SHARP HARDY SPACE ESTIMATES FOR MULTIPLIERS

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Abstract. We provide an improvement of Calderón and Torchinsky's version [5] of the Hörmander multiplier theorem on Hardy spaces H^p $(0 < p < \infty)$, substituting the Sobolev space $L^2_s(A_0)$ by the Lorentz-Sobolev space $L_s^{\tau^{(s,p)},\min(1,p)}(A_0)$, where $\tau^{(s,p)} = \frac{n}{s-(n/\min(1,p)-n)}$ and A_0 is the annulus $\{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$. Our theorem also extends that of Grafakos and Slavíková [10] to the range $0 < p \leqslant 1$. Our result is sharp in the sense that the preceding Lorentz-Sobolev space cannot be replaced by a larger Lorentz-Sobolev space $L_s^{r,q}(A_0)$ with $r < \tau^{(s,p)}$ or $q > \min(1,p)$.

1. Introduction

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$. For the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ we use the definition $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}dx$ and denote by $f^\vee(\xi) := \widehat{f}(-\xi)$ the inverse Fourier transform of f. We also extend these transforms to the space of tempered distributions.

Given a bounded function σ on $\mathbb{R}^n$, the multiplier operator T_σ is defined as

$$T_\sigma f(x) := \int_{\mathbb{R}^n} \sigma(\xi)\widehat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $\langle x,\xi\rangle$ is the dot product of x and ξ in $\mathbb{R}^n$. The classical Mikhlin multiplier theorem [15] states that if a function σ, defined on $\mathbb{R}^n$, satisfies

$$\big|\partial_\xi^\alpha \sigma(\xi)\big| \lesssim_\alpha |\xi|^{-|\alpha|}, \qquad |\alpha| \leqslant \big[n/2\big] + 1,$$

then the operator T_σ admits a bounded extension in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In [13] Hörmander sharpened Mikhlin's result, using the weaker condition

$$\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^2_s(A_0)} < \infty \tag{1.1}$$

for $s > n/2$, where L^2_s denotes the standard L^2-based Sobolev space on $\mathbb{R}^n$, Ψ is a Schwartz function on $\mathbb{R}^n$ whose Fourier transform is supported in the annulus $A_0 = \{\xi : 1/2 < |\xi| < 2\}$ and satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$, $\xi \neq 0$. Calderón and Torchinsky [5] proved that if (1.1) holds for $s > n/p - n/2$, then σ is a Fourier multiplier of Hardy space $H^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$. A different proof was given by Taibleson and Weiss [22]. It turns out that the condition $s > n/\min(1,p) - n/2$ is optimal for boundedness to hold and it is natural to ask whether condition (1.1) can be

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weakened. Baernstein and Sawyer [1] obtained endpoint $H^p(\mathbb{R}^n)$ estimates by using Herz space conditions for $\big(\sigma(2^j\cdot)\widehat{\Psi}\big)^\vee$. An endpoint $H^1 - L^{1,2}$ estimate involving Besov space was given by Seeger [17, 18] and these estimates were improved and extended to Triebel-Lizorkin spaces by Seeger [19] and Park [16]. Grafakos, He, Honzík, and Nguyen [11] substituted $L^2_s(\mathbb{R}^n)$, $s > n/2$ in (1.1) by $L^r_s(\mathbb{R}^n)$, $s > n/r$, while Grafakos and Slavíková [10] recently improved this, replacing (1.1) by

$$\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s,1}_s(A_0)} < \infty$$

where $L^{n/s,1}_s$ is a Lorentz-type Sobolev space (defined in (1.2)).

Before stating our results, we recall the definition of Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ and Lorentz-Sobolev spaces $L^{p,q}_s(\mathbb{R}^n)$. For any measurable function f defined on $\mathbb{R}^n$, the decreasing rearrangement of f is defined by

$$f^*(t) := \inf\big\{s > 0 : d_f(s) \leqslant t\big\}, \qquad t > 0$$

where $d_f(s) := \big|\{x \in \mathbb{R}^n : |f(x)| > s\}\big|$. Here we adopt the convention that the infimum of the empty set is ∞. Then for $0 < p, q \leqslant \infty$ we define

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases} \Big(\displaystyle\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \frac{dt}{t}\Big)^{1/q}, & q < \infty \\ \displaystyle\sup_{t>0} t^{1/p} f^*(t), & q = \infty. \end{cases}$$

The set of all f with $\|f\|_{L^{p,q}(\mathbb{R}^n)} < \infty$ is called the Lorentz space and is denoted by $L^{p,q}(\mathbb{R}^n)$. For $s > 0$ let $(I - \Delta)^{s/2}$ be the inhomogeneous fractional Laplacian operator, defined by

$$(I - \Delta)^{s/2} f := \big((1 + 4\pi^2|\cdot|^2)^{s/2}\widehat{f}\big)^\vee.$$

Then for $0 < p, q \leqslant \infty$ and $s > 0$ let

$$\|f\|_{L^{p,q}_s(\mathbb{R}^n)} := \big\|(I - \Delta)^{s/2} f\big\|_{L^{p,q}(\mathbb{R}^n)}. \tag{1.2}$$

Theorem A. [10] Let $1 < p < \infty$ and $0 < s < n$ satisfy

$$s > \big|n/p - n/2\big|. \tag{1.3}$$

Then there exists $C > 0$ such that

$$\|T_\sigma f\|_{L^p(\mathbb{R}^n)} \leqslant C \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s,1}_s(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Moreover, a counterexample showing that condition (1.3) is optimal can be found in Slavíková [21]; this means that L^p boundedness could fail on the line $\big|n/p - n/2\big| = s$.

The purpose of this paper is to extend Theorem A to Hardy spaces $H^p(\mathbb{R}^n)$ for $0 < p < \infty$. Let Φ be a Schwartz function satisfying $\int_{\mathbb{R}^n} \Phi(x)dx = 1$ and $\mathrm{Supp}(\widehat{\Phi}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$, and $\Phi_k := 2^{kn}\Phi(2^k\cdot)$. We define $H^p(\mathbb{R}^n)$ to be the collection of all tempered distributions f satisfying

$$\|f\|_{H^p(\mathbb{R}^n)} := \big\|\sup_{k\in\mathbb{Z}} |\Phi_k * f|\big\|_{L^p(\mathbb{R}^n)} < \infty.$$

Let

$$\tau^{(s,p)} := \frac{n}{s - (n/\min(1,p) - n)}.$$

The first main result of this paper is the following:

Theorem 1.1. *Let $0 < p < \infty$ and $0 < s < n/\min(1,p)$ satisfy (1.3). Then there exists $C > 0$ such that*

$$\|T_\sigma f\|_{H^p(\mathbb{R}^n)} \leqslant C \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},\min(1,p)}(\mathbb{R}^n)} \|f\|_{H^p(\mathbb{R}^n)}. \tag{1.4}$$

The above theorem coincides with Theorem A if $1 < p < \infty$ because $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and so we mainly deal with the case $0 < p \leqslant 1$ in the paper. However, a complex interpolation argument between H^1- and L^2-boundedness yields the result for $1 < p < 2$; this recovers Theorem A by a duality argument, as our proof for $0 < p \leqslant 1$ is in fact independent of that in Theorem A. We will give a sketch of this in the appendix. Actually the construction of analytic family of operators and interpolation techniques are very similar to those used in [10].

Remark. As a result of Baernstein and Sawyer [1, Corollary 1 (Chapter 3)], for $0 < p < 1$ and $s \geqslant n/p - n/2$ we have

$$\|T_\sigma f\|_{H^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{B^{s,p}_{\tau^{(s,p)}}(\mathbb{R}^n)} \|f\|_{H^p(\mathbb{R}^n)} \tag{1.5}$$

where $\Psi_k := 2^{kn}\Psi(2^k\cdot)$ and $B^{s,q}_p(\mathbb{R}^n)$ is the Besov space with (quasi-)norms

$$\|g\|_{B^{s,q}_p(\mathbb{R}^n)} := \|\Phi * g\|_{L^p(\mathbb{R}^n)} + \Big(\sum_{k=1}^{\infty} 2^{skq}\big\|\Psi_k * g\big\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q}.$$

Then the case $0 < p < 1$ in (1.4) could be also obtained as a consequence of (1.5) and of the embedding

$$B^{s_0,p}_{\tau^{(s_0,p)}}(\mathbb{R}^n) \hookrightarrow L^{\tau^{s_1,p},p}_{s_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,p}_{\tau^{(s_2,p)}}(\mathbb{R}^n), \quad s_2 < s_1 < s_0 \ \text{ and } \ \tau^{(s_1,p)} > 1, \tag{1.6}$$

which follows from the recent generalization of the Franke-Jawerth embedding theorem for Triebel-Lizorkin-Lorentz spaces of Seeger and Trebels [20]. Conversely, our result also implies (1.5) for $s > n/p - n/2$ via the embedding (1.6) as Theorem 1.1 will be proved in a different way, based on the Littlewood-Paley theory for Hardy spaces and some inequalities in Lorentz spaces. We note that when $s = n/p - n/2$, (1.5) holds while (1.4) fails as mentioned below.

On the other hand, a certain weight condition is required in [1] when we extend (1.5) to H^1-boundedness. To be specific, we have

$$\|T_\sigma f\|_{H^1(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{B^{s,1}_{n/s}(\omega)} \|f\|_{H^1(\mathbb{R}^n)}, \quad s \geqslant n/2 \tag{1.7}$$

where $\{\omega(k)^{-1}\}_{k\in\mathbb{N}} \in \ell^2$ and

$$\|g\|_{B^{s,1}_{n/s}(\omega)} := \|\Phi * g\|_{L^{n/s}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} \omega(k) 2^{sk}\big\|\Psi_k * g\big\|_{L^{n/s}(\mathbb{R}^n)}.$$

However, a sharp endpoint H^1- boundedness holds by using Lorentz-Sobolev conditions without weights in Theorem 1.1. This, combined with the embedding (1.6), improves (1.7) by replacing $B^{s,1}_{n/s}(\omega)$ by $B^{s,1}_{n/s}$ for $s > n/2$. When $s = n/2$, the optimality of $\{\omega(k)^{-1}\}_{k\in\mathbb{N}} \in \ell^2$ for (1.7) remains open, but it is known in Park [16, Theorem 3.4] that $B^{n/2,1}_2(\omega)$ in (1.7) cannot be substituted by $B^{n/2,1}_2$.

We now turn our attention to the sharpness of Theorem 1.1. We point out that the example of Slavíková [21] is still applicable to the case $0 < p \leqslant 1$ with the dilation property $\|f(\epsilon\cdot)\|_{H^p(\mathbb{R}^n)} = \epsilon^{-n/p}\|f\|_{H^p(\mathbb{R}^n)}$, and therefore (1.3) is sharp in Theorem 1.1. We now consider the optimality of different parameters. Note that for $0 < r_1 < r_2 < \infty$ and $0 < q_1, q_2 \leqslant \infty$

$$\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{r_1,q_1}_s(\mathbb{R}^n)} \lesssim \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{r_2,q_2}_s(\mathbb{R}^n)} \quad \text{uniformly in } \; j,$$

which follows from the Hölder inequality with even integers s, complex interpolation technique, and a proper embedding theorem. Moreover, if $q_1 \geqslant q_2$, then the embedding $L^{r,q_2}_s(\mathbb{R}^n) \hookrightarrow L^{r,q_1}_s(\mathbb{R}^n)$ yields that

$$\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q_1}_s(\mathbb{R}^n)} \lesssim \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q_2}_s(\mathbb{R}^n)} \quad \text{uniformly in } \; j.$$

Consequently, we may replace $L^{\tau^{(s,p)},\min(1,p)}_s(\mathbb{R}^n)$ in Theorem 1.1 by $L^{r,q}_s(\mathbb{R}^n)$ for $r > \tau^{(s,p)}$ and $0 < q \leqslant \infty$, or by $L^{\tau^{(s,p)},q}_s(\mathbb{R}^n)$ for $0 < q < \min(1,p)$.

The second main result of this paper is the sharpness of the parameters $\tau^{(s,p)}$ and $\min(1,p)$. That is, Theorem 1.1 is sharp in the sense that $\tau^{(s,p)}$ cannot be replaced by any smaller number r, and if $r = \tau^{(s,p)}$, then $\min(1,p)$ cannot be replaced by any larger number q.

Theorem 1.2. *Let $0 < p < \infty$ and $|n/p - n/2| < s < n/\min(1,p)$.*

(1) For any $0 < r < \tau^{(s,p)}$ and $0 < q \leqslant \infty$, there exists a function σ that satisfies

$$\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q}_s(\mathbb{R}^n)} < \infty$$

such that T_σ is unbounded on $H^p(\mathbb{R}^n)$.

(2) For any $q > \min(1,p)$, there exists a function σ that satisfies

$$\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{\tau^{(s,p)},q}_s(\mathbb{R}^n)} < \infty$$

such that T_σ is unbounded on $H^p(\mathbb{R}^n)$.

The paper is organized as follows. Section 2 is dedicated to preliminaries, mostly extensions of inequalities in Lebesgue spaces to Lorentz spaces thanks to a real interpolation technique. We address the case $0 < p \leqslant 1$ of Theorem 1.1 in Section 3 and the proof of Theorem 1.2 is given in Section 4. In the appendix, a complex interpolation method is discussed whose purpose is to establish the L^p-boundedness for $1 < p < 2$.

2. Preliminaries

The Lorentz spaces are generalization of Lebesgue spaces, which occur as intermediate spaces for the real interpolation, so called K-method. For $0 < p, p_0, p_1 < \infty$, $0 < r \leqslant \infty$, and $0 < \theta < 1$ satisfying $p_0 \neq p_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$,

$$(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta,r} = L^{p,r}(\mathbb{R}^n). \tag{2.1}$$

This remains valid for vector-valued spaces. For $0 < p, p_0, p_1 < \infty$, $0 < q, r \leqslant \infty$, and $0 < \theta < 1$ satisfying $p_0 \neq p_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$,

$$\big(L^{p_0}(\ell^q), L^{p_1}(\ell^q)\big)_{\theta,r} = L^{p,r}(\ell^q), \quad \big(\ell^q(L^{p_0}), \ell^q(L^{p_1})\big)_{\theta,r} = \ell^q(L^{p,r}). \tag{2.2}$$

We remark that $\big((L^{p_0}(\ell^{q_0}), L^{p_1}(\ell^{q_1})\big)_{\theta,r} \neq L^{p,r}(\ell^q)$, $\big(\ell^{q_0}(L^{p_0}), \ell^{q_1}(L^{p_1})\big)_{\theta,r} \neq \ell^q(L^{p,r})$ for $q_0 \neq q_1$ with $1/q = (1-\theta)/q_0 + \theta/q_1$. See [2, 3, 6, 7] for more details.

Then many inequalities in Lebesgue spaces can be extended to Lorentz spaces from the following real interpolation method, which appears in [2, 3, 7, 12].

Proposition B. Let $\mathcal{A}$ and $\mathcal{B}$ be two topological vector spaces. Suppose (A_0, A_1) and (B_0, B_1) be couples of quasi-normed spaces continuously embedded into $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $0 < \theta < 1$ and $0 < r \leqslant \infty$. If T is a linear operator such that

$$T : A_0 \to B_0, \qquad T : A_1 \to B_1,$$

with the quasi-norms M_0 and M_1, respectively, then

$$T : (A_0, A_1)_{\theta,r} \to (B_0, B_1)_{\theta,r}$$

is also continuous, and for its quasi-norm we have

$$\|T\|_{(A_0,A_1)_{\theta,r} \to (B_0,B_1)_{\theta,r}} \leqslant M_0^{1-\theta} M_1^{\theta}.$$

As applications of Proposition B, we shall extend Young inequality, Hausdorff-Young inequality, Minkowski inequality, and Kato-Ponce type inequality into Lorentz spaces.

Lemma 2.1. *Let $1 < p \leqslant r < \infty$, $1 \leqslant q < r$, and $0 < t \leqslant \infty$ satisfy $1/r + 1 = 1/p + 1/q$. Then*

$$\|f * g\|_{L^{r,t}(\mathbb{R}^n)} \leqslant \|f\|_{L^{p,t}(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Proof. For a fixed $g \in \mathcal{S}(\mathbb{R}^n)$, we define the linear operator T_g by

$$T_g f := f * g.$$

Choose r_1, θ, and p_1 such that $r < r_1 < \infty$, $0 < \theta < 1$, $p < p_1 < \infty$, $1/r = (1-\theta)/q + \theta/r_1$, and $1/r_1 + 1 = 1/p_1 + 1/q$. Then note that $1/p = 1 - \theta + \theta/p_1$. By using Young inequality, we obtain that

$$\|T_g f\|_{L^q(\mathbb{R}^n)} \leqslant \|g\|_{L^q} \|f\|_{L^1(\mathbb{R}^n)}$$

and

$$\|T_g f\|_{L^{r_1}(\mathbb{R}^n)} \leqslant \|g\|_{L^q} \|f\|_{L^{p_1}(\mathbb{R}^n)}.$$

Then Proposition B with (2.1) completes the proof. □

Lemma 2.2. *Let $2 < p < \infty$ and $0 < r \leqslant \infty$. Then*

$$\|\widehat{f}\|_{L^{p,r}(\mathbb{R}^n)} \leqslant \|f\|_{L^{p',r}(\mathbb{R}^n)}$$

where $1/p + 1/p' = 1$.

Proof. It follows immediately from Hausdorff-Young inequality and Proposition B with (2.1). □

Lemma 2.3. *Let $1 < p < \infty$, $0 < r \leqslant \infty$, and $s > 0$. For any $\vartheta \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|\vartheta \cdot f\|_{L_s^{p,r}(\mathbb{R}^n)} \lesssim_{n,s,p,r,\vartheta} \|f\|_{L_s^{p,r}(\mathbb{R}^n)}. \tag{2.3}$$

Proof. Pick p_0, p_1 satisfying $1 < p_0 < p < p_1 < \infty$ and let T be the linear operator defined by

$$Tf := (I - \Delta)^{s/2}\big(\vartheta \cdot (I - \Delta)^{-s/2} f\big).$$

Then we apply the Kato-Ponce inequality [14] to obtain

$$\|Tf\|_{L^{p_j}} \lesssim \|f\|_{L^{p_j}} \quad \text{for} \quad j = 0, 1.$$

Then (2.3) follows from Proposition B and (2.1). □

Lemma 2.4. *Let $1 \leqslant q < p < \infty$ and $0 < r \leqslant \infty$. Then*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^q \Big)^{1/q} \Big\|_{L^{p,r}(\mathbb{R}^n)} \lesssim \Big(\sum_{k\in\mathbb{Z}} \|f_k\|_{L^{p,r}(\mathbb{R}^n)}^q \Big)^{1/q}$$

Proof. We select $p_1 > 0$ and $0 < \theta < 1$ so that $p < p_1 < \infty$ and $1/p = (1-\theta)/p_1 + \theta/q$. Using Minkowski inequality we write $\big\|\{f_k\}_{k\in\mathbb{Z}}\big\|_{L^{p_1}(\ell^q)} \lesssim \big\|\{f_k\}_{k\in\mathbb{Z}}\big\|_{\ell^q(L^{p_1})}$ and we interpolate this with $\big\|\{f_k\}_{k\in\mathbb{Z}}\big\|_{L^q(\ell^q)} = \big\|\{f_k\}_{k\in\mathbb{Z}}\big\|_{\ell^q(L^q)}$ to obtain

$$\big\|\{f_k\}_{k\in\mathbb{Z}}\big\|_{(L^{p_1}(\ell^q), L^q(\ell^q))_{\theta,r}} \lesssim \big\|\{f_k\}_{k\in\mathbb{Z}}\big\|_{(\ell^q(L^{p_1}, \ell^q(L^q)))_{\theta,r}}.$$

Then the proof is completed in view of (2.2). □

The next ingredient we need is Hölder's inequality in Lorentz spaces, which is an immediate consequence of the Hardy-Littlewood inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant \int_0^\infty f^*(t)g^*(t)dt$$

and Hölder's inequality for Lebesgue spaces.

Lemma 2.5. *Let $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Then*

$$\int_{\mathbb{R}^n} \big|f(x)g(x)\big|dx \leqslant \|f\|_{L^{p,q}(\mathbb{R}^n)} \|g\|_{L^{p',q'}(\mathbb{R}^n)}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$.

The following Lorentz space variant of the Sobolev embedding theorem can be easily obtained from the classical Sobolev embedding theorem combined with the Marcinkiewicz interpolation theorem; the proof is omitted.

Lemma 2.6. *Let $s_0, s_1 \in \mathbb{R}$, $1 < p_0, p_1 < \infty$, and $0 < r_0, r_1 \leqslant \infty$. Then the embedding*

$$L^{p_0,r_0}_{s_0}(\mathbb{R}^n) \hookrightarrow L^{p_1,r_1}_{s_1}(\mathbb{R}^n)$$

holds if $p_0 = p_1$, $s_0 \geqslant s_1$, $r_0 \leqslant r_1$, or if $s_0 - s_1 = n/p_0 - n/p_1 > 0$.

We remark that a generalization of the preceding lemma can be found in the recent work of Seeger and Trebels [20].

Finally, we describe the behavior of decreasing rearrangement of radial functions.

Lemma 2.7. *Suppose f is a radial function with $f(x) = g(|x|)$ for $x \in \mathbb{R}^n$. Then*

$$f^*(t) = g^*\big((t/\Omega_n)^{1/n}\big)$$

where Ω_n stands for the volume of the unit ball in $\mathbb{R}^n$.

Proof. We observe that

$$\begin{aligned} d_f(s) = \big|\big\{x \in \mathbb{R}^n : |f(x)| > s\big\}\big| &= \big|\big\{r\theta \in \mathbb{R}^n : |g(r)| > s, \theta \in \mathbb{S}^{n-1}\big\}\big| \\ &= \Omega_n\big|\big\{r > 0 : |g(r)| > s\big\}\big|^n \\ &= \Omega_n\big(d_g(s)\big)^n \end{aligned}$$

and this proves that

$$\begin{aligned} f^*(t) = \inf\big\{s > 0 : d_f(s) \leqslant t\big\} &= \inf\big\{s > 0 : \Omega_n\big(d_g(s)\big)^n \leqslant t\big\} \\ &= \inf\big\{s > 0 : d_g(s) \leqslant (t/\Omega_n)^{1/n}\big\} \\ &= g^*\big((t/\Omega_n)^{1/n}\big). \end{aligned}$$

□

3. Proof of Theorem 1.1

The set of Schwartz functions whose Fourier transform is compactly supported away from the origin is dense in $H^p(\mathbb{R}^n)$; this is a consequence of Littlewood-Paley theory for H^p as one can approximate $f \in H^p$ by

$$f^{(N)} := \sum_{k=-N}^{N} 2^{kn}\Psi(2^k\cdot) * f \to f \quad \text{ in } \; H^p(\mathbb{R}^n) \qquad \text{ as } \; N \to \infty.$$

See [24] for more details. Thus we may work with such Schwartz functions. Let f be a Schwartz function with compact support away from the origin in frequency space and suppose $\sigma \in L^\infty(\mathbb{R}^n)$ satisfies

$$\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{\tau^{(s,p)},p}_s(\mathbb{R}^n)} < \infty.$$

Let $\Lambda \in \mathcal{S}(\mathbb{R}^n)$ have the properties that $\mathrm{Supp}(\Lambda) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\}$ and $\int_{\mathbb{R}^n} \Lambda(\xi)d\xi = 1$. For $0 < \epsilon < 1/100$, we introduce

$$\sigma^\epsilon(\xi) := \sum_{j\in\mathbb{Z}} \big(\sigma\widehat{\Psi}(\cdot/2^j)\big) * \Lambda^{j,\epsilon}(\xi)$$

where $\Lambda^{j,\epsilon} := (2^j\epsilon)^{-n}\Lambda(\cdot/2^j\epsilon)$. Then since $\widehat{f}$ has compact support away from the origin,

$$T_{\sigma^\epsilon}f = \sum_{j\in\mathbb{Z}}\Big(\big[\big(\sigma\widehat{\Psi}(\cdot/2^j)\big)*\Lambda^{j,\epsilon}\big]\widehat{f}\Big)^{\vee}$$

is a finite sum and thus, using the argument of approximation of identity, for each $k\in\mathbb{Z}$

$$\lim_{\epsilon\to 0}\Phi_k*\big(T_{\sigma^\epsilon}f\big)(x) = \Phi_k*\big(T_\sigma f\big)(x).$$

This proves that

$$\big\|T_\sigma f\big\|_{H^p(\mathbb{R}^n)} \leqslant \big\|\liminf_{\epsilon\to 0}\sup_{k\in\mathbb{Z}}\big|\Phi_k*(T_{\sigma^\epsilon}f)\big|\big\|_{L^p(\mathbb{R}^n)} \leqslant \liminf_{\epsilon\to 0}\big\|T_{\sigma^\epsilon}f\big\|_{H^p(\mathbb{R}^n)}$$

where we applied Fatou's lemma in the last inequality. Therefore, it suffices to show that

$$\|T_{\sigma^\epsilon}f\|_{H^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}\|f\|_{H^p(\mathbb{R}^n)}, \qquad \text{uniformly in } \epsilon. \tag{3.1}$$

Now there exist a sequence of L^∞-atoms $\{a_l\}_{l=1}^\infty$ for $H^p(\mathbb{R}^n)$, and a sequence of scalars $\{\lambda_l\}_{l=1}^\infty$ so that

$$f = \sum_{l=1}^\infty \lambda_l a_l \qquad \text{in } \mathcal{S}'$$

and

$$\Big(\sum_{l=1}^\infty |\lambda_l|^p\Big)^{1/p} \approx \|f\|_{H^p(\mathbb{R}^n)},$$

where L^∞-atom a_l for $H^p(\mathbb{R}^n)$ means that there exists a cube Q_l such that a_l is supported in Q_l, $|a_l|\leqslant |Q_l|^{-1/p}$, and $\int_{\mathbb{R}^n}x^\gamma a_l(x)dx = 0$ for all multi-indices γ with $|\gamma|\leqslant [n/p-n]$.

We note that T_{σ^ϵ} maps $\mathcal{S}(\mathbb{R}^n)$ to itself, which implies that T_{σ^ϵ} is well-defined on $\mathcal{S}'(\mathbb{R}^n)$ using duality argument, and actually, $T_{\sigma^\epsilon}:\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$. This yields that

$$T_{\sigma^\epsilon}f = \sum_{l=1}^\infty \lambda_l(T_{\sigma^\epsilon}a_l) \qquad \text{in the sense of tempered distribution.}$$

Hence we have

$$\|T_{\sigma^\epsilon}f\|_{H^p(\mathbb{R}^n)} \leqslant \Big(\sum_{l=1}^\infty |\lambda_l|^p\big\|T_{\sigma^\epsilon}a_l\big\|_{H^p(\mathbb{R}^n)}^p\Big)^{1/p},$$

using subadditive property of $\|\cdot\|_{H^p(\mathbb{R}^n)}^p$.

Moreover, due to support assumptions and dilations, for each $j\in\mathbb{Z}$, we have

$$\sigma^\epsilon(2^j\xi)\widehat{\Psi}(\xi) = \sum_{l=j-2}^{j+2}\big(\sigma\widehat{\Psi}(\cdot/2^l)\big)*\Lambda^{l,\epsilon}(2^j\xi)\widehat{\Psi}(\xi) = \sum_{l=-2}^{2}\big(\sigma(2^j\cdot)\widehat{\Psi}(\cdot/2^l)\big)*\Lambda^{l,\epsilon}(\xi)\widehat{\Psi}(\xi),$$

from which it follows

$$\sup_{j\in\mathbb{Z}}\big\|\big(\sigma^{\epsilon}(2^j\cdot)\widehat{\Psi}\big)\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)} \lesssim \sum_{l=-2}^{2}\sup_{j\in\mathbb{Z}}\Big\|(I-\Delta)^{s/2}\Big(\big(\sigma(2^j\cdot)\widehat{\Psi}(\cdot/2^l)\big)*\Lambda^{l,\epsilon}\Big)\Big\|_{L^{\tau^{(s,p)},p}(\mathbb{R}^n)}$$

$$\lesssim \sum_{l=-2}^{2}\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}(\cdot/2^l)\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)} \leqslant \sum_{l=-2}^{2}C_l\sup_{j\in\mathbb{Z}}\big\|\sigma(2^{j+l}\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}$$

$$\lesssim \sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}$$

uniformly in ϵ; here we applied Lemmas 2.3 and 2.1 combined with the fact that $\|\Lambda^{l,\epsilon}\|_{L^1(\mathbb{R}^n)}=\|\Lambda\|_{L^1(\mathbb{R}^n)}$.

Therefore, the proof of (3.1) is reduced to the following proposition.

Proposition 3.1. *Let $0<p\leqslant 1$ and a be a H^p-atom, associated with a cube Q in $\mathbb{R}^n$. Then we have*

$$\|T_\sigma a\|_{H^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}$$

where the constant in the inequality is independent of σ and a.

Proof. Introducing the function Θ satisfying $\widehat{\Theta}(\xi):=\widehat{\Psi}(\xi/2)+\widehat{\Psi}(\xi)+\widehat{\Psi}(2\xi)$ so that $\widehat{\Theta}=1$ on the support of $\widehat{\Psi}$, let $\mathcal{L}_j$ and $\mathcal{L}_j^{\Theta}$ be the Littlewood-Paley operators associated with Ψ and Θ, respectively. Let Q^* and Q^{**} denote the concentric dilates of Q with side length $10l(Q)$ and $100l(Q)$, respectively. Then we write

$$\|T_\sigma a\|_{H^p(\mathbb{R}^n)} \approx \Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{L}_jT_\sigma a|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim_p \Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{L}_jT_\sigma a|^2\Big)^{1/2}\Big\|_{L^p(Q^{**})}+\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{L}_jT_\sigma a|^2\Big)^{1/2}\Big\|_{L^p((Q^{**})^c)}.$$

In view of Hölder's inequality, the first part is controlled by

$$|Q^{**}|^{1/p-1/2}\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{L}_jT_\sigma a|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)} \lesssim_n |Q|^{1/p-1/2}\|T_\sigma a\|_{L^2(\mathbb{R}^n)}$$

and we see that

$$\|T_\sigma a\|_{L^2(\mathbb{R}^n)} \leqslant \|\sigma\|_{L^\infty(\mathbb{R}^n)}\|a\|_{L^2(\mathbb{R}^n)} \leqslant \sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^\infty(\mathbb{R}^n)}|Q|^{-(1/p-1/2)}.$$

Now using Lemma 2.5, 2.2, and 2.6 with $1<\tau^{(s,p)}<2$, we obtain

$$\begin{aligned}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^\infty(\mathbb{R}^n)} &\leqslant \big\|\big(\sigma(2^j\cdot)\widehat{\Psi}\big)^\vee\big\|_{L^1(\mathbb{R}^n)}\\ &\lesssim \big\|\big(1+4\pi^2|\cdot|^2\big)^{(s-(n/p-n))/2}\big(\sigma(2^j\cdot)\widehat{\Psi}\big)^\vee\big\|_{L^{(\tau^{(s,p)})',1}(\mathbb{R}^n)}\\ &\leqslant \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_{s-(n/p-n)}^{\tau^{(s,p)},1}(\mathbb{R}^n)} \lesssim \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)},\end{aligned}$$

which finishes the proof of

$$\Big\Vert \Big(\sum_{j\in\mathbb{Z}} |\mathcal{L}_j T_\sigma a|^2 \Big)^{1/2} \Big\Vert_{L^p(Q^{**})} \lesssim \sup_{j\in\mathbb{Z}} \Vert \sigma(2^j\cdot)\widehat{\Psi}\Vert_{L^{\tau^{(s,p)},p}_s(\mathbb{R}^n)}.$$

To verify

$$\Big\Vert \Big(\sum_{j\in\mathbb{Z}} |\mathcal{L}_j T_\sigma a|^2 \Big)^{1/2} \Big\Vert_{L^p((Q^{**})^c)} \lesssim \sup_{j\in\mathbb{Z}} \big\Vert \sigma(2^j\cdot)\widehat{\Psi}\big\Vert_{L^{\tau^{(s,p)},p}_s(\mathbb{R}^n)}, \tag{3.2}$$

we notice that $\mathcal{L}_j T_\sigma a(x)$ can be written as $\big(\sigma\widehat{\Psi}(\cdot/2^j)\big)^\vee \ast (\mathcal{L}_j^\Theta a)(x)$. We decompose the left-hand side of (3.2) to

$$\mathcal{I} := \Big\Vert \Big(\sum_{j:2^j l(Q)<1} \big| (\sigma\widehat{\Psi}(\cdot/2^j))^\vee \ast (\mathcal{L}_j^\Theta a)\big|^2 \Big)^{1/2} \Big\Vert_{L^p((Q^{**})^c)}$$

and

$$\mathcal{J} := \Big\Vert \Big(\sum_{j:2^j l(Q)\geqslant 1} \big| (\sigma\widehat{\Psi}(\cdot/2^j))^\vee \ast (\mathcal{L}_j^\Theta a)\big|^2 \Big)^{1/2} \Big\Vert_{L^p((Q^{**})^c)}.$$

In view of the embedding $\ell^p \hookrightarrow \ell^2$

$$\mathcal{I} \leqslant \Big(\sum_{j:2^j l(Q)<1} \big\Vert (\sigma\widehat{\Psi}(\cdot/2^j))^\vee \ast (\mathcal{L}_j^\Theta a)\big\Vert_{L^p(\mathbb{R}^n)}^p \Big)^{1/p}$$

and Bernstein's inequality, we obtain

$$\big\Vert (\sigma\widehat{\Psi}(\cdot/2^j))^\vee \ast (\mathcal{L}_j^\Theta a)\big\Vert_{L^p(\mathbb{R}^n)} \lesssim 2^{jn(1/p-1)} \big\Vert (\sigma\widehat{\Psi}(\cdot/2^j))^\vee\big\Vert_{L^p(\mathbb{R}^n)} \Vert \mathcal{L}_j^\Theta a\Vert_{L^p(\mathbb{R}^n)}.$$

Using dilation, Lemma 2.5 and 2.2, we have

$$\begin{aligned}
2^{jn(1/p-1)} \Vert (\sigma\widehat{\Psi}(\cdot/2^j))^\vee\Vert_{L^p(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n} \big| \big(\sigma(2^j\cdot)\widehat{\Psi}\big)^\vee(x)\big|^p dx \Big)^{1/p} \\
&\lesssim \Big\Vert \big| \big(1+4\pi^2|\cdot|^2\big)^{s/2} \big(\sigma(2^j\cdot)\widehat{\Psi}\big)^\vee \big|^p \Big\Vert_{L^{(n/(sp))',1}(\mathbb{R}^n)}^{1/p} \\
&= \Big\Vert \big(1+4\pi^2|\cdot|^2\big)^{s/2} \big(\sigma(2^j\cdot)\widehat{\Psi}\big)^\vee \Big\Vert_{L^{p(n/(sp))',p}(\mathbb{R}^n)} \\
&\leqslant \big\Vert \sigma(2^j\cdot)\widehat{\Psi}\big\Vert_{L^{\tau^{(s,p)},p}_s(\mathbb{R}^n)}
\end{aligned} \tag{3.3}$$

since $2 < p(n/(sp))' < \infty$ and $\tau^{(s,p)} = \big(p(n/(sp))'\big)'$. Moreover, for any $M > 0$

$$|\mathcal{L}_j^\Theta a(x)| \lesssim_M |Q|^{1-1/p} \big(2^j l(Q)\big)^{[n/p-n]+1} \frac{2^{jn}}{(1+2^j|x-c_Q|)^M},$$

using standard arguments in [9, Appendix B] with $2^j l(Q) < 1$ and the fact that

$$|a(x)| \lesssim_{n,M} |Q|^{-1/p} \frac{1}{\big(1+|x-c_Q|/l(Q)\big)^M}, \qquad \int_{\mathbb{R}^n} x^\alpha a(x)dx = 0 \ \text{ for } \ |\alpha| \leqslant [n/p-n],$$

$$\big|\partial^\alpha\big(2^{jn}\Psi(2^j\cdot)\big)(x)\big| \lesssim 2^{j|\alpha|} 2^{jn} \frac{1}{(1+2^j|x|)^M} \quad \text{for} \quad \alpha\in\mathbb{Z}^n$$

where c_Q denotes the center of Q. Selecting $M > n/p$, we have

$$\|\mathcal{L}_j a\|_{L^p} \lesssim \big(2^j l(Q)\big)^{[n/p]+1-n/p}$$

and thus

$$\begin{aligned}\mathcal{I} &\lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)} \Big(\sum_{j:2^j l(Q)<1} \big(2^j l(Q)\big)^{p([n/p]+1-n/p)}\Big)^{1/p} \\ &\lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)},\end{aligned}$$

since $[n/p]+1-n/p > 0$.

To estimate $\mathcal{J}$ we further separate into two terms

$$\mathcal{J}_1 := \Big\|\Big(\sum_{j:2^j l(Q)\geqslant 1} \big|\big(\sigma\widehat{\Psi}(\cdot/2^j)\big)^{\vee} * \big(\chi_{(Q^*)^c}\mathcal{L}_j^{\Theta} a\big)\big|^2\Big)^{1/2}\Big\|_{L^p((Q^{**})^c)}$$

and

$$\mathcal{J}_2 := \Big\|\Big(\sum_{j:2^j l(Q)\geqslant 1} \big|\big(\sigma\widehat{\Psi}(\cdot/2^j)\big)^{\vee} * \big(\chi_{Q^*}\mathcal{L}_j^{\Theta} a\big)\big|^2\Big)^{1/2}\Big\|_{L^p((Q^{**})^c)}.$$

Using the embedding $\ell^p \hookrightarrow \ell^2$, Bernstein inequality with

$$\big(\sigma\widehat{\Psi}(\cdot/2^j)\big)^{\vee} * \big(\chi_{(Q^*)^c}\mathcal{L}_j^{\Theta} a\big)(x) = \big(\sigma\widehat{\Psi}(\cdot/2^j)\big)^{\vee} * \big[\mathcal{L}_j^{\Theta}\big(\chi_{(Q^*)^c}\mathcal{L}_j^{\Theta} a\big)\big](x),$$

and the inequality (3.3), we have

$$\mathcal{J}_1 \lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)} \Big(\sum_{j:2^j l(Q)\geqslant 1} \big\|\mathcal{L}_j^{\Theta}\big(\chi_{(Q^*)^c}\mathcal{L}_j^{\Theta} a\big)\big\|_{L^p(\mathbb{R}^n)}^p\Big)^{1/p}.$$

We see that for $x \in (Q^*)^c$ and $M > n/p$

$$\begin{aligned}|\mathcal{L}_j^{\Theta} a(x)| &\lesssim_M |Q|^{-1/p} \int_{y\in Q} \frac{2^{jn}}{(1+2^j|x-y|)^{2M}}dy \lesssim_M |Q|^{-1/p}\frac{1}{(2^j|x-c_Q|)^M} \\ &\lesssim_M |Q|^{-1/p}(2^j l(Q))^{-M}\frac{1}{(1+|x-c_Q|/l(Q))^M}\end{aligned}$$

since $|x-y| \geqslant \frac{9}{10}|x-c_Q|$. Then

$$\begin{aligned}&\big\|\mathcal{L}_j^{\Theta}\big(\chi_{(Q^*)^c}\mathcal{L}_j^{\Theta} a\big)\big\|_{L^p(\mathbb{R}^n)} \\ &\lesssim |Q|^{-1/p}(2^j l(Q))^{-M}\Big[\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} |2^{jn}\Theta(2^j(x-y))|\frac{1}{(1+|x-c_Q|/l(Q))^M}dy\Big)^p dx\Big]^{1/p}.\end{aligned}$$

Standard manipulations with $2^j l(Q) \geqslant 1$ in [9, Appendix B] yield that the last expression is less than a constant times

$$|Q|^{-1/p}(2^j l(Q))^{-M}\Big(\int_{\mathbb{R}^n}\frac{1}{(1+|x-c_Q|/l(Q))^{Mp}}dx\Big)^{1/p} \lesssim (2^j l(Q))^{-M}.$$

Accordingly,

$$\mathcal{J}_1 \lesssim \sup_{j\in\mathbb{Z}} \big\| \sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)} \Big(\sum_{k:2^k l(Q)\geqslant 1} (2^k l(Q))^{-Mp}\Big)^{1/p} \lesssim \sup_{j\in\mathbb{Z}} \big\| \sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}.$$

We now consider $\mathcal{J}_2$. Choose $n/p-n/2<N<s$ so that $n/2<Np<sp<n$ and $2<p\big(n/(Np)\big)'<\infty$. For notational convenience we write

$$\mathcal{E}_j^N\sigma(x) := \big(1+4\pi^2(2^j|x|)^2\big)^{N/2}\big(\sigma\widehat{\Psi}(\cdot/2^j)\big)^{\vee}(x).$$

Observe that if $x\in(Q^{**})^c$ and $y\in Q^*$, then $|x-c_Q|\lesssim|x-y|$ and thus

$$|x-c_Q|^N\big|\big(\sigma\widehat{\Psi}(\cdot/2^j)\big)^{\vee} * \big(\chi_{Q^*}\mathcal{L}_j^{\Theta}a\big)(x)\big| \lesssim 2^{-jN}\big|\mathcal{E}_j^N\sigma\big| * \big|\chi_{Q^*}\mathcal{L}_j^{\Theta}a\big|(x).$$

This proves that $\mathcal{J}_2$ is less than a constant times

$$\begin{aligned}
&\Big\| \frac{1}{|x-c_Q|^N}\Big(\sum_{j:2^j l(Q)\geqslant 1} 2^{-2jN}\Big(\big|\mathcal{E}_j^N\sigma\big| * \big|\chi_{Q^*}\mathcal{L}_j^{\Theta}a\big|\Big)^2\Big)^{1/2}\Big\|_{L^p((Q^{**})^c)}\\
&\lesssim \Big\| \Big(\sum_{j:2^j l(Q)\geqslant 1} 2^{-2jN}\Big(\big|\mathcal{E}_j^N\sigma\big| * \big|\chi_{Q^*}\mathcal{L}_j^{\Theta}a\big|\Big)^2\Big)^{p/2}\Big\|^{1/p}_{L^{(n/(Np))',1}(\mathbb{R}^n)}\\
&= \Big\| \Big(\sum_{j:2^j l(Q)\geqslant 1} 2^{-2jN}\Big(\big|\mathcal{E}_j^N\sigma\big| * \big|\chi_{Q^*}\mathcal{L}_j^{\Theta}a\big|\Big)^2\Big)^{1/2}\Big\|_{L^{p(n/(Np))',p}(\mathbb{R}^n)},
\end{aligned}$$

where we made use of Lemma 2.5 with $n/(Np)>1$. Now using Lemma 2.4 with $p(n/(Np))'>2$, the preceding expression is dominated by a constant multiple of

$$\Big(\sum_{j:2^j l(Q)\geqslant 1} 2^{-2jN}\Big\|\big|\mathcal{E}_j^N\sigma\big| * \big|\chi_{Q^*}\mathcal{L}_j^{\Theta}a\big|\Big\|^2_{L^{p(n/(Np))',p}(\mathbb{R}^n)}\Big)^{1/2}$$

and Lemma 2.1 yields that

$$\Big\|\big|\mathcal{E}_j^N\sigma\big| * \big|\chi_{Q^*}\mathcal{L}_j^{\Theta}a\big|\Big\|_{L^{p(n/(Np))',p}(\mathbb{R}^n)} \lesssim \big\|\mathcal{E}_j^N\sigma\big\|_{L^{p(n/(Np))',p}(\mathbb{R}^n)}\|\mathcal{L}_j^{\Theta}a\|_{L^1(Q^*)}.$$

We see that

$$\begin{aligned}
\big\|\mathcal{E}_j^N\sigma\big\|_{L^{p(n/(Np))',p}(\mathbb{R}^n)} &\lesssim 2^{-j(n/p-n)}2^{jN}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_N^{\tau^{(N,p)},p}(\mathbb{R}^n)}\\
&\lesssim 2^{-j(n/p-n)}2^{jN}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}
\end{aligned}$$

by applying dilation, Lemma 2.2 with $(p(n/(Np))')'=\tau^{(N,p)}$, and Lemma 2.6 with $s>N$. Combining with the estimate $\|\mathcal{L}_j^{\Theta}a\|_{L^1(Q^*)}\lesssim|Q|^{1/2}\|\mathcal{L}_j^{\Theta}a\|_{L^2(\mathbb{R}^n)}$, we finally obtain

$$\begin{aligned}
\mathcal{J}_2 &\lesssim \sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}|Q|^{1/2}\Big(\sum_{j:2^j l(Q)\geqslant 1} 2^{-2j(n/p-n)}\|\mathcal{L}_j^{\Theta}a\|^2_{L^2(\mathbb{R}^n)}\Big)^{1/2}\\
&\lesssim \sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}|Q|^{1/p-1/2}\big\|\big\{\mathcal{L}_j^{\Theta}a\big\}_{j\in\mathbb{Z}}\big\|_{L^2(\ell^2)}\\
&\lesssim \sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{\tau^{(s,p)},p}(\mathbb{R}^n)}
\end{aligned}$$

because $\big\|\big\{\mathcal{L}_j^{\Theta}a\big\}_{j\in\mathbb{Z}}\big\|_{L^2(\ell^2)} \approx \|a\|_{L^2(\mathbb{R}^n)} \leqslant |Q|^{-1/p+1/2}$.

This concludes the proof of the proposition. □

4. Proof of Theorem 1.2

The construction of our counterexamples is based on the idea in [16] and the following lemma is crucial in the proof.

Lemma 4.1. *Let $0 < s, \gamma < \infty$ and define the function on $\mathbb{R}^n$*

$$\mathcal{H}^{(s,\gamma)}(x) := \frac{1}{(1+4\pi^2|x|^2)^{s/2}}\frac{1}{(1+\ln(1+4\pi^2|x|^2))^{\gamma/2}}. \tag{4.1}$$

Then $\widehat{\mathcal{H}^{(s,\gamma)}}$ is a positive radial function and there exist $c_{s,\gamma,n}, d_{s,\gamma,n} > 0$ such that

$$\widehat{\mathcal{H}^{(s,\gamma)}}(\xi) \leqslant c_{s,\gamma,n}e^{-|\xi|/2} \qquad \textit{when} \quad |\xi| \geqslant 1 \tag{4.2}$$

and

$$\frac{1}{d_{s,\gamma,n}} \leqslant \frac{\widehat{\mathcal{H}^{(s,\gamma)}}(\xi)}{\mathfrak{T}^{(s,\gamma)}(\xi)} \leqslant d_{s,\gamma,n} \qquad \textit{when} \quad |\xi| \leqslant 1$$

where

$$\mathfrak{T}^{(s,\gamma)}(\xi) := \begin{cases} |\xi|^{-(n-s)}(1+2\ln|\xi|^{-1})^{-\gamma/2} & \textit{for} \quad 0 < s < n \\ 1 & \textit{for} \quad s \geqslant n. \end{cases}$$

Proof. It is known that the Fourier transform of $(1+4\pi^2|x|^2)^{-s/2}$ is the Bessel potential $G_s(\xi)$. Recall that G_s is a postive radial function, $\|G_s\|_{L^1(\mathbb{R}^n)} = 1$, and there exist $C_{s,n}, D_{s,n} > 0$ such that

$$G_s(\xi) \leqslant C_{(s,n)}e^{-|\xi|/2} \quad \text{for} \quad |\xi| \geqslant 1, \tag{4.3}$$

and

$$\frac{1}{D_{(s,n)}} \leqslant \frac{G_s(\xi)}{\mathfrak{G}_s(\xi)} \leqslant D_{(s,n)} \quad \text{for} \quad |\xi| \leqslant 1 \tag{4.4}$$

where

$$\mathfrak{G}_s(\xi) := \begin{cases} |\xi|^{-(n-s)} & \text{for} \quad 0 < s < n \\ \ln(2|\xi|^{-1}) & \text{for} \quad s = n \\ 1 & \text{for} \quad s > n. \end{cases}$$

Here we note that for any $\epsilon > 0$

$$C_{(s,n)}, D_{(s,n)} \lesssim_{\epsilon,n} e^{\epsilon|s-n|}. \tag{4.5}$$

We refer to [9, Ch. 1.2.2] for more details.

Using the identity

$$A^{-\gamma/2} = \frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-tA}t^{\gamma/2}\frac{dt}{t},$$

which is valid for $A > 0$, we write

$$\big(1+\log(1+4\pi^2|x|^2)\big)^{-\gamma/2} = \frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}e^{-t\log(1+4\pi^2|x|^2)}t^{\gamma/2}\frac{dt}{t}$$
$$= \frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}\frac{1}{(1+4\pi^2|x|^2)^t}t^{\gamma/2}\frac{dt}{t}.$$

We obtain from this that the Fourier transform of $\big(1+\log(1+4\pi^2|x|^2)\big)^{-\gamma/2}$ is

$$\frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}G_{2t}(\xi)t^{\gamma/2}\frac{dt}{t}$$

and consequently,

$$\widehat{\mathcal{H}^{(s,\gamma)}}(\xi) = G_s * \Big(\frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}G_{2t}(\cdot)t^{\gamma/2}\frac{dt}{t}\Big)(\xi) = \frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}G_{2t+s}(\xi)t^{\gamma/2}\frac{dt}{t}.$$

Clearly, $\widehat{\mathcal{H}^{(s,\gamma)}}$ is a positive radial function since so is G_{2t+s}.

Suppose $|\xi| \geqslant 1$. Then using (4.3) and (4.5) with $0 < \epsilon < 1/100$,

$$|\widehat{\mathcal{H}^{(s,\gamma)}}(\xi)| \lesssim_{\epsilon,n} \frac{1}{\Gamma(\gamma/2)}\Big(\int_0^\infty e^{-t}e^{\epsilon|2t+s-n|}t^{\gamma/2}\frac{dt}{t}\Big)e^{-|\xi|/2} \lesssim_{s,n,\gamma} e^{-|\xi|/2},$$

which proves (4.2).

Now we assume that $|\xi| \leqslant 1$. When $0 < s < n$

$$\widehat{\mathcal{H}^{(s,\gamma)}}(\xi) = \frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t}G_{2t+s}(\xi)t^{\gamma/2}\frac{dt}{t} + \frac{1}{\Gamma(\gamma/2)}\int_{\frac{n-s}{2}}^\infty e^{-t}G_{2t+s}(\xi)t^{\gamma/2}\frac{dt}{t}.$$

Then using (4.4), (4.5), and change of variables,

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t}G_{2t+s}(\xi)t^{\gamma/2}\frac{dt}{t}\\
&\lesssim_{n,\epsilon} |\xi|^{-(n-s)}\frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t}|\xi|^{2t}e^{\epsilon(n-2t-s)}t^{\gamma/2}\frac{dt}{t}\\
&\leqslant e^{\epsilon(n-s)}|\xi|^{-(n-s)}\frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t(1+2\ln(|\xi|^{-1}))}\frac{dt}{t}\\
&\leqslant e^{\epsilon(n-s)}|\xi|^{-(n-s)}(1+2\ln(|\xi|^{-1}))^{-\gamma/2}\frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}t^{\gamma/2}\frac{dt}{t}\\
&\lesssim_{s,n,\gamma} |\xi|^{-(n-s)}(1+2\ln(|\xi|^{-1}))^{-\gamma/2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t}G_{2t+s}(\xi)t^{\gamma/2}\frac{dt}{t}\\
&\gtrsim_{n,\epsilon} |\xi|^{-(n-s)}\frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t}|\xi|^{2t}e^{-\epsilon(n-2t-s)}t^{\gamma/2}\frac{dt}{t}\\
&\geqslant e^{-\epsilon(n-s)}|\xi|^{-(n-s)}\frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t(1+2\ln(|\xi|^{-1}))}\frac{dt}{t}\\
&\geqslant e^{-\epsilon(n-s)}|\xi|^{-(n-s)}(1+2\ln(|\xi|^{-1}))^{-\gamma/2}\frac{1}{\Gamma(\gamma/2)}\int_0^{\frac{n-s}{2}} e^{-t}t^{\gamma/2}\frac{dt}{t}\\
&\gtrsim_{s,n,\gamma} |\xi|^{-(n-s)}(1+2\ln(|\xi|^{-1}))^{-\gamma/2}.
\end{aligned}$$

Similarly, we can also prove that

$$\frac{1}{\Gamma(\gamma/2)}\int_{\frac{n-s}{2}}^{\infty} e^{-t}G_{2t+s}(\xi)t^{\gamma/2}\frac{dt}{t} \approx_{s,n,\gamma} 1.$$

A similar computation, together with (4.4) and (4.5), will lead to an estimate for $s \geqslant n$, in which $\widehat{\mathcal{H}^{(s,\gamma)}} \approx_{s,\gamma,n} 1$ for $|\xi| \leqslant 1$. We leave this to the reader to avoid unnecessary repetition. $\square$

In what follows let $\eta, \widetilde{\eta}$ denote Schwartz functions so that $\eta \geqslant 0$, $\eta(x) \geqslant c$ on $\{x \in \mathbb{R}^n : |x| \leqslant 1/100\}$ for some $c > 0$, $\mathrm{Supp}(\widehat{\eta}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 1/1000\}$, $\widehat{\widetilde{\eta}}(\xi) = 1$ for $|\xi| \leqslant 1/1000$, and $\mathrm{Supp}(\widehat{\widetilde{\eta}}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 1/100\}$. Let $e_1 := (1, 0, \dots, 0) \in \mathbb{Z}^n$ and $0 < t, \gamma < \infty$. Define $\mathcal{H}^{(t,\gamma)}$ as in (4.1),

$$K^{(t,\gamma)}(x) := \mathcal{H}^{(t,\gamma)} * \widetilde{\eta}(x)e^{2\pi i\langle x, e_1\rangle},$$

and

$$\sigma^{(t,\gamma)}(\xi) := \widehat{K^{(t,\gamma)}}(\xi).$$

We investigate an upper bound of $\sup_{j\in\mathbb{Z}} \big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q}_s(\mathbb{R}^n)}$ and a lower bound of $\|T_{\sigma^{(t,\gamma)}}\|_{H^p(\mathbb{R}^n)\to H^p(\mathbb{R}^n)}$ when $t - n < s$.

4.1. **Upper bound of** $\sup_{j\in\mathbb{Z}} \big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q}_s(\mathbb{R}^n)}$**.** Note that, due to the supports of $\sigma^{(t,\gamma)}$ and $\widehat{\Psi}$, we have

$$\sigma^{(t,\gamma)}(2^j\xi)\widehat{\Psi}(\xi) = \begin{cases} \widehat{K^{(t,\gamma)}}(2^j\xi)\widehat{\Psi}(\xi), & -2 \leqslant j \leqslant 2\\ 0, & \text{otherwise.}\end{cases}$$

For $-2 \leqslant j \leqslant 2$ and $t - n < s$,

$$\big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q}_s(\mathbb{R}^n)} \lesssim \big\|\sigma^{(t,\gamma)}\big\|_{L^{r,q}_s(\mathbb{R}^n)} \lesssim \big\|\widehat{\mathcal{H}^{(t,\gamma)}}\big\|_{L^{r,q}_s(\mathbb{R}^n)} = \big\|\widehat{\mathcal{H}^{(t-s,\gamma)}}\big\|_{L^{r,q}(\mathbb{R}^n)}$$

where Lemma 2.3 is applied.

For $u>0$ define

$$\mathcal{T}^{(t-s,\gamma)}(u) := \begin{cases} u^{-(n-t+s)}(1+2\ln u^{-1})^{-\gamma/2} & \text{for } \ u\leqslant 1\\ e^{-u/2+1/2} & \text{for } \ u>1. \end{cases}$$

Then $\mathcal{T}^{(t-s,\gamma)}$ is a positive decreasing function and this implies that

$$\big(\mathcal{T}^{(t-s,\gamma)}\big)^*(u) = \mathcal{T}^{(t-s,\gamma)}(u). \tag{4.6}$$

We first assume $0<q<\infty$. By using Lemma 4.1, we have

$$\widehat{\mathcal{H}^{(t-s,\gamma)}}(\xi) \lesssim_{s,t,\gamma,n} \mathcal{T}^{(t-s,\gamma)}(|\xi|),$$

from which

$$\begin{aligned}\big\|\widehat{\mathcal{H}^{(t-s,\gamma)}}\big\|_{L^{r,q}(\mathbb{R}^n)} &\lesssim_{s,t,\gamma.n} \big\|\mathcal{T}^{(t-s,\gamma)}\big(|\cdot|\big)\big\|_{L^{r,q}(\mathbb{R}^n)}\\ &= \Big(\int_0^\infty \Big(\mathcal{T}^{(t-s,\gamma)}\big((u/\Omega_n)^{1/n}\big)u^{1/r}\Big)^q \frac{du}{u}\Big)^{1/q}\\ &= \Omega_n^{1/r} n^{1/q}\Big(\int_0^\infty \big(\mathcal{T}^{(t-s,\gamma)}(u)\big)^q u^{nq/r}\frac{du}{u}\Big)^{1/q}\end{aligned}$$

where Lemma 2.7 is applied with (4.6). Furthermore,

$$\begin{aligned}\Big(\int_0^1 \big(\mathcal{T}^{(t-s,\gamma)}(u)\big)^q u^{nq/r}\frac{du}{u}\Big)^{1/q} &= \Big(\int_0^1 \frac{1}{u^{n-t+s-n/r}}\frac{1}{(1+2\ln u^{-1})^{\gamma q/2}}\frac{du}{u}\Big)^{1/q}\\ &= \Big(\int_1^\infty u^{(n-t+s-n/r)q}\frac{1}{(1+2\ln u)^{\gamma q/2}}\frac{du}{u}\Big)^{1/q}\end{aligned}$$

and

$$\Big(\int_1^\infty \big(\mathcal{T}^{(t-s,\gamma)}(u)\big)^q u^{nq/r}\frac{du}{u}\Big)^{1/q} = e^{1/2}\Big(\int_1^\infty e^{-uq/2}u^{nq/r}\frac{du}{u}\Big)^{1/q} \lesssim_{q,r,n} 1$$

Finally, we conclude that

$$\sup_{j\in\mathbb{Z}} \big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q}_s(\mathbb{R}^n)} \lesssim_{s,\gamma,n,q,r} 1+\Big(\int_1^\infty u^{(n-t+s-n/r)q}\frac{1}{(1+2\ln u)^{\gamma q/2}}\frac{du}{u}\Big)^{1/q} \tag{4.7}$$

and with the usual modification if $q=\infty$ we may also obtain

$$\sup_{j\in\mathbb{Z}} \big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,\infty}_s(\mathbb{R}^n)} \lesssim_{s,\gamma,n,r} 1+\sup_{u>1}\frac{u^{n-t+s-n/r}}{(1+2\ln u)^{\gamma/2}}. \tag{4.8}$$

4.2. **Lower bound of $\|T_{\sigma^{(t,\gamma)}}\|_{H^p(\mathbb{R}^n)\to H^p(\mathbb{R}^n)}$.** If $1\leqslant p<\infty$, then

$$\|T_{\sigma^{(t,\gamma)}}\|_{H^p(\mathbb{R}^n)\to H^p(\mathbb{R}^n)} \geqslant \|\sigma^{(t,\gamma)}\|_{L^\infty(\mathbb{R}^n)} \geqslant |\sigma^{(t,\gamma)}(e_1)| \gtrsim \|\mathcal{H}^{(t,\gamma)}\|_{L^1(\mathbb{R}^n)}.$$

Moreover, for $0<p<1$, define $f(x):=\eta(x)e^{2\pi i\langle x,e_1\rangle}$. Observe that $\big|T_{\sigma^{(t,\gamma)}}f(x)\big| = \big|\mathcal{H}^{(t,\gamma)}*\eta(x)\big|$ and thus

$$\begin{aligned}\|T_{\sigma^{(t,\gamma)}}\|_{H^p(\mathbb{R}^n)\to H^p(\mathbb{R}^n)} &\gtrsim \|T_{\sigma^{(t,\gamma)}}f\|_{H^p(\mathbb{R}^n)} \geqslant \|T_{\sigma^{(t,\gamma)}}f\|_{L^p(\mathbb{R}^n)}\\ &= \|\mathcal{H}^{(t,\gamma)}*\eta\|_{L^p(\mathbb{R}^n)} \gtrsim \|\mathcal{H}^{(t,\gamma)}\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

where the last inequality follows from the fact that $\mathcal{H}^{(t,\gamma)}, \eta \geqslant 0$ and $\mathcal{H}^{(t,\gamma)}(x-y) \geqslant \mathcal{H}^{(t,\gamma)}(x)\mathcal{H}^{(t,\gamma)}(y)$.

Consequently, for any $0 < p < \infty$,

$$\|T_{\sigma^{(t,\gamma)}}\|_{H^p(\mathbb{R}^n)\to H^p(\mathbb{R}^n)} \gtrsim \|\mathcal{H}^{(t,\gamma)}\|_{L^{\min(1,p)}(\mathbb{R}^n)}$$

$$= \Big(\int_{\mathbb{R}^n} \frac{1}{(1+4\pi^2|x|^2)^{t\min(1,p)/2}} \frac{1}{(1+\ln(1+4\pi^2|x|^2))^{\min(1,p)\gamma/2}} dx \Big)^{1/\min(1,p)}. \tag{4.9}$$

4.3. **Completion of the proof of Theorem 1.2.** We are only concerned with the case $0 < p \leqslant 2$ as the other cases follow by a duality argument. Suppose $n/p - n/2 < s < n/\min(1,p)$.

We first assume $r < \tau^{(s,p)}$ and $0 < q \leqslant \infty$. Then we can choose $t < \frac{n}{\min(1,p)}$ so that

$$r < \frac{n}{s-(t-n)} < \frac{n}{s-(n/\min(1,p)-n)} = \tau^{(s,p)}.$$

Note that $t-n < s$ and $n-t+s-n/r < 0$, from which

$$\sup_{j\in\mathbb{Z}} \big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q}_s(\mathbb{R}^n)} \lesssim_{s,\gamma,n,q,r} 1$$

due to (4.7) and (4.8). Moreover, since $t\min(1,p) < n$

$$\|T_{\sigma^{(t,\gamma)}}\|_{H^p(\mathbb{R}^n)\to H^p(\mathbb{R}^n)} = \infty,$$

using (4.9).

Now suppose $r = \tau^{(s,p)}$ and $\min(1,p) < q$. Choose

$$2/q < \gamma \leqslant 2/\min(1,p) \tag{4.10}$$

and let $t = \frac{n}{\min(1,p)}$ such that $n-t+s-n/r = 0$. Then

$$\sup_{j\in\mathbb{Z}} \big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,q}_s(\mathbb{R}^n)} \lesssim_{s,\gamma,n,q} 1 + \Big(\int_1^\infty \frac{1}{(1+2\ln u)^{\gamma q/2}}\frac{du}{u}\Big)^{1/q} \lesssim 1$$

because of (4.10) for $0 < q < \infty$, and similarly, $\sup_{j\in\mathbb{Z}} \big\|\sigma^{(t,\gamma)}(2^j\cdot)\widehat{\Psi}\big\|_{L^{r,\infty}_s(\mathbb{R}^n)} \lesssim_{s,\gamma,n} 1$ for $q = \infty$. On the other hand, $\|T_{\sigma^{(t,\gamma)}}\|_{H^p(\mathbb{R}^n)\to H^p(\mathbb{R}^n)}$ is bounded below by

$$\Big(\int_{\mathbb{R}^n} \frac{1}{(1+4\pi^2|x|^2)^{n/2}} \frac{1}{(1+\ln(1+4\pi^2|x|^2))^{\min(1,p)\gamma/2}} dx \Big)^{1/\min(1,p)},$$

which diverges for the choice of γ in (4.10).

Appendix A. Complex Interpolation of H^1- and L^2-boundedness

In this section, we review the complex interpolation method of Calderón-Torchinsky [5] and Triebel [23], which is a generalization of the well-known method of Calderón [4] and Fefferman and Stein [8].

Let $A := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ be a strip in the complex plane $\mathbb{C}$ and $\overline{A}$ denote its closure. We say that the mapping $z \mapsto f_z \in \mathcal{S}'(\mathbb{R}^n)$ is a $\mathcal{S}'$-analytic function on A if the following properties are satisfied:

(1) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $g(x,z) := \big(\varphi \widehat{f_z}\big)(x)$ is a uniformly continuous and bounded function on $\mathbb{R}^n \times \overline{A}$.
(2) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support and any fixed $x \in \mathbb{R}^n$, $h_x := \big(\varphi \widehat{f_z}\big)^\vee$ is an analytic function on A.

Let $0 < p_0, p_1 < \infty$. Then we define $F\big(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)\big)$ to be the collection of all $\mathcal{S}'$-analytic functions f_z on A such that

$$f_{it} \in H^{p_0}(\mathbb{R}^n), \qquad f_{1+it} \in H^{p_1}(\mathbb{R}^n) \quad \text{for any} \;\; t \in \mathbb{R}$$

and

$$\sup_{t\in\mathbb{R}} \|f_{l+it}\|_{H^{p_l}(\mathbb{R}^n)} < \infty \qquad \text{for each} \;\; l = 1,2.$$

Moreover,

$$\|f_z\|_{F(H^{p_0}(\mathbb{R}^n),H^{p_1}(\mathbb{R}^n))} := \max\Big(\sup_{t\in\mathbb{R}} \|f_{it}\|_{H^{p_0}(\mathbb{R}^n)}, \sup_{t\in\mathbb{R}} \|f_{1+it}\|_{H^{p_1}(\mathbb{R}^n)} \Big).$$

For $0 < \theta < 1$ the intermediate space $(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))_\theta$ is defined by

$$\big(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)\big)_\theta := \big\{ g : \exists f_z \in F\big(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)\big) \text{ so that } g = f_\theta \big\}$$

and the (quasi-)norm in the space is

$$\|g\|_{(H^{p_0}(\mathbb{R}^n),H^{p_1}(\mathbb{R}^n))_\theta} := \inf_{f_z \in F(H^{p_0}(\mathbb{R}^n),H^{p_1}(\mathbb{R}^n)) : g = f_\theta} \|f_z\|_{F(H^{p_0}(\mathbb{R}^n),H^{p_1}(\mathbb{R}^n))}$$

where the infimum is taken over all admissible functions f_z in the sense that $f_z \in F\big(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)\big)$ and $g = f_\theta$. It is known in [5, 23] that for any $0 < p_0, p_1 < \infty$ and $0 < \theta < 1$

$$\big(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)\big)_\theta = H^p(\mathbb{R}^n) \quad \text{when} \;\; 1/p = (1-\theta)/p_0 + \theta/p_1. \tag{A.1}$$

We now use this method to interpolate H^1- and L^2-boundedness of the multiplier operator T_σ to obtain L^p estimates for $1 < p < 2$. Note that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Since most arguments are very similar to that used in the proof of [10, Theorem 3.1], we shall provide only the outline of the proof, omitting the details.

We may consider a Schwartz function f whose Fourier transform is compactly supported via a density argument. Suppose that $1 < p < 2$ and $n/p - n/2 < s < n$. Let $0 < \theta < 1$ satisfy $1/p = (1-\theta)/1 + \theta/2$. Then we have $s > n/p - n/2 = (1-\theta)n/2$. Pick $s_0 > n/2$ so that

$$s > (1-\theta)s_0 > (1-\theta)n/2$$

and let $s_1 := \frac{s-(1-\theta)s_0}{\theta} > 0$ which implies

$$s = (1-\theta)s_0 + \theta s_1.$$

Since $f \in L^p(\mathbb{R}^n) = H^p(\mathbb{R}^n) = \big(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n)\big)_\theta$, by definition, for any $\epsilon > 0$, there exists $f_z^\epsilon \in F\big(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n)\big)$ such that $f = f_\theta^\epsilon$ and

$$\|f_z^\epsilon\|_{F(H^1(\mathbb{R}^n),H^2(\mathbb{R}^n))} < \|f\|_{(H^1(\mathbb{R}^n),H^2(\mathbb{R}^n))_\theta} + \epsilon. \tag{A.2}$$

Now let $\widehat{\Theta}(\xi) := \widehat{\Psi}(\xi/2) + \widehat{\Psi}(\xi) + \widehat{\Psi}(2\xi)$ as before, and $\sigma^{j,s} := (I-\Delta)^{s/2}\big(\sigma(2^j\cdot)\widehat{\Psi}\big)$ for each $j \in \mathbb{Z}$. We define, as in [10, (3.18)],

$$\sigma_z(\xi) := \frac{(1+\theta)^{n+1}}{(1+z)^{n+1}} \sum_{j\in\mathbb{Z}} (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \Big(\sigma^{j,s} h_{j,s}^{\frac{s-(1-z)s_0-zs_1}{n}}\Big)(\xi/2^j)\widehat{\Theta}(\xi/2^j)$$

where $h_{j,s} : \mathbb{R}^n \to (0,\infty)$ is a measure preserving transformation so that $|\sigma^{j,s}| = (\sigma^{j,s})^* \circ h_{j,s}$. Then we note that $\sigma_\theta = \sigma$ and $F_z := T_{\sigma_z} f_z^\epsilon$ is a $\mathcal{S}'$-analytic function on A. Moreover,

$$\begin{aligned}\|T_\sigma f\|_{H^p(\mathbb{R}^n)} &\approx \big\|T_{\sigma_\theta} f_\theta^\epsilon\big\|_{(H^1(\mathbb{R}^n),H^2(\mathbb{R}^n))_\theta} = \|F_\theta\|_{(H^1(\mathbb{R}^n),H^2(\mathbb{R}^n))_\theta} \\ &\leqslant \|F_z\|_{F(H^1(\mathbb{R}^n),H^2(\mathbb{R}^n))} = \max\Big(\sup_{t\in\mathbb{R}} \|F_{it}\|_{H^1(\mathbb{R}^n)}, \sup_{t\in\mathbb{R}} \|F_{1+it}\|_{H^2(\mathbb{R}^n)}\Big).\end{aligned}$$

By using Theorem 1.1 for $p = 1$, we have

$$\begin{aligned}\|F_{it}\|_{H^1(\mathbb{R}^n)} = \|T_{\sigma_{it}} f_{it}^\epsilon\|_{H^1(\mathbb{R}^n)} &\lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma_{it}(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s_0,1}_{s_0}(\mathbb{R}^n)} \|f_{it}^\epsilon\|_{H^1(\mathbb{R}^n)} \\ &\lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma_{it}(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s_0,1}_{s_0}(\mathbb{R}^n)} \Big(\|f\|_{(H^1(\mathbb{R}^n),H^2(\mathbb{R}^n))_\theta} + \epsilon\Big),\end{aligned}$$

where (A.2) is applied in the last inequality. Similarly, with L^2-boundedness,

$$\begin{aligned}\|F_{1+it}\|_{H^2(\mathbb{R}^n)} = \|T_{\sigma_{1+it}} f_{1+it}^\epsilon\|_{H^2(\mathbb{R}^n)} &\lesssim \|\sigma_{1+it}\|_{L^\infty(\mathbb{R}^n)} \|f_{1+it}^\epsilon\|_{H^2(\mathbb{R}^n)} \\ &\lesssim \sup_{j\in\mathbb{Z}} \big\|\sigma_{1+it}(2^j\cdot)\widehat{\Psi}\big\|_{L^\infty(\mathbb{R}^n)} \Big(\|f\|_{(H^1(\mathbb{R}^n),H^2(\mathbb{R}^n))_\theta} + \epsilon\Big).\end{aligned}$$

Therefore, once we prove

$$\big\|\sigma_{it}(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s_0,1}_{s_0}(\mathbb{R}^n)}, \big\|\sigma_{1+it}(2^j\cdot)\widehat{\Psi}\big\|_{L^\infty(\mathbb{R}^n)} \lesssim \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s,1}_s(\mathbb{R}^n)} \tag{A.3}$$

uniformly in j, then we are done by using (A.1) and taking $\epsilon \to 0$.

Let us prove (A.3). We first observe that

$$\begin{aligned}&\sigma_z(2^j\xi)\widehat{\Psi}(\xi) \\ &= \frac{(1+\theta)^{n+1}}{(1+z)^{n+1}} \sum_{k\in\mathbb{Z}} (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \Big(\sigma^{k,s} h_{k,s}^{\frac{s-(1-z)s_0-zs_1}{n}}\Big)(\xi/2^{k-j})\widehat{\Theta}(\xi/2^{k-j})\widehat{\Psi}(\xi)\end{aligned}$$

is actually finite sum over k near j due to the supports of $\widehat{\Theta}$ and $\widehat{\Psi}$, and for simplicity, we may therefore take $k = j$ in the calculation below.

Using Lemma 2.3, we have

$$\big\|\sigma_{it}(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s_0,1}_{s_0}(\mathbb{R}^n)} \lesssim \frac{1}{(1+|t|^2)^{(n+1)/2}} \Big\|(I-\Delta)^{\frac{(s_0-s_1)it}{2}} \Big(\sigma^{j,s} h_{j,s}^{\frac{s-s_0+(s_0-s_1)it}{n}}\Big)\Big\|_{L^{n/s_0,1}(\mathbb{R}^n)}.$$

Then we apply [10, Lemma 3.5, 3.7] to bound this by

$$\begin{aligned}\Big\|\sigma^{j,s} h_{j,s}^{\frac{s-s_0+(s_0-s_1)it}{n}}\Big\|_{L^{n/s_0,1}(\mathbb{R}^n)} &\lesssim \big\|(\sigma^{j,s})^*(r) r^{(s-s_0)/n}\big\|_{L^{n/s_0,1}(0,\infty)} \\ &\lesssim \|(\sigma^{j,s})^*\|_{L^{n/s,1}(0,\infty)} \lesssim \|\sigma^{j,s}\|_{L^{n/s,1}(\mathbb{R}^n)} = \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^{n/s,1}_s(\mathbb{R}^n)}.\end{aligned}$$

On the other hand, using [10, Lemma 3.4, 3.5, 3.7],

$$\begin{aligned}
&\big\|\sigma_{1+it}(2^j\cdot)\widehat{\Psi}\big\|_{L^\infty(\mathbb{R}^n)}\\
&\lesssim \frac{1}{(1+|t|^2)^{(n+1)/2}}\Big\|(I-\Delta)^{-s_1/2}(I-\Delta)^{(s_0-s_1)it/2}\Big(\sigma^{j,s}h_{j,s}^{\frac{s-s_1+(s_0-s_1)it}{n}}\Big)\Big\|_{L^\infty(\mathbb{R}^n)}\\
&\lesssim \frac{1}{(1+|t|^2)^{(n+1)/2}}\Big\|(I-\Delta)^{(s_0-s_1)it/2}\Big(\sigma^{j,s}h_{j,s}^{\frac{s-s_1+(s_0-s_1)it}{n}}\Big)\Big\|_{L^{n/s_1,1}(\mathbb{R}^n)}\\
&\lesssim \Big\|\sigma^{j,s}h_{j,s}^{\frac{s-s_1+(s_0-s_1)it}{n}}\Big\|_{L^{n/s_1,1}(\mathbb{R}^n)} \lesssim \big\|(\sigma^{j,s})^*(r)r^{(s-s_1)/n}\big\|_{L^{n/s_1,1}(0,\infty)}\\
&\lesssim \|(\sigma^{j,s})^*\|_{L^{n/s,1}(0,\infty)} \lesssim \|\sigma^{j,s}\|_{L^{n/s,1}(\mathbb{R}^n)} = \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L_s^{n/s,1}(\mathbb{R}^n)},
\end{aligned}$$

which finishes the proof of (A.3).

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