup_{j_1,\dots,j_n\in\mathbb{Z}}\Big\|\Gamma(s_1,\dots,s_n)\Big[\sigma(2^J\xi)\prod_{\ell=1}^n\widehat{\psi}(\xi_\ell)\Big]\Big\|_{L^r}\|f\|_{L^p}.\tag{5.2}$$

Assuming Theorem 5.1, we complete the proof of Theorem 4.2 as follows:

Proof. Given $1 \le r \le \infty$ and $\gamma_\ell > 1/r$, $\ell = 1, \dots, n$, fix $p \in (1, \infty)$ satisfying (4.3). In fact, we can assume that $p \in (1, 2)$, since the case $p \in (2, \infty)$ follows by duality and the case $p = 2$ is a consequence of Plancherel's theorem and of a Sobolev embedding into L^∞. In addition, assume first that $\min_\ell \gamma_\ell \le \frac{1}{2}$. In view of (4.3), there is $\tau \in (0, 1)$ such that

$$\frac{1}{p} - \frac{1}{2} < \tau \min_\ell \gamma_\ell. \tag{5.3}$$

Set $p_1 = \frac{2}{\tau+1}$, $r_1 = 2r \min_\ell \gamma_\ell$ and $\gamma_\ell^1 = \frac{1}{2} + \varepsilon$, $\ell = 1, \dots, n$, where $\varepsilon > 0$ is a real number whose exact value will be specified later. Since $p_1 > 1$ and $r_1 \gamma_\ell^1 > 2\gamma_\ell^1 > 1$, $\ell = 1, \dots, n$, Proposition 4.4 yields that

$$\|T_\sigma(f)\|_{L^{p_1}} \le C_1 \sup_{j_1, \dots, j_n \in \mathbb{Z}} \left\| \Gamma(\gamma_1^1, \dots, \gamma_n^1) \Big[\sigma(2^J \xi) \prod_{\ell=1}^n \widehat{\psi}_\ell(\xi_\ell) \Big] \right\|_{L^{r_1}} \|f\|_{L^{p_1}}. \tag{5.4}$$

Pick $p_0 = 2$. Let θ be the real number satisfying

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

namely, $\theta = \frac{2}{\tau}(\frac{1}{p} - \frac{1}{2})$. Observe that, by (5.3), $0 < \theta < 2 \min_\ell \gamma_\ell \le 1$. Finally, choose real numbers r_0 and γ_ℓ^0, $\ell = 1, \dots, n$, in such a way that

$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \gamma_\ell = (1-\theta)\gamma_\ell^0 + \theta \gamma_\ell^1. \tag{5.5}$$

We claim that, for a suitable choice of $\varepsilon > 0$, one has $r_0 > 2$ and $r_0 \gamma_\ell^0 > 1$, $\ell = 1, \dots, n$. Indeed, since $\min_\ell \gamma_\ell \le \frac{1}{2}$, we have $r_1 \le r$, and thus, by (5.5), $r_0 \ge r \ge r_1 > 2$. Further,

$$r_0 \gamma_\ell^0 = \frac{r_1 r(\gamma_\ell - \theta \gamma_\ell^1)}{r_1 - \theta r} = \frac{r \min_k \gamma_k (\gamma_\ell - \frac{\theta}{2} - \theta\varepsilon)}{\min_k \gamma_k - \frac{\theta}{2}} \ge \frac{r \min_k \gamma_k (\min_k \gamma_k - \frac{\theta}{2} - \theta\varepsilon)}{\min_k \gamma_k - \frac{\theta}{2}}.$$

Since $r \min_k \gamma_k > 1$ and $\min_k \gamma_k - \frac{\theta}{2} > 0$, one gets $r_0 \gamma_\ell^0 > 1$ if $\varepsilon > 0$ is small enough. Consequently, the space $\{g : \Gamma(\gamma_1^0, \dots, \gamma_n^0) g \in L^{r_0}\}$ embeds in L^∞, and we thus have

$$\|T_\sigma(f)\|_{L^2} \le C_1 \sup_{j_1, \dots, j_n \in \mathbb{Z}} \left\| \Gamma(\gamma_1^0, \dots, \gamma_n^0) \Big[\sigma(2^J \xi) \prod_{\ell=1}^n \widehat{\psi}_\ell(\xi_{S_\ell}) \Big] \right\|_{L^{r_0}} \|f\|_{L^2}. \tag{5.6}$$

The boundedness of T_σ on $L^p(\mathbb{R}^n)$ for any σ satisfying (4.2) thus follows from Theorem 5.1. Finally, if $\min_\ell \gamma_\ell > \frac{1}{2}$, then the required assertion follows directly from Proposition 4.4. □

Proof of Theorem 5.1. The proof of Theorem 5.1 follows closely that of [3, Theorem 4.7] and for this reason we only provide an outline of its proof with few details. Throughout

the proof we shall use the notation introduced at the beginning of the previous section. Also, whenever $J \in \mathbb{Z}^n$, we denote

$$\varphi_J = \Gamma(s_1, \dots, s_n)\Big[\sigma(2^J\xi)\widehat{\psi}(\xi)\Big],$$

and for z with real part in $[0,1]$ we define

$$\sigma_z(\xi) = \sum_{J\in\mathbb{Z}^n} \Gamma\Big(\{-s_\ell^0(1-z) - s_\ell^1 z\}_{\ell=1}^n\Big)\Big[|\varphi_J|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_J)}\Big]\,(2^{-J}\xi)\widehat{\theta}(2^{-J}\xi). \tag{5.7}$$

For any given $\xi \in \mathbb{R}^n$, this sum has only finitely many terms and one can show that

$$\|\sigma_{\tau+it}\|_{L^\infty} \lesssim (1+|t|)^{\frac{3n}{2}} \Big(\sup_{J\in\mathbb{Z}^n} \big\|\Gamma(s_1,\dots,s_n)[\sigma(2^J\xi)\widehat{\psi}(\xi)]\big\|_{L^r}\Big)^{\frac{r}{r_\tau}}, \tag{5.8}$$

where r_τ is the real number satisfying $\frac{1}{r_\tau} = \frac{1-\tau}{r_0} + \frac{\tau}{r_1}$.

Let T_z be the family of operators associated to the multipliers σ_z. Fix $f, g \in \mathscr{C}_0^\infty$ and $1 < p_0 < p < p_1 < \infty$. Given $\varepsilon > 0$ there exist functions f_z^ε and g_z^ε such that $\|f_\theta^\varepsilon - f\|_{L^p} < \varepsilon$, $\|g_\theta^\varepsilon - g\|_{L^{p'}} < \varepsilon$, and that

$$\|f_{it}^\varepsilon\|_{L^{p_0}} \le \big(\|f\|_{L^p}^p + \varepsilon\big)^{\frac{1}{p_0}}, \quad \big\|f_{1+it}^\varepsilon\big\|_{L^{p_1}} \le \big(\|f\|_{L^p}^p + \varepsilon\big)^{\frac{1}{p_1}},$$
$$\|g_{it}^\varepsilon\|_{L^{p_0'}} \le \big(\|g\|_{L^{p'}}^{p'} + \varepsilon\big)^{\frac{1}{p_0'}}, \quad \big\|g_{1+it}^\varepsilon\big\|_{L^{p_1'}} \le \big(\|g\|_{L^{p'}}^{p'} + \varepsilon\big)^{\frac{1}{p_1'}}.$$

The existence of f_z^ε and g_z^ε is folklore and is omitted; for a similar construction see [3, Theorem 3.3]. Let $F(z) = \int T_{\sigma_z}(f_z^\varepsilon) g_z^\varepsilon\, dx$. Then $F(z)$ is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sigma_z(\xi)\widehat{f_z^\varepsilon}(\xi)\widehat{g_z^\varepsilon}(\xi)\, d\xi \\
&= \sum_{J\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \Gamma\Big(\{-s_\ell^0(1-z) - s_\ell^1 z\}_{\ell=1}^n\Big)\Big[|\varphi_J|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_J)}\Big]\,(2^{-j}\xi)\widehat{\theta}(2^{-J}\xi)\widehat{f_z^\varepsilon}(\xi)\widehat{g_z^\varepsilon}(\xi)\, d\xi \\
&= \sum_{J\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \Big[|\varphi_J|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_J)}\Big]\,(2^{-J}\xi)\Gamma\Big(\{-s_\ell^0(1-z) - s_\ell^1 z\}_{\ell=1}^n\Big)\Big[\widehat{\theta}(2^{-J}\cdot)\widehat{f_z^\varepsilon}\widehat{g_z^\varepsilon}\Big](\xi)\, d\xi.
\end{aligned}$$

The function $F(z)$ is analytic on the strip $0 < \Re(z) < 1$ and continuous up to the boundary. Notice that $\sigma_{it}(2^K\cdot)\widehat{\psi}$ picks up only the terms of (5.7) for which J differs from K in some coordinate by at most one unit. For simplicity we may therefore take $K = J$ in the calculation below. Using the Kato-Ponce inequality we may "remove" the factor $\widehat{\theta}$ and write

$$\begin{aligned}
&\|T_{\sigma_{it}}(f_{it}^\varepsilon)\|_{L^{p_0}} \\
&\le K_0 \sup_{K\in\mathbb{Z}^n} \Big\|\Gamma(s_1^0,\dots,s_n^0)\big[\sigma_{it}(2^K\cdot)\widehat{\psi}\big]\Big\|_{L^{r_0}} \|f_{it}^\varepsilon\|_{L^{p_0}} \\
&\le K_0 \sup_{K\in\mathbb{Z}^n} \Big\|\Gamma\big(\{s_\ell^0 - s_\ell^0(1-it) - s_\ell^1 it\}_{\ell=1}^n\big)\big[|\varphi_K|^{r(\frac{1-it}{r_0}+\frac{it}{r_1})} e^{i\mathrm{Arg}\,(\varphi_K)}\big]\Big\|_{L^{r_0}} \|f_{it}^\varepsilon\|_{L^{p_0}}
\end{aligned}$$

$$\lesssim (1+|t|)^{\frac{3n}{2}} K_0 \sup_{K\in\mathbb{Z}^N} \|\varphi_K\|_{L^r}^{\frac{r}{r_0}} \big(\|f\|_{L^p}^p + \varepsilon\big)^{\frac{1}{p_0}}.$$

Using Hölder's inequality $|F(it)| \le \|T_{\sigma_{it}}(f^\varepsilon_{it})\|_{L^{p_0}} \|g^\varepsilon_{it}\|_{L^{p_0'}}$, we may therefore write

$$|F(it)| \le C(1+|t|)^{\frac{3n}{2}} K_0 \sup_{J\in\mathbb{Z}^n} \big\|\Gamma(\{s_\ell\}_{\ell=1}^n)[\sigma(2^J\cdot)\widehat{\psi}]\big\|_{L^r}^{\frac{r}{r_0}} \big(\|f\|_{L^p}^p + \varepsilon\big)^{\frac{1}{p_0}} \big(\|g\|_{L^{p'}}^{p'} + \varepsilon\big)^{\frac{1}{p_0'}}$$

for some constant $C = C(n, r_0, s^0_\ell, s^1_\ell)$. Similarly, for some constant $C = C(n, r_1, s^0_\ell, s^1_\ell)$ we obtain

$$|F(1+it)| \le C(1+|t|)^{\frac{3n}{2}} K_1 \sup_{J\in\mathbb{Z}^n} \big\|\Gamma(\{s_\ell\}_{\ell=1}^n)[\sigma(2^J\cdot)\widehat{\psi}]\big\|_{L^r}^{\frac{r}{r_1}} \big(\|f\|_{L^p}^p + \varepsilon\big)^{\frac{1}{p_1}} \big(\|g\|_{L^{p'}}^{p'} + \varepsilon\big)^{\frac{1}{p_1'}}.$$

Thus for $z = \tau + it$, $t \in \mathbb{R}$ and $0 \le \tau \le 1$ it follows from (5.8) and from the definition of $F(z)$ that

$$|F(z)| \le C''(1+\ |t|)^{\frac{3n}{2}} \Big(\sup_{J\in\mathbb{Z}^n} \big\|\Gamma(s_1,\dots,s_n)\big[\sigma(2^J\cdot)\widehat{\psi}\big]\big\|_{L^r}\Big)^{\frac{r}{r_\tau}} \|f^\varepsilon_z\|_{L^2}\|g^\varepsilon_z\|_{L^2} = A_\tau(t)\,,$$

noting that $\|f^\varepsilon_z\|_{L^2}\|g^\varepsilon_z\|_{L^2}$ is bounded above by constants independent of t and τ. Since $A_\tau(t) \le \exp(Ae^{a|t|})$, the hypotheses of three lines lemma are valid. It follows that

$$|F(\theta)| \le C\, K_0^{1-\theta} K_1^\theta \sup_{J\in\mathbb{Z}^n} \big\|\Gamma(\{s_\ell\}_{\ell=1}^n)[\sigma(2^J\cdot)\widehat{\psi}]\big\|_{L^r} \big(\|f\|_{L^p}^p + \varepsilon\big)^{\frac{1}{p}} \big(\|g\|_{L^{p'}}^{p'} + \varepsilon\big)^{\frac{1}{p'}}.$$

Taking the supremum over all functions $g \in L^{p'}$ with $\|g\|_{L^{p'}} \le 1$, a simple density argument yields for some $C_* = C_*(n, r_1, r_2, s^0_\ell, s^1_\ell)$

$$\|T_\sigma(f)\|_{L^p} \le C_*\, K_0^{1-\theta} K_1^\theta \sup_{J\in\mathbb{Z}^n} \big\|\Gamma(s_1,\dots,s_n)[\sigma(2^J\cdot)\widehat{\psi}]\big\|_{L^r} \|f\|_{L^p}\,.$$

This completes the proof of the sufficiency part of Theorem 4.2. The proof of the necessity part is postponed to section 6. □

6. The sharpness of the Marcinkiewicz multiplier theorem

In this section we discuss examples indicating the sharpness of Theorem 4.2. We first consider the multiplier

$$\sigma(\xi) = e^{i|\xi|^2} \prod_{\ell=1}^n \phi(\xi_\ell)|\xi_\ell|^{-2\gamma_\ell},$$

where ϕ is a smooth function on $\mathbb{R}$ which vanishes in a neighborhood of the origin and is equal to 1 near infinity, and $\gamma > 0$. This multiplier can be obtained by taking tensor products of the functions $\sigma_{a,b}$ introduced at the end of Section 1, with $a = 2$ and $b = 2\gamma_\ell$. Then σ satisfies (4.2) when $r > 1$ and $\min_{1\le\ell\le n} \gamma_\ell > \frac{1}{r}$. In addition, if boundedness holds for T_σ from $L^p(\mathbb{R}^n)$ to itself, then by testing on tensor type functions, we must necessarily have that each $m_{2,2\gamma_\ell}(\xi_\ell)$ is bounded from $L^p(\mathbb{R})$ to itself and thus we must have $\big|\frac{1}{2} - \frac{1}{p}\big| \le \gamma_\ell$ for all $\ell = 1, \dots, n$.

Next we discuss an example indicating that Theorem 4.2 does not hold in the limiting case when $\left|\frac{1}{2}-\frac{1}{p}\right| = \gamma_\ell$ for some $\ell = 1, \dots, n$. Such an example appeared, at least in the two-dimensional case with both smoothness parameters equal, in Carbery and Seeger [5, remark after Proposition 6.1]. We provide an example in the spirit of theirs, given by an explicit closed-form expression and valid in all dimensions $n \geq 2$.

Example 6.1 ([32])**.** *Given $\alpha \in (0,1)$, consider the function*

$$\sigma(\xi,\eta) = \varphi(|\eta|)e^{-\frac{\xi^2}{2}}|\eta|^{i\xi}(\log|\eta|)^{-\alpha}, \quad (\xi,\eta) \in \mathbb{R}\times\mathbb{R}^{n-1} = \mathbb{R}^n,$$

where φ is a smooth function on the line such that $0 \leq \varphi \leq 1$, $\varphi = 0$ on $(-\infty, 8]$ and $\varphi = 1$ on $[9,\infty)$. Then
(i) *σ satisfies (4.2), with r large enough, whenever $\gamma_1 = \alpha$ and $\gamma_2, \dots, \gamma_n$ are arbitrary positive real numbers;*
(ii) *σ is an L^p Fourier multiplier for a given $1 < p < \infty$ if and only if $\alpha > |\frac{1}{p}-\frac{1}{2}|$.*

The previous example indicates that condition (4.2) does not guarantee that T_σ is L^p bounded unless all indices $\gamma_1, \dots, \gamma_n$ in (4.2) are larger than $|\frac{1}{p}-\frac{1}{2}|$. In particular, for a given $i \in \{1, \dots, n\}$, one does not have boundedness on the critical line $\gamma_i = |\frac{1}{p}-\frac{1}{2}|$, no matter how large the remaining parameters are.

Let us now verify the statement of part (i) of Example 6.1. We shall first prove that

$$\sup_{k,\ell\in\mathbb{Z}} \|(I-\partial_\xi^2)^{\frac{\alpha}{2}}(I-\Delta_\eta)^{\frac{s}{2}}[\widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi, 2^\ell\eta)]\|_{L^r} < \infty \tag{6.1}$$

for any $s > 0$ and $r > 1$. Here, Φ denotes a Schwartz function on $\mathbb{R}^{n-1}$ whose Fourier transform is supported in the set $\{\eta \in \mathbb{R}^{n-1} : \frac{1}{2} \leq |\eta| \leq 2\}$ and which satisfies $\sum_{\ell\in\mathbb{Z}} \widehat{\Phi}(2^\ell\eta) = 1$ for all $\eta \neq 0$. Indeed, for any $k, \ell \in \mathbb{Z}$, $\ell \geq 3$, and for any given nonnegative integer m, we have

$$\|\widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi, 2^\ell\eta)\|_{L^r} \leq C\ell^{-\alpha}$$

and

$$\|(I-\partial_\xi)^{\frac{1}{2}}(I-\Delta_\eta)^{\frac{m}{2}}[\widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi, 2^\ell\eta)]\|_{L^r} \leq C\ell\cdot\ell^{-\alpha},$$

where the constant C is independent of k and ℓ. Interpolating between these two estimates, we obtain

$$\|(I-\partial_\xi)^{\frac{\alpha}{2}}(I-\Delta_\eta)^{\frac{\alpha m}{2}}[\widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi, 2^\ell\eta)]\|_{L^r} \leq C.$$

Notice also that the last inequality in fact holds for all integers k, ℓ, since the function $\widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi, 2^\ell\eta)$ is identically equal to 0 if $\ell \le 2$. Hence, we have

$$\sup_{k,\ell\in\mathbb{Z}} \|(I-\partial_\xi)^{\frac{\alpha}{2}}(I-\Delta_\eta)^{\frac{\alpha m}{2}}[\widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi, 2^\ell\eta)]\|_{L^r} \le C,$$

and interpolating between variants of this estimate corresponding to different values of m, we obtain (6.1) for any $s > 0$. Now, part (i) of Example 6.1 follows by an application of Theorem 7.1, which will be stated and proved in the next section, in the variable η.

Let us finally focus on part (ii) of Example 6.1. If $\alpha > |\frac{1}{p} - \frac{1}{2}|$, then σ is an L^p Fourier multiplier thanks to (i) and Theorem 4.2. Let us now prove that T_σ is not L^p bounded if $\alpha \le |\frac{1}{p} - \frac{1}{2}|$. By duality, it suffices to discuss only the case when $1 < p < 2$. Furthermore, one can make use of the result of Herz and Rivière [34] which asserts that if T_σ is L^p bounded then it is necessarily bounded also on the mixed norm space $L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1}))$, defined as

$$\|f\|_{L^p(\mathbb{R};L^2(\mathbb{R}^{n-1}))} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |f(\xi,\eta)|^2 d\,\eta\right)^{\frac{p}{2}} d\xi\right)^{\frac{1}{2}}.$$

Thus, the proof will be complete if we show that T_σ is not bounded on $L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1}))$.

Let f be the function on $\mathbb{R}^n$ whose Fourier transform satisfies

$$\widehat{f}(\xi,\eta) = e^{-\frac{\xi^2}{2}}\varphi(|\eta|)|\eta|^{\frac{1-n}{2}}(\log|\eta|)^{-\frac{1}{2}}(\log\log|\eta|)^{-\beta}, \quad (\xi,\eta)\in\mathbb{R}\times\mathbb{R}^{n-1}.$$

Using Plancherel's theorem in the variable η, it is easy to check that $f \in L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1}))$ whenever $\beta > \frac{1}{2}$. Our next goal is to prove that $T_\sigma f = \mathcal{F}^{-1}(\sigma\widehat{f})$ does not belong to $L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1}))$ if $\beta \in (\frac{1}{2}, \frac{1}{p}]$. Using Plancherel's theorem in the variable η again, this is equivalent to showing that $\mathcal{F}_\xi^{-1}(\sigma\widehat{f})$ is not in $L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1}))$, where $\mathcal{F}_\xi^{-1}$ stands for the inverse Fourier transform in the ξ variable.

Observe that

$$\begin{aligned}\mathcal{F}_\xi^{-1}(\sigma\widehat{f})(x,\eta) &= Ce^{-\frac{1}{4}(2\pi x+\log|\eta|)^2}\varphi^2(|\eta|)|\eta|^{\frac{1-n}{2}}(\log|\eta|)^{-\alpha-\frac{1}{2}}(\log\log|\eta|)^{-\beta}\\ &\ge C\chi_{\{(x,\eta):\ x<-2,\ e^{-2\pi x-1}<|\eta|<e^{-2\pi x}\}}(x,\eta)e^{2\pi x\frac{n-1}{2}}(-x)^{-\alpha-\frac{1}{2}}(\log(-x))^{-\beta}.\end{aligned}$$

Therefore,

$$\|\mathcal{F}_\xi^{-1}(\sigma\widehat{f})\|_{L^p(\mathbb{R};L^2(\mathbb{R}^{n-1}))} \ge C\left(\int_{-\infty}^{-2}(-x)^{(-\alpha-\frac{1}{2})p}(\log(-x))^{-\beta p}\,dx\right)^{\frac{1}{p}} = \infty,$$

which yields the desired conclusion.

7. Comparison between the Hörmander and Marcinkiewicz multiplier theorems

In this section we show that the class of multipliers which satisfies the assumptions of Theorem 4.2 is strictly larger than the set of multipliers treated by the version of the Hörmander multiplier theorem due to Calderón and Torchinsky [3, Theorem 4.6]; see Section 1 for the statement of this theorem. Before we come to the proof we would like to emphasize that such a comparison is not possible for the classical versions of these two theorems (see (1.2) for the Hörmander condition and (4.1) for the Marcinkiewicz condition). Indeed, while condition (1.2) requires the multiplier to have more than $n/2$ derivatives in each variable, condition (4.1) assumes n derivatives in total, but only one in each variable. Therefore, there are multipliers satisfying (1.2) but not (4.1), and also multipliers satisfying (4.1) but not (1.2).

To compare the fractional versions of the Hörmander and Marcinkiewicz multiplier theorems, we first notice that Theorem 4.2 assumes the multiplier σ to have $1/r+\varepsilon$ derivatives in each variable, while the Hörmander multiplier theorem requires more than n/r derivatives in all variables, and so there are multipliers which can be treated by Theorem 4.2, but not by [3, Theorem 4.6]. On the other hand, it is an easy consequence of the following theorem that every multiplier satisfying the assumptions of the Hörmander multiplier theorem falls under the scope of Theorem 4.2 as well.

Theorem 7.1. *Let ψ be a Schwartz function on the line whose Fourier transform is supported in the set $\{\xi : \frac{1}{2} \le |\xi| \le 2\}$ and which satisfies $\sum_{k\in\mathbb{Z}} \widehat{\psi}(2^k\xi) = 1$ for every $\xi \neq 0$. Also, let Φ be a Schwartz function on $\mathbb{R}^n$ having analogous properties. If $1 < r < \infty$ and $\gamma_1, \dots, \gamma_n$ are real numbers larger than $\frac{1}{r}$, then*

$$\sup_{j_1,\dots,j_n\in\mathbb{Z}} \left\| (I-\partial_1^2)^{\frac{\gamma_1}{2}} \cdots (I-\partial_n^2)^{\frac{\gamma_n}{2}} \left[\sigma(2^{j_1}\xi_1,\dots,2^{j_n}\xi_n) \prod_{\ell=1}^n \widehat{\psi}(\xi_\ell) \right] \right\|_{L^r} \le C \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{\gamma_1+\cdots+\gamma_n}{2}} \left[\sigma(2^j\xi)\widehat{\Phi}(\xi) \right] \right\|_{L^r}. \tag{7.1}$$

Crucial ingredients needed for the proof of Theorem 7.1 are two one-dimensional inequalities contained in the following lemma.

Lemma 7.2. *Let ψ be as in Theorem 7.1. If $k \in \mathbb{Z}$, $\gamma > 0$ and $1 < r < \infty$ are such that $\gamma r > 1$, then*

$$\left\| f(2^k\cdot)\widehat{\psi} \right\|_{L^r(\mathbb{R})} \le C \left\| (I-\partial^2)^{\frac{\gamma}{2}} f \right\|_{L^r(\mathbb{R})} \tag{7.2}$$

and

$$\big\|(-\partial^2)^{\frac{\gamma}{2}}\big[f(2^k\cdot)\widehat{\psi}\big]\big\|_{L^r(\mathbb{R})} \le C\big(1+2^{k(\gamma-\frac{1}{r})}\big)\big\|(I-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})}. \tag{7.3}$$

Proof. Since $\gamma r > 1$, the Sobolev embedding theorem yields

$$|f(2^k x)| \le \|f\|_{L^\infty(\mathbb{R})} \le C\big\|(I-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})} \quad \text{for a.e. } x\in\mathbb{R}. \tag{7.4}$$

Therefore,

$$\big\|f(2^k\cdot)\widehat{\psi}\,\big\|_{L^r(\mathbb{R})} \le C\big\|(I-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})}\big\|\widehat{\psi}\,\big\|_{L^r(\mathbb{R})} = C'\big\|(I-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})}.$$

This proves (7.2).

Further, using the Kato-Ponce inequality [29], the estimate (7.4) and the fact that $\widehat{\psi}$ is smooth and with compact support, we obtain

$$\begin{aligned}
&\big\|(-\partial^2)^{\frac{\gamma}{2}}\big[f(2^k\cdot)\widehat{\psi}\big]\big\|_{L^r(\mathbb{R})}\\
&\qquad\le C\Big(\big\|(-\partial^2)^{\frac{\gamma}{2}}\big[f(2^k\cdot)\big]\big\|_{L^r(\mathbb{R})}\big\|\widehat{\psi}\,\big\|_{L^\infty(\mathbb{R})} + \big\|f(2^k\cdot)\big\|_{L^\infty(\mathbb{R})}\big\|(-\partial^2)^{\frac{\gamma}{2}}\widehat{\psi}\big\|_{L^r(\mathbb{R})}\Big)\\
&\qquad\le C\Big(\big\|(-\partial^2)^{\frac{\gamma}{2}}\big[f(2^k\cdot)\big]\big\|_{L^r(\mathbb{R})} + \big\|(I-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})}\Big)\\
&\qquad= C\Big(2^{k(\gamma-\frac{1}{r})}\big\|(-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})} + \big\|(I-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})}\Big)\\
&\qquad\le C\big(2^{k(\gamma-\frac{1}{r})}+1\big)\big\|(I-\partial^2)^{\frac{\gamma}{2}}f\big\|_{L^r(\mathbb{R})},
\end{aligned}$$

namely, (7.3). □

Proof of Theorem 7.1. Set $F(\xi) = \sum_{\kappa=-n}^{n}\widehat{\Phi}(2^\kappa\xi)$, $\xi\in\mathbb{R}^n$. Then $F(\xi)=1$ for any ξ satisfying $\frac{1}{2^n}\le|\xi|\le 2^n$. Therefore, if $j_1,\dots,j_n$ are integers and $j:=\max\{j_1,\dots,j_n\}$, then $F(2^{j_1-j}\xi_1,\dots,2^{j_n-j}\xi_n)=1$ on $\{(\xi_1,\dots,\xi_n) : \frac{1}{2}\le|\xi_1|\le 2,\dots,\frac{1}{2}\le|\xi_n|\le 2\}$. Consequently,

$$\prod_{\ell=1}^{n}\widehat{\psi}(\xi_\ell) = F(2^{j_1-j}\xi_1,\dots,2^{j_n-j}\xi_n)\prod_{\ell=1}^{n}\widehat{\psi}(\xi_\ell).$$

Using this, we can write

$$\begin{aligned}
&\Big\|(I-\partial_1^2)^{\frac{\gamma_1}{2}}\cdots(I-\partial_n^2)^{\frac{\gamma_n}{2}}\Big[\sigma(2^{j_1}\xi_1,\dots,2^{j_n}\xi_n)\prod_{\ell=1}^{n}\widehat{\psi}(\xi_\ell)\Big]\Big\|_{L^r}\\
&=\Big\|(I-\partial_1^2)^{\frac{\gamma_1}{2}}\cdots(I-\partial_n^2)^{\frac{\gamma_n}{2}}\Big[\sigma(2^{j_1}\xi_1,\dots,2^{j_n}\xi_n)F(2^{j_1-j}\xi_1,\dots,2^{j_n-j}\xi_n)\prod_{\ell=1}^{n}\widehat{\psi}(\xi_\ell)\Big]\Big\|_{L^r}\\
&\le C\sum_{a=-n}^{n}\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}
\end{aligned}$$

$$\left\|(-\partial_{i_1}^2)^{\frac{\gamma_{i_1}}{2}}\cdots(-\partial_{i_k}^2)^{\frac{\gamma_{i_k}}{2}}\left[\sigma(2^{j_1}\xi_1,\dots,2^{j_n}\xi_n)\widehat{\Phi}(2^{j_1-j+a}\xi_1,\dots,2^{j_n-j+a}\xi_n)\prod_{\ell=1}^n\widehat{\psi}(\xi_\ell)\right]\right\|_{L^r}.$$

Using the estimate (7.3) in variables $i_1,\dots,i_k$ and inequality (7.2) in the remaining variables, we estimate the corresponding term in the last expression by a constant multiple of

$$\begin{aligned}&\left[\prod_{s=1}^n\left(1+2^{(j_{i_s}-j+a)(\gamma_{i_s}-\frac{1}{r})}\right)\right]\left\|(I-\partial_1^2)^{\frac{\gamma_1}{2}}\cdots(I-\partial_n^2)^{\frac{\gamma_n}{2}}\left[\sigma(2^{j-a}\xi)\widehat{\Phi}(\xi)\right]\right\|_{L^r}\\&\le C\left(1+2^{n\max_{\ell=1,\dots,n}(\gamma_\ell-\frac{1}{r})}\right)^n\left\|(I-\Delta)^{\frac{\gamma_1+\dots+\gamma_n}{2}}\left[\sigma(2^{j-a}\xi)\widehat{\Phi}(\xi)\right]\right\|_{L^r}\\&\le C\sup_{m\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{\gamma_1+\dots+\gamma_n}{2}}\left[\sigma(2^m\xi)\widehat{\Phi}(\xi)\right]\right\|_{L^r}.\end{aligned}$$

This implies (7.1). □

8. A boundedness criterion for bilinear Fourier multiplier operators

As physical and natural phenomena depend on numerous inputs, it natural to develop theories that model dependencies on many variables. Multilinear Fourier Analysis provides a framework to study operations that depend linearly on several input functions. Multilinear multiplier operators are special kinds of multilinear operators in which the product of frequencies is jointly altered by a common symbol. Based on this definition, bilinear multiplier operators are given by

$$T_m(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi+\eta)}\,d\xi d\eta\,,$$

where f,g are Schwartz functions and m is a bounded function on $\mathbb{R}^{2n}$. These are exactly the bilinear operators that commute with simultaneous translations of functions. The study of general bilinear operators was initiated by Coifman and Meyer [9], [10] but since the turn of the present century this area has been enjoying a resurgence of activity. We refer to [11], [33], and [19] for general material related to the multilinear operators. A classical by now criterion for boundedness of bilinear multiplier operators says that if m satisfies

$$|\partial^\alpha\partial^\beta\sigma(\xi,\eta)|\le C_{\alpha,\beta}(|\xi|+|\eta|)^{-|\alpha|-|\beta|}$$

for sufficiently large multiindices α, β, then the associated bilinear operator T_m admits a bounded extension from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1/p_1+1/p_2=1/p$, $1<p_1,p_2\le\infty$ and $1/2<p<\infty$. This was proved by Coifman and Meyer [10] in the case when $p>1$ and was extended to the case $p\le 1$ by Grafakos and Torres [33] and

independently by Kenig and Stein [38]. This theorem is essentially saying that linear Mikhlin multipliers on $\mathbb{R}^{2n}$ are bounded bilinear multipliers on $\mathbb{R}^n \times \mathbb{R}^n$. It should be noted that the analogue to the Marcinkiewicz condition

$$|\partial^\alpha \partial^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta} |\xi|^{-|\alpha|} |\eta|^{-|\beta|}$$

does not suffice to guarantee boundedness of T_m on any product of Lebesgue spaces; a counterexample to indicate this fact was constructed in Grafakos and Kalton [25].

Extensions of the Coifman-Meyer–Kenig-Stein–Grafakos-Torres result for bilinear multipliers that satisfy Hörmander's [37] classical Sobolev space weakening of Mikhlin's condition for linear operators are available in the literature as well. They were initiated by Tomita [50] and subsequently further investigated by Grafakos, Fujita, Miyachi, Nguyen, Si, and Tomita among others; see [30] [17], [26], [41], [42], [28], [27].

As we have seen, bilinear Fourier multiplier operators may map $L^{p_1} \times L^{p_2}$ to L^p when $1/p_1 + 1/p_2 = 1/p$ for a variety of parameters p_1 and p_2 but in this note, we only focus on the $L^2 \times L^2 \to L^1$ boundedness of such operators. Such estimates are central and play the same role in bilinear theory as the L^2 boundedness plays in linear multiplier theory. As Plancherel's identity $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$ does not hold on L^1, there does not seem to be a straightforward way to characterize the boundedness of bilinear multiplier operators from $L^2 \times L^2 \to L^1$; however, different types of sharp sufficient conditions are available. For instance, a bilinear variant of the Hörmander multiplier theorem asserts that if the functions $m(2^k \cdot)\phi$ have s derivatives in $L^r(\mathbb{R}^{2n})$ $(1 < r < \infty)$ uniformly in $k \in \mathbb{Z}$, with ϕ being a suitable smooth bump supported in $1/2 < |\xi| < 2$, then T_m is bounded from $L^2 \times L^2$ to L^1 when $s > s_0 = \max(n/2, 2n/r)$ and s_0 cannot be replaced by any smaller number; see [21]. Thus more than $n/2$ derivatives of $m(2^k \cdot)\phi$ in $L^4(\mathbb{R}^{2n})$ uniformly in k are required of a generic multiplier m for T_m to map $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

In this note we restrict our attention to multipliers whose derivatives are merely bounded. We introduce the space

$$\mathscr{L}^\infty(\mathbb{R}^{2n}) = \big\{ m : \mathbb{R}^{2n} \to \mathbb{C} : \ \partial^\alpha m \text{ exist for all } \alpha \text{ and } \|\partial^\alpha m\|_{L^\infty} < \infty \big\}.$$

In the linear setting we have $m \in L^\infty$ if and only if the corresponding linear operator is bounded on L^2. So one may guess that a bilinear operator T_m is bounded from $L^2 \times L^2$ to L^1 when m lies in $\mathscr{L}^\infty$. However Bényi and Torres [2] provided an example of a function $m \in \mathscr{L}^\infty$ for which the associated bilinear operator T_m is unbounded from $L^{p_1} \times L^{p_2}$ to L^p for any $1 \le p_1, p_2 < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$. The counterexample of Bényi and Torres is also complemented by a positive result of the same authors [2] involving

mixed norm spaces and a subsequent positive result of Grafakos, He and Honzík [20, Corollary 8], who showed that the mere L^2 integrability of functions in $\mathscr{L}^\infty$ suffices to yield the $L^2 \times L^2 \to L^1$ boundedness of T_m.

It turns out that if $m \in \mathscr{L}^\infty$ then a sharp criterion for the boundedness of the bilinear multiplier operator T_m from $L^2 \times L^2 \to L^1$ can be formulated in terms of the magnitude of integrability of the function m. We provide a proof of the main direction of this result, the one that yields the boundedness of the operator. In what follows, $\mathscr{C}^M(\mathbb{R}^{2n})$ denotes the class of all functions on $\mathbb{R}^{2n}$ whose partial derivatives of order up to and including order M are continuous.

Theorem 8.1. [24] *Let $1 \le q < 4$ and set $M_q = \left\lfloor \frac{2n}{4-q} \right\rfloor + 1$. Let m be a function in $L^q(\mathbb{R}^{2n}) \cap \mathscr{C}^{M_q}(\mathbb{R}^{2n})$ satisfying*

$$\|\partial^\alpha m\|_{L^\infty} \le C_0 < \infty \quad \text{for all multiindices } \alpha \text{ with } |\alpha| \le M_q. \tag{8.1}$$

Then there is a constant C depending on n and q such that the bilinear operator T_m with multiplier m satisfies

$$\|T_m\|_{L^2\times L^2 \to L^1} \le C\, C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{4}}. \tag{8.2}$$

Additionally, we are aware of examples indicating that for any $q \ge 4$ there exist functions $m \in L^q(\mathbb{R}^{2n}) \cap \mathscr{L}^\infty(\mathbb{R}^{2n})$ such that the associated operator T_m does not map $L^2 \times L^2$ to L^1; see [24] for $q > 4$ and [47] for $q = 4$. These counterexamples are discussed in Section 11.

9. Bumps centered at lattice points

Before we prove Theorem 8.1 we present the idea of its proof in a simpler context. We examine the situation where the bilinear operator is given as a finite sum of products of smooth bumps supported in small discs centered at some lattice points in $\mathbb{R}^2$. As the set of lattice points may not be a product of subsets of $\mathbb{Z}$, the associated bilinear operator cannot be written as a product of linear operators and an alternative approach needs to be employed for its boundedness.

Let us define a linear operator as follows:

$$S_k(f)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\phi(\xi - k)e^{2\pi i\xi x} d\xi, \tag{9.1}$$

where ϕ is a smooth function on the line supported in the interval $(-1/10, 1/10)$.

Let E be a subset of $\mathbb{Z}^2$ of size N. We denote by E_1 the set of all $k \in \mathbb{Z}$ with the property that there exists an $l \in \mathbb{Z}$ such that the point $(k, l) \in E$. That is E_1 is the set

of all first coordinates of elements of E. We think of the set E as a union of columns Col_k indexed by $k \in E_1$ and we write

$$E = \bigcup_{k \in E_1} Col_k$$

and

$$T_{\sigma_N}(f,g) = \sum_{k \in E_1} S_k(f) \sum_{l:\,(k,l) \in Col_k} S_l(g).$$

We split the columns in large and small. Precisely, we write

$$E = E^{large} \cup E^{small},$$

where E^{large} contains all columns of size $\geq K$ and E^{small} contains all columns of size $< K$, for some K to be chosen later. Analogously we split

$$E_1 = E_1^{large} \cup E_1^{small},$$

where E_1^{large} and E_1^{small} is the set of all first coordinates of columns in E^{large} and E^{small}, respectively. Correspondingly we define:

$$T_{\sigma_N}^{large}(f,g) = \sum_{k \in E_1^{large}} S_k(f) \sum_{l:\,(k,l) \in Col_k} S_l(g)$$

and

$$\begin{aligned} T_{\sigma_N}^{small}(f,g) &= \sum_{k \in E_1^{small}} S_k(f) \sum_{l:\,(k,l) \in Col_k} S_l(g) \\ &= \sum_{l:\,\exists k\,(k,l) \in E^{small}} S_l(g) \sum_{k:\,(k,l) \in E^{small}} S_k(f) \end{aligned}$$

so that

$$T_{\sigma_N}(f,g) = T_{\sigma_N}^{large}(f,g) + T_{\sigma_N}^{small}(f,g).$$

We start with $T_{\sigma_N}^{large}$. We have

$$\begin{aligned} \big\|T_{\sigma_N}^{large}(f,g)\big\|_{L^1} &\leq \sum_{k \in E_1^{large}} \Big\|S_k(f) \sum_{l:\,(k,l) \in Col_k} S_l(g)\Big\|_{L^1} \\ &\leq \sum_{k \in E_1^{large}} \big\|S_k(f)\big\|_{L^2} \Big\| \sum_{l:\,(k,l) \in Col_k} S_l(g)\Big\|_{L^2} \\ &\leq \Big(\sum_{k \in E_1^{large}} \big\|S_k(f)\big\|_{L^2}^2 \Big)^{\frac{1}{2}} \Big(\sum_{k \in E_1^{large}} \Big\| \sum_{l:\,(k,l) \in Col_k} S_l(g)\Big\|_{L^2}^2 \Big)^{\frac{1}{2}} \\ &\leq \|\phi\|_{L^\infty} \|f\|_{L^2} (\#E_1^{large})^{\frac{1}{2}} \|\phi\|_{L^\infty} \|g\|_{L^2}, \end{aligned}$$

exploiting the orthogonality of S_k's on L^2.

Notice that as there are N points in E and each column in E^{large} has at least K elements, this means that there are at most N/K columns in E^{large}. We conclude that

$$\left\|T^{large}_{\sigma_N}(f,g)\right\|_{L^1} \leq (N/K)^{\frac{1}{2}}\|\phi\|^2_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}. \tag{9.2}$$

We continue with $T^{small}_{\sigma_N}$. We have

$$\begin{aligned}
&\left\|T^{small}_{\sigma_N}(f,g)\right\|_{L^1}\\
&\quad=\Big\|\sum_{l:\,\exists k\,(k,l)\in E^{small}} S_l(g)\sum_{k:\,(k,l)\in E^{small}} S_k(f)\Big\|_{L^1}\\
&\quad\leq \sum_{l:\,\exists k\,(k,l)\in E^{small}} \Big\|S_l(g)\sum_{k:\,(k,l)\in E^{small}} S_k(f)\Big\|_{L^1}\\
&\quad\leq \sum_{l:\,\exists k\,(k,l)\in E^{small}} \left\|S_l(g)\right\|_{L^2}\Big\|\sum_{k:\,(k,l)\in E^{small}} S_k(f)\Big\|_{L^2}\\
&\quad\leq\Big(\sum_{l:\,\exists k\,(k,l)\in E^{small}} \left\|S_l(g)\right\|^2_{L^2}\Big)^{\frac{1}{2}}\Big(\sum_{l:\,\exists k\,(k,l)\in E^{small}}\Big\|\sum_{k:\,(k,l)\in E^{small}} S_k(f)\Big\|^2_{L^2}\Big)^{\frac{1}{2}}\\
&\quad\leq\|\phi\|_{L^\infty}\|g\|_{L^2}\Big(\sum_{l:\,\exists k\,(k,l)\in E^{small}}\sum_{k:\,(k,l)\in E^{small}}\left\|S_k(f)\right\|^2_{L^2}\Big)^{\frac{1}{2}}\\
&\quad=\|\phi\|_{L^\infty}\|g\|_{L^2}\Big(\sum_{k\in E^{small}_1}\sum_{l:\,(k,l)\in Col_k}\left\|S_k(f)\right\|^2_{L^2}\Big)^{\frac{1}{2}}\\
&\quad\leq\|\phi\|_{L^\infty}\|g\|_{L^2}K^{\frac{1}{2}}\Big(\sum_{k\in E^{small}_1}\left\|S_k(f)\right\|^2_{L^2}\Big)^{\frac{1}{2}}\\
&\quad\leq\|\phi\|_{L^\infty}\|g\|_{L^2}K^{\frac{1}{2}}\|\phi\|_{L^\infty}\|f\|_{L^2},
\end{aligned}$$

as all columns in E^{small} have size at most K. This yields

$$\left\|T^{small}_{\sigma_N}(f,g)\right\|_{L^1} \leq K^{\frac{1}{2}}\|\phi\|^2_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}. \tag{9.3}$$

In view of (9.2) and (9.3), the optimal choice of $K = N^{1/2}$. This proves

$$\left\|T_{\sigma_N}(f,g)\right\|_{L^1} \leq N^{\frac{1}{4}}\|\phi\|^2_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}. \tag{9.4}$$

10. Proof of Theorem 8.1

We plan to outline the proof of Theorem 8.1. This is based on the product-type wavelet method initiated in [20]. Our approach here incorporates several crucial combinatorial improvements compared to [20].

We recall some facts related to product-type wavelets that will be crucial in our approach of proving Theorem 8.1 For a fixed $M \in \mathbb{N}$ there exist real-valued compactly supported functions ψ_F, ψ_M in $\mathcal{C}^k(\mathbb{R})$, called *father wavelet* and *mother wavelet*, respectively, that satisfy

$$\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$$

and

$$\int_{\mathbb{R}} x^k \psi_M(x)dx = 0 \qquad \text{for all } 0 \le k \le M.$$

Then the family of functions of the variables $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\begin{aligned}
&\bigcup_{\mu_1,\mu_2\in\mathbb{Z}^n} \Big\{\psi_F(x_1-\mu_1)\psi_F(x_2-\mu_2)\Big\} \\
&\cup \bigcup_{\mu_1,\mu_2\in\mathbb{Z}^n} \bigcup_{\lambda=0}^{\infty} \Big\{2^{\frac{n\lambda}{2}}\psi_F(2^\lambda x_1-\mu_1)2^{\frac{n\lambda}{2}}\psi_M(2^\lambda x_2-\mu_2)\Big\} \\
&\cup \bigcup_{\mu_1,\mu_2\in\mathbb{Z}^n} \bigcup_{\lambda=0}^{\infty} \Big\{2^{\frac{n\lambda}{2}}\psi_M(2^\lambda x_1-\mu_1)2^{\frac{n\lambda}{2}}\psi_F(2^\lambda x_2-\mu_2)\Big\} \\
&\cup \bigcup_{\mu_1,\mu_2\in\mathbb{Z}^n} \bigcup_{\lambda=0}^{\infty} \Big\{2^{\frac{n\lambda}{2}}\psi_M(2^\lambda x_1-\mu_1)2^{\frac{n\lambda}{2}}\psi_M(2^\lambda x_2-\mu_2)\Big\}
\end{aligned}$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$. This result is due to Triebel [52].

We denote by $\mathcal{J}$ the set of all pairs (λ, G) such that either $\lambda = 0$ and $G = (F, F)$, or λ is a nonnegative integer and G has the form (F, M), (M, F), or (M, M). For $(\lambda, G) \in \mathcal{J}$ and $(\mu_1, \mu_2) \in \mathbb{Z}^{2n}$ we set

$$\Psi^{\lambda,G}_{\mu_1,\mu_2}(x_1, x_2) = 2^{\frac{n\lambda}{2}}\psi_{G_1}(2^\lambda x_1-\mu_1)2^{\frac{n\lambda}{2}}\psi_{G_2}(2^\lambda x_2-\mu_2).$$

for $(x_1, x_2) \in \mathbb{R}^{2n}$, where $G = (G_1, G_2)$ and $(\lambda, G) \in \mathcal{J}$.

The cancellation of wavelets is manifested in the following result.

Lemma 10.1. *Let M be a positive integer. Assume that $m \in \mathscr{C}^{M+1}$ is a function on $\mathbb{R}^{2n}$ such that*

$$\sup_{|\alpha|\le M+1} \|\partial^\alpha m\|_{L^\infty} \le C_0 < \infty\,.$$

Then for $(\lambda, G) \in \mathcal{J}$ and $(\mu_1, \mu_2) \in \mathbb{Z}^{2n}$ we have

$$|\langle \Psi^{\lambda,G}_{\mu_1,\mu_2}, m\rangle| \le CC_0 2^{-(M+n+1)\lambda}\,, \tag{10.1}$$

provided that ψ_M has M vanishing moments.

This lemma can be easily proved and is essentially a restatement of Lemma 7 in [20]. Note that if $G = (F, F)$ there is no cancellation, however, there is no decay claimed in (10.1), as $\lambda = 0$ in this case.

Proof of Theorem 8.1. To prove the theorem we use the product type wavelets introduced. We begin by fixing a large number M to be determined later, which denotes the number of vanishing moments of the mother wavelet.

For $(\lambda, G) \in \mathcal{J}$ and $\mu \in \mathbb{Z}^{2n}$ we denote the wavelet coefficient by

$$b_\mu^{\lambda,G} = \langle \Psi_\mu^{\lambda,G}, m\rangle.$$

By [51, Theorem 1.64] and by the fact that L^q coincides with the Triebel-Lizorkin space $F^0_{q,2}$, we obtain

$$\|m\|_{L^q(\mathbb{R}^{2n})} \approx \Big\|\Big(\sum_{(\lambda,G)\in\mathcal{J}}\sum_{\mu\in\mathbb{Z}^{2n}} |b_\mu^{\lambda,G} 2^\lambda \chi_{Q_{\lambda\mu}}|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^{2n})}, \tag{10.2}$$

where $Q_{\lambda\mu}$ is the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{1-\lambda}$.

Now, let us fix $(\lambda, G) \in \mathcal{J}$. For notational simplicity, we write b_μ instead of $b_\mu^{\lambda,G}$ in what follows. We also denote by $\tilde{Q}_{\lambda\mu}$ the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{-\lambda}$. Noting that these cubes are pairwise disjoint in μ (for the fixed value of λ), the equivalence (10.2) yields

$$\begin{aligned}
\|m\|_{L^q(\mathbb{R}^{2n})} &\gtrsim 2^{n\lambda}\Big\|\Big(\sum_{\mu\in\mathbb{Z}^2} |b_\mu|^2 \chi_{Q_{\lambda\mu}}\Big)^{\frac12}\Big\|_{L^q(\mathbb{R}^{2n})}\\
&\geq 2^{n\lambda}\Big\|\Big(\sum_{\mu\in\mathbb{Z}^2} |b_\mu|^2 \chi_{\tilde{Q}_{\lambda\mu}}\Big)^{\frac12}\Big\|_{L^q(\mathbb{R}^{2n})}\\
&= 2^{n\lambda}\Big\|\sum_{\mu\in\mathbb{Z}^{2n}} |b_\mu| \chi_{\tilde{Q}_{\lambda\mu}}\Big\|_{L^q(\mathbb{R}^{2n})}\\
&= 2^{n\lambda(1-\frac{2}{q})}\Big(\sum_{\mu\in\mathbb{Z}^2} |b_\mu|^q\Big)^{\frac1q}.
\end{aligned}$$

Setting $b = (b_\mu)_{\mu\in\mathbb{Z}^{2n}}$, the preceding sequence of inequalities yields

$$\|b\|_{\ell^q} \leq C 2^{-n\lambda(1-\frac{2}{q})}\|m\|_{L^q}. \tag{10.3}$$

Also, Lemma 10.1 implies that

$$\|b\|_{\ell^\infty} \leq C C_0 2^{-\lambda(M+n+1)}, \tag{10.4}$$

where M is the number of vanishing moments of ψ_M.

We have an infinite $\times$ infinite matrix of wavelet coefficients indexed by $\mathbb{Z}^{2n}$. To better organize these coefficients, define

$$U_r = \{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n = \mathbb{Z}^{2n} : \ 2^{-r-1}\|b\|_{\ell^\infty} < |b_{(k,l)}| \le 2^{-r}\|b\|_{\ell^\infty}\},$$

where r is a nonnegative integer. Also, we write U_r as a union of the following two disjoint sets:

$$U_r^1 = \{(k,l) \in U_r : \mathrm{card}\{s : (k,s) \in U_r\} \ge K\};$$
$$U_r^2 = \{(k,l) \in U_r : \mathrm{card}\{s : (k,s) \in U_r\} < K\},$$

where K is a positive number to be determined. Thinking of U_r as an infinite $\times$ infinite matrix with integer entries, in this splitting, we placed in U_r^1 all columns of U^r that have size greater than or equal to K and in U_r^2 the remaining ones. We call U_r^1 the long columns of U_r and U_r^1 the short columns. Let us denote

$$E = \{k \in \mathbb{Z}^n : (k,l) \in U_r^1 \text{ for some } l \in \mathbb{Z}^n\}.$$

This set is exactly the set of projections of all long columns. Then

$$(\#E)\, K \left[2^{-(r+1)}\|b\|_{\ell^\infty}\right]^q \le \sum_{(k,l)\in U_r^1} |b_{(k,l)}|^q \le \|b\|_{\ell^q}^q,$$

and therefore

$$\#E \le K^{-1}\left[2^{-(r+1)}\|b\|_{\ell^\infty}\right]^{-q}\|b\|_{\ell^q}^q. \tag{10.5}$$

Having broken down the wavelet coefficients in groups we proceed with the analysis of the sums of the decomposition associated with these groups. Given $(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n$, it follows from the definition of $\Psi_{(k,l)}^{\lambda,G}$ that $\Psi_{(k,l)}^{\lambda,G}$ can be written in the tensor product form

$$\Psi_{(k,l)}^{\lambda,G}(x_1,x_2) = \omega_{1,k}(x_1)\omega_{2,l}(x_2)$$

and

$$\|\omega_{1,k}\|_{L^\infty} \approx \|\omega_{2,l}\|_{L^\infty} \approx 2^{\frac{n\lambda}{2}}.$$

Define

$$m^{r,1} = \sum_{(k,l)\in U_r^1} b_{(k,l)}\Psi_{(k,l)}^{\lambda,G} = \sum_{(k,l)\in U_r^1} b_{(k,l)}\omega_{1,k}\omega_{2,l}.$$

Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Then

$$\begin{aligned}\left\|T_{m^{r,1}}(f,g)\right\|_{L^1} &\le \Big\| \sum_{(k,l)\in U_r^1} b_{(k,l)}\mathcal{F}^{-1}(\omega_{1,k}\widehat{f}\,)\mathcal{F}^{-1}(\omega_{2,l}\widehat{g}\,)\Big\|_{L^1} \\ &\le \sum_{k\in E}\left\|\omega_{1,k}\widehat{f}\,\right\|_{L^2}\Big\| \sum_{l:(k,l)\in U_r^1} b_{(k,l)}\omega_{2,l}\widehat{g}\,\Big\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}2^{\frac{n\lambda}{2}}2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2}\\
&\leq C\Big(\sum_{k\in E}1\Big)^{1/2}\Big(\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\Big)^{\frac{1}{2}}2^{\frac{n\lambda}{2}}2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2}\\
&\leq C\Big\{K^{-\frac{1}{2}}\big[2^{-(r+1)}\|b\|_{\ell^\infty}\big]^{-\frac{q}{2}}\|b\|_{\ell^q}^{\frac{q}{2}}\Big\}\Big\{2^{\frac{n\lambda}{2}}2^{-r}\|b\|_{\ell^\infty}\Big\}2^{\frac{n\lambda}{2}}\|f\|_{L^2}\|g\|_{L^2},
\end{aligned}
$$

where we used estimate (10.5) and the property that the supports of the functions $\omega_{1,k}$ and $\omega_{2,l}$ have finite overlap.

Now define

$$
m^{r,2}=\sum_{(k,l)\in U_r^2}b_{(k,l)}\omega_{1,k}\omega_{2,l}.
$$

Then

$$
\begin{aligned}
\|T_{m^{r,2}}(f,g)\|_{L^1}&\leq\Big\|\sum_{(k,l)\in U_r^2}b_{(k,l)}\mathcal{F}^{-1}(\omega_{1,k}\widehat{f}\,)\mathcal{F}^{-1}(\omega_{2,l}\widehat{g}\,)\Big\|_{L^1}\\
&\leq\sum_{l:\,\exists k\,(k,l)\in U_r^2}\big\|\omega_{2,l}\widehat{g}\,\big\|_{L^2}\Big\|\sum_{k:(k,l)\in U_r^2}b_{(k,l)}\omega_{1,k}\widehat{f}\,\Big\|_{L^2}\\
&\leq\Big(\sum_{l\in\mathbb{Z}^n}\big\|\omega_{2,l}\widehat{g}\,\big\|_{L^2}^2\Big)^{\frac{1}{2}}\Big(\sum_{l:\,\exists k\,(k,l)\in U_r^2}\Big\|\sum_{k:(k,l)\in U_r^2}b_{(k,l)}\omega_{1,k}\widehat{f}\,\Big\|_{L^2}^2\Big)^{\frac{1}{2}}\\
&\leq C2^{\frac{\lambda}{2}}\|g\|_{L^2}\Big(\sum_{k:\,\exists l\,(k,l)\in U_r^2}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\sum_{l:(k,l)\in U_r^2}|b_{(k,l)}|^2\Big)^{\frac{1}{2}}\\
&\leq C2^{\frac{n\lambda}{2}}\|g\|_{L^2}2^{-r}\|b\|_{\ell^\infty}K^{\frac{1}{2}}\Big(\sum_{k\in\mathbb{Z}}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\Big)^{\frac{1}{2}}\\
&\leq C2^{\frac{n\lambda}{2}}2^{-r}\|b\|_{\ell^\infty}K^{\frac{1}{2}}2^{\frac{n\lambda}{2}}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}
$$

We have now obtained the estimates for an unknown quantity K:

$$
\|T_{m^{r,1}}(f,g)\|_{L^1}\leq CK^{-\frac{1}{2}}\big[2^{-(r+1)}\|b\|_{\ell^\infty}\big]^{-\frac{q}{2}}\|b\|_{\ell^q}^{\frac{q}{2}}2^{n\lambda}2^{-r}\|b\|_{\ell^\infty}\|f\|_{L^2}\|g\|_{L^2}, \tag{10.6}
$$

$$
\|T_{m^{r,2}}(f,g)\|_{L^1}\leq C2^{n\lambda}2^{-r}\|b\|_{\ell^\infty}K^{\frac{1}{2}}\|f\|_{L^2}\|g\|_{L^2}. \tag{10.7}
$$

We choose K optimally so that the two quantities on the right in (10.6) and (10.7) are equal. The optimal choice of K is

$$
K=\left(\frac{2^r\|b\|_{\ell^q}}{\|b\|_{\ell^\infty}}\right)^{\frac{q}{2}}.
$$

This choice of K yields for

$$
m^r=\sum_{(k,l)\in U_r}b_{(k,l)}\omega_{1,k}\omega_{2,l}=m^{r,1}+m^{r,2}
$$

the estimate

$$\|T_{m^r}\|_{L^2\times L^2\to L^1} \le C\, 2^{n\lambda}\, 2^{-r(1-\frac{q}{4})}\|b\|_{\ell^\infty}^{1-\frac{q}{4}}\|b\|_{\ell^q}^{\frac{q}{4}}. \tag{10.8}$$

Using (10.3) and (10.4) in (10.8) we obtain

$$\|T_{m^r}\|_{L^2\times L^2\to L^1} \le CC_0^{1-\frac{q}{4}} 2^{n\lambda-\lambda(1-\frac{q}{4})(M+n+1)+n(\frac{2}{q}-1)\frac{q}{4}\lambda} 2^{-r(1-\frac{q}{4})}\|m\|_{L^q}^{\frac{q}{4}}.$$

Notice that when $q<4$ we have $1-\frac{q}{4}<0$, hence we can sum in $r\in\mathbb{Z}^+$. Also,

$$2^{n\lambda-\lambda(1-\frac{q}{4})(M+n+1)+n(\frac{2}{q}-1)\frac{q}{4}\lambda} = 2^{\lambda[\frac{n}{2}-\frac{4-q}{4}(M+1)]}$$

and the exponent is negative only when $M+1>\frac{2n}{4-q}$. Thus, if we choose $M=\lfloor\frac{2n}{4-q}\rfloor$, we can sum first over $\lambda\in\mathbb{Z}^+\cup\{0\}$ when $G\in\{(F,M),(M,M),(M,F)\}$. For $G=(F,F)$ there is no need to sum over λ. This yields (8.2) for any G and completes the proof of Theorem 8.1. □

11. The sharpness of the condition $q<4$

In this section we discuss optimality of the assumption $q<4$ of Theorem 8.1. The main result is the following.

Theorem 11.1. *Suppose that $q\ge 4$. Then there exists a function $m\in L^q(\mathbb{R}^{2n})\cap\mathcal{L}^\infty(\mathbb{R}^{2n})$ such that the associated operator T_m does not map $L^2\times L^2$ to L^1.*

Theorem 11.1 was proved in [24] in the case $q>4$. The limiting case $q=4$ is discussed in [47]. We do not include the full proof here as it is somewhat technical but we will describe the main ideas needed to reach the conclusion. To further simplify the presentation, we will assume that $n=1$.

We start by studying a randomized variant of the operator T_{σ_N} from Section 9. Namely, we fix a (large) positive integer K and consider the set $E=\{(j,k)\in\mathbb{N}^2 : j+k\le K\}$. We observe that $N:=\#E=\frac{K(K-1)}{2}$.

Let $(r_j(t))_{j=0}^\infty$ denote the sequence of Rademacher functions; see Section 4 for the definition. Assume that ϕ is a smooth function on $\mathbb{R}$ supported in the interval $[-1/10,1/10]$ assuming value 1 in $[-1/20,1/20]$. Let S_k be the operator given by (9.1). We fix $t\in[0,1]$ and define the operator $T_{K,t}$ as

$$T_{K,t}(f,g)=\sum_{(j,k)\in E} r_{j+k}(t)S_j(f)S_k(g)=\sum_{j=1}^{K-1}\sum_{k=1}^{K-j} r_{j+k}(t)S_j(f)S_k(g).$$

This operator is associated with the multiplier

$$m_{K,t}(\xi,\eta) = \sum_{j=1}^{K-1}\sum_{k=1}^{K-j} r_{j+k}(t)\phi(\xi-j)\phi(\eta-k).$$

Let φ be a Schwartz function on $\mathbb{R}$ whose Fourier transform is supported in the interval $[-1/100, 1/100]$. We define functions f and g on $\mathbb{R}$ in terms of their Fourier transform by

$$\widehat{f}(\xi) = K^{-\frac{1}{2}} \sum_{j=1}^{K} \widehat{\varphi}(\xi-j) \tag{11.1}$$

and

$$\widehat{g}(\eta) = K^{-\frac{1}{2}} \sum_{k=1}^{K} \widehat{\varphi}(\eta-k). \tag{11.2}$$

Then both f and g are Schwartz functions whose L^2-norms are bounded by a constant independent of K. We observe that

$$\begin{aligned} T_{m_{K,t}}(f,g)(x) &= K^{-1} \sum_{j=1}^{K-1}\sum_{k=1}^{K-j} r_{j+k}(t)(\varphi(x))^2 e^{2\pi i x(j+k)} \\ &= K^{-1} \sum_{l=2}^{K} (l-1) r_l(t) e^{2\pi i x l}(\varphi(x))^2. \end{aligned}$$

Using Fubini's theorem and Khintchine's inequality (4.7), we obtain

$$\begin{aligned} \int_0^1 \|T_{m_{K,t}}(f,g)\|_{L^1}\, dt &= K^{-1} \int_{\mathbb{R}} \int_0^1 \left| \sum_{l=2}^{K} (l-1) r_l(t) e^{2\pi i x l}(\varphi(x))^2 \right| dt\, dx \\ &\approx K^{-1} \int_{\mathbb{R}} \left(\sum_{l=2}^{K} (l-1)^2 |\varphi(x)|^4 \right)^{\frac{1}{2}} dx \\ &\approx K^{\frac{1}{2}} \approx N^{\frac{1}{4}}. \end{aligned}$$

This indicates the sharpness of the estimate in Section 9. Let us now observe that this calculation is also relevant for proving Theorem 11.1. We fix $q \in [1,\infty)$ and consider the function

$$M_{K,t} = K^{-\frac{2}{q}} m_{K,t}.$$

Then the L^q norm of $M_{K,t}$ is bounded uniformly in K and t, and the same is true for the L^∞ norms of all partial derivatives of $M_{K,t}$. The calculation above yields

$$\int_0^1 \|T_{M_{K,t}}(f,g)\|_{L^1}\, dt \approx K^{\frac{1}{2}-\frac{2}{q}}.$$

We see that the right-hand side approaches infinity as $K \to \infty$ if $q > 4$, which implies that T_m is not bounded from $L^2 \times L^2$ to L^1 with a constant depending only on the L^q norm of m and on the L^∞ norms of its derivatives in this case. A modification of this example can then be used to construct a single function $m \in L^q(\mathbb{R}^{2n}) \cap \mathcal{L}^\infty(\mathbb{R}^{2n})$ for which the associated operator T_m does not map $L^2 \times L^2$ to L^1.

References

[1] C. Bennett, R. Sharpley. *Interpolation of operators*, Academic Press, Boston, 1988.

[2] A. Bényi, R. Torres. *Almost orthogonality and a class of bounded bilinear pseudodifferential operators.* Math. Res. Lett. **11** (2004), 1–11.

[3] A. P. Calderón, A. Torchinsky. *Parabolic maximal functions associated with a distribution, II,* Adv. Math. **24** (1977), 101–171.

[4] A. Carbery. *Differentiation in lacunary directions and an extension of the Marcinkiewicz multiplier theorem*, Ann. de l' Institut Fourier (Grenoble) **38** (1988), 157–168.

[5] A. Carbery, A. Seeger. *H^p- and L^p-Variants of multiparameter Calderón-Zygmund theory*, Trans. AMS **334** (1992), 719–747.

[6] A. Carbery, A. Seeger. *Homogeneous Fourier multipliers of Marcinkiewicz type*, Ark. Mat. **33** (1995), 45–80.

[7] A. Cianchi and L. Pick. *Sobolev embeddings into BMO, VMO and L^∞*, Ark. Mat. **36** (1998), 317 – 340.

[8] A. Cianchi, L. Pick, L. Slavíková. *Higher-order Sobolev embeddings and isoperimetric inequalities*, Adv. Math. **273** (2015), 568–650.

[9] R. R. Coifman, Y. Meyer. *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[10] R. R. Coifman, Y. Meyer. *Commutateurs d' intégrales singulières et opérateurs multilinéaires*, Ann. de l' Inst. Fourier, Grenoble **28** (1978), 177–202.

[11] R. R. Coifman, Y. Meyer. *Au-delà des opérateurs pseudo-différentiels*, Astérisque **57**, 1978.

[12] W. C. Connett, A. L. Schwartz. *The theory of ultraspherical multipliers*, Mem. Amer. Math. Soc. **9** (1977), no. 183, iv+92 pp.

[13] W. C. Connett, A. L. Schwartz. *A remark about Calderón's upper s method of interpolation*, Interpolation spaces and allied topics in analysis (Lund, 1983), 48–53, Lecture Notes in Math., 1070, Springer, Berlin, 1984.

[14] J. Duoandikoetxea. *Fourier Analysis*, Grad. Studies in Math., Vol 29, AMS, Providence, RI, 2001.

[15] D. E. Edmunds, B. Opic. *Boundedness of fractional maximal operators between classical and weak type Lorentz spaces*, Dissertationes Math. (Rozprawy Mat.) **410** (2002), 50 pp.

[16] C. Fefferman, E. M. Stein. *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

[17] M. Fujita, N. Tomita. *Weighted norm inequalities for multilinear Fourier multipliers.* Trans. Amer. Math. Soc. **364** (2012), 6335–6353.

[18] L. Grafakos. *Classical Fourier Analysis*, 3rd edition, GTM **249**, Springer, NY 2014.

[19] L. Grafakos. *Modern Fourier Analysis*, 3rd edition, GTM **250**, Springer, NY 2014.

[20] L. Grafakos, D. He, P. Honzík. *Rough bilinear singular integrals.* Adv. Math. **326** (2018), 54–78.

[21] L. Grafakos, D. He, P. Honzík. *The Hörmander multiplier theorem, II: The bilinear local L^2 case.* Math. Zeit. **289** (2018), 875–887.

[22] L. Grafakos, D. He, P. Honzík, H. V. Nguyen. *The Hörmander multiplier theorem I: The linear case revisited*, Illinois J. Math. **61** (2017), 25–35.

[23] L. Grafakos, D. He, H. V. Nguyen, L. Yan. *Multilinear multiplier theorems and applications*, J. Fourier. Anal. Appl., to appear.

[24] L. Grafakos, D. He, L. Slavíková. *$L^2 \times L^2 \to L^1$ boundedness criteria*, Math. Ann., to appear.

[25] L. Grafakos, N. Kalton. *The Marcinkiewicz multiplier condition for bilinear operators*, Studia Math. **146** (2001), 115–156.
[26] L. Grafakos, A. Miyachi, N. Tomita. *On multilinear Fourier multipliers of limited smoothness.* Canad. J. Math. **65** (2013), 299–330.
[27] L. Grafakos, A. Miyachi, H. V. Nguyen, N. Tomita. *Multilinear Fourier multipliers with minimal Sobolev regularity, II.* J. Math. Soc. Japan **69** (2017), 529–562.
[28] L. Grafakos, H. V. Nguyen. *Multilinear Fourier multipliers with minimal Sobolev regularity, I.* Colloq. Math. **144** (2016), 1–30.
[29] L. Grafakos, S. Oh. *The Kato-Ponce inequality*, Comm. in PDE **39** (2014), 1128–1157.
[30] L. Grafakos, Z. Si. *The Hörmander multiplier theorem for multilinear operators.* J. Reine Angew. Math. **668** (2012), 133–147.
[31] L. Grafakos, L. Slavíková. *A sharp version of the Hörmander multiplier theorem*, Int. Math. Res. Not., to appear.
[32] L. Grafakos, L. Slavíková. *The Marcinkiewicz multiplier theorem revisited*, Arch. Math. **112** (2019), 191 - 203.
[33] L. Grafakos, R. H. Torres. *Multilinear Calderón-Zygmund Theory.* Adv. Math. **165** (1999), 124–164.
[34] C. Herz, N. M. Rivière. *Estimates for translation invariant operators on spaces with mixed norms*, Studia Math. **44** (1972), 511–515.
[35] I. I. Jr. Hirschman. *A convexity theorem for certain groups of transformations*, J. Analyse Math. **2** (1953), 209–218.
[36] I. I. Jr. Hirschman. *On multiplier transformations*, Duke Math. J. **26** (1959), 221–242.
[37] L. Hörmander. *Estimates for translation invariant operators in L^p spaces,* Acta Math. **104** (1960), 93–139.
[38] C. Kenig, E. M. Stein. *Multilinear estimates and fractional integration.* Math. Res. Lett. **6** (1999), 1–15.
[39] S. G. Mikhlin. *On the multipliers of Fourier integrals.* (Russian) Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 701–703.
[40] A. Miyachi. *On some Fourier multipliers for $H^p(\mathbb{R}^n)$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 157–179.
[41] A. Miyachi, N. Tomita. *Minimal smoothness conditions for bilinear Fourier multipliers.* Rev. Mat. Iber. **29** (2013), 495–530.
[42] A. Miyachi, N. Tomita. *Boundedness criterion for bilinear Fourier multiplier operators.* Tohoku Math. J. **66** (2014), 55–76.
[43] A. Seeger. *A limit case of the Hörmander multiplier theorem*, Monatsh. Math. **105** (1988), 151–160.
[44] A. Seeger, *Estimates near L^1 for Fourier multipliers and maximal functions*, Arch. Math. (Basel) **53** (1989), 188–193.
[45] A. Seeger. *Remarks on singular convolution operators*, Studia Math. **97** (1990), no. 2, 91 – 114.
[46] L. Slavíková. *On the failure of the Hörmander multiplier theorem in a limiting case*, Rev. Mat. Iberoam., to appear.
[47] L. Slavíková. A manuscript.
[48] E. M. Stein. *Singular integral and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.
[49] E. M. Stein. *Editor's note: The differentiability of functions in $\mathbb{R}^n$*, Ann. of Math. (2) **113** (1981), no. 2, 383 – 385.
[50] N. Tomita. *A Hörmander type multiplier theorem for multilinear operators.* J. Funct. Anal. **259** (2010), 2028–2044.
[51] H. Triebel. *Theory of function spaces. III.* Monographs in Mathematics, **100**, Birkhäuser Verlag, Basel, 2006.
[52] H. Triebel. *Bases in function spaces, sampling, discrepancy, numerical integration.* EMS Tracts in Mathematics, **11**, European Mathematical Society (EMS), Zürich, 2010.
[53] S. Wainger. *Special trigonometric series in k-dimensions*, Mem. Amer. Math. Soc. **59** (1965), 1–102.

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