NONUNIFORM SOBOLEV SPACES

TING CHEN, LOUKAS GRAFAKOS, AND WENCHANG SUN

Abstract. We study nonuniform Sobolev spaces, i.e., spaces of functions whose partial derivatives lie in possibly different Lebesgue spaces. Although standard proofs do not apply, we show that nonuniform Sobolev spaces share similar properties as the classical ones. These spaces arise naturally in the study of certain PDEs. For instance, we illustrate that nonuniform fractional Sobolev spaces are useful in the study of local estimates for solutions of heat equations and the convergence of Schrödinger operators. In this work we extend recent advances on local energy estimates for solutions of heat equations and the convergence of Schrödinger operators to nonuniform fractional Sobolev spaces.

1. Introduction

Given a positive integer m, a positive number $p \in [1, \infty]$ and an open set $\Omega \subset \mathbb{R}^N$, we denote by

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), |\alpha| \le m\},$$

the classical Sobolev space, where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$, and D^α denotes the weak partial derivatives, that is, partial derivatives in the sense of distributions. The theory of Sobolev spaces plays an import role in the study of partial differential equations and many other fields. We refer to text books by Adams and Fournier [1], Demengel and Demengel [16], Evans [20], Grafakos [25, 26], Leoni [29], Pişkin and Okutmuştur [34] for an overview of Sobolev spaces and applications in PDEs and in harmonic analysis.

Since a function and its derivatives might have different properties, it is not necessary for them to lie in the same Lebesgue space. In this paper, we consider *nonuniform Sobolev spaces* on $\mathbb{R}^N$, i.e., spaces for which a function and its derivatives belong to different Lebesgue spaces.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. For $k \ge 1$ and $\vec{p} = (p_0, \dots, p_k) \in [1, \infty]^{k+1}$, the nonuniform Sobolev space $W_k^{\vec{p}}(\Omega)$ consists of all measurable functions f for which $\partial^\alpha f \in L^{p_{|\alpha|}}(\Omega)$, where $|\alpha| \le k$. For $f \in W_k^{\vec{p}}$, define its norm by*

$$\|f\|_{W_k^{\vec{p}}} = \sum_{|\alpha| \le k} \|D^\alpha f\|_{L^{p_{|\alpha|}}}.$$

2020 *Mathematics Subject Classification.* 46E35.

Key words and phrases. Sobolev spaces, Nonuniform Sobolev spaces, fractinoal Sobolev spaces, embedding theorems, heat equations, Schrödinger operators.

T. Chen was supported by the National Natural Science Foundation of China (No. 12271267) and the Fundamental Research Funds for the Central Universities.

L. Grafakos was supported by a Simons Grant (No. 624733).

W. Sun was supported by the National Natural Science Foundation of China (No. 12171250 and U21A20426).

These spaces naturally arise in the study of certain PDEs. In particular, some equations have solutions in nonuniform Sobolev spaces but have no solution in the classical Sobolev spaces. For example, consider positive solutions of the critical p-Laplace equation

$$\Delta_p u + u^{p^*-1} = 0 \tag{1.1}$$

in $\mathbb{R}^N$, where $1 < p < N$, $p^* = Np/(N-p)$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

This equation is well studied in the literature. It was shown by Damascelli, Merchán, Montoro and Sciunzi [15], Sciunzi [38] and Vétois [43] that if a function u in the class

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N)\}$$

is a solution of (1.1), then it is of the form

$$u(x) = U_{\lambda,x_0}(x) = \left(\frac{\lambda^{1/(p-1)} N^{1/p}((N-p)/(p-1))^{(p-1)/p}}{\lambda^{p/(p-1)} + |x-x_0|^{p/(p-1)}}\right)^{(N-p)/p},$$

where λ is a positive constant and x_0 is a point in $\mathbb{R}^N$. Moreover, Catino, Monticelli and Roncoroni [10] provided the classification of positive solutions to the critical p-Laplace equation. See also Ciraolo, Figalli and Roncoroni [12] for solutions in convex cones.

Observe that $\mathcal{D}^{1,p}$ itself is the nonuniform Sobolev space $W_1^{(p^*,p)}$. Since $|U_{\lambda,x_0}(x)| \approx |x|^{-(N-p)/(p-1)}$ whenever $|x|$ is large enough, it is easy to see that (1.1) has a positive solution in $W^{1,p}$ if and only if $1 < p < N^{1/2}$. However, we can increase the integrability index p of the solution if we consider nonuniform Sobolev spaces. Indeed, we obtain that (1.1) has a positive solution in $W_1^{(p_0,p)}$ whenever $1 < p < N$ and $p_0 > N(p-1)/(N-p)$.

On the other hand, for $1 \le p_1 < N$ and f in the classical Sobolev space W^{1,p_1}, the Sobolev inequality says that

$$\|f\|_{L^{Np_1/(N-p_1)}} \le C_{N,p_1} \|\nabla f\|_{L^{p_1}}. \tag{1.2}$$

This is proved by showing that

$$\|f\|_{L^{Np_1/(N-p_1)}}^{(N-1)p_1/(N-p_1)} \le C_{N,p_1} \|\nabla f\|_{L^{p_1}} \|f\|_{L^{Np_1/(N-p_1)}}^{N(p_1-1)/(N-p_1)} \tag{1.3}$$

for compactly supported differentiable functions f. We refer to [16] for details.

For the nonuniform case, that is, for $f \in C^1 \cap W_k^{\vec{p}}$ with $\vec{p} = (p_0, p_1)$, both (1.2) and (1.3) are still true. However, their proofs are quite different from the uniform case. In the classical uniform case, one first obtains the density of compactly supported differentiable functions, then derives (1.2) from (1.3) for such functions, and finally extends (1.2) to all functions in W^{1,p_1} by density. For the nonuniform case, compactly supported differentiable functions are still dense in $W_1^{\vec{p}}$. However, the embedding inequality is required in the proof. So we have to prove (1.2) directly for functions which is not compactly supported. In this case, (1.2) is not a straightforward consequence of (1.3): we have to show first that $f \in L^{Np_1/(N-p_1)}$.

In some applications, only the derivatives are concerned. For example, Fefferman, Israel and Luli [21, 22] studied Sobolev extension operators for homogeneous Sobolev spaces. In this case, it is natural to consider nonuniform Sobolev spaces.

Another application we discuss concerns local estimates for solutions of heat equations. We obtain local energy estimates for initial data in nonuniform Sobolev spaces; this extends the local estimates by Fefferman, McCormick, Robinson and Rodrigo [23].

Moreover, nonuniform Sobolev spaces are also useful in the study of the convergence of Schrödinger operators defined by

$$e^{it(-\Delta)^{a/2}}f(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{i(x\cdot\omega+t|\omega|^a)}\hat{f}(\omega)\mathrm{d}\omega,$$

where $a>1$ is a constant. It is well known that for f nice enough, $e^{it(-\Delta)^{a/2}}f(x)$ is the solution of the fractional Schrödinger equation

$$\begin{cases} i\partial_t u + (-\Delta_x)^{a/2}u = 0, & (x,t)\in\mathbb{R}^N\times R,\\ u(x,0)=f(x), & x\in\mathbb{R}^N.\end{cases}$$

For the case $N=1$ and $a=2$, Carleson [9] studied the convergence of $e^{-it\Delta}f(x)$ as t tends to 0 for functions f in the Sobolev space

$$H^s(\mathbb{R}^N) := \{f\in L^2 : (1+|\omega|^2)^{s/2}\hat{f}(\omega)\in L^2\}.$$

It was shown [9] that when $a=2$,

$$\lim_{t\to 0} e^{it(-\Delta)^{a/2}}f(x) = f(x), \quad a.e. \tag{1.4}$$

for all $f\in H^s(\mathbb{R})$ if $s\geq 1/4$.

Since then, many works have appeared on this topic. Dahlberg and Kenig [14] provided counterexamples indicating that the range $s\geq 1/4$ is sharp for $N=1$. And Sjölin [40] extended this result to the case $a>1$.

For $a=2$ and higher dimensions $N\geq 2$, Sjölin [40] and Vega [42] proved that (1.4) is true when $s>1/2$. For the case $N=2$, this result was improved to $s>3/8$ by Lee [28]. Bourgain [3, 4] proved that $s>1/2-1/(4N)$ is sufficient and $s\geq 1/2-1/(2N+2)$ is necessary for the convergence. Du, Guth and Li [18] proved that $s>1/2-1/(2N+2)$ is sufficient for the dimension $N=2$ and Du and Zhang [19] showed that it is also true for general $N\geq 3$.

For the general case $a>1$ and $N\geq 2$, Sjölin [40] proved that (1.4) is valid when $s>1/2$. Prestini [35] showed that $s\geq 1/4$ is necessary and sufficient for radial functions f and dimensions $N\geq 2$. Cho and Ko [11] proved that $s>1/3$ is sufficient for the dimension $N=2$.

Related works also include the non-tangential convergence by Shiraki [39], Yuan, Zhao and Zheng [45], Li, Wang and Yan [32, 33], the convergence along curves by Cao and Miao [8] and Zheng [46], the Hausdorff dimension of the divergence set by Li, Li and Xiao [31], and the convergence rate by Cao, Fan and Wang [7]; see also the works by Cowling [13], Walther [44], and Rogers and Villarroya [36] for the case $a<1$.

Although the range $s\geq 1/4$ of the index s is sharp for $N=1$, and $s>1/2-1/(2N+2)$ is also sharp up to the endpoint for $N\geq 2$, we show that the result can be further extended when nonuniform Sobolev spaces are considered.

Suppose that $0<s<1$ and $1\leq p<\infty$. Recall that the classical fractional Sobolev space W^p_s consists of all measurable functions f for which

$$\|f\|_{W^p_s} := \|f\|_{L^p} + [f]_{W^p_s} < \infty,$$

where

$$[f]_{W^p_s} := \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{1/p}$$

is the Gagliardo seminorm of f. Fractional Sobolev spaces were introduced by Aronszajn [2], Gagliardo [24] and Slobodeckiĭ [41]. We refer to the text book by Leoni [30] and papers by Brezis and Mironescu [5] and Di Nezza [17] for an overview of fractional Sobolev spaces. See also recent papers by Brezis, Van Schaftingen and Yung [6] and Gu and Yung [27] for the limit of norms of fractional Sobolev spaces.

We now provide a formal definition of nonuniform fractional Sobolev spaces.

Definition 1.2. *Given $0 < s < 1$ and $\vec{p} = (p_0, p_1) \in [1, \infty)^2$, the nonuniform fractional Sobolev space is defined by*

$$W_s^{\vec{p}}(\mathbb{R}^N) = \{f : \|f\|_{W_s^{\vec{p}}} := \|f\|_{L^{p_0}} + [f]_{W_s^{p_1}} < \infty\}.$$

For the case $s > 1$, set $\nu_s = s - \lfloor s \rfloor$ and $\vec{p} = (p_0, \dots, p_{\lceil s \rceil})$. If s is not an integer, define

$$W_s^{\vec{p}}(\mathbb{R}^N) = \Big\{ f : \|f\|_{W_s^{\vec{p}}} := \sum_{l=0}^{\lfloor s \rfloor} \sum_{|\alpha|=l} \|D^\alpha f\|_{L^{p_l}} + \sum_{|\alpha|=\lfloor s \rfloor} [D^\alpha f]_{W_{\nu_s}^{p_{\lceil s \rceil}}} < \infty \Big\}.$$

Here $\lfloor s \rfloor$ stands for the greatest integer which is less than or equal to s, and $\lceil s \rceil$ stands for the ceiling, i.e., the least integer which is greater than or equal to s.

Recall that if $p_0 = \ldots = p_{\lceil s \rceil} = 2$, then $W_s^{(2,\dots,2)}(\mathbb{R}^N)$ coincides with $H^s(\mathbb{R}^N)$. We refer to [16, Proposition 4.17] for a proof.

The introduction and study of these spaces is motivated by the fact that they provide stronger results than classical Sobolev spaces. As an application we show that if the convergence of Schrödinger operators holds for all functions in H^s for some $s > 0$, then it also holds for all functions in $W_s^{\vec{p}}$ with the same index s if $p_{\lceil s \rceil} = 2$. Moreover, we also obtain convergence results in the case $1 < p_{\lceil s \rceil} < 2$.

The paper is organized as follows. In Section 2, we show that compactly supported infinitely differentiable functions are dense in nonuniform Sobolev spaces. We also obtain the Sobolev inequality for functions in nonuniform Sobolev spaces. In Section 3, we focus on nonuniform fractional Sobolev spaces and present an embedding theorem for such spaces. And in Section 4, we give two applications of nonuniform Sobolev spaces. We give local energy estimates for solutions of the heat equation with initial data in nonuniform Sobolev spaces. And we prove the almost everywhere convergence of Schrödinger operators for a large class of functions, extending known results.

Symbols and Notations. $\{e_i : 1 \le i \le N\}$ stands for the canonical basis for $\mathbb{R}^N$, that is, $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^N$, where only the i-th component is 1 and all others are 0. For any $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, we have $x = \sum_{i=1}^N x_i e_i$. $B(0, R)$ stands for the ball $\{x \in \mathbb{R}^N : |x| < R\}$.

For a tempered distribution f and a function φ in the Schwartz class $\mathscr{S}$, the notation $\langle f, \varphi \rangle$ stands for the value of the action of f on φ.

2. Embedding Inequalities for Nonuniform Sobolev Spaces

Denote by $C_c^\infty(\mathbb{R}^N)$ the function space consisting of all compactly supported infinitely differentiable functions. In this section we obtain the density of $C_c^\infty(\mathbb{R}^N)$ in nonuniform Sobolev spaces. We then obtain a Sobolev embedding theorem for nonuniform Sobolev spaces; and the aforementioned density is a crucial tool in this embedding. Both of these results seem to require proofs that are quite different from the classical case. For the nonuniform case, we need first to prove the density for special indices. Then we deduce an embedding result, with which we finally obtain the density of $C_c^\infty(\mathbb{R}^N)$ for full indices.

We begin with a simple lemma.

Lemma 2.1. *For any $k \geq 1$ and $\vec{p} \in [1,\infty)^{k+1}$, the space $C^\infty \cap W_k^{\vec{p}}(\mathbb{R}^N)$ is dense in $W_k^{\vec{p}}(\mathbb{R}^N)$.*

Proof. Take some $f \in W_k^{\vec{p}}$ and $\varphi \in C_c^\infty$ such that φ is nonnegative and $\|\varphi\|_{L^1} = 1$. Set

$$g_\lambda(x) = \int_{\mathbb{R}^N} f(y)\frac{1}{\lambda^N}\varphi(\frac{x-y}{\lambda})\mathrm{d}y, \qquad \lambda > 0.$$

We have $g_\lambda \in C^\infty$. For any multi-index α with $|\alpha| \leq k$, we have

$$\begin{aligned}\partial^\alpha g_\lambda(x) &= \int_{\mathbb{R}^N} \partial^\alpha f(y)\frac{1}{\lambda^N}\varphi(\frac{x-y}{\lambda})\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \partial^\alpha f(x-\lambda y)\varphi(y)\mathrm{d}y.\end{aligned} \tag{2.1}$$

It follows from Minkowski's inequality and Lebesgue's dominated convergence theorem that

$$\lim_{\lambda\to 0} \|\partial^\alpha g_\lambda - \partial^\alpha f\|_{L^{p_{|\alpha|}}} \leq \lim_{\lambda\to 0}\int_{\mathbb{R}^N} \|\partial^\alpha f(\cdot - \lambda y) - \partial^\alpha f\|_{L^{p_{|\alpha|}}}\varphi(y)\mathrm{d}y = 0.$$

This completes the proof. □

The proof of the density of C_c^∞ in nonuniform Sobolev spaces is split in two cases. First, we consider special indices.

Lemma 2.2. *Suppose that $k \geq 1$ and $\vec{p} = (p_0, \ldots, p_k)$ with $1/p_i \leq 1/p_{i-1} + 1/N$, $1 \leq i \leq k$. Then the space $C_c^\infty(\mathbb{R}^N)$ is dense in $W_k^{\vec{p}}(\mathbb{R}^N)$.*

Proof. By Lemma 2.1, for any $f \in W_k^{\vec{p}}$ and $\varepsilon > 0$, there is some $g \in C^\infty \cap W_k^{\vec{p}}$ such that

$$\|g - f\|_{W_k^{\vec{p}}} < \varepsilon.$$

Moreover, if follows from (2.1) and Young's inequality that we may choose the function g such that

$$\partial^\gamma g \in L^r, \qquad \forall\ r \geq p_{|\gamma|}. \tag{2.2}$$

The proof will be complete if we can show that there is some $\tilde{g} \in C_c^\infty$ such that

$$\|g - \tilde{g}\|_{W_k^{\vec{p}}} < C'\varepsilon. \tag{2.3}$$

Take some $\psi \in C_c^\infty$ such that $\psi(x) = 1$ whenever $|x| < 1$. Set $g_n(x) = \psi(x/n)g(x)$. It remains to show that the sequence $\{g_n : n \geq 1\}$ converges to g in $W_k^{\vec{p}}$.

We see from the choice of ψ that $\{g_n : n \geq 1\}$ converges to g in L^{p_0}. On the other hand, fix some multi-index α with $1 \leq |\alpha| \leq k$. Then $\partial^\alpha g_n(x)$ is the sum of $\psi(x/n)\partial^\alpha g(x)$ and terms like $(1/n^{|\beta|})\partial^\beta\psi(x/n)\partial^\gamma g(x)$, where $|\beta| + |\gamma| = |\alpha|$ and $|\beta| \geq 1$.

If $p_{|\alpha|} \geq p_{|\gamma|}$, we deduce from (2.2) that

$$\begin{aligned}\Big\|\frac{1}{n^{|\beta|}}\partial^\beta\psi(\frac{\cdot}{n})\partial^\gamma g\Big\|_{L^{p_{|\alpha|}}} &\leq \frac{1}{n^{|\beta|}}\cdot\|\partial^\beta\psi\|_{L^\infty}\|\partial^\gamma g 1_{\{|x|\geq n\}}\|_{L^{p_{|\alpha|}}} \\ &\to 0, \qquad \text{as } n\to\infty.\end{aligned}$$

If $p_{|\alpha|} < p_{|\gamma|}$, there is some $r > 1$ such that

$$\frac{1}{p_{|\alpha|}} = \frac{1}{p_{|\gamma|}} + \frac{1}{r}.$$

Applying Hölder's inequality, we have

$$\begin{aligned}\Big\|\frac{1}{n^{|\beta|}}\partial^\beta\psi(\frac{\cdot}{n})\partial^\gamma g\Big\|_{L^{p_{|\alpha|}}} &\le \frac{1}{n^{|\beta|}}\cdot\|\partial^\beta\psi(\frac{\cdot}{n})\|_{L^r}\|\partial^\gamma g1_{\{|x|\ge n\}}\|_{L^{p_{|\gamma|}}}\\ &= \frac{1}{n^{|\beta|-N/r}}\cdot\|\partial^\beta\psi\|_{L^r}\|\partial^\gamma g1_{\{|x|\ge n\}}\|_{L^{p_{|\gamma|}}}. \end{aligned}\tag{2.4}$$

Since $1/p_i \le 1/p_{i-1}+1/N$ for all $1\le i\le k$, we have

$$\frac{1}{r}=\frac{1}{p_{|\alpha|}}-\frac{1}{p_{|\gamma|}}\le\frac{|\alpha|-|\gamma|}{N}=\frac{|\beta|}{N}.$$

Consequently, $|\beta|-N/r\ge 0$. It follows from (2.4) and (2.2) that

$$\Big\|\frac{1}{n^{|\beta|}}\partial^\beta\psi(\frac{\cdot}{n})\partial^\gamma g\Big\|_{L^{p_{|\alpha|}}}\to 0,\qquad \text{as } n\to\infty.\tag{2.5}$$

Hence the sequence $\{g_n : n\ge 1\}$ converges to g in $W_k^{\vec{p}}$ and (2.3) is valid. ☐

To prove the density of C_c^∞ for full indices, we first establish an embedding theorem. For classical Sobolev spaces, the embedding inequality

$$\|\varphi\|_{L^{N/(N-1)}}\le C\|\nabla\varphi\|_{L^1}$$

is first proved for functions in C_c^∞, and then for general functions in $W^{1,1}$ by the density of C_c^∞ in $W^{1,1}$. We refer to [16, eq. (2.38)] for details.

In the nonuniform case, we have no such density result at the moment. So we need to prove it directly for functions in $C^\infty\cap L^{p_0}(\mathbb{R}^N)$ for some $p_0>0$.

Lemma 2.3. *Suppose that $N\ge 2$.*

(i) *For any $\varphi\in C^\infty\cap L^{p_0}(\mathbb{R}^N)$ with $0<p_0<\infty$,*

$$\|\varphi\|_{L^{N/(N-1)}}\le\frac{1}{N}\sum_{|\alpha|=1}\|\partial^\alpha\varphi\|_{L^1}.\tag{2.6}$$

Moreover, the above inequality is also true for any $f\in W_1^{\vec{p}}$ with $\vec{p}=(p_0,1)$ and $1\le p_0<\infty$.

(ii) *Suppose that $\vec{p}=(p_0,p_1)$, where $1\le p_0<\infty$ and $1\le p_1<N$. For any $f\in W_1^{\vec{p}}(\mathbb{R}^N)$, we have*

$$\|f\|_{L^{Np_1/(N-p_1)}}\le C_{N,p_1}\|\nabla f\|_{L^{p_1}}.\tag{2.7}$$

Hence for all q between p_0 and $Np_1/(N-p_1)$, we have

$$\|f\|_{L^q}\le C\|f\|_{W_1^{\vec{p}}}.\tag{2.8}$$

Proof. (i) The proof is similar to that for [16, eq. (2.38)]. Here we only provide a sketch with emphasis on the difference.

Fix some $\varphi\in C^\infty\cap L^{p_0}$. If the right-hand side of (2.6) equals infinity, then (2.6) is certainly valid. So we only need to consider the case $\|\partial^\alpha\varphi\|_{L^1}<\infty$ for all multi-indices α with $|\alpha|=1$.

For each index i, denote $\tilde{x}_i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)$. Since $\varphi\in L^{p_0}$, we have

$$\int_{\mathbb{R}^{N-1}}\left(\int_{\mathbb{R}}|\varphi(x)|^{p_0}\mathrm{d}x_i\right)\mathrm{d}\tilde{x}_i=\|\varphi\|_{L^{p_0}}^{p_0}<\infty.$$

Hence there are measurable sets $A_i \subset \mathbb{R}^{N-1}$ of measure zero such that

$$\int_{\mathbb{R}} |\varphi(x)|^{p_0} \mathrm{d}x_i < \infty, \qquad \forall\, \tilde{x}_i \in \mathbb{R}^{N-1} \setminus A_i.$$

Consequently, for each $\tilde{x}_i$ there is a sequence $\{a_n : n \geq 1\} \subset \mathbb{R}$ (depending on $\tilde{x}_i$) such that $\lim_{n\to\infty} a_n = -\infty$ and $\lim_{n\to\infty} \varphi(x + (a_n - x_i)e_i) = 0$. Since for each x_i we have

$$\varphi(x) - \varphi(x + (a_n - x_i)e_i) = \int_{a_n}^{x_i} \partial_{x_i}\varphi(x + (t - x_i)e_i)\mathrm{d}t,$$

by the Fundamental Theorem of Calculus, letting $n \to \infty$, we obtain

$$\varphi(x) = \int_{-\infty}^{x_i} \partial_{x_i}\varphi(x + (t - x_i)e_i)\mathrm{d}t, \qquad \forall\, x_i \in \mathbb{R},\ \forall\, \tilde{x}_i \in \mathbb{R}^{N-1} \setminus A_i.$$

Here and henceforth $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 only on the ith entry and 0 elsewhere. It follows that

$$|\varphi(x)| \leq \int_{\mathbb{R}} |\partial_{x_i}\varphi(x + (t - x_i)e_i)|\, \mathrm{d}t, \qquad \forall\, x \in \mathbb{R}^N \setminus B_i, \tag{2.9}$$

where

$$B_i = \{x \in \mathbb{R}^N : x_i \in \mathbb{R}, \tilde{x}_i \in A_i\}$$

is of measure zero in $\mathbb{R}^N$. Observe that the integral in (2.9) is independent of x_i.

For $1 \leq i \leq N$, define the function F_i on $\mathbb{R}^{N-1}$ by

$$F_i(\tilde{x}_i) = \int_{\mathbb{R}} |\partial_{x_i}\varphi(x + (t - x_i)e_i)|\, \mathrm{d}t.$$

We have

$$|\varphi(x)|^{N/(N-1)} \leq \prod_{i=1}^{N} F_i(\tilde{x}_i)^{1/(N-1)}, \qquad \forall x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{N} B_i.$$

Now following the same arguments as that in [16, Page 76] we obtain

$$\|\varphi\|_{L^{N/(N-1)}} \leq \frac{1}{N} \sum_{|\alpha|=1} \|\partial^\alpha \varphi\|_{L^1}.$$

For the general case, i.e., $f \in W_1^{\vec{p}}$ with $p_1 = 1$, since $C^\infty \cap W_1^{\vec{p}}$ is dense in $W_1^{\vec{p}}$, there is a sequence $\{\varphi_n : n \geq 1\} \subset C^\infty \cap W_1^{\vec{p}}$ which is convergent to f in $W_1^{\vec{p}}$, as well as on $\mathbb{R}^N$ almost everywhere.

For each n, we have

$$\|\varphi_n\|_{L^{N/(N-1)}} \leq \frac{1}{N} \sum_{|\alpha|=1} \|\partial^\alpha \varphi_n\|_{L^1}.$$

Letting $n \to \infty$, we apply Fatou's lemma to finally deduce

$$\|f\|_{L^{N/(N-1)}} \leq \frac{1}{N} \sum_{|\alpha|=1} \|\partial^\alpha f\|_{L^1}.$$

(ii) We see from the embedding theorem for Sobolev's spaces (see [16, eq. (2.46)]) that for any $f \in C_c^\infty(\mathbb{R}^N)$,

$$\|f\|_{L^{Np_1/(N-p_1)}} \leq C\|\nabla f\|_{L^{p_1}}.$$

If $1/p_1 \leq 1/p_0 + 1/N$, then C_c^∞ is dense in $W_1^{\vec{p}}$, thanks to Lemma 2.2. Similar arguments as the previous case we get that the above inequality is true for all $f \in W_1^{\vec{p}}$.

It remains to consider the case $1/p_1 > 1/p_0 + 1/N$.

Take $f \in W_1^{\vec{p}}$. Set $\vec{\tilde{p}} = (\tilde{p}_0, p_1)$, where $1/\tilde{p}_0 = 1/p_1 - 1/N$. It is easy to see that $1 < \tilde{p}_0 < p_0$. We conclude that $f \in W_1^{\vec{\tilde{p}}}$, for which we only need to show that $f \in L^{\tilde{p}_0}$.

To this end, set $r = p_0/p_1' + 1$ and $h = |f|^{r-1}f$. We have $\nabla h = r|f|^{r-1}\nabla f$. Since $f \in L^{p_0}$, $\nabla f \in L^{p_1}$ and $(r-1)/p_0 + 1/p_1 = 1$, we see from Hölder's inequality that

$$\|\nabla h\|_1 \le r\||f|^{r-1}\|_{L^{p_0/(r-1)}}\|\nabla f\|_{L^{p_1}} = r\|f\|_{L^{p_0}}^{r-1}\|\nabla f\|_{L^{p_1}}.$$

It follows from (2.6) that

$$\|h\|_{L^{N/(N-1)}} \le C\|\nabla h\|_{L^1} \le Cr\|f\|_{L^{p_0}}^{r-1}\|\nabla f\|_{L^{p_1}}.$$

Denote $q = rN/(N-1)$. Observe that $\|h\|_{L^{N/(N-1)}} = \|f\|_{L^q}^r$. We have

$$\|f\|_{L^q} \le \|f\|_{L^{p_0}}^{1-1/r}\Big(Cr\|\nabla f\|_{L^{p_1}}\Big)^{1/r}. \tag{2.10}$$

Since $1/p_1 > 1/p_0 + 1/N$, we have

$$\frac{q}{p_0} = \frac{N}{N-1}\Big(\frac{1}{p_0} + 1 - \frac{1}{p_1}\Big) < 1.$$

On the other hand, it follows from $1/p_0 < 1/p_1 - 1/N$ that

$$\frac{1}{p_1'} < \frac{p_0}{p_1'}\Big(\frac{1}{p_1} - \frac{1}{N}\Big).$$

Hence,

$$1 - \frac{1}{N} < \frac{1}{p_1} - \frac{1}{N} + \frac{p_0}{p_1'}\Big(\frac{1}{p_1} - \frac{1}{N}\Big).$$

Therefore we have

$$\frac{1}{q} = \frac{N-1}{N}\cdot\frac{1}{1 + p_0/p_1'} < \frac{1}{p_1} - \frac{1}{N}.$$

Set $q_0 = p_0$. For $n \ge 1$, define r_n and q_n recursively by

$$r_n = \frac{q_{n-1}}{p_1'} + 1 \quad \text{and} \quad q_n = \frac{r_n N}{N-1}.$$

We see from above arguments that both $\{q_n : n \ge 1\}$ and $\{r_n : n \ge 1\}$ are decreasing, $1/p_1 > 1/q_n + 1/N$ and

$$\|f\|_{L^{q_n}} \le \|f\|_{L^{q_{n-1}}}^{1-1/r_n}\Big(Cr_n\|\nabla f\|_{L^{p_1}}\Big)^{1/r_n}. \tag{2.11}$$

Since q_n is bounded from below, the limits $\tilde{r} := \lim_{n\to\infty} r_n$ and $\tilde{q} := \lim_{n\to\infty} q_n$ exist. Moreover, $\tilde{q} = \tilde{r}N/(N-1) > \tilde{r} \ge 1$.

Take some constant ε small enough such that $\varepsilon \cdot Cr_1\|\nabla f\|_{L^{p_1}} < 1$. Since $r_n \le r_1$, we have $\varepsilon \cdot Cr_n\|\nabla f\|_{L^{p_1}} < 1$ for all $n \ge 1$. Set $\tilde{f} = \varepsilon f$ and substitute $\tilde{f}$ for f in (2.11), we get

$$\|\tilde{f}\|_{L^{q_n}} \le \|\tilde{f}\|_{L^{q_{n-1}}}^{1-1/r_n}, \qquad n \ge 1.$$

Applying the above inequality recursively, we obtain

$$\begin{aligned}\|\tilde{f}\|_{L^{q_n}} &\le \|\tilde{f}\|_{L^{q_{n-2}}}^{(1-1/r_n)(1-1/r_{n-1})} \\ &\le \cdots \\ &\le \|\tilde{f}\|_{L^{q_0}}^{\prod_{i=1}^n(1-1/r_i)} \\ &\le \max\{\|\tilde{f}\|_{L^{q_0}}, 1\}.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^N} |\tilde{f}(x)|^{q_n} \mathrm{d}x \le \max\{\|\tilde{f}\|_{L^{q_0}}, 1\}^{q_n}.$$

Letting $n \to \infty$, we see from Fatou's lemma that $\tilde{f} \in L^{\tilde{q}}$. Consequently $f \in L^{\tilde{q}}$.

Observe that

$$q_n = \frac{r_n N}{N-1} = \frac{N}{N-1}\Big(\frac{q_{n-1}}{p_1'} + 1\Big).$$

Letting $n \to \infty$, we get

$$\tilde{q} = \frac{N}{N-1}\Big(\frac{\tilde{q}}{p_1'} + 1\Big).$$

Hence $1/p_1 = 1/\tilde{q} + 1/N$. Therefore, $\tilde{q} = \tilde{p}_0$ and $f \in L^{\tilde{p}_0}$.

Set $r = \tilde{p}_0/p_1' + 1$. Then (2.10) turns out to be

$$\|f\|_{L^{\tilde{p}_0}} \le \|f\|_{L^{\tilde{p}_0}}^{1-1/r} \Big(Cr\|\nabla f\|_{L^{p_1}}\Big)^{1/r}.$$

Finally, we have

$$\|f\|_{L^{\tilde{p}_0}} \le Cr\|\nabla f\|_{L^{p_1}}$$

and this establishes (2.7) since $1/p_1 - 1/N = 1/\tilde{q} = 1/\tilde{p}_0$.

Moreover, (2.7) implies that $\|f\|_{L^{Np_1/(N-p_1)}} \le C\|f\|_{W_1^{\vec{p}}}$. Since $\|f\|_{L^{p_0}} \le C'\|f\|_{W_1^{\vec{p}}}$, by interpolation, we get that for any q between p_0 and $Np_1/(N-p_1)$,

$$\|f\|_{L^q} \le C\|f\|_{W_1^{\vec{p}}}.$$

This completes the proof. □

The following is an immediate consequence.

Corollary 2.4. *Suppose that $\vec{p} = (p_0, p_1)$, $\vec{q} = (q_0, q_1)$, $1 \le p_0, p_1, q_0, q_1 < \infty$ and $p_1 < N$. If $q_1 = p_1$ and q_0 lies between p_0 and $Np_1/(N-p_1)$, then*

$$W_1^{\vec{p}}(\mathbb{R}^N) \hookrightarrow W_1^{\vec{q}}(\mathbb{R}^N).$$

Remark 2.5. The range for q in the inequality (2.8) is the best possible, which can be shown as that for classical Sobolev spaces.

For example, take some $f \in W_1^{\vec{p}}$. Suppose that (2.8) is true for some q. Replacing $f(\cdot/\lambda)$ for f in (2.8), we obtain

$$\big\|f\big(\frac{\cdot}{\lambda}\big)\big\|_{L^q} \lesssim \big\|f\big(\frac{\cdot}{\lambda}\big)\big\|_{L^{p_0}} + \frac{1}{\lambda}\big\|\nabla f\big(\frac{\cdot}{\lambda}\big)\big\|_{L^{p_1}}.$$

Thus we have

$$\lambda^{N/q}\|f\|_{L^q} \lesssim \lambda^{N/p_0}\|f\|_{L^{p_0}} + \frac{1}{\lambda^{1-N/p_1}}\|\nabla f\|_{L^{p_1}}$$

and therefore

$$1 \lesssim \lambda^{N(1/p_0 - 1/q)} + \lambda^{N(1/p_1 - 1/N - 1/q)}.$$

First, we assume that $p_0 \le Np_1/(N-p_1)$. If $q > Np_1/(N-p_1)$, then both $1/p_1 - 1/N - 1/q$ and $1/p_0 - 1/q$ are positive. Letting $\lambda \to 0$, we get a contradiction.

If $q < p_0$, then both $1/p_1 - 1/N - 1/q$ and $1/p_0 - 1/q$ are negative. Letting $\lambda \to \infty$, we also get a contradiction.

Next, we assume that $p_0 > Np_1/(N-p_1)$. With similar arguments we get a contradiction. Hence (2.8) is true if and only if q is between p_0 and $Np_1/(N-p_1)$.

With the help of Lemma 2.3, we prove the density of compactly supported infinitely many differentiable functions in nonuniform Sobolev spaces with general indices.

Theorem 2.6. *For any $k \ge 1$ and $\vec{p} \in [1,\infty)^{k+1}$, the space $C_c^\infty(\mathbb{R}^N)$ is dense in $W_k^{\vec{p}}(\mathbb{R}^N)$.*

Proof. By Lemma 2.2, it suffices to consider the case $1/p_{i_0} > 1/p_{i_0-1} + 1/N$ for some i_0 with $1 \le i_0 \le k$.

In this case, we have $N \ge 2$. Let $\vec{q} = (q_0, \ldots, q_N)$, where q_n is defined recursively by

$$q_N = p_N,$$
$$q_n = \begin{cases} p_n, & \text{if } 1/p_n \ge 1/q_{n+1} - 1/N, \\ Nq_{n+1}/(N - q_{n+1}), & \text{otherwise}, \end{cases} \qquad n = N-1, \ldots, 0.$$

By Lemma 2.3 (ii), we have $f \in W_k^{\vec{q}}$. Since $1/p_n \le 1/q_n \le 1/q_{n-1} + 1/N$ for all $1 \le n \le N$, we have

$$\frac{1}{p_n} \le \frac{1}{q_n} \le \frac{1}{q_m} + \frac{n-m}{N}, \quad n > m. \tag{2.12}$$

It suffices to show that (2.5) is also true in this case.

In the sequel we adopt the notation introduced in the proof of Lemma 2.2. Recall that $\partial^\gamma f \in L^{q_{|\gamma|}}$. As in the proof of Lemma 2.2, we may assume that $\partial^\gamma g \in L^r$ for any $r > q_{|\gamma|}$. If $p_{|\alpha|} \ge q_{|\gamma|}$, then we get (2.5) as in the proof of Lemma 2.2. For the case $p_{|\alpha|} < q_{|\gamma|}$, we see from (2.12) that

$$\frac{1}{p_{|\alpha|}} \le \frac{1}{q_{|\gamma|}} + \frac{|\beta|}{N}.$$

Hence there is some $r \ge N/|\beta|$ such that

$$\frac{1}{p_{|\alpha|}} = \frac{1}{q_{|\gamma|}} + \frac{1}{r}.$$

Applying Hölder's inequality, we get

$$\Big\| \frac{1}{n^{|\beta|}} \partial^\beta \psi(\frac{\cdot}{n}) \partial^\gamma g \Big\|_{L^{p_{|\alpha|}}} \le \frac{1}{n^{|\beta| - N/r}} \cdot \|\partial^\beta \psi\|_{L^r} \|\partial^\gamma g 1_{\{|x| \ge n\}}\|_{L^{q_{|\gamma|}}} \to 0.$$

This completes the proof. □

Recall that for an integer $n \ge 0$, the space $C_b^n(\mathbb{R}^N)$ consists of all functions f such that f is n times continuously differentiable and for any multi-index α with $|\alpha| \le n$, $D^\alpha f \in L^\infty$.

For an integer $n \ge 0$ and a positive number $\nu \in (0,1)$, the space $C^{n,\nu}(\mathbb{R}^N)$ consists of all functions f in C_b^n such that for any multi-index α with $|\alpha| = n$,

$$|D^\alpha f(x) - D^\alpha f(y)| \le C_{n,\nu} |x-y|^\nu, \quad \forall x, y \in \mathbb{R}^N.$$

Below is an embedding theorem for higher-order nonuniform Sobolev spaces.

Theorem 2.7. *Let k be a positive integer and $\vec{p} = (p_0, \ldots, p_k) \in [1,\infty)^{k+1}$.*

(i) *If $kp_k < N$, then for any q between p_0 and $Np_k/(N - kp_k)$, we have*

$$W_k^{\vec{p}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N).$$

(ii) *If $kp_k = N$, then for any q with $p_0 \le q < \infty$, we have*

$$W_k^{\vec{p}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N).$$

(iii) *If $kp_k > N$, then $W_k^{\vec{p}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. More precisely, if $kp_k > N$ and $N/p_k \notin \mathbb{Z}$, then there is some integer k_0 such that $(k_0-1)p_k < N < k_0 p_k$ and*

$$W_k^{\vec{p}}(\mathbb{R}^N) \hookrightarrow C_b^{k-k_0, k_0 - N/p_k}(\mathbb{R}^N).$$

If $N/p_k \in \mathbb{Z}$ and $k \geq k_0 := N/p_k + 1$, then for any $0 < \lambda < 1$,

$$W_k^{\vec{p}}(\mathbb{R}^N) \hookrightarrow C_b^{k-k_0,\lambda}(\mathbb{R}^N).$$

Proof. (i) First, we consider the case $kp_k < N$. We see from Lemma 2.3 that the conclusion is true for $k = 1$.

For the case $k \geq 2$, applying (2.7) recursively, we get

$$\begin{aligned}
\|f\|_{L^{Np_k/(N-kp_k)}} &\leq C_1 \sum_{|\alpha|=1} \|\partial^\alpha f\|_{L^{Np_k/(N-(k-1)p_k)}} \\
&\leq C_2 \sum_{|\alpha|=2} \|\partial^\alpha f\|_{L^{Np_k/(N-(k-2)p_k)}} \\
&\leq \dots \\
&\leq C_k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^{p_k}}. \qquad (2.13)
\end{aligned}$$

By interpolation, we get that for any q between p_0 and $Np_k/(N-kp_k)$, $W_k^{\vec{p}} \hookrightarrow L^q$.

(ii) Next, we consider the case $kp_k = N$.

First, we assume that $k = 1$ and $N > 1$. Take some $f \in W_1^{\vec{p}}$, where $\vec{p} = (p_0, N)$. Set $r = p_0(N-1)/N + 1$ and $h = |f|^{r-1} f$. We have $\nabla h = r|f|^{r-1}\nabla f$. Since $(r-1)/p_0 + 1/N = 1$, we see from Hölder's inequality that

$$\|\nabla h\|_{L^1} \leq r\|f\|_{L^{p_0}}^{r-1} \|\nabla f\|_{L^N} < \infty.$$

Hence $\nabla h \in L^1$. By Lemma 2.3, $h \in L^{N/(N-1)}$. Therefore, $f \in L^{rN/(N-1)} = L^{p_0 + N/(N-1)}$.

Replacing p_0 by $p_0 + N/(N-1)$ in the preceding argument, we obtain that f lies in $L^{p_0 + 2N/(N-1)}$. Repeating this procedure yields that $f \in L^{p_0 + nN/(N-1)}$ for any $n \geq 1$. Hence $f \in L^q$ for any q satisfying $p_0 \leq q < \infty$.

For the case $k = N = 1$, we see from (2.9) that for smooth functions f,

$$\|f\|_{L^\infty} \lesssim \|f'\|_{L^1}.$$

Applying the density of $C_c(\mathbb{R})$ in $W_1^{\vec{p}}(\mathbb{R})$, we get that the above inequality is valid for all $f \in W_1^{\vec{p}}$. Hence $f \in L^q$ for any q satisfying $p_0 \leq q < \infty$.

Next we assume that $kp_k = N$ for some $k \geq 2$. Take some $f \in W_k^{\vec{p}}$, where $\vec{p} = (p_0, \dots, p_k)$. For any multi-index α with $|\alpha| = 1$, we have $\partial^\alpha f \in W_{k-1}^{(p_1,\dots,p_k)}$. Note that $(k-1)p_k < N$. We see from (i) that $\partial^\alpha f \in L^{Np_k/(N-(k-1)p_k)} = L^N$. Hence $f \in W_1^{(p_0,N)}$. Now we see from arguments in the case $k = 1$ that $f \in L^q$ for any q satisfying $p_0 \leq q < \infty$.

(iii) Finally, we consider the case $kp_k > N$.

First, we show that $W_k^{\vec{p}} \subset C_b^0$. We prove it by induction on k. For $k = 1$, the arguments in [16, pages 80-82] work well for the nonuniform case with minor changes. Specifically, applying the L^{p_1} norm for derivatives and the L^{p_0} norm for the function itself, the same arguments yield that $W_1^{\vec{p}} \hookrightarrow C_b^0$. That is, functions in $W_1^{\vec{p}}$ are continuous and bounded.

Now we assume that the conclusion is true for the cases 1, …, $k-1$. Consider the case k. Fix some $f \in W_k^{\vec{p}}$. There are two cases:

(a) $(k-1)p_k \geq N$.

If $(k-1)p_k > N$, then we see from the inductive assumption that for any multi-index α with $|\alpha| = 1$,

$$\|\partial^\alpha f\|_{L^\infty} \lesssim \|\partial^\alpha f\|_{W_{k-1}^{(p_1,\ldots,p_k)}} \leq \|f\|_{W_k^{\vec{p}}}.$$

If $(k-1)p_k = N$, then we see from (ii) that for any multi-index α with $|\alpha| = 1$, $\partial^\alpha f \in L^q$ for all q with $p_0 \leq q < \infty$.

Hence for $(k-1)p_k \geq N$, there is some $q > \max\{N, p_0\}$ such that

$$\|\nabla f\|_{L^q} \lesssim \|f\|_{W_k^{\vec{p}}}.$$

Consequently, $f \in W_1^{(p_0,q)}$. Applying the inductive assumption for the case $k = 1$, we get that f is continuous and

$$\|f\|_{L^\infty} \lesssim \|f\|_{W_1^{(p_0,q)}} \lesssim \|f\|_{W_k^{\vec{p}}}.$$

(b) $(k-1)p_k < N$.

In this case, we see from (i) that $f \in W_1^{(p_0,q_1)}$ with $q_1 = Np_k/(N-(k-1)p_k)$. Since $kp_k > N$, we have $q_1 > N$. Applying the inductive assumption for the case $k = 1$ again, we get that f is continuous and

$$\|f\|_{L^\infty} \lesssim \|f\|_{W_1^{(p_0,q_1)}} \lesssim \|f\|_{W_k^{\vec{p}}}.$$

By induction, the embedding $W_k^{\vec{p}} \hookrightarrow C_b^0$ is valid for all $k \geq 1$.

Next we prove the Hölder continuity.

First, we assume that N/p_k is not an integer. When $k = 1$, we have $p_1 > N$. Take some $f \in W_1^{\vec{p}}$. Let φ and g_λ be defined as in Lemma 2.1. We have

$$g_\lambda(x+y) - g_\lambda(x) = \int_0^1 \nabla g_\lambda(x+ty)\cdot y\,\mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} \nabla f(x+ty-z)\cdot y\varphi(z)\,\mathrm{d}z\mathrm{d}t, \quad \forall x, y \in \mathbb{R}^N.$$

It follows from Minkowski's inequality that

$$\|g_\lambda(\cdot+y) - g_\lambda\|_{L^{p_1}} \leq |y|\cdot\|\nabla f\|_{L^{p_1}}.$$

Since f is continuous and $\lim_{\lambda\to 0} g_\lambda(x) = f(x)$ for all $x \in \mathbb{R}^N$, letting $\lambda \to 0$ in the above inequality, we see from Fatou's lemma that

$$\|f(\cdot+y) - f\|_{L^{p_1}} \leq |y|\cdot\|\nabla f\|_{L^{p_1}}.$$

On the other hand, it is easy to see that

$$\|\nabla(f(\cdot+y) - f)\|_{L^{p_1}} \leq 2\|\nabla f\|_{L^{p_1}}.$$

Applying the fact for classical Sobolev spaces that if $u, |\nabla u| \in L^{p_1}$ and $p_1 > N$, then

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^{p_1}}^{1-N/p_1}\|\nabla u\|_{L^{p_1}}^{N/p_1}, \tag{2.14}$$

we get

$$\|f(\cdot+y) - f\|_{L^\infty} \leq |y|^{1-N/p_1}\cdot\|\nabla f\|_{L^{p_1}}.$$

Hence $f \in C_b^{0,1-N/p_1}$.

When $k \geq 2$, there is some positive integer $k_0 \leq k$ such that $(k_0-1)p_k < N < k_0 p_k$. It follows that for any multi-index α with $|\alpha| = k - k_0$, we have $D^\alpha f \in W_{k_0}^{(p_{k-k_0},\ldots,p_k)}$. Now we see from (i) that for any multi-index β with $|\beta| = 1$,

$$D^{\alpha+\beta} f \in L^{Np_k/(N-(k_0-1)p_k)}.$$

Since $Np_k/(N-(k_0-1)p_k) > N$, applying the conclusion for $k = 1$, we get $D^\alpha f \in C_b^{0,1-N/(Np_k/(N-(k_0-1)p_k))} = C_b^{0,k_0-N/p_k}$.

Moreover, by the embedding inequality we have proved, $D^\beta f \in L^q$ whenever $1 \leq |\beta| \leq k - k_0$ and q is large enough. In particular, $D^\beta f \in L^q$ for some $q > N$. Applying the conclusion for $k = 1$ again, we get $D^\alpha f \in C_b^0$ when $|\alpha| \leq k - k_0 - 1$. Hence $f \in C_b^{k-k_0,k_0-N/p_k}$.

It remains to consider the case $N/p_k \in \mathbb{Z}$. When $(k_0 - 1) = N < k_0 p_k$, for any multi-index α with $|\alpha| = k - k_0$, we have $D^\alpha f \in W_{k_0}^{(p_{k-k_0},\ldots,p_k)}$. Now we see from (ii) that for q large enough and $|\beta| = 1$,

$$D^{\alpha+\beta} f \in L^q.$$

The arguments for the case $k = 1$ show that for q large enough, $D^\alpha f \in C_b^{0,1-N/q}$. Consequently, $D^\alpha f \in C_b^{0,\lambda}$ whenever $|\alpha| = k - k_0$ and $0 < \lambda < 1$.

On the other hand, the same arguments as those for the case $N/p_k \notin \mathbb{Z}$ show that $D^\alpha f \in C_b^0$ when $|\alpha| \leq k - k_0 - 1$. Hence $f \in C_b^{k-k_0,\lambda}$ for any $0 < \lambda < 1$. □

Theorem 2.7 has some interesting consequences. In fact, the following corollary can be proved with similar arguments as in the proof of Theorem 2.7 (iii), for which we leave the details to interested readers.

Corollary 2.8. *Let k be a positive integer and $\vec{p} = (p_0, \ldots, p_k) \in [1,\infty)^{k+1}$. If $p_k > N$, then*

$$W_k^{\vec{p}} \hookrightarrow C_b^{k-1,1-N/p_k}.$$

On the other hand, the following result shows that the Sobolev space $W_k^{\vec{q}}$ with $1 \leq q_k < N/k$ and $1/q_i = 1/q_k - (k-i)/N$ is the largest one among all Sobolev spaces $W_k^{\vec{p}}$ with $p_k = q_k$.

Corollary 2.9. *Suppose that $\vec{p} = (p_0, \ldots, p_k) \in [1,\infty)^{k+1}$ with $1 \leq p_k < N/k$. Let $\vec{q} = (q_0, \ldots, q_k)$ be such that $q_k = p_k$ and $q_i = Np_k/(N-(k-i)p_k)$ for $0 \leq i \leq k-1$. Then we have*

$$W_k^{\vec{p}} \hookrightarrow W_k^{\vec{q}}.$$

Moreover, set $\vec{q}^{(i)} = (q_0, \ldots, q_i)$, $1 \leq i \leq k$. We have

$$W_k^{\vec{q}^{(k)}} \hookrightarrow W_{k-1}^{\vec{q}^{(k-1)}} \hookrightarrow \cdots \hookrightarrow W_1^{\vec{q}^{(1)}} \hookrightarrow L^{q_0}.$$

Proof. We see from (2.13) that for any $f \in W_k^{\vec{p}}$,

$$\|f\|_{W_k^{\vec{q}}} \leq C \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^{p_k}} \leq C\|f\|_{W_k^{\vec{p}}}.$$

Hence $W_k^{\vec{p}} \hookrightarrow W_k^{\vec{q}}$. The second conclusion is obvious. This completes the proof. □

3. Nonuniform Fractional Sobolev Spaces

Recall that nonuniform fractional Sobolev spaces are introduced in Definition 1.2. We point out that for any $0 < s < 1$ and $\vec{p} = (p_0, p_1) \in [1, \infty)^2$, $W_s^{\vec{p}}(\mathbb{R}^N)$ is a Banach space, a fact that can be proved with almost the same arguments as that used in the proof of [16, Proposition 4.24].

3.1. Embedding Theorem for Nonuniform Fractional Sobolev Spaces.

The main result in this subsection is the following embedding theorem.

Theorem 3.1. *Suppose that $\vec{p} = (p_0, p_1)$ with $1 \le p_0 < \infty$, $1 < p_1 < \infty$ and $0 < s < 1$. When $p_0 \le p_1$, we have*

(i) *If $sp_1 < N$, then $W_s^{\vec{p}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $p_0 \le q \le Np_1/(N - sp_1)$.*
(ii) *If $sp_1 = N$, then $W_s^{\vec{p}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $p_0 \le q < \infty$.*
(iii) *If $sp_1 > N$, then $W_s^{\vec{p}}(\mathbb{R}^N) \hookrightarrow C_b^{0,s-N/p_1}(\mathbb{R}^N)$.*

For the case $p_0 > p_1$, if $sp_1 < N$ and $p_0 < Np_1/(N - sp_1)$, we also have $W_s^{\vec{p}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $p_0 \le q \le Np_1/(N - sp_1)$.

To prove Theorem 3.1, we need some preliminary results.

We see from [16, Lemma 4.33] that for any $f \in L^p$,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x \, \mathrm{d}y < \infty$$

is equivalent to

$$\iint_{\mathbb{R}^N \times \mathbb{R}} \frac{|f(x) - f(x + ae_j)|^p}{|a|^{1+ps}} \mathrm{d}x \, \mathrm{d}a < \infty, \qquad \forall\ 1 \le j \le N.$$

Checking the arguments in the proof of [16, Lemma 4.33], we find that the hypothesis $f \in L^p$ is not used. Moreover, with the same arguments we get the following result.

Proposition 3.2. *Let $0 < s < 1$ and $1 \le p < \infty$. Then there exist positive constants C_1 and C_2 such that for any measurable function f which is finite almost everywhere,*

$$C_1 [f]_{W_s^p}^p \le \sum_{j=1}^{N} \iint_{\mathbb{R}^N \times \mathbb{R}} \frac{|f(x) - f(x + ae_j)|^p}{|a|^{1+ps}} \mathrm{d}x \, \mathrm{d}a \le C_2 [f]_{W_s^p}^p, \qquad \forall\ 1 \le j \le N.$$

The following construction is used in the proof of the embedding theorem. Unlike the classical case, here we need that φ equals 1 in a neighborhood of 0 to ensure that its derivatives vanish near 0.

Lemma 3.3. *Suppose that $1 \le p_0, p_1 < \infty$, $0 < s \le 1$, $f \in W_s^{\vec{p}}(\mathbb{R}^N)$, $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(t) = 1$ for $|t| < 1$ and $\varphi(t) = 0$ for $|t| > A$, where A is a constant. Let $\alpha = 1 - 1/p_1 - s$ for $0 < s < 1$ and $\alpha = -1/p_1 + \eta$ for $s = 1$, where $0 < \eta < 1$. Set*

$$g(t, x) = \frac{\varphi(t)}{t^N} \int_{[0,t]^N} f(x + y) \, \mathrm{d}y, \quad t > 0, \tag{3.1}$$

$$g(0, x) = \lim_{t \to 0} g(t, x). \tag{3.2}$$

We have

$$\|t^\alpha \nabla_x g(t, x)\|_{L^{p_1}(\mathbb{R}^{N+1})} \lesssim \|f\|_{W_s^{\vec{p}}}. \tag{3.3}$$

Moreover, there are two functions h_0 and h_1 such that $\partial_t g(t,x) = h_0(t,x) + h_1(t,x)$ and

$$\|t^\alpha h_0(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})} \lesssim \|f\|_{W_s^{\vec{p}}}, \tag{3.4}$$

$$\|h_1(t,\cdot)\|_{L^{p_0}(\mathbb{R}^N)} \lesssim \|f\|_{L^{p_0}} 1_{[1,A]}(t). \tag{3.5}$$

Furthermore, if $p_0 \le p_1$, we have

$$\|t^\alpha \nabla g(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})} \lesssim \|f\|_{W_s^{\vec{p}}}. \tag{3.6}$$

Proof. First, we estimate $\|\partial_{x_i} g(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}$, $1 \le i \le N$. Denote

$$\hat{x}_i = x - x_i e_i \quad \text{and} \quad \mathrm{d}\hat{x}_i = \prod_{j\neq i} \mathrm{d}x_j.$$

Observe

$$g(t,x) = \frac{\varphi(t)}{t^N} \int_{x_i}^{x_i+t} \int_{[0,t]^{N-1}} f(\hat{x}_i + \hat{y}_i + y_i e_i)\,\mathrm{d}\hat{y}_i\,\mathrm{d}y_i.$$

We get

$$\begin{aligned}
\partial_{x_i} g(t,x) &= \frac{\varphi(t)}{t^N} \int_{[0,t]^{N-1}} \Big(f(\hat{x}_i + \hat{y}_i + (x_i+t)e_i) - f(\hat{x}_i + \hat{y}_i + x_i e_i) \Big) \mathrm{d}\hat{y}_i \\
&= \frac{\varphi(t)}{t^N} \int_{[0,t]^{N-1}} \Big(f(x + te_i + \hat{y}_i) - f(x + \hat{y}_i) \Big) \mathrm{d}\hat{y}_i \\
&= \frac{\varphi(t)}{t} \int_{[0,1]^{N-1}} \Big(f(x + te_i + t\hat{y}_i) - f(x + t\hat{y}_i) \Big) \mathrm{d}\hat{y}_i.
\end{aligned}$$

Hence for $0 < s < 1$,

$$\begin{aligned}
\|t^\alpha \partial_{x_i} g(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}^{p_1} &\le \|\varphi(t)\|_{L^\infty}^{p_1} \int_0^\infty \int_{\mathbb{R}^N} \int_{[0,1]^{N-1}} \frac{|f(x+te_i+t\hat{y}_i) - f(x+t\hat{y}_i)|^{p_1}}{t^{1+sp_1}} \mathrm{d}\hat{y}_i\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim [f]_{W_s^{p_1}}^{p_1}.
\end{aligned}$$

For the case $s = 1$, we have $\alpha = -1/p_1 + \eta$. We see from (3.1) that

$$\nabla_x g(t,x) = \frac{\varphi(t)}{t^N} \int_{[0,t]^N} \nabla f(x+y)\mathrm{d}y.$$

Thus we obtain

$$\|t^\alpha \nabla_x g(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})} \le \|t^\alpha \varphi(t)\|_{L^{p_1}} \|\nabla f\|_{L^{p_1}} < \infty$$

proving (3.3).

Next we deal with $\partial_t g(t,x)$. A simple computation shows that

$$\begin{aligned}
\partial_t g(t,x) &= \varphi(t) \Big(\frac{-N}{t^{N+1}} \int_{[0,t]^N} f(x+y)\mathrm{d}y + \frac{1}{t^N} \sum_{i=1}^N \int_{[0,t]^{N-1}} f(x + \hat{y}_i + te_i)\mathrm{d}\hat{y}_i \Big) \\
&\quad + \frac{\varphi'(t)}{t^N} \int_{[0,t]^N} f(x+y)\mathrm{d}y \\
&= \frac{\varphi(t)}{t^{N+1}} \sum_{i=1}^N \int_{[0,t]^N} \Big(f(x + \hat{y}_i + te_i) - f(x+y) \Big)\mathrm{d}y + \varphi'(t) \int_{[0,1]^N} f(x+ty)\mathrm{d}y
\end{aligned}$$

$$= \frac{\varphi(t)}{t}\sum_{i=1}^{N}\int_{[0,1]^N}\Big(f(x+t\hat{y}_i+te_i)-f(x+ty)\Big)\mathrm{d}y+\varphi'(t)\int_{[0,1]^N}f(x+ty)\mathrm{d}y$$
$$:= h_0(t,x)+h_1(t,x).$$

Next, we prove (3.4). For $0<s<1$, we have

$$\|t^\alpha h_0(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}^{p_1} \lesssim \|\varphi\|_{L^\infty}^{p_1}\sum_{i=1}^{N}\int_0^\infty\int_{[0,1]^N}\int_{\mathbb{R}^N}\frac{|f(x+t\hat{y}_i+te_i)-f(x+ty)|^{p_1}}{t^{1+sp_1}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

By the change of variables $x\to x+ty$, we write

$$\begin{aligned}\|t^\alpha h_0(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}^{p_1} &\lesssim \|\varphi\|_{L^\infty}^{p_1}\sum_{i=1}^{N}\int_0^\infty\int_{[0,1]^N}\int_{\mathbb{R}^N}\frac{|f(x+t(1-y_i)e_i)-f(x)|^{p_1}}{t^{1+sp_1}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &= \|\varphi\|_{L^\infty}^{p_1}\sum_{i=1}^{N}\int_0^\infty\int_0^1(1-y_i)^{sp_1}\int_{\mathbb{R}^N}\frac{|f(x+te_i)-f(x)|^{p_1}}{t^{1+sp_1}}\mathrm{d}x\,\mathrm{d}y_i\,\mathrm{d}t\\ &\lesssim [f]_{W_s^{p_1}}^{p_1},\end{aligned}\tag{3.7}$$

having used the change of variables $t\to(1-y_i)t$. And for the case $s=1$, we have

$$\begin{aligned}\|t^\alpha h_0(t,x)\|_{L_x^{p_1}} &\le |t^{\alpha-1}\varphi(t)|\sum_{i=1}^{N}\int_{[0,1]^N}\Big\|f(x+t\hat{y}_i+te_i)-f(x+ty)\Big\|_{L_x^{p_1}}\mathrm{d}y\\ &= |t^{\alpha-1}\varphi(t)|\sum_{i=1}^{N}\int_{[0,1]^N}\Big\|\int_{ty_i}^{t}\partial_{x_i}f(x+t\hat{y}_i+\tau e_i)\mathrm{d}\tau\Big\|_{L_x^{p_1}}\mathrm{d}y\\ &\lesssim |t^\alpha\varphi(t)|\cdot\|\nabla f\|_{L^{p_1}}.\end{aligned}$$

Hence

$$\|t^\alpha h_0(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}\lesssim\|\nabla f\|_{L^{p_1}}.\tag{3.8}$$

To prove (3.5), note that $\varphi'(t)=0$ for $0<t<1$ or $t>A$. For $1\le t\le A$, we see from Minkowski's inequality that

$$\|h_1(t,\cdot)\|_{L^{p_0}(\mathbb{R}^N)}\le|\varphi'(t)|\int_{[0,1]^N}\|f(\cdot+ty)\|_{L^{p_0}}\mathrm{d}y\le|\varphi'(t)|\cdot\|f\|_{L^{p_0}}.\tag{3.9}$$

Hence (3.5) is valid.

Finally, we consider the case $p_0\le p_1$. To prove (3.6), it suffices to show that $\|t^\alpha h_1(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}\lesssim\|f\|_{L^{p_0}}$, thanks to (3.3) and (3.4).

Recall that $\varphi'(t)=0$ for $0<t<1$. For $t\ge1$, we have

$$\begin{aligned}|h_1(t,x)| &\le|\varphi'(t)|\Big(\int_{[0,1]^N}|f(x+ty)|^{p_0}\mathrm{d}y\Big)^{1/p_0}\\ &=|\varphi'(t)|\Big(\frac{1}{t^N}\int_{[0,t]^N}|f(x+y)|^{p_0}\mathrm{d}y\Big)^{1/p_0}\\ &\le|\varphi'(t)|\cdot\|f\|_{L^{p_0}}.\end{aligned}$$

Hence

$$\|t^\alpha h_1(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}^{p_1}\le\int_{\mathbb{R}^N}\int_1^\infty|t^\alpha\varphi'(t)|^{p_1}\|f\|_{L^{p_0}}^{p_1-p_0}\Big(\int_{[0,1]^N}|f(x+ty)|\mathrm{d}y\Big)^{p_0}\mathrm{d}t\,\mathrm{d}x$$

$$\leq \int_{\mathbb{R}^N} \int_1^\infty |t^\alpha \varphi'(t)|^{p_1} \|f\|_{L^{p_0}}^{p_1-p_0} \int_{[0,1]^N} |f(x+ty)|^{p_0}\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}x$$
$$\leq \|f\|_{L^{p_0}}^{p_1}.$$

This completes the proof. □

For the case $1 \leq p_0 \leq p_1 < \infty$, the nonuniform Sobolev space $W_s^{(p_0,p_1)}$ is a subspace of the classical one $W_s^{p_1}$. Moreover, the inclusion is in fact an embedding and is also valid for $s = 1$.

Lemma 3.4. *Suppose that* $0 < s \leq 1$ *and* $1 \leq p_0 \leq p_1 < \infty$. *We have*

$$W_s^{\vec{p}} \hookrightarrow W_s^{p_1}.$$

Moreover, the inclusion is not true if $p_0 > p_1$.

Proof. Let φ and ψ be functions in $C_c^\infty(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R})$, respectively, with values between 0 and 1 and equal to 1 in neighborhoods of 0. Set $\Phi(t,x) = \varphi(x)\psi(t)$. Take some $A > 0$ such that $\operatorname{supp}\Phi \subset [-A, A]^{N+1}$.

Fix some $f \in W_s^{\vec{p}}$. Let α and g be defined as in Lemma 3.3, δ_0 be the Dirac measure at 0, and

$$E(t,x) = \begin{cases} \dfrac{c_N}{(t^2+|x|^2)^{(N-1)/2}}, & N \geq 2, \\ c_1 \log(t^2+|x|^2), & N = 1, \end{cases}$$

be the fundamental solution of the Laplacian satisfying $\Delta E = \delta_0$.

Recall that for a tempered distribution Λ and a function $h \in C^\infty$, for which the function itself and all of its derivatives have at most polynomial growth at infinity, the product $h\Lambda$ is the tempered distribution [25, Definition 2.3.15] defined by

$$\langle h\Lambda, u\rangle = \langle \Lambda, hu\rangle, \qquad \forall u \in \mathscr{S}. \tag{3.10}$$

Since $\Phi(t,x)$ equals 1 in a neighbourhood of 0, we have

$$\Phi\Delta E = \Phi\delta_0 = \delta_0.$$

Hence

$$\begin{aligned} g &= \delta_0 * g \\ &= \Delta(\Phi E) * g - 2(\nabla\Phi \cdot \nabla E) * g - ((\Delta\Phi)E) * g \\ &= ((\nabla\Phi)E) * \nabla g + (\Phi\nabla E) * \nabla g - 2(\nabla\Phi \cdot \nabla E) * g - ((\Delta\Phi)E) * g, \end{aligned} \tag{3.11}$$

where we use the notation

$$u * v = \sum_{i=1}^{N+1} u_i * v_i$$

to denote the convolution of two $\mathbb{C}^{N+1}$-valued functions.

Since $g(0,x) = f(x)$ a.e., it suffices to show that $\|g(0,\cdot)\|_{L^{p_1}} \lesssim \|f\|_{W_s^{\vec{p}}}$. We prove it in the following steps:

(S1) First, we estimate $\|((\nabla\Phi)E) * \nabla g(0,\cdot)\|_{L^{p_1}}$. For $N \geq 2$, we have

$$((\nabla\Phi)E) * \nabla g(0,x) = \int_0^A \int_{\mathbb{R}^N} \frac{(\nabla\Phi)(-t,y)\cdot\nabla g(t,x-y)}{(t^2+|y|^2)^{(N-1)/2}}\mathrm{d}y\,\mathrm{d}t.$$

Applying Minkovski's inequality, we get

$$\begin{aligned}
\|((\nabla\Phi)E)*\nabla g(0,\cdot)\|_{L^{p_1}} &\le \|\nabla\Phi\|_{L^\infty}\int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\int_{[-A,A]^N}\frac{\mathrm{d}y}{(t+|y|)^{N-1}}\,\mathrm{d}t\\
&\lesssim \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\int_{[0,A]^2}\frac{\mathrm{d}y_1\mathrm{d}y_2}{t+y_1+y_2}\,\mathrm{d}t\\
&\approx \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\int_0^A \log\frac{t+y_1+A}{t+y_1}\mathrm{d}y_1\,\mathrm{d}t\\
&\lesssim \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\int_0^A \frac{1}{(t+y_1)^\varepsilon}\mathrm{d}y_1\,\mathrm{d}t \qquad (0<\varepsilon<1)\\
&\lesssim \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\mathrm{d}t\\
&\lesssim \|t^{-\alpha}\cdot 1_{[0,A]}(t)\|_{L^{p_1'}}\|t^\alpha\nabla g(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}\\
&\approx \|t^\alpha\nabla g(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}\\
&\lesssim \|f\|_{W_s^{\vec{p}}},
\end{aligned}$$

where we applied (3.6) in the last step. When $N\ge 3$, the first inequality is obtained by successively integrating the variables $y_N, y_{N-1},\dots,y_3$ over $[0,\infty)$; the effect of this is the reduction of the exponent by 1 in each integration, so after $N-2$ integrations, the exponent becomes $(N-1)-(N-2)=1$.

For the case $N=1$, we have $|E(t,y)|=|c_N\log(t^2+y^2)|\lesssim(|t|+|y|)^{-\varepsilon}$ for any $\varepsilon>0$. Hence

$$\begin{aligned}
\|((\nabla\Phi)E)*\nabla g(0,\cdot)\|_{L^{p_1}} &\lesssim \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\int_0^A \frac{1}{(t+y_1)^\varepsilon}\mathrm{d}y_1\,\mathrm{d}t \qquad (0<\varepsilon<1)\\
&\lesssim \|f\|_{W_s^{\vec{p}}}.
\end{aligned}$$

(S2) Next, we estimate $\|(\Phi\nabla E)*\nabla g(0,\cdot)\|_{L^{p_1}}$. Observe that for $(t,y)\in[-A,A]^{N+1}$,

$$|\nabla E(t,y)|\lesssim\frac{1}{(|t|+|y|)^N}.$$

Similar arguments as the previous case show that

$$\begin{aligned}
\|(\Phi\nabla E)*\nabla g(0,\cdot)\|_{L^{p_1}} &\lesssim \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\int_{[-A,A]^N}\frac{\mathrm{d}y}{(t+|y|)^N}\,\mathrm{d}t\\
&\approx \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\int_0^A \frac{\mathrm{d}y_1}{t+y_1}\,\mathrm{d}t\\
&\approx \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}\log\frac{t+A}{t}\mathrm{d}t\\
&\lesssim \int_0^A \|\nabla g(t,\cdot)\|_{L^{p_1}}t^{-\varepsilon}\mathrm{d}t\\
&\lesssim \|t^{-(\alpha+\varepsilon)}\cdot 1_{[0,A]}(t)\|_{L^{p_1'}}\|t^\alpha\nabla g(t,x)\|_{L^{p_1}(\mathbb{R}^{N+1})}\\
&\lesssim \|f\|_{W_s^{\vec{p}}},
\end{aligned}$$

where $0<\varepsilon<s$ for $s<1$ and $0<\varepsilon<1-\eta$ for $s=1$. Again, we apply (3.6) in the last step.

(S3) We deal with the last two terms. Observe that both $\nabla\Phi$ and $\Delta\Phi$ vanish in a neighborhood of 0. Hence both $(\nabla\Phi\cdot\nabla E)$ and $(\Delta\Phi)E$ are compactly supported bounded functions. Hence

$$F(x) := |2(\nabla\Phi\cdot\nabla E) * g(0,x)| + |((\Delta\Phi)E) * g(0,x)| \lesssim \int_0^A \int_{[-A,A]^N} |g(t,x-y)|\mathrm{d}y\,\mathrm{d}t.$$

For any $q > p_0$, there is some $r > 1$ such that

$$\frac{1}{r} + \frac{1}{p_0} = \frac{1}{q} + 1.$$

Observe that

$$\|g(t,\cdot)\|_{L^{p_0}} = |\varphi(t)| \cdot \Big\| \int_{[0,1]^N} f(\cdot + ty)\mathrm{d}y \Big\|_{L^{p_0}} \lesssim \|\varphi\|_{L^\infty} \cdot \|f\|_{L^{p_0}}. \tag{3.12}$$

We see from Young's inequality that

$$\|F\|_{L^q} \lesssim \|f\|_{L^{p_0}}.$$

Setting $q = p_1$, we get $\|F\|_{L^{p_1}} \lesssim \|f\|_{L^{p_0}}$.

Combing results in (S1), (S2), and (S3), yields

$$W_s^{(p_0,p_1)} \hookrightarrow W_s^{(p_1,p_1)} = W_s^{p_1}.$$

For the case $p_0 > p_1$, we provide a counterexample in Example 3.6. This completes the proof. □

The following is an immediate consequence.

Corollary 3.5. *Suppose that $\vec{p} = (p_0, p_1)$ with $1 \le p_0 \le p_1 < \infty$. For any $0 < \tilde{s} < s < 1$, we have*

$$W_1^{\vec{p}} \hookrightarrow W_s^{\vec{p}} \hookrightarrow W_{\tilde{s}}^{\vec{p}}. \tag{3.13}$$

Proof. First, we prove that $W_1^{\vec{p}} \hookrightarrow W_s^{\vec{p}}$. Take some $f \in W_1^{\vec{p}}$. As for functions in classical Sobolev spaces, for almost all $x \in \mathbb{R}^N$, we have

$$|f(x) - f(x+ae_j)|^{p_1} = \Big| \int_0^a \partial_{x_j} f(x+te_j)\mathrm{d}t \Big|^{p_1} \le |a|^{p_1-1} \int_0^a |\partial_{x_j} f(x+te_j)|^{p_1}\mathrm{d}t.$$

Hence

$$\int_{\mathbb{R}^N} |f(x) - f(x+ae_j)|^{p_1}\mathrm{d}x \le |a|^{p_1} \|\partial_{x_j} f\|_{L^{p_1}}^{p_1}.$$

Therefore,

$$\int_{|a|<1}\int_{\mathbb{R}^N} \frac{|f(x) - f(x+ae_j)|^{p_1}}{|a|^{1+sp_1}}\mathrm{d}x\,\mathrm{d}a \le \int_{|a|<1} \frac{|a|^{p_1}}{|a|^{1+sp_1}} \|\partial_{x_j} f\|_{L^{p_1}}^{p_1}\mathrm{d}a \lesssim \|\partial_{x_j} f\|_{L^{p_1}}^{p_1}.$$

On the other hand, since $p_0 \le p_1$, we have $W_1^{\vec{p}} \hookrightarrow W_1^{p_1}$, thanks to Lemma 3.4. Hence

$$\int_{|a|\ge 1}\int_{\mathbb{R}^N} \frac{|f(x) - f(x+ae_j)|^{p_1}}{|a|^{1+sp_1}}\mathrm{d}x\,\mathrm{d}a \lesssim \int_{|a|\ge 1} \frac{\|f\|_{L^{p_1}}^{p_1}}{|a|^{1+sp_1}}\mathrm{d}a \lesssim \|f\|_{L^{p_1}}^{p_1}.$$

Consequently, $\|f\|_{W_s^{\vec{p}}} \lesssim \|f\|_{W_1^{\vec{p}}}$.

Next, we show that $W_s^{\vec{p}} \hookrightarrow W_{\tilde{s}}^{\vec{p}}$. Fix some $f \in W_s^{\vec{p}}$. Since $\tilde{s} < s$, we have

$$\int_{|a|<1}\int_{\mathbb{R}^N} \frac{|f(x) - f(x+ae_j)|^{p_1}}{a^{1+p_1\tilde{s}}}\mathrm{d}x\,\mathrm{d}a \le \int_{|a|<1}\int_{\mathbb{R}^N} \frac{|f(x) - f(x+ae_j)|^{p_1}}{a^{1+p_1 s}}\mathrm{d}x\,\mathrm{d}a \lesssim [f]_{W_s^{p_1}}^{p_1}.$$

On the other hand, by Lemma 3.4, $f \in L^{p_1}$. Hence

$$\int_{|a|\geq 1}\int_{\mathbb{R}^N}\frac{|f(x)-f(x+ae_j)|^{p_1}}{a^{1+p_1\tilde{s}}}\mathrm{d}x\,\mathrm{d}a \leq \int_{|a|\geq 1}\frac{C_{p_1}\|f\|_{L^{p_1}}^{p_1}}{a^{1+p_1\tilde{s}}}\mathrm{d}a \lesssim \|f\|_{L^{p_1}}^{p_1}.$$

This completes the proof. □

We point out that the inclusion in Corollary 3.5 is not true in general for the case $p_0 > p_1$, which is quite different from the classical case, since for any $1 \leq p < \infty$ and $0 < \tilde{s} < s < 1$,

$$W_1^p \hookrightarrow W_s^p \hookrightarrow W_{\tilde{s}}^p.$$

Below is a counterexample.

Example 3.6. *Suppose that $0 < s \leq 1$, $1 \leq p_0, p_1 < \infty$ and $p_1\delta < N < p_0\delta$. Let $f(x) = (1+|x|^2)^{-\delta/2}$, $x \in \mathbb{R}^N$ and let $\vec{p} = (p_0, p_1)$. We have $f \in W_s^{\vec{p}}(\mathbb{R}^N)$ if and only if $p_1(\delta+s) > N$. Consequently, for any $0 < s \leq 1$ and $p_0 > p_1$, $W_s^{\vec{p}} \not\subset W_s^{p_1}$.*

Moreover, if $0 < \tilde{s} < s \leq 1$ and $\tilde{s} < N/p_1 - N/p_0$, we have $W_s^{\vec{p}} \not\subset W_{\tilde{s}}^{\vec{p}}$.

Proof. (i) We show that $f \in W_s^{\vec{p}}(\mathbb{R}^N)$ if and only if $p_1(\delta+s) > N$, for which we only need to consider the case $0 < s < 1$ since the other case $s = 1$ is obvious.

First, we assume that $p_1(\delta+s) > N$. Let us estimate the integral

$$\int_{\mathbb{R}^N}|f(x)-f(x+ae_j)|^{p_1}\mathrm{d}x, \qquad 1 \leq j \leq N.$$

Denote $\hat{x}_j = x - x_je_j$ and $\mathrm{d}\hat{x}_j = \prod_{i\neq j}\mathrm{d}x_i$. Observe that

$$|f(x)-f(x+ae_j)|^{p_1} = \Big|\int_0^a \partial_{x_j}f(x+te_j)\mathrm{d}t\Big|^{p_1} \leq |a|^{p_1-1}\Big|\int_0^a|\partial_{x_j}f(x+te_j)|^{p_1}\mathrm{d}t\Big|. \tag{3.14}$$

When $|a| > 1$, $|t| \leq |a|$ and $|x| > 2|a|$, we have $|x+te_j| > |a|$. Hence

$$\begin{aligned}
&\int_{|x|>2|a|}|f(x)-f(x+ae_j)|^{p_1}\mathrm{d}x\\
&\lesssim |a|^{p_1-1}\Big|\int_0^a\int_{|x|>2|a|}\frac{|x_j+t|^{p_1}}{(1+|\hat{x}_j|^2+|x_j+t|^2)^{(\delta+2)p_1/2}}\mathrm{d}x\,\mathrm{d}t\Big|\\
&\leq |a|^{p_1-1}\Big|\int_0^a\int_{|x|>|a|}\frac{|x_j|^{p_1}}{(1+|\hat{x}_j|^2+|x_j|^2)^{(\delta+2)p_1/2}}\mathrm{d}x\,\mathrm{d}t\Big|\\
&= |a|^{p_1}\bigg(\int_{|x_j|>|a|}\int_{\mathbb{R}^{N-1}}\frac{|x_j|^{p_1}}{(1+|\hat{x}_j|^2+|x_j|^2)^{(\delta+2)p_1/2}}\mathrm{d}\hat{x}_j\,\mathrm{d}x_j\\
&\qquad+\int_{|x_j|<|a|}\int_{|\hat{x}_j|>(a^2-x_j^2)^{1/2}}\frac{|x_j|^{p_1}}{(1+|\hat{x}_j|^2+|x_j|^2)^{(\delta+2)p_1/2}}\mathrm{d}\hat{x}_j\,\mathrm{d}x_j\bigg)\\
&\approx |a|^{p_1}\bigg(\int_{|x_j|>|a|}\frac{|x_j|^{p_1}\mathrm{d}x_j}{(1+|x_j|)^{(\delta+2)p_1-(N-1)}}+\int_{|x_j|<|a|}\frac{|x_j|^{p_1}\mathrm{d}x_j}{(1+|a|)^{(\delta+2)p_1-(N-1)}}\bigg)\\
&\approx |a|^{N-p_1\delta},
\end{aligned} \tag{3.15}$$

where we use the fact that $(\delta+1)p_1 > (\delta+s)p_1 > N$. On the other hand,

$$\int_{|x|\leq 2|a|}|f(x)-f(x+ae_j)|^{p_1}\mathrm{d}x$$

$$
\begin{aligned}
&\lesssim \int_{|x|\le 2|a|} \Big(\frac{1}{(1+|x|^2)^{p_1\delta/2}} + \frac{1}{(1+|x+ae_j|^2)^{p_1\delta/2}}\Big)\mathrm{d}x \\
&\le \int_{|x|\le 2|a|} \frac{1}{(1+|x|^2)^{p_1\delta/2}}\mathrm{d}x + \int_{|x+ae_j|\le 3|a|} \frac{1}{(1+|x+ae_j|^2)^{p_1\delta/2}}\mathrm{d}x \\
&\lesssim |a|^{N-p_1\delta}. \qquad (3.16)
\end{aligned}
$$

Putting (3.15) and (3.16) together, we get

$$
\int_{\mathbb{R}^N} |f(x)-f(x+ae_j)|^{p_1}\mathrm{d}x \lesssim |a|^{N-p_1\delta}, \quad |a|>1.
$$

For $|a|\le 1$, since $|\partial_{x_j} f(x)| \lesssim 1/(1+|x|^2)^{(\delta+1)/2}$ and $p_1(\delta+1) > p_1(\delta+s) > N$, we have $\partial_{x_j} f \in L^{p_1}$. It follows from (3.14) that

$$
\int_{\mathbb{R}^N} |f(x)-f(x+ae_j)|^{p_1}\mathrm{d}x \le |a|^{p_1}\|\partial_{x_j} f\|_{L^{p_1}}^{p_1}.
$$

Combining the preceding estimates we deduce

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^N} \frac{|f(x)-f(x+ae_j)|^{p_1}}{|a|^{1+p_1 s}}\mathrm{d}x\,\mathrm{d}a &= \int_{|a|\le 1}\int_{\mathbb{R}^N} \frac{|f(x)-f(x+ae_j)|^{p_1}}{|a|^{1+p_1 s}}\mathrm{d}x\,\mathrm{d}a \\
&\quad + \int_{|a|>1}\int_{\mathbb{R}^N} \frac{|f(x)-f(x+ae_j)|^{p_1}}{|a|^{1+p_1 s}}\mathrm{d}x\,\mathrm{d}a \\
&\lesssim \int_{|a|\le 1} |a|^{p_1-1-p_1 s}\mathrm{d}a + \int_{|a|>1} \frac{\mathrm{d}a}{|a|^{p_1(\delta+s)-N+1}} \\
&< \infty.
\end{aligned}
$$

Hence $f \in W_s^{\vec{p}}$.

Next we show that $f \notin W_s^{\vec{p}}$ whenever $p_1(\delta+s) \le N$.

When $a>2$, $a^2 < |x|^2 < 5a^2/4$ and $x_j > 0$, we have $1+|x|^2 < 3a^2/2$ and $1+|x+ae_j|^2 = 1+|x|^2+2ax_j+a^2 > 2a^2$. Hence

$$
\begin{aligned}
\int_{\mathbb{R}^N} |f(x)-f(x+ae_j)|^{p_1}\mathrm{d}x &\ge \int_{\substack{a^2<|x|^2<5a^2/4\\ x_j>0}} |f(x)-f(x+ae_j)|^{p_1}\mathrm{d}x \\
&\gtrsim \int_{\substack{a^2<|x|^2<5a^2/4\\ x_j>0}} \frac{\mathrm{d}x}{a^{p_1\delta}} \\
&\approx a^{N-p_1\delta}.
\end{aligned}
$$

Therefore,

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^N} \frac{|f(x)-f(x+ae_j)|^{p_1}}{|a|^{1+p_1 s}}\mathrm{d}x\,\mathrm{d}a \gtrsim \int_{a\ge 2} \frac{\mathrm{d}a}{a^{p_1(\delta+s)-N+1}} = \infty.
$$

(ii) Now assume that $p_0 > p_1$. Take some $\varepsilon > 0$ such that $\varepsilon < \min\{s, N/p_1 - N/p_0\}$. Set $\delta = N/p_1 - \varepsilon$. We have $p_1\delta < N < p_0\delta$ and $p_1(\delta+s) = N + p_1(s-\varepsilon) > N$. Hence $f \in W_s^{\vec{p}} \setminus W_s^{p_1}$.

(iii) When $\tilde{s} < N/p_1 - N/p_0$, we have $p_0 > p_1$. Take some constant δ such that

$$
\max\Big\{\frac{N}{p_0}, \frac{N}{p_1} - s\Big\} < \delta < \frac{N}{p_1} - \tilde{s}.
$$

Then $p_0\delta > N > p_1\delta$, $p_1(\delta+s) > N$ and $p_1(\delta+\tilde{s}) < N$. Hence $f \in W_s^{\vec{p}} \setminus W_{\tilde{s}}^{\vec{p}}$. □

Proof of Theorem 3.1. For the case $p_0 \le p_1$, the conclusion follows from Lemma 3.4 and the embedding theorem for classical fractional Sobolev spaces.

Now we assume that $p_0 > p_1$. It suffices to show that if $f \in W_s^{\vec{p}}$, then $f \in L^q$ with $q = Np_1/(N - sp_1)$.

We adopt the notation used in the proof of Lemma 3.4. We rewrite (3.11) as

$$\begin{aligned} g = {} & ((\nabla_x \Phi)E) * \nabla_x g + (\Phi \nabla_x E) * \nabla_x g + ((\partial_t \Phi)E) * h_0 + (\Phi \partial_t E) * h_0 \\ & + ((\partial_t \Phi)E) * h_1 + (\Phi \partial_t E) * h_1 - 2(\nabla\Phi \cdot \nabla E) * g - ((\Delta\Phi)E) * g. \end{aligned} \tag{3.17}$$

Checking the proof of [16, Proposition 4.47], we find that

$$\left\| \Big(((\nabla_x \Phi)E) * \nabla_x g + (\Phi \nabla_x E) * \nabla_x g + ((\partial_t \Phi)E) * h_0 + (\Phi \partial_t E) * h_0 \Big)(0,\cdot) \right\|_{L^q} \lesssim \|f\|_{W_s^{\vec{p}}}. \tag{3.18}$$

In fact, the only property of g used in the proof of [16, Proposition 4.47] is that $t^\alpha \nabla g(t,x)$ lies in L^{p_1}. In our case, $\partial_t g$ is replaced by h_0. By Lemma 3.3, $t^\alpha h_0(t,x) \in L^{p_1}$. So (3.18) is true.

It remains to consider the last four terms in (3.17). Since Φ is compactly supported, we have

$$|(\partial_t \Phi)E(t,x)| \lesssim \frac{1}{(|t|+|x|)^N},$$

and

$$|\Phi \partial_t E(t,x)| \lesssim \frac{1}{(|t|+|x|)^N}.$$

Observe that $h_1(t,x) = 0$ for $t < 1$ or $t > A$. We have

$$\left| \Big(((\partial_t \Phi)E) * h_1 + (\Phi \partial_t E) * h_1 \Big)(0,x) \right| \lesssim \int_1^A \int_{[-A,A]^N} \frac{|h_1(t,x-y)|}{(t+|y|)^N} \mathrm{d}y \, \mathrm{d}t.$$

Take some $r > 1$ such that $1/p_0 + 1/r = 1/q + 1$. Applying Young's inequality, we deduce from (3.5) that

$$\left\| \Big(((\partial_t \Phi)E) * h_1 + (\Phi \partial_t E) * h_1 \Big)(0,\cdot) \right\|_{L^q} \lesssim \int_1^A \|h_1(t,\cdot)\|_{L^{p_0}} \left\| \frac{1}{(t+|\cdot|)^N} \right\|_{L^r} \mathrm{d}t \lesssim \|f\|_{L^{p_0}}.$$

For the last two terms in (3.17), we show in (S3) of the proof of Lemma 3.4 that

$$\left\| \Big(|(\nabla\Phi \cdot \nabla E) * g| + |((\Delta\Phi)E) * g| \Big)(0,\cdot) \right\|_{L^q} \lesssim \|f\|_{L^{p_0}}.$$

Combining these facts we derive the desired conclusion. □

3.2. **Density of Compactly Supported Infinitely Differentiable Functions.** In this subsection, we show that compactly supported infinitely differentiable functions are dense in fractional nonuniform Sobolev spaces for certain indices.

First, we show that smooth functions are dense in nonuniform fractional Sobolev spaces.

Lemma 3.7. *For any $0 < s < 1$ and $\vec{p} = (p_0, p_1)$ with $1 \le p_0, p_1 < \infty$, $C^\infty \cap W_s^{\vec{p}}(\mathbb{R}^N)$ is dense in $W_s^{\vec{p}}(\mathbb{R}^N)$.*

Proof. As in the proof of Lemma 2.1, take some $\varphi \in C_c^\infty$ such that φ is nonnegative and $\|\varphi\|_{L^1} = 1$. Fix some $f \in W_s^{\vec{p}}$ and set

$$g_\lambda(x) = \int_{\mathbb{R}^N} f(y) \frac{1}{\lambda^N} \varphi(\frac{x-y}{\lambda}) \mathrm{d}y, \qquad \lambda > 0.$$

We have $g_\lambda \in C^\infty$. Moreover, since

$$g_\lambda(x) = \int_{\mathbb{R}^N} f(x - \lambda y) \varphi(y) \mathrm{d}y,$$

we have

$$g_\lambda(x) - f(x) = \int_{\mathbb{R}^N} (f(x - \lambda y) - f(x)) \varphi(y) \mathrm{d}y.$$

Now we derive from Minkowski's inequality and the continuity of translation operators in Lebesgue spaces that

$$\lim_{\lambda \to 0} \|g_\lambda - f\|_{L^{p_0}} \le \lim_{\lambda \to 0} \int_{\mathbb{R}^N} \|f(\cdot - \lambda y) - f\|_{L^{p_0}} \varphi(y) \mathrm{d}y = 0 \tag{3.19}$$

and

$$\begin{aligned}
&\lim_{\lambda \to 0} [g_\lambda - f]_{W_s^{p_1}} \\
&\le \lim_{\lambda \to 0} \int_{\mathbb{R}^N} \left\| \frac{\big(f(x - \lambda y) - f(x)\big) - \big(f(z - \lambda y) - f(z)\big)}{|x - z|^{N/p_1 + s}} \right\|_{L^{p_1}_{(x,z)}(\mathbb{R}^N \times \mathbb{R}^N)} \varphi(y) \mathrm{d}y \\
&= \lim_{\lambda \to 0} \int_{\mathbb{R}^N} \left\| \frac{f(x - \lambda y) - f(z - \lambda y)}{|(x - \lambda y) - (z - \lambda y)|^{N/p_1 + s}} - \frac{f(x) - f(z)}{|x - z|^{N/p_1 + s}} \right\|_{L^{p_1}_{(x,z)}(\mathbb{R}^N \times \mathbb{R}^N)} \varphi(y) \mathrm{d}y \\
&= 0,
\end{aligned}$$

having applied the fact that $(f(x) - f(z))/|x - z|^{N/p_1 + s} \in L^{p_1}_{(x,z)}$. Hence $\lim_{\lambda \to 0} \|g_\lambda - f\|_{W_s^{\vec{p}}} = 0$. □

As in the classical case, to prove the density of compactly supported smooth functions, we first approximate a function in fractional nonuniform Sobolev spaces by its truncation, then approximate the truncation by its regularization. However, since p_0 needs not to be identical to p_1, the technique details are quite different.

Theorem 3.8. *Suppose that $0 < s < 1$, $\vec{p} = (p_0, p_1) \in [1, \infty)^2$ and $s/N \ge 1/p_1 - 1/p_0$. Then $C_c^\infty(\mathbb{R}^N)$ is dense in $W_s^{\vec{p}}(\mathbb{R}^N)$.*

Proof. Take some nonnegative function $\varphi \in C_c^\infty$ such that $\varphi(x) = 1$ when $|x| < 1$ and $\varphi(x) = 0$ when $|x| > 2$.

Fix some $f \in W_s^{\vec{p}}$. First, we show that $\varphi(\cdot/n) f$ tends to f in $W_s^{\vec{p}}$ as n tends to the infinity. Since $\varphi(\cdot/n) f$ tends to f in L^{p_0}, thanks to Lebesgue's dominated convergence theorem, it suffices to show that $\lim_{n \to \infty} [\varphi(\cdot/n) f - f]_{W_s^{p_1}} = 0$.

If $p_0 \le p_1$, by Lemma 3.4, we have $W_s^{\vec{p}} \hookrightarrow W_s^{p_1}$. Now we see from the density result in the classical fractional Sobolev spaces that $\lim_{n \to \infty} [\varphi(\cdot/n) f - f]_{W_s^{p_1}} = 0$.

It remains to consider the case $p_0 > p_1$. Observe that

$$\Big(\varphi(\frac{x}{n}) - 1\Big) f(x) - \Big(\varphi(\frac{y}{n}) - 1\Big) f(y) = \Big(\varphi(\frac{x}{n}) - 1\Big)(f(x) - f(y)) + \Big(\varphi(\frac{x}{n}) - \varphi(\frac{y}{n})\Big) f(y).$$

We have

$$[\varphi(\frac{\cdot}{n})f-f]_{W_s^{p_1}}^{p_1}\lesssim \iint_{\mathbb{R}^{2N}}\frac{|(\varphi(x/n)-1)(f(x)-f(y))|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y + \iint_{\mathbb{R}^{2N}}\frac{|(\varphi(x/n)-\varphi(y/n))f(y)|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y.$$

Since $f\in W_s^{\vec{p}}$, we see from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|(\varphi(x/n)-1)(f(x)-f(y))|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y=0.$$

Now we only need to show that

$$\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|(\varphi(x/n)-\varphi(y/n))f(y)|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y=0. \tag{3.20}$$

We split the integral into two parts,

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|(\varphi(x/n)-\varphi(y/n))f(y)|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y\\&=\Big(\iint_{|y|\le 4n}+\iint_{|y|>4n}\Big)\frac{|(\varphi(x/n)-\varphi(y/n))f(y)|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y\\&=I+II.\end{aligned}$$

We have

$$\begin{aligned}I&=\Big(\iint_{\substack{|y|\le 4n\\|x-y|\le 8n}}+\iint_{\substack{|y|\le 4n\\|x-y|>8n}}\Big)\frac{|(\varphi(x/n)-\varphi(y/n))f(y)|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y\\&=I_1+I_2,\end{aligned}$$

where

$$I_1\le\frac{\|\nabla\varphi\|_{L^\infty}^{p_1}}{n^{p_1}}\iint_{\substack{|y|\le 4n\\|x-y|\le 8n}}\frac{|x-y|^{p_1}|f(y)|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y\lesssim\frac{1}{n^{sp_1}}\int_{|y|\le 4n}|f(y)|^{p_1}\mathrm{d}y,$$

and

$$I_2=\iint_{\substack{|y|\le 4n\\|x-y|>8n}}\frac{|\varphi(y/n)f(y)|^{p_1}}{|x-y|^{N+sp_1}}\mathrm{d}x\,\mathrm{d}y\lesssim\frac{1}{n^{sp_1}}\int_{|y|\le 4n}|f(y)|^{p_1}\mathrm{d}y.$$

Hence

$$I\lesssim\frac{1}{n^{sp_1}}\int_{|y|\le 4n}|f(y)|^{p_1}\mathrm{d}y.$$

Denote $r=p_0/p_1$. For any $k\ge 1$, there is some n_0 such that

$$\|f\cdot 1_{\{|y|\le n_0\}}\|_{L^{p_0}}\ge\Big(1-\frac{1}{k}\Big)\|f\|_{L^{p_0}}.$$

Applying Hölder's inequality, we get

$$\begin{aligned}I&\lesssim\frac{1}{n^{sp_1}}\Big(\int_{|y|\le n_0}|f(y)|^{p_1}\mathrm{d}y+\int_{n_0<|y|\le 4n}|f(y)|^{p_1}\mathrm{d}y\Big)\\&\lesssim\frac{1}{n^{sp_1}}\Big(n_0^{N/r'}\||f|^{p_1}\|_{L^r}+n^{N/r'}\||f|^{p_1}\cdot 1_{\{|y|>n_0\}}\|_{L^r}\Big)\\&\le\frac{n_0^{N/r'}}{n^{sp_1}}\|f\|_{L^{p_0}}^{p_1}+\frac{1}{k^{p_1}}\cdot\frac{1}{n^{Np_1(s/N-1/p_1+1/p_0)}}\|f\|_{L^{p_0}}^{p_1}.\end{aligned}$$

Since $s/N \geq 1/p_1 - 1/p_0$, letting $n \to \infty$ and $k \to \infty$ successively, we get

$$\lim_{n\to\infty} I = 0.$$

Next we deal with II. Recall that $\varphi(x) = 0$ when $|x| > 2$. We have

$$\begin{aligned} II &= \iint_{|y|>4n} \frac{|(\varphi(x/n) - \varphi(y/n))f(y)|^{p_1}}{|x-y|^{N+sp_1}} \mathrm{d}x\mathrm{d}y \\ &= \int_{|x|\leq 2n} \int_{|y|>4n} \frac{|\varphi(x/n)f(y)|^{p_1}}{|x-y|^{N+sp_1}} \mathrm{d}x\mathrm{d}y \\ &\lesssim \int_{|x|\leq 2n} |\varphi(\frac{x}{n})|^{p_1} \mathrm{d}x \int_{|y|>4n} \frac{|f(y)|^{p_1}}{|y|^{N+sp_1}} \mathrm{d}y \\ &\lesssim n^N \cdot \|f \cdot 1_{\{|y|\geq 4n\}}\|_{L^{p_0}}^{p_1} \cdot \frac{1}{n^{sp_1+N/r}} \\ &= \frac{1}{n^{Np_1(s/N-1/p_1+1/p_0)}} \|f \cdot 1_{\{|y|\geq 4n\}}\|_{L^{p_0}}^{p_1}. \end{aligned}$$

Hence $\lim_{n\to\infty} II = 0$. Therefore, (3.20) is true.

The above arguments show that the sequence $\{\varphi(\cdot/n)f \ n \geq 1\}$ is a subset of $W_s^{\vec{p}}$ and is convergent to f in $W_s^{\vec{p}}$.

To finish the proof, it suffices to show that for any compactly supported function $f \in W_s^{\vec{p}}$, f can be approximated by functions in C_c^∞, which can be achieved with the same arguments as in the proof of Lemma 3.7. □

4. Applications

4.1. **Local estimates for solutions of heat equations.** Consider the classical solution of the heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = 0, & t > 0, \\ u(0,x) = u_0(x). \end{cases}$$

That is,

$$u(t,x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/(4t)} u_0(y)\mathrm{d}y, \qquad t > 0, x \in \mathbb{R}^N. \tag{4.1}$$

Fefferman, McCormick, Robinson, and Rodrigo [23] studied local energy estimates for $u(t,x)$. They proved that if the initial data u_0 belongs to the Sobolev space H^s, then for any $T > 0$, the classical solution of the heat equation satisfies that $u \in L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+1})$ and $t^{1/2}u(t,x) \in L^2(0,T;H^{s+2})$. As a result, $u \in L^q(0,T;H^{s+2})$ for any $0 < q < 1$.

In this subsection, we show that the local estimates for solutions of heat equations are also valid when the initial data belong to nonuniform Sobolev spaces.

Theorem 4.1. *Suppose that $s > 0$, $T > 0$ and $\vec{p} = (p_0, \dots, p_{\lceil s \rceil})$ with $1 < p_i < \infty$ for $0 \leq i \leq \lceil s \rceil$. Set $\vec{r} = (p_0, \dots, p_{\lfloor s \rfloor}, 2)$ for $s \notin \mathbb{Z}$ and $\vec{r} = \vec{p}$ for $s \in \mathbb{Z}$. Let u be the classical solution of the heat equation with initial data $u_0 \in W_s^{\vec{p}}(\mathbb{R}^N)$. Then $u \in L^\infty(0,T;W_s^{\vec{p}}(\mathbb{R}^N))$ with*

$$\sup_{0\leq t\leq T} \|u(t,\cdot)\|_{W_s^{\vec{p}}} \leq C_{s,\vec{p},N} \|u_0\|_{W_s^{\vec{p}}}. \tag{4.2}$$

Moreover, if $1 < p_{\lfloor s\rfloor}, p_{\lceil s\rceil} \le 2$*, then*

$$\int_0^T t^{\varrho}\|u(t,\cdot)\|^2_{W^{(\vec{r},2)}_{s+1}}\mathrm{d}t \le C_{s,\vec{p},N}(1+T^{\theta_1})\|u_0\|^2_{W^{\vec{p}}_s}, \tag{4.3}$$

$$\int_0^T t^{1+\varrho}\|u(t,\cdot)\|^2_{W^{(\vec{r},2,2)}_{s+2}}\mathrm{d}t \le C_{s,\vec{p},N}(1+T^{\theta_2})\|u_0\|^2_{W^{\vec{p}}_s}, \tag{4.4}$$

where

$$\varrho = (2-p_s)\sigma, \quad p_s = \min\{p_{\lfloor s\rfloor}, p_{\lceil s\rceil}\}, \quad \sigma = \frac{N}{2p_{\lfloor s\rfloor}} + \frac{1}{2},$$

and θ_1, θ_2 *are constants. Consequently, for any* $0 < q < 2/(2+\varrho)$, $u \in L^q(0,T; W^{(\vec{r},2,2)}_{s+2}(\mathbb{R}^N))$ *and*

$$\int_0^T \|u(t,\cdot)\|^q_{W^{(\vec{r},2,2)}_{s+2}}\mathrm{d}t \le C_{s,\vec{p},N}T^{1-(1+\varrho/2)q}(1+T^{\theta_2})^{q/2}\|u_0\|^q_{W^{\vec{p}}_s}. \tag{4.5}$$

Before giving a proof of Theorem 4.1, we discuss a lemma.

Lemma 4.2. *Suppose that* $f \in C^2_b(\mathbb{R})$ *and* $f, f'' \in L^p(\mathbb{R})$ *for some* $1 < p < \infty$*. Then* $f''|f|^{p-2}f \in L^1(\mathbb{R})$ *and*

$$\int_{\mathbb{R}} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x = -(p-1)\int_{\mathbb{R}} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x,$$

where we apply the convention that $|f'(x)|^2|f(x)|^{p-2} = 0$ *whenever* $f'(x) = 0$*.*

Proof. Since $f, f'' \in L^p$, we see from Hölder's inequality that $f''|f|^{p-2}f \in L^1$. To prove the conclusion, it suffices to show the following equations,

$$\int_0^\infty f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x = -f'(0)|f(0)|^{p-2}f(0) - (p-1)\int_0^\infty |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x, \tag{4.6}$$

$$\int_{-\infty}^0 f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x = f'(0)|f(0)|^{p-2}f(0) - (p-1)\int_{-\infty}^0 |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x. \tag{4.7}$$

We prove only the first equation, and the second one can be proved similarly.

Since $f \in L^p(\mathbb{R})$, there exists a sequence $\{x_n : n \ge 1\} \subset (0,\infty)$ such that

$$\lim_{n\to\infty} x_n = \infty \quad \text{and} \quad \lim_{n\to\infty} f(x_n) = 0. \tag{4.8}$$

When $p \ge 2$, $|f(x)|^{p-2}f(x)$ is continuously differentiable. Integrating by parts we obtain

$$\begin{aligned}
&\int_0^\infty f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x \\
&= \lim_{n\to\infty}\int_0^{x_n} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x \\
&= \lim_{n\to\infty}\Big(f'(x_n)|f(x_n)|^{p-2}f(x_n) - f'(0)|f(0)|^{p-2}f(0) - (p-1)\int_0^{x_n}|f'(x)|^2|f(x)|^{p-2}\mathrm{d}x\Big) \\
&= -f'(0)|f(0)|^{p-2}f(0) - (p-1)\int_0^\infty |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x.
\end{aligned}$$

Hence (4.6) is valid.

It remains to consider the case $1 < p < 2$. Since f is continuous, the set $E := \{x > 0 : f(x) \ne 0\}$ is open in $(0,\infty)$. Consequently, E is the union of at most countable pairwise disjoint intervals (a_i, b_i), $i \in I$. There are three cases:

(i) $b_i < \infty$ for each $i \in I$.

For each interval (a_i, b_i) and $\varepsilon > 0$ small enough, $|f(x)|^{p-2}f(x)$ is continuously differentiable on $[a_i+\varepsilon, b_i-\varepsilon]$. Integrating by parts again, we write

$$\begin{aligned}
&\int_{a_i}^{b_i} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\int_{a_i+\varepsilon}^{b_i-\varepsilon} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\Big((f'|f|^{p-2}f)(b_i-\varepsilon) - (f'|f|^{p-2}f)(a_i+\varepsilon) - (p-1)\int_{a_i+\varepsilon}^{b_i-\varepsilon} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x\Big)\\
&= -(f'|f|^{p-2}f)(a_i) - (p-1)\int_{a_i}^{b_i} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x, \qquad (4.9)
\end{aligned}$$

where $f(a_i) = 0$ if $a_i \neq 0$. Moreover, if $0 \notin \{a_i : i \in I\}$, then $f(0) = 0$. It follows that

$$\begin{aligned}
\int_0^\infty f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x &= \sum_{i\in I}\int_{a_i}^{b_i} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x\\
&= -f'(0)|f(0)|^{p-2}f(0) - \sum_{i\in I}(p-1)\int_{a_i}^{b_i} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x. \qquad (4.10)
\end{aligned}$$

Next we show that the set $F := \{x > 0 : f(x) = 0 \text{ and } f'(x) \neq 0\}$ is at most countable.

Take some $x_0 \in F$. If for any $\varepsilon > 0$, $((x_0-\varepsilon, x_0+\varepsilon)\cap F)\setminus\{x_0\} \neq \emptyset$, then there exists a sequence $\{y_k : k \geq 1\} \subset F \setminus \{x_0\}$ such that $\lim_{k\to\infty} y_k = x_0$. Consequently,

$$f'(x_0) = \lim_{k\to\infty}\frac{f(y_k) - f(x_0)}{y_k - x_0} = 0,$$

which contradicts the fact $f'(x_0) \neq 0$. Hence for any $x \in F$, there is some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon)\cap F = \{x\}$. Consequently, there exist rational numbers r_x and R_x such that $r_x < x < R_x$ and $(r_x, R_x)\cap F = \{x\}$. Since $\{(r_x, R_x) : x \in F\}$ is at most countable, $F = \cup_{x\in F}(r_x, R_x)\cap F$ is also at most countable.

Using the convention that $|f'(x)|^2|f(x)|^{p-2} = 0$ whenever $f'(x) = 0$, we see from (4.10) that

$$\begin{aligned}
\int_0^\infty f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x &= -f'(0)|f(0)|^{p-2}f(0) - (p-1)\int_{\{x:\, f(x)\neq 0\}} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x\\
&\quad - (p-1)\int_{\{x:\, f(x)=f'(x)=0\}} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x\\
&= -f'(0)|f(0)|^{p-2}f(0) - (p-1)\int_0^\infty |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x.
\end{aligned}$$

Hence (4.6) is valid.

(ii) $b_{i_0} = \infty$ for some $i_0 \in I$ and $a_{i_0} > 0$.

In this case, $b_i \le a_{i_0}$ for all $i \in I \setminus \{i_0\}$. Note that $(f'|f|^{p-2}f)(a_{i_0}) = 0$. Similar arguments as to those in Case (i) yield that

$$\int_0^{a_{i_0}} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x = -(f'|f|^{p-2}f)(0) - (p-1)\int_0^{a_{i_0}} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x. \quad (4.11)$$

Recall that there is a sequence $\{x_n : n \ge 1\}$ satisfying (4.8). In analogy with (4.9) we obtain

$$\begin{aligned}
&\int_{a_{i_0}}^{\infty} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x \\
&= \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} \int_{a_{i_0}+\varepsilon}^{x_n} f''(x)|f(x)|^{p-2}f(x)\mathrm{d}x \\
&= \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} \Big((f'|f|^{p-2}f)(x_n) - (f'|f|^{p-2}f)(a_{i_0}+\varepsilon) - (p-1)\int_{a_{i_0}}^{x_n} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x\Big) \\
&= -(p-1)\int_{a_{i_0}}^{\infty} |f'(x)|^2|f(x)|^{p-2}\mathrm{d}x. \qquad (4.12)
\end{aligned}$$

Now (4.6) follows from (4.11) and (4.12).

(iii) $E = (0, \infty)$.

Similar arguments as in (4.12) yield (4.6). This completes the proof. □

Proof of Theorem 4.1. First, we assume that $s \notin \mathbb{Z}$. We have the following sequence of steps.

(S1) We prove that

$$\|\partial_x^\alpha u(t,\cdot)\|_{L^{p_{|\alpha|}}} \le \|\partial^\alpha u_0\|_{L^{p_{|\alpha|}}}, \qquad |\alpha| \le \lfloor s \rfloor, t > 0. \quad (4.13)$$

Without loss of generality, we assume that u and u_0 are real functions. Since $\partial_t u = \Delta_x u$, we have

$$\int_{\mathbb{R}^N} (\partial_t u(t,x))|u(t,x)|^{p_0-2}u(t,x)\mathrm{d}x = \int_{\mathbb{R}^N} (\Delta_x u(t,x))|u(t,x)|^{p_0-2}u(t,x)\mathrm{d}x. \quad (4.14)$$

From (4.1) we see that for fixed $t > 0$ and $0 < \varepsilon < t/2$, all of $(u(t+\varepsilon,x) - u(t,x))/\varepsilon$, $u(t+\varepsilon,x)$ and $u(t,x)$ are bounded by some function $F \in L^{p_0}(\mathbb{R}^N)$, which is independent of ε. Applying the inequality

$$\Big||a|^{p_0} - |b|^{p_0}\Big| \le C_{p_0}|a-b|(|a|^{p_0-1} + |b|^{p_0-1}),$$

we get

$$\begin{aligned}
\left|\frac{|u(t+\varepsilon,x)|^{p_0} - |u(t,x)|^{p_0}}{\varepsilon}\right| &\le C_{p_0}\frac{|u(t+\varepsilon,x) - u(t,x)|}{\varepsilon} \cdot (|u(t+\varepsilon,x)|^{p_0-1} + |u(t,x)|^{p_0-1}) \\
&\le C_{p_0}|F(x)|^{p_0}.
\end{aligned}$$

Note that $\partial_t|u(t,x)|^{p_0} = p_0|u(t,x)|^{p_0-2}u(t,x)\partial_t u(t,x)$. We see from the Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}^N} (\partial_t u(t,x))|u(t,x)|^{p_0-2}u(t,x)\mathrm{d}x = \frac{1}{p_0}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_{L^{p_0}}^{p_0}.$$

On the other hand, applying Lemma 4.2 yields that

$$\int_{\mathbb{R}^N}(\Delta_x u(t,x))|u(t,x)|^{p_0-2}u(t,x)\mathrm{d}x = -(p_0-1)\int_{\mathbb{R}^N}|\nabla_x u(t,x)|^2|u(t,x)|^{p_0-2}\mathrm{d}x.$$

Now we see from (4.14) that

$$\frac{1}{p_0}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_{L^{p_0}}^{p_0} + (p_0-1)\int_{\mathbb{R}^N}|\nabla_x u(t,x)|^2|u(t,x)|^{p_0-2}\mathrm{d}x = 0.$$

Integrating with respect to t, we obtain

$$\frac{1}{p_0}\|u(t,\cdot)\|_{L^{p_0}}^{p_0} + (p_0-1)\int_0^t\int_{\mathbb{R}^N}|\nabla_x u(\tau,x)|^2|u(\tau,x)|^{p_0-2}\mathrm{d}x\,\mathrm{d}\tau = \frac{1}{p_0}\|u_0\|_{L^{p_0}}^{p_0}. \tag{4.15}$$

Note that for any multi-index α with $|\alpha|\le\lfloor s\rfloor$, $\partial_x^\alpha u(t,x)$ meets the heat equation with initial data $\partial^\alpha u_0$. Replacing $\partial_x^\alpha u$ for u and $p_{|\alpha|}$ for p_0 in (4.15), respectively, we get when $|\alpha|\le\lfloor s\rfloor$,

$$\begin{aligned}&\frac{1}{p_{|\alpha|}}\|\partial_x^\alpha u(t,\cdot)\|_{L^{p_{|\alpha|}}}^{p_{|\alpha|}} + (p_{|\alpha|}-1)\int_0^t\int_{\mathbb{R}^N}|\nabla_x\partial_x^\alpha u(\tau,x)|^2|\partial_x^\alpha u(\tau,x)|^{p_{|\alpha|}-2}\mathrm{d}x\,\mathrm{d}\tau\\ &= \frac{1}{p_{|\alpha|}}\|\partial^\alpha u_0\|_{L^{p_{|\alpha|}}}^{p_{|\alpha|}}.\end{aligned} \tag{4.16}$$

Hence (4.13) is true.

(S2) We prove (4.2).

Set $g(t,a,x) := \partial_x^\alpha u(t,x+ae_j) - \partial_x^\alpha u(t,x)$, where $a\in\mathbb{R}$, $|\alpha|=\lfloor s\rfloor$ and $1\le j\le N$. We have

$$\partial_t g(t,a,x) - \Delta_x g(t,a,x) = 0.$$

Replacing $g(t,a,x)$ for $u(t,x)$ and $p_{\lceil s\rceil}$ for p_0 in (4.15), respectively, we get

$$\begin{aligned}&\frac{1}{p_{\lceil s\rceil}}\|g(t,a,\cdot)\|_{L^{p_{\lceil s\rceil}}}^{p_{\lceil s\rceil}} + (p_{\lceil s\rceil}-1)\int_0^t\int_{\mathbb{R}^N}|\nabla_x g(\tau,a,x)|^2|g(\tau,a,x)|^{p_{\lceil s\rceil}-2}\mathrm{d}\tau\,\mathrm{d}x\\ &= \frac{1}{p_{\lceil s\rceil}}\|g(0,a,\cdot)\|_{L^{p_{\lceil s\rceil}}}^{p_{\lceil s\rceil}},\quad |\alpha|=\lfloor s\rfloor.\end{aligned} \tag{4.17}$$

Recall that $\nu_s = s-\lfloor s\rfloor$. Multiplying both sides by $1/|a|^{1+\nu_s p_{\lceil s\rceil}}$ and integrating with respect to $a\in\mathbb{R}$ yields, for all $t>0$,

$$\left[\partial_x^\alpha u(t,\cdot)\right]_{W_{\nu_s}^{p_{\lceil s\rceil}}}^{p_{\lceil s\rceil}} \le C_{s,\vec{p},N}\left[\partial^\alpha u_0\right]_{W_{\nu_s}^{p_{\lceil s\rceil}}}^{p_{\lceil s\rceil}}, \tag{4.18}$$

where we applied Proposition 3.2. It follows from (4.13) and (4.18) that

$$\begin{aligned}\sup_{0\le t\le T}\|u(t,\cdot)\|_{W_s^{\vec{p}}} &= \sup_{0\le t\le T}\left(\sum_{|\alpha|\le\lfloor s\rfloor}\|\partial_x^\alpha u(t,\cdot)\|_{L^{p_{|\alpha|}}} + \sum_{|\alpha|=\lfloor s\rfloor}\left[\partial_x^\alpha u(t,\cdot)\right]_{W_{\nu_s}^{p_{\lceil s\rceil}}}\right)\\ &\le \sum_{|\alpha|\le\lfloor s\rfloor}\|\partial^\alpha u_0\|_{L^{p_{|\alpha|}}} + C_{s,\vec{p},N}\sum_{|\alpha|=\lfloor s\rfloor}\left[\partial^\alpha u_0\right]_{W_{\nu_s}^{p_{\lceil s\rceil}}}\\ &\le C'_{s,\vec{p},N}\|u_0\|_{W_s^{\vec{p}}},\end{aligned}$$

which proves (4.2).

(S3) We prove (4.3).

Taking derivatives on both sides of (4.1), we obtain

$$\partial_x^\alpha u(t,x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/(4t)} \partial^\alpha u_0(y)\mathrm{d}y, \qquad \forall t>0, x\in\mathbb{R}^N, |\alpha| = \lfloor s\rfloor.$$

It follows from Hölder's inequality that

$$|\partial_x^\alpha u(t,x)| \le \frac{C}{t^{N/(2p_{\lfloor s\rfloor})}} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}, \quad \forall x\in\mathbb{R}^N. \tag{4.19}$$

Since $p_{\lfloor s\rfloor} \le 2$, we have

$$|\partial_x^\alpha u(t,x)|^{p_{\lfloor s\rfloor}-2} \ge \Big(\frac{C}{t^{N/(2p_{\lfloor s\rfloor})}} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}\Big)^{p_{\lfloor s\rfloor}-2}, \quad \forall x\in\mathbb{R}^N.$$

As a consequence of (4.16) we obtain

$$\int_0^T \int_{\mathbb{R}^N} |\nabla_x \partial_x^\alpha u(t,x)|^2 \Big(\frac{C}{t^{N/(2p_{\lfloor s\rfloor})}} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}\Big)^{p_{\lfloor s\rfloor}-2} \mathrm{d}x\,\mathrm{d}t \le \frac{1}{p_{\lfloor s\rfloor}(p_{\lfloor s\rfloor}-1)} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}^{p_{\lfloor s\rfloor}}.$$

Hence,

$$\int_0^T t^{(2-p_{\lfloor s\rfloor})N/(2p_{\lfloor s\rfloor})} \|\nabla_x \partial_x^\alpha u(t,\cdot)\|_{L^2}^2 \mathrm{d}t \le C' \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}^2, \quad |\alpha| = \lfloor s\rfloor. \tag{4.20}$$

Therefore,

$$\int_0^T t^{(2-p_{\lfloor s\rfloor})\sigma} \|\nabla_x \partial_x^\alpha u(t,\cdot)\|_{L^2}^2 \mathrm{d}t \le C T^{1-p_{\lfloor s\rfloor}/2} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}^2, \quad |\alpha| = \lfloor s\rfloor.$$

Consequently,

$$\int_0^T t^{\varrho} \|\nabla_x \partial_x^\alpha u(t,\cdot)\|_{L^2}^2 \mathrm{d}t \le C T^{(p_{\lfloor s\rfloor}-p_s)\sigma} T^{1-p_{\lfloor s\rfloor}/2} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}^2, \quad |\alpha| = \lfloor s\rfloor. \tag{4.21}$$

Next we estimate $[\partial_x^{\alpha+\beta} u(t,\cdot)]_{W^2_{\nu_s}}$. Setting $t=T$ in (4.17), we get

$$\int_0^T \int_{\mathbb{R}^N} |\nabla_x g(t,a,x)|^2 |g(t,a,x)|^{p_{\lceil s\rceil}-2} \mathrm{d}t\,\mathrm{d}x \le \frac{1}{p_{\lceil s\rceil}(p_{\lceil s\rceil}-1)} \|g(0,a,\cdot)\|_{L^{p_{\lceil s\rceil}}}^{p_{\lceil s\rceil}}, \quad |\alpha| = \lfloor s\rfloor. \tag{4.22}$$

Thus,

$$\int_0^T \int_{\mathbb{R}^N} \frac{|\nabla_x g(t,a,x)|^2}{|a|^{1+2\nu_s}} \cdot \frac{|g(t,a,x)|^{p_{\lceil s\rceil}-2}}{|a|^{(p_{\lceil s\rceil}-2)\nu_s}} \mathrm{d}t\,\mathrm{d}x \le C \frac{\|g(0,a,\cdot)\|_{L^{p_{\lceil s\rceil}}}^{p_{\lceil s\rceil}}}{|a|^{1+p_{\lceil s\rceil}\nu_s}}, \quad |\alpha| = \lfloor s\rfloor. \tag{4.23}$$

If $|a|\ge 1$, we see from (4.19) that

$$\frac{|g(t,a,x)|}{|a|^{\nu_s}} \le \frac{C}{t^{N/(2p_{\lfloor s\rfloor})}} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}} \le \frac{CT^{1/2}}{t^{N/(2p_{\lfloor s\rfloor})+1/2}} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}, \quad \forall 0<t<T, x\in\mathbb{R}^N.$$

If $|a|<1$, we have

$$\frac{|g(t,a,x)|}{|a|^{\nu_s}} = \frac{1}{|a|^{\nu_s}} \Big| \int_0^a \partial_{x_j} \partial_x^\alpha u(t, x+\tau e_j)\mathrm{d}\tau \Big| \le \|\partial_{x_j}\partial_x^\alpha u(t,\cdot)\|_{L^\infty}.$$

By (4.1),

$$\partial_{x_j}\partial_x^\alpha u(t,x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} \frac{-(x_j-y)}{2t^{1/2}} e^{-|x-y|^2/(4t)} \partial^\alpha u_0(y)\mathrm{d}y.$$

Therefore, for $|a| < 1$,

$$\frac{|g(t,a,x)|}{|a|^{\nu_s}} \le \frac{C}{t^{N/(2p_{\lfloor s\rfloor})+1/2}} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}, \quad \forall t > 0, x \in \mathbb{R}^N.$$

Consequently

$$\frac{|g(t,a,x)|^{p_{\lceil s\rceil}-2}}{|a|^{(p_{\lceil s\rceil}-2)\nu_s}} \ge \left(\frac{C\max\{1,T^{1/2}\}}{t^{N/(2p_{\lfloor s\rfloor})+1/2}} \|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}\right)^{p_{\lceil s\rceil}-2}, \quad \forall 0 < t < T,\ a \in \mathbb{R},\ x \in \mathbb{R}^N.$$

It follows from (4.23) that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} t^{(2-p_{\lceil s\rceil})(N/(2p_{\lfloor s\rfloor})+1/2)} \frac{|\nabla_x g(t,a,x)|^2}{|a|^{1+2\nu_s}} \mathrm{d}t\,\mathrm{d}x \\
&\qquad \le \left(C\max\{1,T^{1/2}\}\|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}\right)^{2-p_{\lceil s\rceil}} \frac{\|g(0,a,\cdot)\|_{L^{p_{\lceil s\rceil}}}^{p_{\lceil s\rceil}}}{|a|^{1+p_{\lceil s\rceil}\nu_s}}, \quad |\alpha| = \lfloor s\rfloor.
\end{aligned}$$

Integrating with respect to $a \in \mathbb{R}$ yields

$$\begin{aligned}
&\int_0^T t^{(2-p_{\lceil s\rceil})(N/(2p_{\lfloor s\rfloor})+1/2)} \left[\nabla_x \partial_x^\alpha u(t,\cdot)\right]_{W^2_{\nu_s}}^2 \mathrm{d}t \\
&\qquad \le \left(C\max\{1,T^{1/2}\}\|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}\right)^{2-p_{\lceil s\rceil}} \left[\partial^\alpha u_0\right]_{W^{p_{\lceil s\rceil}}_{\nu_s}}^{p_{\lceil s\rceil}} \\
&\qquad \le C'\max\{1,T^{1-p_{\lceil s\rceil}/2}\}\|u_0\|^2_{W^{\vec{p}}_s}, \quad |\alpha| = \lfloor s\rfloor. \qquad (4.24)
\end{aligned}$$

Thus, we obtain

$$\int_0^T t^\varrho \left[\nabla_x \partial_x^\alpha u(t,\cdot)\right]_{W^2_{\nu_s}}^2 \mathrm{d}t \le C' T^{(p_{\lceil s\rceil}-p_s)\sigma} \max\{1,T^{1-p_{\lceil s\rceil}/2}\}\|u_0\|^2_{W^{\vec{p}}_s}, \quad |\alpha| = \lfloor s\rfloor. \quad (4.25)$$

On the other hand, applying (4.13) yields that

$$\int_0^T t^\varrho \|\partial_x^\alpha u(t,\cdot)\|^2_{L^{p_{|\alpha|}}} \mathrm{d}t \le T^{1+\varrho}\|\partial^\alpha u_0\|^2_{L^{p_{|\alpha|}}}, \quad \text{when } |\alpha| \le \lfloor s\rfloor. \quad (4.26)$$

Summing up (4.21), (4.25) and (4.26), we get

$$\int_0^T t^\varrho \|u(t,\cdot)\|^2_{W^{(\vec{r},2)}_{s+1}} \mathrm{d}t \le C(1 + T^{\max\{1+\varrho,1-p_{\lceil s\rceil}/2+|p_{\lceil s\rceil}-p_{\lfloor s\rfloor}|\sigma\}})\|u_0\|^2_{W^{\vec{p}}_s}.$$

Hence (4.3) is valid.

(S4) We prove (4.4).

Set $h(t,a,x) := \partial_x^{\alpha+\beta} u(t,x+ae_j) - \partial_x^{\alpha+\beta} u(t,x)$, where $|\alpha| = \lfloor s\rfloor$, $|\beta| = 1$ and $1 \le j \le N$. We have

$$\partial_t h(t,a,x) - \Delta_x h(t,a,x) = 0.$$

Taking the L^2_x inner product with $t^{1+\varrho} h(t,a,x)$, we get

$$\int_{\mathbb{R}^N} (\partial_t h(t,a,x)) t^{1+\varrho} h(t,a,x)\mathrm{d}x - \int_{\mathbb{R}^N} (\Delta_x h(t,a,x)) t^{1+\varrho} h(t,a,x)\mathrm{d}x = 0.$$

Hence

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(t^{1+\varrho}\|h(t,a,\cdot)\|^2_{L^2}) + t^{1+\varrho}\|\nabla_x h(t,a,\cdot)\|^2_{L^2} = \frac{1+\varrho}{2} t^\varrho \|h(t,a,\cdot)\|^2_{L^2}.$$

Integrating with respect to $t \in [0,T]$ yields

$$\frac{T^{1+\varrho}}{2}\|h(T,a,\cdot)\|_{L^2}^2 + \int_0^T t^{1+\varrho}\|\nabla_x h(t,a,\cdot)\|_{L^2}^2 \mathrm{d}t = \frac{1+\varrho}{2}\int_0^T t^{\varrho}\|h(t,a,\cdot)\|_{L^2}^2 \mathrm{d}t. \tag{4.27}$$

Consequently,

$$\begin{aligned}\frac{T^{1+\varrho}}{2}\Big[\partial_x^{\alpha+\beta}u(T,\cdot)\Big]_{W^2_{\nu_s}}^2 + \int_0^T t^{1+\varrho}\Big[\nabla_x\partial_x^{\alpha+\beta}u(t,\cdot)\Big]_{W^2_{\nu_s}}^2 \mathrm{d}t &= \frac{1+\varrho}{2}\int_0^T t^{1+\varrho}\Big[\partial_x^{\alpha+\beta}u(t,\cdot)\Big]_{W^2_{\nu_s}}^2 \mathrm{d}t \\ &\le C_{s,\vec{p},N}T^{1+(p_{\lceil s\rceil}-p_s)\sigma}\max\{1,T^{1-p_{\lceil s\rceil}/2}\}\|u_0\|_{W_s^{\vec{p}}}^2,\end{aligned}$$

applying (4.25). Hence for $|\alpha| = \lfloor s\rfloor$ and $|\beta| = 1$,

$$\int_0^T t^{1+\varrho}\Big[\nabla_x\partial_x^{\alpha+\beta}u(t,\cdot)\Big]_{W^2_{\nu_s}}^2 \mathrm{d}t \le C_{s,\vec{p},N}T^{1+(p_{\lceil s\rceil}-p_s)\sigma}\max\{1,T^{1-p_{\lceil s\rceil}/2}\}\|u_0\|_{W_s^{\vec{p}}}^2. \tag{4.28}$$

On the other hand, substituting $\partial_x^{\alpha+\beta}u(t,x)$ for $h(t,a,x)$ in (4.27), where $|\beta| = 1$ and $|\alpha| = \lfloor s\rfloor$, we get

$$\begin{aligned}\frac{T^{1+\varrho}}{2}\|\partial_x^{\alpha+\beta}u(T,\cdot)\|_{L^2}^2 + \int_0^T t^{1+\varrho}\|\nabla_x\partial_x^{\alpha+\beta}u(t,\cdot)\|_{L^2}^2 \mathrm{d}t &= \frac{1+\varrho}{2}\int_0^T t^{\varrho}\|\partial_x^{\alpha+\beta}u(t,\cdot)\|_{L^2}^2 \mathrm{d}t \\ &\le CT^{(p_{\lfloor s\rfloor}-p_s)\sigma}T^{1-p_{\lfloor s\rfloor}/2}\|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}^2,\end{aligned}$$

applying (4.21). Hence

$$\int_0^T t^{1+\varrho}\|\nabla_x\partial_x^{\alpha+\beta}u(t,\cdot)\|_{L^2}^2 \mathrm{d}t \le CT^{(p_{\lfloor s\rfloor}-p_s)\sigma}T^{1-p_{\lfloor s\rfloor}/2}\|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}^2. \tag{4.29}$$

Moreover, we see from (4.21) that when $|\alpha| = \lfloor s\rfloor$ and $|\beta| = 1$,

$$\int_0^T t^{1+\varrho}\|\partial_x^{\alpha+\beta}u(t,\cdot)\|_{L^2}^2 \mathrm{d}t \le CT^{1+(p_{\lfloor s\rfloor}-p_s)\sigma}T^{1-p_{\lfloor s\rfloor}/2}\|\partial^\alpha u_0\|_{L^{p_{\lfloor s\rfloor}}}^2. \tag{4.30}$$

When $|\alpha| \le \lfloor s\rfloor$, we apply (4.13) to obtain

$$\int_0^T t^{1+\varrho}\|\partial_x^{\alpha}u(t,\cdot)\|_{L^{p_{|\alpha|}}}^2 \mathrm{d}t \le T^{2+\varrho}\|\partial^\alpha u_0\|_{L^{p_{|\alpha|}}}^2. \tag{4.31}$$

Combining (4.28), (4.29), (4.30) and (4.31) we deduce

$$\int_0^T t^{1+\varrho}\|u(t,\cdot)\|_{W_{s+2}^{(\vec{r},2,2)}}^2 \mathrm{d}t \le C_{s,\vec{p},N}\left(T^{(p_{\lfloor s\rfloor}-p_s)\sigma+1-p_{\lfloor s\rfloor}/2} + T^{2+\varrho}\right)\|u_0\|_{W_s^{\vec{p}}}^2.$$

Hence (4.4) is true.

This completes the proof of (4.2), (4.3) and (4.4) in the case $s \notin \mathbb{Z}$.

When $s \in \mathbb{Z}$, we apply similar arguments; we only provide a sketch. First, (4.2) follows from (4.13). Then we get (4.3) by (4.13) and (4.20). Finally, (4.4) follows from (4.3) and (4.29). In both cases, (4.2), (4.3) and (4.4) are valid. It follows that for any $0 < q < 2/(2+\varrho)$,

$$\begin{aligned}\int_0^T \|u(t,\cdot)\|_{W_{s+2}^{(\vec{r},2,2)}}^q \mathrm{d}t &= \int_0^T t^{-q(1+\varrho)/2}t^{q(1+\varrho)/2}\|u(t,\cdot)\|_{W_{s+2}^{(\vec{r},2,2)}}^q \mathrm{d}t \\ &\le \Big(\int_0^T t^{-q(1+\varrho)/(2-q)}\mathrm{d}t\Big)^{1-q/2}\Big(\int_0^T t^{(1+\varrho)}\|u(t,\cdot)\|_{W_{s+2}^{(\vec{r},2,2)}}^2 \mathrm{d}t\Big)^{q/2} \\ &\le C_{s,\vec{p},N}T^{1-(1+\varrho/2)q}(1+T^{\theta_2})^{q/2}\|u_0\|_{W_s^{\vec{p}}}^q.\end{aligned}$$

This completes the proof. □

Remark 4.3.

(i) Whenever $p_{\lfloor s \rfloor} = p_{\lceil s \rceil} = 2$, we have $\varrho = 0$. In this case, the local estimate (4.5) is valid for all $0 < q < 1$, which coincides with [23, Lemma 2.1].

Moreover, (4.3), (4.4) and (4.5) now turn out to be

$$\int_0^T \|u(t,\cdot)\|^2_{W^{(\vec{p},2)}_{s+1}} \mathrm{d}t \le C_{s,\vec{p},N}(1+T)\|u_0\|^2_{W^{\vec{p}}_s},$$
$$\int_0^T t\|u(t,\cdot)\|^2_{W^{(\vec{p},2,2)}_{s+2}} \mathrm{d}t \le C_{s,\vec{p},N}(1+T^2)\|u_0\|^2_{W^{\vec{p}}_s},$$
$$\int_0^T \|u(t,\cdot)\|^q_{W^{(\vec{p},2,2)}_{s+2}} \mathrm{d}t \le C_{s,\vec{p},N}T^{1-q}(1+T^2)^{q/2}\|u_0\|^q_{W^{\vec{p}}_s}, \quad 0<q<1.$$

(ii) Theorem 4.1 extends the local estimates for initial data in classical Sobolev spaces H^s. For example, consider the initial data

$$u_0(x) = \frac{1}{(1+|x|^2)^{\delta/2}}.$$

When $\max\{0, N/2-1\} < \delta < N/2$ and $1<s<2$, $u_0 \in W_s^{(p_0,2,2)}$ for all $p_0 > N/\delta$. Now Theorem 4.1 gives local energy estimates for the heat equation with initial data u_0, while one has no local estimates with classical Sobolev spaces since $u_0 \notin L^2$.

4.2. **Convergence of Schrödinger Operators.** In this subsection, we study the convergence of Schrödinger operators.

Take some function φ such that $\hat{\varphi} \in C_c^\infty$, $0 \le \hat{\varphi}(\omega) \le 1$, $\hat{\varphi}(\omega) = 1$ for $|\omega| < 1$ and $\hat{\varphi}(\omega) = 0$ for $|\omega| > 2$. Set $\hat{f_1} = \hat{\varphi} \cdot \hat{f}$ and $f_2 = f - f_1$.

Note that $(f * \varphi)\hat{} = \hat{\varphi}\hat{f}$ (for a proof, see [37, Theorem 7.19]). Hence $f_1 = f * \varphi$ and $f_2 = f - f * \varphi$. Now we rewrite $e^{it(-\Delta)^{a/2}}$ as

$$e^{it(-\Delta)^{a/2}} f = e^{it(-\Delta)^{a/2}} f_1 + e^{it(-\Delta)^{a/2}} f_2. \tag{4.32}$$

If $f \in H^s$, then $\hat{f}$ is locally integrable. Hence the convergence for

$$\lim_{t\to 0} e^{it(-\Delta)^{a/2}} f_1(x) = \lim_{t \to 0} \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i(x\cdot\omega + t|\omega|^a)} \hat{f}_1(\omega) \mathrm{d}\omega$$

is obvious.

However, for $f \in W_s^{\vec{p}}$ with $p_0 > 2$, we do not know whether $\hat{f}$ is locally integrable. So we have to deal with the term $e^{it(-\Delta)^{a/2}} f_1$ with new method.

For the case $p_{\lceil s \rceil} = 2$, we show that if $e^{it(-\Delta)^{a/2}} f$ is convergent as t tends to zero for all functions in H^s for some $s > 0$, then the same is true for all functions in $W_s^{\vec{p}}$ with the same index s.

Theorem 4.4. *Let $s > 0$, $a > 1$, $\vec{p} = (p_0, \ldots, p_{\lceil s \rceil})$ with $p_{\lceil s \rceil} = 2$ and $1 \le p_l < \infty$ for all $0 \le l \le \lceil s \rceil - 1$. Suppose that for all functions $f \in H^s(\mathbb{R}^N)$,*

$$\lim_{t \to 0} e^{it(-\Delta)^{a/2}} f(x) = f(x), \quad a.e. \tag{4.33}$$

We have:

(i) *If $0 < s < N/2$, then (4.33) is valid for all functions $f \in W_s^{\vec{p}}$.*

(ii) *If $s \geq N/2$ and $a = 2$, then (4.33) is also true if we interpret the operator $e^{-it\Delta}$ as*
$$e^{-it\Delta} f(x) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i(x\cdot\omega + t|\omega|^2)} \Big(1 - \varphi_0(\frac{\omega}{\varepsilon})\Big) \hat{f}(\omega) \mathrm{d}\omega, \tag{4.34}$$
where $\varphi_0 \in C_c^\infty(\mathbb{R})$ satisfies that $\varphi_0(x) = 1$ for $|x| < 1$ and $\varphi_0(x) = 0$ for $|x| > 2$, the limit exists almost everywhere, and the limit is independent of the choice of φ_0 with the aforementioned properties.

For the case $1 < p_{\lceil s \rceil} < 2$ and $a = 2$, we have similar results.

Theorem 4.5. *Let $s > 0$, $\vec{p} = (p_0, \ldots, p_{\lceil s \rceil})$ with $1 < p_{\lceil s \rceil} < 2$ and $1 \leq p_l < \infty$ for all $0 \leq l \leq \lceil s \rceil - 1$. Let*
$$\beta_s = \begin{cases} s, & s \in \mathbb{Z}, \\ \lfloor s \rfloor + \nu_s(p_{\lceil s \rceil} - 1), & s \notin \mathbb{Z}. \end{cases} \tag{4.35}$$
We have:

(i) *If*
$$\frac{1}{p_{\lceil s \rceil}} - \frac{N}{2(N+1)} < \frac{\beta_s}{N} < \frac{1}{p_{\lceil s \rceil}}, \tag{4.36}$$
then for any $f \in W_s^{\vec{p}}$,
$$\lim_{t \to 0} e^{-it\Delta} f(x) = f(x), \quad a.e. \tag{4.37}$$

(ii) *If $\beta_s \geq N/p_{\lceil s \rceil}$, then (4.37) is also true if we interpret the operator $e^{-it\Delta}$ as in (4.34).*

Remark 4.6. *If $0 < s < 1$, $p_0 > N/\delta > p_1 = 2$ and $2(\delta + s) > N$, then the function f defined in Example 3.6 satisfies that $f \in W_s^{(p_0,2)} \setminus W_s^{(2,2)}$. That is, $W_s^{(p_0,2)}$ is not contained in $W_s^{(2,2)} = H^s$.*

For the case $p_0 > 2$, the Fourier transform of functions in $W_s^{\vec{p}}$ might be distributions. In this case, we show that $\hat{f}$ is the distributional limit of a sequence of locally integrable functions. Moreover, we prove that $\hat{f}$ is locally integrable if $\beta_s/N < 1/p_{\lceil s \rceil}$.

Lemma 4.7. *Suppose that $s > 0$ and $\vec{p} = (p_0, \ldots, p_{\lceil s \rceil})$ with $1 < p_{\lceil s \rceil} \leq 2$ and $1 \leq p_l < \infty$ for $0 \leq l \leq \lceil s \rceil - 1$. Let β_s be defined by (4.35). For any $f \in W_s^{\vec{p}}$, $\hat{f}$ coincides with a function λ in the domain $\mathbb{R}^N \setminus \{0\}$ such that $|x|^{\beta_s} \lambda(x) \in L^{p'_{\lceil s \rceil}}$ and $\hat{f} = \lim_{\varepsilon \to 0} (1 - \varphi_0(x/\varepsilon))\lambda(x)$ in the distributional sense, where $\varphi_0 \in C^\infty(\mathbb{R}^N)$ satisfies that $\varphi_0(x) = 1$ for $|x| < 1$ and $\varphi_0(x) = 0$ for $|x| > 2$.*

Consequently, if $\beta_s p_{\lceil s \rceil} < N$, then λ is locally integrable and $\hat{f} = \lambda$.

Proof. First, we show that there is some function λ such that $|x|^{\beta_s} \lambda(x) \in L^{p'_{\lceil s \rceil}}$ and
$$\langle \hat{f}, \varphi \rangle = \int_{\mathbb{R}^N} \lambda(x) \varphi(x) \mathrm{d}x, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{4.38}$$

There are three cases:

(A1) $0 < s < 1$.

In this case, $\vec{p} = (p_0, p_1)$. We see from Proposition 3.2 that
$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|f(x) - f(y)|^{p_1}}{|x - y|^{N + p_1 s}} \mathrm{d}x\, \mathrm{d}y < \infty$$

is equivalent to

$$\iint_{\mathbb{R}^N\times\mathbb{R}} \frac{|f(x)-f(x+ae_j)|^{p_1}}{|a|^{1+p_1 s}}\mathrm{d}x\,\mathrm{d}a<\infty, \qquad \forall 1\le j\le N. \tag{4.39}$$

Applying Fubini's theorem, we get that for almost all real numbers a, $f-f(\cdot+ae_j)\in L^{p_1}$. It is easy to see that $f(\cdot-ae_j)\hat{}=\phi_j(a\cdot)\hat{f}$ in the distributional sense, where

$$\phi_j(x)=e^{-ix_j}.$$

Hence

$$(f-f(\cdot+ae_j))\hat{}=(1-\phi_j(a\cdot))\hat{f}.$$

Since $f-f(\cdot+ae_j)\in L^{p_1}$ for almost all $a>0$, by the Hausdorff-Young inequality, there exist functions $g_a\in L^{p_1'}$ such that

$$(f-f(\cdot+ae_j))\hat{}=g_a.$$

Hence for any $\varphi\in C_c^\infty(B(0,1/a)\setminus\{0\})$,

$$\langle(1-\phi_j(a\cdot))\hat{f},\varphi\rangle=\int_{\mathbb{R}^N} g_a(x)\varphi(x)\mathrm{d}x.$$

Therefore,

$$\langle|1-\phi_j(a\cdot)|^2\hat{f},\varphi\rangle=\langle(1-\phi_j(a\cdot))\hat{f},(1-\phi_j(a\cdot))^*\varphi\rangle \;=\int_{\mathbb{R}^N}(1-\phi_j(ax))^* g_a(x)\varphi(x)\mathrm{d}x,$$

where z^* denotes the conjugate of a complex number z. It follows that

$$\Big\langle\sum_{j=1}^N|1-\phi_j(a\cdot)|^2\hat{f},\varphi\Big\rangle=\int_{\mathbb{R}^N}\sum_{j=1}^N(1-\phi_j(ax))^* g_a(x)\varphi(x)\mathrm{d}x,$$

Set

$$\lambda_a(x)=\frac{\sum_{j=1}^N(1-\phi_j(ax))^* g_a(x)}{\sum_{j=1}^N|1-\phi_j(ax)|^2}. \tag{4.40}$$

Note that $\sum_{j=1}^N|1-\phi_j(a\cdot)|^2$ has no zero in the area $0<|x|<1/a$. Hence

$$\frac{\varphi}{\sum_{j=1}^N|1-\phi_j(a\cdot)|^2}\in C_c^\infty(\mathbb{R}^N\setminus\{0\}).$$

It follows that

$$\Big\langle\sum_{j=1}^N|1-\phi_j(a\cdot)|^2\hat{f},\frac{\varphi}{\sum_{j=1}^N|1-\phi_j(a\cdot)|^2}\Big\rangle=\int_{\mathbb{R}^N}\lambda_a(x)\varphi(x)\mathrm{d}x.$$

On the other hand,

$$\Big\langle\sum_{j=1}^N|1-\phi_j(a\cdot)|^2\hat{f},\frac{\varphi}{\sum_{j=1}^N|1-\phi_j(a\cdot)|^2}\Big\rangle=\langle\hat{f},\varphi\rangle.$$

Hence

$$\langle\hat{f},\varphi\rangle=\int_{\mathbb{R}^N}\lambda_a(x)\varphi(x)\mathrm{d}x, \qquad \forall\varphi\in C_c^\infty(B(0,1/a)\setminus\{0\}). \tag{4.41}$$

That is, $\hat{f}$ coincides with a function λ_a in the domain $B(0,1/a)\setminus\{0\}$. By the uniqueness, we have

$$\lambda_a(x)=\lambda_{a'}(x), \qquad a'<a, 0<|x|<\frac{1}{a}.$$

Take a sequence $\{a_n : n \geq 1\}$ such that $a_n \to 0$. Then the limit

$$\lambda(x) := \lim_{n\to\infty} \lambda_{a_n}(x)$$

exists for almost all $x \neq 0$. Now we see from (4.41) that (4.38) is true.

Since $f - f(\cdot + ae_j) \in L^{p_1}$, we see from the above equation that for all $a \in \mathbb{R}$,

$$\Big(f - f(\cdot + ae_j)\Big)\hat{}\,(x) = \lambda(x)(1 - \phi_j(ax)), \qquad \forall \varepsilon < |x| < A.$$

Applying the Hausdorff Young inequality, we get

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}} \frac{|f(x) - f(x + ae_j)|^{p_1}}{|a|^{1+p_1 s}} \mathrm{d}x\,\mathrm{d}a \\
&\geq \iint_{\mathbb{R}^N\times\mathbb{R}} \frac{|(f - f(\cdot + ae_j))\hat{}\,(x)|^{p_1'}}{|a|^{1+p_1 s}} \mathrm{d}x\,\mathrm{d}a \\
&= \lim_{\substack{\varepsilon\to 0\\ A\to\infty}} \int_{\varepsilon<|x|<A} \int_{\mathbb{R}} \frac{|(f - f(\cdot + ae_j))\hat{}\,(x)|^{p_1'}}{|a|^{1+p_1 s}} \mathrm{d}a\,\mathrm{d}x \\
&= \lim_{\substack{\varepsilon\to 0\\ A\to\infty}} \int_{\varepsilon<|x|<A} \int_{\mathbb{R}} \frac{|\lambda(x)(1 - \phi_j(ax))|^{p_1'}}{|a|^{1+p_1 s}} \mathrm{d}a\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}} \frac{|\lambda(x)|^{p_1'} |2\sin(ax_j/2)|^{p_1'}}{|a|^{1+p_1 s}} \mathrm{d}a\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |x_j|^{p_1 s} |\lambda(x)|^{p_1'} \int_{\mathbb{R}} \frac{|2\sin(a/2)|^{p_1'}}{|a|^{1+p_1 s}} \mathrm{d}a\,\mathrm{d}x \\
&= C_{s,j} \int_{\mathbb{R}^N} |x_j|^{p_1 s} |\lambda(x)|^{p_1'} \mathrm{d}x. \qquad (4.42)
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^N} |x_j|^{p_1 s} |\lambda(x)|^{p_1'} \mathrm{d}x \leq \frac{1}{C_{s,j}} \iint_{\mathbb{R}^N\times\mathbb{R}} \frac{|f(x) - f(x + ae_j)|^{p_1}}{|a|^{1+p_1 s}} \mathrm{d}x\,\mathrm{d}a < \infty. \qquad (4.43)$$

That is, $|x_j|^{s(p_1-1)}\lambda(x) \in L^{p_1'}$ for all $1 \leq j \leq N$. Consequently, $|x|^{s(p_1-1)}\lambda(x) \in L^{p_1'}$.

(A2) $s = k \geq 1$ is an integer.

For all multi-index α with $|\alpha| = k$, let h_α be the Fourier transform of $D^\alpha f$ and $\psi_\alpha(x) = (ix)^\alpha$. Then $h_\alpha = \psi_\alpha \hat{f}$ is a function in $L^{p_k'}$. Consequently,

$$|\omega|^s |\hat{f}(\omega)| \approx \sum_{|\alpha|=k} |h_\alpha(\omega)| \in L^{p_k'}.$$

For any $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi(x) = 0$ in a neighbourhood of 0, since $h_\alpha = \psi_\alpha \hat{f}$ is a function in $L^{p_k'}$, we have

$$\langle \psi_\alpha \hat{f}, \varphi \rangle = \int_{\mathbb{R}^N} h_\alpha(x)\varphi(x)\mathrm{d}x.$$

Hence

$$\langle \psi_\alpha^2 \hat{f}, \varphi \rangle = \langle \psi_\alpha \hat{f}, \psi_\alpha \varphi \rangle = \int_{\mathbb{R}^N} h_\alpha(x)\psi_\alpha(x)\varphi(x)\mathrm{d}x.$$

Therefore,

$$\Big\langle \sum_{|\alpha|=k} \psi_\alpha^2 \hat{f}, \varphi \Big\rangle = \int_{\mathbb{R}^N} \sum_{|\alpha|=k} h_\alpha(x)\psi_\alpha(x)\varphi(x)\mathrm{d}x.$$

Note that $\sum_{|\alpha|=k} \psi_\alpha^2(x)$ has no zero other than $x = 0$. Substituting $\varphi/\sum_{|\alpha|=k} \psi_\alpha^2$ for φ in the above equation, we get

$$\langle \hat{f}, \varphi \rangle = \Big\langle \sum_{|\alpha|=k} \psi_\alpha^2 \hat{f}, \frac{\varphi}{\sum_{|\alpha|=k} \psi_\alpha^2} \Big\rangle = \int_{\mathbb{R}^N} \frac{\sum_{|\alpha|=k} h_\alpha(x)\psi_\alpha(x)}{\sum_{|\alpha|=k} \psi_\alpha(x)^2} \varphi(x)\mathrm{d}x.$$

Let

$$\lambda(x) = \frac{\sum_{|\alpha|=k} h_\alpha(x)\psi_\alpha(x)}{\sum_{|\alpha|=k} \psi_\alpha(x)^2}.$$

We get (4.38) and $|x|^s|\lambda(x)| \lesssim \sum_\alpha |h_\alpha(x)| \in L^{p'_{\lceil s\rceil}}$.

(A3) $s > 1$ is not an integer.

For any multi-index α with $|\alpha| = \lfloor s \rfloor$, set $\psi_\alpha(x) = (ix)^\alpha$. Substituting $(p_{\lceil s\rceil-1}, p_{\lceil s\rceil})$, ν_s and $D^\alpha f$ for (p_0, p_1), s and f respectively in (A1), we get $|\omega|^{\nu_s(p_{\lceil s\rceil}-1)}(D^\alpha f)\hat{}(\omega) \in L^{p'_{\lceil s\rceil}}$ and there exists a function λ_α such that $|\omega|^{\nu_s(p_{\lceil s\rceil}-1)}\lambda_\alpha(\omega) \in L^{p'_{\lceil s\rceil}}$ and for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi(x) = 0$ in a neighbourhood of 0,

$$\langle \psi_\alpha \hat{f}, \varphi \rangle = \langle (D^\alpha f)\hat{}, \varphi \rangle = \int_{\mathbb{R}^N} \lambda_\alpha(x)\varphi(x)\mathrm{d}x.$$

With similar arguments as in the previous case, we get

$$\langle \hat{f}, \varphi \rangle = \int_{\mathbb{R}^N} \frac{\sum_{|\alpha|=\lfloor s\rfloor} \lambda_\alpha(x)\psi_\alpha(x)}{\sum_{|\alpha|=\lfloor s\rfloor} \psi_\alpha(x)^2} \varphi(x)\mathrm{d}x.$$

Let

$$\lambda(x) = \frac{\sum_{|\alpha|=\lfloor s\rfloor} \lambda_\alpha(x)\psi_\alpha(x)}{\sum_{|\alpha|=\lfloor s\rfloor} \psi_\alpha(x)^2}.$$

We get (4.38) and $|x|^{\lfloor s\rfloor+\nu_s(p_{\lceil s\rceil}-1)}\lambda(x)| \lesssim \sum_\alpha |x|^{\nu_s(p_{\lceil s\rceil}-1)}|\lambda_\alpha(x)| \in L^{p'_{\lceil s\rceil}}$.

Recall that C_c^∞ is dense in $\mathscr{S}$. For any $\varphi \in \mathscr{S}$ with $\varphi(x) = 0$ in a neighbourhood of 0, we see from (4.38) that

$$\langle \hat{f}, \varphi \rangle = \int_{\mathbb{R}^N} \lambda(x)\varphi(x)\mathrm{d}x. \tag{4.44}$$

Take some $\varphi_0 \in C^\infty(\mathbb{R}^N)$ such that $\varphi_0(x) = 1$ for $|x| < 1$ and $\varphi_0(x) = 0$ for $|x| > 2$. We conclude that

$$\lim_{\varepsilon\to 0} \varphi_0\big(\frac{\cdot}{\varepsilon}\big)\hat{f} = 0, \qquad \text{in} \quad \mathscr{S}'.$$

It suffices to show that for any $\varphi \in \mathscr{S}$,

$$\lim_{\varepsilon\to 0} \langle \varphi_0\big(\frac{\cdot}{\varepsilon}\big)\hat{f}, \varphi \rangle = 0. \tag{4.45}$$

In fact,

$$\begin{aligned} |\langle \varphi_0\big(\frac{\cdot}{\varepsilon}\big)\hat{f}, \varphi \rangle| &= |\langle \hat{f}, \varphi_0\big(\frac{\cdot}{\varepsilon}\big)\varphi \rangle| \\ &= \Big|\Big\langle f, \Big(\varphi_0\big(\frac{\cdot}{\varepsilon}\big)\varphi\Big)\hat{} \Big\rangle\Big| \end{aligned}$$

$$
\begin{aligned}
&= \left|\left\langle f, \varepsilon^N \hat{\varphi}_0(\varepsilon\cdot) * \hat{\varphi}\right\rangle\right| \\
&\le \|f\|_{L^{p_0}} \cdot \|\varepsilon^N \hat{\varphi}_0(\varepsilon\cdot) * \hat{\varphi}\|_{L^{p_0'}} \\
&\le \|f\|_{L^{p_0}} \cdot \|\varepsilon^N \hat{\varphi}_0(\varepsilon\cdot)\|_{L^{p_0'}} \cdot \|\hat{\varphi}\|_{L^1} \\
&\to 0.
\end{aligned}
$$

Hence (4.45) is true. Consequently,

$$
\hat{f} = \lim_{\varepsilon\to 0}\Big(1-\varphi_0\big(\frac{\cdot}{\varepsilon}\big)\Big)\hat{f} \qquad \text{in} \quad \mathscr{S}'.
$$

It follows that for any $\varphi \in \mathscr{S}$,

$$
\begin{aligned}
\langle \hat{f}, \varphi\rangle &= \lim_{\varepsilon\to 0}\Big\langle \Big(1-\varphi_0\big(\frac{\cdot}{\varepsilon}\big)\Big)\hat{f}, \varphi\Big\rangle \\
&= \lim_{\varepsilon\to 0}\Big\langle \hat{f}, \Big(1-\varphi_0\big(\frac{\cdot}{\varepsilon}\big)\Big)\varphi\Big\rangle \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \lambda(x)\Big(1-\varphi_0\big(\frac{x}{\varepsilon}\big)\Big)\varphi(x)\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \lambda(x)\Big(1-\varphi_0\big(\frac{x}{\varepsilon}\big)\Big)\varphi(x)\mathrm{d}x. \qquad (4.46)
\end{aligned}
$$

Hence $\hat{f} = \lim_{\varepsilon\to 0}(1-\varphi_0(x/\varepsilon))\lambda(x)$ in distributional sense.

Recall that $|x|^{\beta_s}\lambda(x) \in L^{p'_{\lceil s\rceil}}$. If $\beta_s p_{\lceil s\rceil} < N$, we see from Hölder's inequality that $\lambda(x) = |x|^{-\beta_s}|x|^{\beta_s}\lambda(x) \in L^1(B(0,R))$ for any $R > 0$. Hence $\lambda(x)$ is locally integrable. This completes the proof. □

We are now ready to prove the main results.

Proof of Theorem 4.4. Fix some $f \in W_s^{\vec{p}}$. By Lemma 4.7, there is some $h \in L^2$ such that $\hat{f}(\omega) = h(\omega)/|\omega|^s$.

Take some function φ such that $\hat{\varphi} \in C_c^\infty$, $0 \le \hat{\varphi}(\omega) \le 1$, $\hat{\varphi}(\omega) = 1$ for $|\omega| < 1$ and $\hat{\varphi}(\omega) = 0$ for $|\omega| > 2$. Set $\hat{f}_1 = \hat{\varphi}\cdot\hat{f}$ and $f_2 = f - f_1$. Then the decomposition (4.32) is true.

First, we consider the case $s < N/2$. In this case, we have

$$
\hat{f}_1(\omega) = \frac{1}{|\omega|^s} 1_{\{|\omega|<2\}}(\omega)\cdot\varphi(\omega)h(\omega) \in L^1.
$$

It follows from the dominated convergence theorem that $e^{it(-\Delta)^{a/2}}f_1(x)$ tends to $f_1(x)$ as t tends to 0.

On the other hand, since $\hat{f}_2(\omega) = (1-\hat{\varphi}(\omega))\hat{f}(\omega) = 0$ for $|\omega| \le 1$, we have

$$
\|\hat{f}_2\|_{L^2} \le \||\omega|^s \hat{f}_2(\omega)\|_{L^2} \le \||\omega|^s \hat{f}(\omega)\|_{L^2} < \infty.
$$

Hence $f_2 \in H^s$. By the hypothesis, we get $\lim_{t\to 0} e^{it(-\Delta)^{a/2}}f_2(x) = f_2(x)$, a.e.

Next we consider the case $s \ge N/2$ and $a = 2$.

Set $\varphi_0 = \varphi$. Since $1 - \varphi_0(\omega/\varepsilon) = 0$ for $|\omega| < \varepsilon$, we rewrite $e^{-it\Delta}f$ as

$$
e^{-it\Delta}f(x) = \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^N}\int_{|\omega|>\varepsilon} e^{i(x\cdot\omega + t|\omega|^2)}\Big(1-\varphi_0\big(\frac{\omega}{\varepsilon}\big)\Big)\hat{f}(\omega)\mathrm{d}\omega. \qquad (4.47)
$$

Recall that $|\omega|^s\hat{f}(\omega) \in L^2$. Hence $\hat{f}\cdot 1_{\{|\omega|>\varepsilon\}} \in L^2$. Therefore, the integral in the above equation is well defined. Let us show that the limit in (4.47) exists almost everywhere. Observe that

$$\begin{aligned}&\int_{|\omega|>\varepsilon} e^{i(x\cdot\omega+t|\omega|^2)}\Big(1-\varphi_0\big(\frac{\omega}{\varepsilon}\big)\Big)\hat{f}(\omega)\mathrm{d}\omega\\&=\int_{|\omega|>\varepsilon} e^{i(x\cdot\omega+t|\omega|^2)}\Big(1-\varphi_0\big(\frac{\omega}{\varepsilon}\big)\Big)\hat{\varphi}(\omega)\hat{f}(\omega)\mathrm{d}\omega+\int_{|\omega|>1} e^{i(x\cdot\omega+t|\omega|^2)}\hat{f}_2(\omega)\mathrm{d}\omega.\end{aligned}$$

By (4.46), we have

$$\begin{aligned}e^{-it\Delta}f_1(x)&=\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^N}\int_{|\omega|>\varepsilon} e^{i(x\cdot\omega+t|\omega|^2)}\Big(1-\varphi_0\big(\frac{\omega}{\varepsilon}\big)\Big)\hat{\varphi}(\omega)\hat{f}(\omega)\mathrm{d}\omega\\&=\frac{1}{(2\pi)^N}\langle\hat{f}, e^{i(x\cdot\omega+t|\omega|^2)}\hat{\varphi}(\omega)\rangle.\end{aligned}$$

That is, the limit in (4.47) exists for almost all x.

Observe that $e^{i(x\cdot\omega+t|\omega|^2)}\hat{\varphi}(\omega)$ tends to $e^{ix\cdot\omega}\hat{\varphi}(\omega)$ in $\mathscr{S}$ as $t\to 0$. We get

$$\begin{aligned}\lim_{t\to 0} e^{-it\Delta}f_1(x)&=\lim_{t\to 0}\frac{1}{(2\pi)^N}\langle\hat{f}, e^{i(x\cdot\omega+t|\omega|^2)}\hat{\varphi}(\omega)\rangle\\&=\frac{1}{(2\pi)^N}\langle\hat{f}, e^{ix\cdot\omega}\hat{\varphi}(\omega)\rangle\\&=\langle f, \varphi(x-\cdot)\rangle\\&=f*\varphi(x).\end{aligned}$$

As in the previous case, $f_2\in H^s$. By the hypothesis, $\lim_{t\to 0} e^{it(-\Delta)^{a/2}}f_2(x)=f_2(x)$, a.e. Hence $\lim_{t\to 0} e^{it(-\Delta)^{a/2}}f(x)=f(x)$, a.e. This completes the proof. □

Proof of Theorem 4.5. Fix some $f\in W_s^{\vec{p}}$. By Lemma 4.7, there is some $h\in L^{p'_{\lceil s\rceil}}$ such that $\hat{f}(\omega)=h(\omega)/|\omega|^{\beta_s}$.

As in the proof of Theorem 4.4, we apply the decomposition (4.32) with $a=2$ for $e^{-it\Delta}f$.

First, we deal with $e^{-it\Delta}f_2(x)$. Since $\hat{f}_2(\omega)=(1-\hat{\varphi}(\omega))\hat{f}(\omega)=0$ for $|\omega|\le 1$, we have

$$\begin{aligned}\|\hat{f}_2\|_{L^2}&=\||\omega|^{-\beta_s}\cdot 1_{\{|\omega|>1\}}\cdot|\omega|^{\beta_s}\hat{f}(\omega)\|_{L^2}\\&\le\||\omega|^{-\beta_s}\cdot 1_{\{|\omega|>1\}}\|_{L^r}\cdot\||\omega|^{\beta_s}\hat{f}(\omega)\|_{L^{p'_{\lceil s\rceil}}}<\infty,\end{aligned}\tag{4.48}$$

where $r>1$ satisfying $1/r=1/2-1/p'_{\lceil s\rceil}=1/p_{\lceil s\rceil}-1/2<\beta_s/N$. More precisely, since

$$\frac{1}{p_{\lceil s\rceil}}-\frac{1}{2}+\frac{1}{2(N+1)}=\frac{1}{p_{\lceil s\rceil}}-\frac{N}{2(N+1)}<\frac{\beta_s}{N}.$$

We have

$$\frac{1}{r}<\frac{\beta_s-N/(2(N+1))}{N}.$$

Consequently, there is some $\tau>N/(2(N+1))$ such that

$$\frac{1}{r}<\frac{\beta_s-\tau}{N}.$$

Applying Hölder's inequality yields

$$\||\omega|^\tau \hat{f}(\omega)\|_{L^2} \leq \left\||\omega|^{-(\beta-\tau)} 1_{\{|\omega|>1\}}\right\|_{L^r} \cdot \left\||\omega|^{\beta_s} \hat{f}(\omega)\right\|_{L^{p'_{\lceil s\rceil}}} < \infty.$$

Thus $f_2 \in H^\tau$.

Recall that it was proved in [9, 18, 19] that $\lim_{t\to 0} e^{-it\Delta} f(x) = f(x)$, a.e., for all $f \in H^\tau$ with $\tau > N/(2(N+1))$. Hence

$$\lim_{t\to 0} e^{-it\Delta} f_2(x) = f_2(x), \quad a.e.$$

Next we deal with $e^{-it\Delta} f_1$. If $\beta_s < N/p_{\lceil s\rceil}$, then we have $\hat{f}_1 \in L^1$. It follows from the dominated convergence theorem that $e^{-it\Delta} f_1(x)$ tends to $f_1(x)$ as t tends to 0.

If $\beta_s \geq N/p_{\lceil s\rceil}$, employing the same arguments as in the case $s \geq N/2$ in the proof of Theorem 4.4, we obtain

$$\lim_{t\to 0} e^{-it\Delta} f_1(x) = f * \varphi(x).$$

Hence $\lim_{t\to 0} e^{-it\Delta} f(x) = f(x)$, a.e. This completes the proof. □

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