MULTIPLIER CONDITIONS FOR BOUNDEDNESS INTO HARDY SPACES

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Abstract. In the present work we find useful and explicit necessary and sufficient conditions for linear and multilinear multiplier operators of Coifman-Meyer type, finite sum of products of Calderón-Zygmund operators, and also of intermediate types to be bounded from a product of Lebesgue or Hardy spaces into a Hardy space. These conditions state that the symbols of the multipliers $\sigma(\xi_1, \dots, \xi_m)$ and their derivatives vanish on the hyperplane $\xi_1 + \cdots + \xi_m = 0$.

1. Introduction

Hardy spaces are spaces of distributions on $\mathbb{R}^n$ whose smooth maximal functions lie in $L^p(\mathbb{R}^n)$, for $0 < p < \infty$. These spaces coincide with $L^p(\mathbb{R}^n)$ if $1 < p < \infty$. Let $0 < p \le 1$ and N be a prescribed integer satisfying $N \ge \lfloor n(\frac{1}{p} - 1) \rfloor + 1$, where $\lfloor s \rfloor$ denotes the largest integer less than or equal to s. An L^∞ function a is said to be a (p, ∞)-atom, if a is supported on some cube Q and satisfies

$$\|a\|_{L^\infty} \le 1, \quad \int_{\mathbb{R}^n} x^\alpha a(x) dx = 0$$

for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le N$, see [7], [22]. The space $H^p(\mathbb{R}^n)$ can be characterized as the set of all tempered distributions which can be expressed as a sum of the form $\sum_{j=1}^\infty \lambda_j a_j$, where a_j are (p, ∞)-atoms and $(\lambda_j)_{j=1}^\infty$ is a sequence of non-negative numbers such that

$$\Big\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \Big\|_{L^p} < \infty.$$

In this note we study linear or multilinear multiplier operators that map products of Hardy spaces into other Hardy spaces. These operators have the form

$$T_\sigma(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} \sigma(\xi_1, \dots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m)\, d\xi_1 \cdots d\xi_m, \tag{1.1}$$

where σ is a bounded function on $\mathbb{R}^{mn}$. Here $\widehat{f}(\xi)$ denotes the Fourier transform of a Schwartz function f defined by $\int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$. We are interested in explicit conditions on the symbol σ that characterize boundedness into a Hardy space. These conditions reflect the amount of oscillation the symbols contain. For instance, the boundedness into $H^1(\mathbb{R}^n)$ for m-linear operators is characterized by the vanishing condition $\sigma(\xi_1, \dots, \xi_m) = 0$ on the hyperplane Δ_n, where Δ_n is given by

$$\Delta_n = \{(\xi_1, \dots, \xi_m) \in \mathbb{R}^{mn} \ :\ \xi_1 + \cdots + \xi_m = 0\}.$$

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For a multiindex $\alpha = (i_1, \dots, i_n)$ we set $\partial_k^\alpha = \partial_{\xi_{k1}}^{i_1} \cdots \partial_{\xi_{kn}}^{i_n}$, where $\xi_k = (\xi_{k1}, \dots, \xi_{kn}) \in \mathbb{R}^n$. A symbol $\sigma(\xi_1, \dots, \xi_m)$ on $\mathbb{R}^{mn}$ is called of Coifman-Meyer type if

$$\left|\partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} \sigma(\xi_1, \dots, \xi_m)\right| \le C_{\alpha_1,\dots,\alpha_m}(|\xi_1| + \cdots + |\xi_m|)^{-(|\alpha_1|+\cdots+|\alpha_m|)} \tag{1.2}$$

for any n-tuples α of nonnegative integers α_j with $|\alpha_j| \le N$, where N is large enough. The α_j are called multiindices. Here $|\alpha| = i_1 + \cdots + i_n$ is the size of a multiindex $\alpha = (i_1, \dots, i_n) \in \mathbb{N}_0^n$. The associated operators T_σ are called multilinear Calderón-Zygmund operators; these were initially introduced in [3] and were extensively studied in [15]. These operators map products $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ of Lebesgue spaces into another Lebesgue space $L^p(\mathbb{R}^n)$, where $1 < p_j < \infty$, $j = 1, 2, \dots, m$, and $0 < p < \infty$ satisfy

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}. \tag{1.3}$$

Boundedness into a Lebesgue space also holds if the initial spaces are Hardy spaces, as shown in [11]; the range $0 < p_i < \infty$ is included in [11]. Additionally, it was shown by the authors [14] that T_σ maps a product of Hardy spaces into another Hardy space if the action of T_σ on atoms has vanishing moments, i.e.

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma(a_1, \dots, a_m)(x)\, dx = 0 \tag{1.4}$$

for all (p_j, ∞)-atoms a_j and for all $|\alpha| \le \lfloor n(\frac{1}{p} - 1) \rfloor$. Condition (1.4) appeared in [5] and also in [1].

Remarkably, the vanishing moment condition (1.4) is only required to hold for all smooth functions with compact support $a_j \in \mathcal{O}_N(\mathbb{R}^n)$, where

$$\mathcal{O}_N(\mathbb{R}^n) = \bigcap_{\beta \in \mathbb{N}_0^n, |\beta| \le N} \left\{ f \in \mathscr{C}_c^\infty(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} x^\beta f(x)\, dx = 0 \right\}.$$

Here, $\mathbb{N}_0$ denotes the set of all nonnegative integers. We have the following theorem concerning operators associated with Coifman-Meyer symbols.

Theorem 1.1. *Let σ be a bounded function on $\mathbb{R}^{mn}$ and $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{mn} \setminus \{(0, \dots, 0)\})$ that satisfies* (1.2). *Fix $0 < p_i \le \infty$, $0 < p \le 1$ that satisfy* (1.3). *Then the following two statements are equivalent:*

(a) T_σ maps $H^{p_1}(\mathbb{R}^n) \times \cdots \times H^{p_m}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$.

(b) For all multiindices α with $|\alpha| \le \lfloor n(\frac{1}{p} - 1) \rfloor$ we have

$$(\partial_m^\alpha \sigma)(\xi_1, \dots, \xi_m) = 0 \tag{1.5}$$

for all $(\xi_1, \dots, \xi_m) \in \Delta_n \setminus \{(0, \dots, 0)\}$.

We also consider symbols of the product form

$$\sigma(\xi_1, \dots, \xi_m) = \sum_{j=1}^{M} \sigma_{j1}(\xi_1) \cdots \sigma_{jm}(\xi_m) \tag{1.6}$$

where the σ_{jk}'s are Fourier transforms of sufficiently smooth Calderón-Zygmund kernels on $\mathbb{R}^n$. For such symbols with $m = 2$ it was shown in [5] (see also [12] and [16]) that the associated operators are bounded from a product of Hardy spaces into another Hardy space if and only if (1.4) holds. For symbols of the form (1.6) we prove the following analogous result:

Theorem 1.2. *Let σ_{jk}, $1 \le j \le M, 1 \le k \le m$, be Fourier transforms of Calderón-Zygmund kernels on $\mathbb{R}^n$, and let σ be a function on $\mathbb{R}^{mn}$ given by* (1.6). *Fix $0 < p_i < \infty$, $0 < p \le 1$ satisfying* (1.3). *Then the following two statements are equivalent:*
(a) T_σ maps $H^{p_1}(\mathbb{R}^n) \times \cdots \times H^{p_m}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$.
(b) For all multiindices α with $|\alpha| \le \lfloor n(\frac{1}{p} - 1)\rfloor$, condition (1.5) *holds, i.e.*

$$(\partial_m^\alpha \sigma)(\xi_1, \dots, \xi_m) = 0$$

for all $(\xi_1, \dots, \xi_m) \in (\mathbb{R}^n \setminus \{0\})^m \cap \Delta_n$.

Note that for symbols of both types (1.2) and (1.6) we always have

$$\left|\partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} \sigma(\xi_1, \dots, \xi_m)\right| \le C_{\alpha_1,\dots,\alpha_m} |\xi_1|^{-|\alpha_1|} \cdots |\xi_m|^{-|\alpha_m|} \tag{1.7}$$

for all $\alpha_j \in \mathbb{N}_0^n$ and all $\xi_j \in \mathbb{R}^n$, $j = 1, \dots, m$, under the assumption that $|\alpha_j| > 0$ if $\xi_j \neq 0$. It turns out that condition (1.7) suffices for verifying the equivalence between (a) and (b) in both Theorems 1.1 and 1.2, although it is not strong enough to imply boundedness on any product of Lebesgue spaces (see [10]).

Remark 1.3. By symmetry, we note that in condition (1.5) the derivative ∂_m^α can be replaced by ∂_k^α for any $k \in \{1, \dots, m-1\}$ in Theorems 1.1 and 1.2.

Boundedness into $H^p(\mathbb{R}^n)$ for operators T_σ is often expressed in terms of cancellation of the action of the operator on tuples of atoms. Let $x^\alpha = x_1^{i_1} \cdots x_n^{i_n}$ if $\alpha = (i_1, \dots, i_n)$. In order for the integral

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma(a_1, \dots, a_m)(x)\, dx$$

to be absolutely convergent, it is necessary for $T_\sigma(a_1, \dots, a_m)(x)$ to have decay, where a_j are (p_j, ∞)-atoms. Precisely, we assume that for any m-tuple of (p_j, ∞)-atom a_j there exists a function $b \in L^{p_j}(\mathbb{R}^n)$ which decays like $|x|^{-mn-N-1}$ as $|x| \to \infty$, such that for all $x \in \mathbb{R}^n$

$$|T_\sigma(a_1, \dots, a_m)(x)| \lesssim b(x). \tag{1.8}$$

We note that condition (1.8) is valid for a large class of multilinear operators such as those in Theorems 1.1 and 1.2. Indeed, for operators with symbols of the form (1.6) we can take

$$b(x) = \sum_{j=1}^M \prod_{k=1}^m \Big[|T_{\sigma_{jk}}(a_k)(x)| \chi_{Q_j^*}(x) + \frac{|Q_k|^{1-\frac{1}{p_k}+\frac{N+1}{nm}} \chi_{(Q_k^*)^c}(x)}{\big(|x - c_k| + \ell(Q_k)\big)^{n+\frac{N+1}{m}}} \Big], \tag{1.9}$$

where Q_k is a cube that contains the support of a_k, $\ell(Q_k)$ denotes the length of Q_k.

Condition (1.8) is also valid for Coifman-Meyer multipliers (1.2). Indeed, we can choose

$$b(x) = |T_\sigma(a_1, \dots, a_m)(x)| \chi_{\cup_{k=1}^m Q_k^*}(x) + \prod_{k=1}^m \frac{|Q_k|^{1-\frac{1}{p_k}+\frac{N+1}{nm}} \chi_{(Q_k^*)^c}(x)}{\big(|x - c_k| + \ell(Q_k)\big)^{n+\frac{N+1}{m}}}. \tag{1.10}$$

See [14] for estimates (1.9) and (1.10).

To state the main equivalence result between cancellation of multipliers and cancellation of the action of an operator on m tuples of atoms we introduce some notation. For $0 < \epsilon < 1$ and $1 \le i \le m$, we denote

$$\Gamma_{i,\epsilon}(\mathbb{R}^{mn}) = \{(\xi_1, \dots, \xi_m) \in \mathbb{R}^{mn} \ : \ |\xi_i| \le \epsilon\}, \quad \Gamma_\epsilon(\mathbb{R}^{mn}) = \bigcup_{i=1}^m \Gamma_{i,\epsilon}(\mathbb{R}^{mn}). \tag{1.11}$$

We also set

$$\Gamma_i(\mathbb{R}^{mn}) = \{(\xi_1, \ldots, \xi_m) \in \mathbb{R}^{mn} \ : \ \xi_i = 0\}, \quad \Gamma(\mathbb{R}^{mn}) = \bigcup_{i=1}^{m} \Gamma_i(\mathbb{R}^{mn}). \tag{1.12}$$

We will derive both Theorems 1.1 and 1.2 via the following general result.

Theorem 1.4. *Let σ in $L^\infty(\mathbb{R}^{mn}) \cap \mathcal{C}^\infty\big(\mathbb{R}^{mn} \setminus \Gamma(\mathbb{R}^{mn})\big)$ satisfy (1.7). Assume that T_σ satisfies (1.8) for all $a_j \in \mathcal{O}_N(\mathbb{R}^n)$ and*

$$0 < p_j < \infty, \quad 1 \le j \le m, \quad 0 < p \le 1, \quad \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Then the following two statements are equivalent:

(a) *For all multiindices α with $|\alpha| \le \lfloor n(\frac{1}{p} - 1) \rfloor$, condition (1.5) holds, i.e.*

$$(\partial_m^\alpha \sigma)(\xi_1, \ldots, \xi_m) = 0, \quad \forall\, (\xi_1, \ldots, \xi_m) \in \Delta_n \setminus \Gamma(\mathbb{R}^{mn}).$$

(b) *For all $a_i \in \mathcal{O}_N(\mathbb{R}^n)$, $1 \le i \le m$, condition (1.4) holds, i.e.*

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma(a_1, \ldots, a_m)(x)\, dx = 0$$

for all α with $|\alpha| \le \lfloor n(\frac{1}{p} - 1) \rfloor$.

Throughout this paper, we denote multiindices by letters α, β, γ, etc and use the abbreviation $\alpha \le \beta$ to denote that $\alpha_j \le \beta_j$ for all j if $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. We also let C denote a constant independent of crucial parameters whose value may vary on different occurrences.

2. The linear case

In the linear case, assumption (1.4) holds automatically via the following lemma:

Lemma 2.1. *For any $a \in \mathcal{O}_N(\mathbb{R}^n)$ and $|\alpha| \le N$, we have that*

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma(a)(x) dx = 0.$$

Proof. We write

$$\left| \int_{\mathbb{R}^n} (-2\pi i x)^\alpha T_\sigma(a)(x) dx \right| = \left| \partial^\alpha \left[\widehat{T_\sigma(a)} \right](0) \right| = \lim_{\epsilon \to 0} \left| \int_{\mathbb{R}^n} \sigma(\xi) \widehat{a}(\xi) \partial^\alpha [\varphi_\epsilon](\xi) d\xi \right|$$

integrating by parts. Now, we notice that by the Taylor expansion and the vanishing moments of a,

$$\widehat{a}(\xi) = \sum_{|\beta| \le |\alpha|} C_\beta \partial^\beta \widehat{a}(0) \xi^\beta + \mathrm{O}(|\xi|^{|\alpha|+1}) = \mathrm{O}(|\xi|^{|\alpha|+1})$$

as $|\xi| \to 0$. Hence, we see that

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} (-2\pi i x)^\alpha T_\sigma(a)(x) dx \right| &\le C_\alpha \lim_{\epsilon \to 0} \int_{Q(0,\epsilon)} \left| \sigma(\xi) |\xi|^{|\alpha|+1} \partial^\alpha [\varphi_\epsilon](\xi)] \right| d\xi \\
&\le C_\alpha \lim_{\epsilon \to 0} \epsilon \int_{Q(0,\epsilon)} |\sigma(\xi) [\partial^\alpha \varphi]_\epsilon(\xi)| \, d\xi \\
&\le C_\alpha \lim_{\epsilon \to 0} \epsilon \|\sigma\|_{L^\infty} \|\partial \varphi\|_{L^1} = 0.
\end{aligned}$$

□

As a result, the linear Fourier multipliers satisfying the suitable decay condition map product of Hardy spaces into Hardy spaces.

3. The bilinear case

For the sake of clarity of exposition, we first discuss the bilinear case of Theorem 1.4.

Theorem 3.1. *Let $\sigma \in L^\infty(\mathbb{R}^n\times\mathbb{R}^n)\cap\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{(\xi,\eta):\ |\xi||\eta|=0\})$ satisfy* (1.7) *and suppose that T_σ satisfies* (1.8)*. Then for a given $N\in\mathbb{N}_0$ the following conditions are equivalent:*

(a) For all $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le N$ and $\xi_1\in\mathbb{R}^n\setminus\{0\}$, we have

$$\partial_2^\alpha\sigma(\xi_1,-\xi_1)=0. \tag{3.1}$$

(b) For any smooth functions $a_1,a_2\in\mathcal{O}_N(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma(a_1,a_2)(x)dx=0,\quad \forall\quad |\alpha|\le N. \tag{3.2}$$

To obtain Theorem 3.1 we need a couple of lemmas. Here and below we denote by $B(x,r)$ the open ball centered at x of radius $r>0$.

Lemma 3.2. *Assume that σ is a bounded function on $\mathbb{R}^n\times\mathbb{R}^n$ and smooth away from the axes that satisfies* (1.7)*. Fix $N\in\mathbb{N}_0$. Then for all $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le N$ there is a constant C_α such that*

$$\sup_{0<\epsilon<1}\ \sup_{\xi_1\in\mathbb{R}^n\setminus B(0,2\epsilon)}\left|\int_{\mathbb{R}^n} g(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)\partial^\alpha[\varphi_\epsilon](\xi)d\xi\right|\le C_\alpha, \tag{3.3}$$

where g is a smooth function with bounded derivatives $\partial^\beta g$ and $\partial^\beta g(0)=0$ for all $|\beta|\le N$.

Proof. Fix any $\epsilon<1$ and any $\xi_1\in\mathbb{R}^n\setminus B(0,2\epsilon)$. We will show that

$$\left|\int_{\mathbb{R}^n} g(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)\partial^\alpha[\varphi_\epsilon](\xi)d\xi\right|\le C_\alpha, \tag{3.4}$$

where C_α is independent of ϵ and ξ_1. Note that the function $\xi\mapsto\sigma(\xi_1,\xi-\xi_1)$ is smooth on the domain of integration $|\xi|<\epsilon$, since $\xi_1\notin B(0,2\epsilon)$ and thus $|\xi-\xi_1|\ge\epsilon$. With this in mind, involving the Taylor expansion of g, we notice that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} g(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)\partial^\alpha[\varphi_\epsilon](\xi)d\xi\right|\\
\le& C\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left|\int_{\mathbb{R}^n}\partial^\beta g(\xi-\xi_1)\partial_2^{\alpha-\beta}\sigma(\xi_1,\xi-\xi_1)\varphi_\epsilon(\xi)d\xi\right|\\
\le& C_\alpha\|\varphi\|_{L^1}\max_{\beta\le\alpha}\ \sup_{\xi\in\mathbb{R}^n\setminus B(\xi_1,\epsilon)}\left|\partial^\beta g(\xi-\xi_1)\partial_2^{\alpha-\beta}\sigma(\xi_1,\xi-\xi_1)\right|\\
\le& C'_{\alpha,\sigma}\|\varphi\|_{L^1}\big[\max_{\beta\le\alpha}\ \sup_{\xi\in\mathbb{R}^n\setminus\{\xi_1\}}|\partial^\beta g(\xi-\xi_1)||\xi-\xi_1|^{|\beta|-|\alpha|}\big]=:C''_{\alpha,\sigma,g,\varphi}<\infty,
\end{aligned}$$

for any $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le N$. Here we used assumption (1.7) and the fact that $\partial^\beta g$ are bounded and vanishing at 0 for all $|\beta|\le N$. $\square$

Lemma 3.3. *Given $a_1,a_2\in\mathcal{O}_N(\mathbb{R}^n)$ and σ in $L^\infty(\mathbb{R}^n\times\mathbb{R}^n)\cap\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{(\xi,\eta):\ |\xi||\eta|=0\})$ that satisfies* (1.7)*, if $T_\sigma(a_1,a_2)$ has sufficient decay* (1.8)*, then we have*

$$\int_{\mathbb{R}^n}(-2\pi ix)^\alpha T_\sigma(a_1,a_2)(x)dx=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^n}\widehat{a_1}(\xi_1)\partial^{\alpha-\beta}\widehat{a_2}(-\xi_1)\partial_2^\beta\sigma(\xi_1,-\xi_1)\,d\xi_1. \tag{3.5}$$

Proof. First, we write

$$\int_{\mathbb{R}^n}(-2\pi i x)^\alpha T_\sigma(a_1,a_2)(x)dx = \partial^\alpha\left[\widehat{T_\sigma(a_1,a_2)}\right](0)$$

$$= \lim_{\epsilon\to 0}(-1)^{|\alpha|}\int_{\mathbb{R}^n}\widehat{T_\sigma(a_1,a_2)}(\xi)\partial^\alpha[\varphi_\epsilon](\xi)d\xi \tag{3.6}$$

using integration by parts. In view of the identity

$$\widehat{T_\sigma(a_1,a_2)}(\xi) = \int_{\mathbb{R}^n}\widehat{a_1}(\xi_1)\widehat{a_2}(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)d\xi_1, \tag{3.7}$$

the expression on the right in (3.6) equals

$$\lim_{\epsilon\to 0}(-1)^{|\alpha|}\int_{\mathbb{R}^n}\widehat{a_1}(\xi_1)\left(\int_{\mathbb{R}^n}\widehat{a_2}(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)\partial^\alpha[\varphi_\epsilon](\xi)d\xi\right)d\xi_1. \tag{3.8}$$

Now, we decompose (3.8) as $\lim_{\epsilon\to 0}(\mathrm{I}_\epsilon+\mathrm{II}_\epsilon)$, where

$$\mathrm{I}_\epsilon := (-1)^{|\alpha|}\int_{B(0,2\epsilon)}\widehat{a_1}(\xi_1)\left(\int_{\mathbb{R}^n}\widehat{a_2}(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)\partial^\alpha[\varphi_\epsilon](\xi)d\xi\right)d\xi_1,$$

$$\mathrm{II}_\epsilon := (-1)^{|\alpha|}\int_{\mathbb{R}^n\setminus B(0,2\epsilon)}\widehat{a_1}(\xi_1)\left(\int_{\mathbb{R}^n}\widehat{a_2}(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)\partial^\alpha[\varphi_\epsilon](\xi)d\xi\right)d\xi_1.$$

For the first term, using the vanishing moment condition for a_1, we have that

$$|\mathrm{I}_\epsilon| \le C\|\widehat{a_2}\|_{L^\infty}\|\sigma\|_{L^\infty}\|\partial^\alpha\varphi\|_{L^1}\int_{B(0,2\epsilon)}|\xi_1|^N\epsilon^{-|\alpha|}d\xi_1 \le C\epsilon^{N-|\alpha|+n}\to 0 \quad (\epsilon\to 0).$$

For the second term, inequality (3.3) gives us

$$\left|\int_{\mathbb{R}^n}\widehat{a_2}(\xi-\xi_1)\sigma(\xi_1,\xi-\xi_1)\partial^\alpha[\varphi_\epsilon](\xi)d\xi\right| \le C_\alpha, \tag{3.9}$$

for any $\epsilon\in(0,1)$ and any $\xi_1\in\mathbb{R}^n\setminus B(0,2\epsilon)$ where the constant C_α is independent of ϵ and ξ_1. Recall ∂_2 is the derivative with respect to the second variable of a function of two variables. Integrating by parts, we rewrite II_ϵ as

$$\mathrm{II}_\epsilon = (-1)^{|\alpha|}\int_{\mathbb{R}^n\setminus B(0,2\epsilon)}\widehat{a_1}(\xi_1)\left(\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^n}\partial^{\alpha-\beta}\widehat{a_2}(\xi-\xi_1)\partial_2^\beta\sigma(\xi_1,\xi-\xi_1)\varphi_\epsilon(\xi)d\xi\right)d\xi_1.$$

The Lebesgue dominated convergence theorem and the approximation to identity, combined with the fact that (3.9) holds and that $\widehat{a_1}\in L^1(\mathbb{R}^n)$, yields

$$\lim_{\epsilon\to 0}\mathrm{II}_\epsilon = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^n}\widehat{a_1}(\xi_1)\partial^{\alpha-\beta}\widehat{a_2}(-\xi_1)\partial_2^\beta\sigma(\xi_1,-\xi_1)\,d\xi_1.$$

This completes the proof of the lemma. □

Lemma 3.4. *There exists a function $\zeta\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ such that*

$$\{\xi\in B(0,1)\,:\,\widehat{\zeta}(\xi)=0\}=\{0\}. \tag{3.10}$$

Proof. The Fourier transform of the function $\left(\frac{\cos|\xi|-1}{|\xi|}\right)^{n+1}$ on $\mathbb{R}^n$ is known to be compactly supported; see [2, Lemma 3.1] and bounded but may not be smooth. Let Φ be a smooth and compactly supported function with non-vanishing integral. Then $\zeta=\Phi*\left(\left(\frac{\cos|\xi|-1}{|\xi|}\right)^{n+1}\right)^\vee$ lies in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and satisfies $\widehat{\zeta}(\xi) \neq 0$ for all $0 \neq \xi$ in a neighborhood of the origin, since $\widehat{\Phi}$ and $\cos|\xi| - 1$ do not vanish near zero and $\cos|\xi| - 1$ vanishes only at zero. It remains to dilate ζ to make it satisfy (3.10). $\square$

Lemma 3.5. *Let $N \in \mathbb{N}$ be fixed and $F \in L^\infty(\mathbb{R}^n)$. Assume for all functions $G \in L^\infty_{\rm c}(\mathbb{R}^n)$ with $\widehat{G} \in L^1(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} x^\alpha G(x)\,dx = 0 \qquad \forall\ |\alpha| \leq N,$$

we have

$$\int_{\mathbb{R}^n} \widehat{G}(\xi) F(\xi)\,d\xi = 0,$$

Then $F = 0$ a.e..

Proof. Denote

$$\Omega_N(\mathbb{R}^n) = \Big\{ f \in L^\infty_{\rm c}(\mathbb{R}^n)\ :\ \widehat{f} \in L^1(\mathbb{R}^n), \int_{\mathbb{R}^n} x^\alpha f(x)\ dx = 0, \quad \forall |\alpha| \leq N \Big\}.$$

First, we observe that if $G \in \Omega_N(\mathbb{R}^n)$, then $G_{x_0} \in \Omega_N(\mathbb{R}^n)$, where $G_{x_0} = G(\cdot - x_0)$ for given $x_0 \in \mathbb{R}^n$. To check this observation for $G \in \Omega_N(\mathbb{R}^n)$, we can easily see that G_{x_0} is a bounded function with bounded support. Also $\widehat{G_{x_0}}(\xi) = e^{2\pi i x_0 \cdot \xi}\widehat{G}(\xi)$; and hence $\widehat{G_{x_0}} \in L^1(\mathbb{R}^n)$, since $\widehat{G} \in L^1(\mathbb{R}^n)$. Next we want to show that

$$\int_{\mathbb{R}^n} x^\alpha G_{x_0}(x)\ dx = 0, \qquad \forall |\alpha| \leq N. \tag{3.11}$$

In fact, we have

$$\begin{aligned}\int_{\mathbb{R}^n} x^\alpha G_{x_0}(x)\ dx &= \int_{\mathbb{R}^n} (x + x_0)^\alpha G(x)\ dx \\ &= \sum_{\beta \leq \alpha} C_{\alpha,\beta}(x_0) \int_{\mathbb{R}^n} x^\beta G(x)\ dx = 0, \quad \forall |\alpha| \leq N.\end{aligned}$$

Thus (3.11) is verified, and we are done with checking that $G_{x_0} \in \Omega_N(\mathbb{R}^n)$.

As a consequence of the above observation, we claim that $\widehat{G}F = 0$ a.e. and for all $G \in \Omega_N(\mathbb{R}^n)$. Indeed, fix $G \in \Omega_N(\mathbb{R}^n)$. For each $x_0 \in \mathbb{R}^n$, the above observation showed that $G_{x_0} = G(\cdot - x_0) \in \Omega_N(\mathbb{R}^n)$. Therefore,

$$\int_{\mathbb{R}^n} \widehat{G}(\xi) F(\xi) e^{2\pi i x_0 \cdot \xi}\ d\xi = \int_{\mathbb{R}^n} \widehat{G_{x_0}}(\xi) F(\xi)\ d\xi = 0,$$

i.e., $\big(\widehat{G}F\big)^\vee(x_0) = 0$ for each $x_0 \in \mathbb{R}^n$, and for all $G \in \Omega_N(\mathbb{R}^n)$. This completes our claim $\widehat{G}F = 0$ a.e. and for all $G \in \Omega_N(\mathbb{R}^n)$.

The rest of the proof is to verify that $F = 0$ a.e. by showing $F = 0$ a.e. on $B(0,1)$. By Lemma 3.4, we can find a function $\zeta \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ such that $\widehat{\zeta}(0) = 0$ and $\widehat{\zeta}(\xi) \neq 0$ for all $0 < |\xi| < 1$. Define

$$G = \underbrace{\zeta * \cdots * \zeta}_{N+1 \text{ times}}.$$

It is clear that $G \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ and

$$\widehat{G}(\xi) = \big[\zeta(\xi)\big]^{N+1},$$

which satisfies condition $\partial^\alpha \widehat{G}(0) = 0$ for all $|\alpha| \leq N$. Thus $G \in \Omega_N(\mathbb{R}^n)$. By our claim, we have $\widehat{G}F = 0$ a.e. Noting that $\widehat{G}(\xi) \neq 0$ for $0 < |\xi| < 1$, we deduce $F = 0$ a.e. on $B(0,1)$. By a suitable dilation, we can show that $F = 0$ a.e. on $\mathbb{R}^n$. $\square$

Proof of Theorem 3.1. We first assume (3.1), and then prove (3.2). This direction can be obtained easily by Lemma 3.3.

Next we consider the inverse implication, i.e., assume (3.2) and then prove (3.1). We first focus on the case of $\alpha = 0$. By Lemma 3.3, condition (3.2) is equivalent to

$$\int_{\mathbb{R}^n} \widehat{a_1}(\xi_1)\widehat{a_2}(-\xi_1)\sigma(\xi_1, -\xi_1)\, d\xi_1 = 0$$

for all H^{p_1}-atoms a_1 and for all H^{p_2}-atoms a_2. Now Lemma 3.5 implies that

$$\widehat{a_2}(-\xi_1)\sigma(\xi_1, -\xi_1) = 0, \quad \forall \xi_1 \neq 0. \tag{3.12}$$

Fix $\xi_1 \in \mathbb{R}^n, \xi_1 \neq 0$. Choose $a_2 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, such that $\widehat{a_2}(-\xi_1) > 0$, and hence (3.12) deduces $\sigma(\xi_1, -\xi_1) = 0$, which implies (3.1) for $\alpha = 0$.

Next, we discuss the case of $|\alpha| \geq 1$ by induction on its order. Indeed, assume inductively that (3.1) holds for all $|\alpha| \leq k < N$. We want to show that it also holds for $|\alpha| = k + 1 \leq N$. The inductive hypothesis together with Lemma 3.3 deduces

$$\int_{\mathbb{R}^n} \widehat{a_1}(\xi_1)\widehat{a_2}(-\xi_1)\partial_2^\alpha\sigma(\xi_1, -\xi_1)\, d\xi_1 = 0.$$

Repeat the argument in the case $\alpha = 0$, we obtain (3.1) for $|\alpha| = k + 1$. The proof of the theorem is now completed. □

4. The multilinear case

In this section we prove Theorem 1.4.

Lemma 4.1. *Let $N \in \mathbb{N}$ and α be a multi-index with $|\alpha| \leq N$. Let σ and a_i be functions as stated in Theorem 1.4. Then we have*

$$\begin{aligned} &\int_{\mathbb{R}^n} (-2\pi i x)^\alpha T_\sigma(a_1, \ldots, a_m)(x)\, dx \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^{(m-1)n}} \widehat{a_1}(\xi_1) \cdots \widehat{a_{m-1}}(\xi_{m-1})\partial^{\alpha-\beta}\widehat{a_m}(-\xi_1 - \cdots - \xi_{m-1}) \times \\ &\qquad \times \partial_m^\beta \sigma(\xi_1, \ldots, \xi_{m-1}, -\xi_1 - \cdots - \xi_{m-1})\, d\xi_1 \cdots d\xi_{m-1}. \end{aligned} \tag{4.1}$$

Proof. Recall that the function φ is supported in the unit ball and $\widehat{\varphi}(0) = 1$. Fix $a_j \in \mathcal{O}(\mathbb{R}^n)$, $1 \leq j \leq m$. Now we have

$$\begin{aligned} &\int_{\mathbb{R}^n} (-2\pi i x)^\alpha T_\sigma(a_1, \ldots, a_m)(x)\, dx \\ &= \partial^\alpha \left[\widehat{T_\sigma(a_1, \ldots, a_m)}\right](0) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \widehat{T_\sigma(a_1, \ldots, a_m)}(\xi)\partial^\alpha[\varphi_\epsilon](\xi) d\xi \\ &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^{mn}} \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m)\sigma(\xi_1, \ldots, \xi_m)\partial^\alpha[\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m. \end{aligned} \tag{4.2}$$

Let

$$\Delta_\epsilon^{m-1} = \{(\xi_1, \ldots, \xi_m) \in \mathbb{R}^{mn} \ : \ |\xi_1 + \cdots + \xi_{m-1}| \leq 2\epsilon\},$$

and denote

$$\Sigma_\epsilon^0 = \left(\cup_{i=1}^{m-1} \Gamma_{i,\epsilon}(\mathbb{R}^{mn})\right) \cup \Delta_\epsilon^{m-1},$$

where $\Gamma_{i,\epsilon}(\mathbb{R}^{mn})$ is defined in (1.11). Also set $\Sigma_\epsilon^1 = \mathbb{R}^{mn} \setminus \Sigma_\epsilon^0$, and hence $\mathbb{R}^{mn} = \Sigma_\epsilon^0 \cup \Sigma_\epsilon^1$. The last integral in (4.2) can be decomposed into two parts: $\mathrm{I}_\epsilon + \mathrm{II}_\epsilon$, where

$$\mathrm{I}_\epsilon = \int_{\Sigma_\epsilon^0} \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m$$

and

$$\mathrm{II}_\epsilon = \int_{\Sigma_\epsilon^1} \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m.$$

Next we will show that $\lim_{\epsilon \to 0} \mathrm{I}_\epsilon = 0$. Indeed, we can estimate

$$\begin{aligned} |\mathrm{I}_\epsilon| \le \sum_{i=1}^{m-1} \Big| \int_{\Gamma_{i,\epsilon}(\mathbb{R}^{mn})} & \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m \Big| \\ & + \Big| \int_{\Delta_\epsilon^{m-1}} \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m \Big|. \end{aligned}$$

Thus, it is enough to show that

$$\lim_{\epsilon \to 0} \int_{\Gamma_{i,\epsilon}(\mathbb{R}^{mn})} \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m = 0, \tag{4.3}$$

for all $1 \le i \le m-1$, and

$$\lim_{\epsilon \to 0} \int_{\Delta_\epsilon^{m-1}} \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m = 0. \tag{4.4}$$

Without loss of generality, we have only to prove (4.3) for $i = 1$. In this case, we have

$$|\widehat{a_1}(\xi)| \le C(a_1) \min(1, |\xi|^{N+1}) \le C(a_1) |\xi|^{|\alpha|+1}$$

and hence

$$\begin{aligned} \Big| \int_{\Gamma_{1,\epsilon}(\mathbb{R}^{mn})} & \widehat{a_1}(\xi_1) \cdots \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) d\xi_1 \cdots d\xi_m \Big| \\ & \le \int_{\Gamma_{1,\epsilon}(\mathbb{R}^{mn})} \Big| \widehat{a_1}(\xi_1) \cdots , \widehat{a_m}(\xi_m) \sigma(\xi_1, \dots, \xi_m) \partial^\alpha [\varphi_\epsilon](\xi_1 + \cdots + \xi_m) \Big| d\xi_1 \cdots d\xi_m \\ & \le C(a_1) \|\partial^\alpha \varphi\|_{L^\infty} \|\widehat{a_2}\|_{L^1} \cdots \|\widehat{a_{m-1}}\|_{L^1} \|\widehat{a_m}\|_{L^1} \|\sigma\|_{L^\infty} \epsilon^{-|\alpha|-n} \int_{B(0,2\epsilon)} |\xi_1|^{|\alpha|+1} \, d\xi_1 \\ & \le C(a_1) \|\partial^\alpha \varphi\|_{L^\infty} \|\widehat{a_2}\|_{L^1} \cdots \|\widehat{a_{m-1}}\|_{L^1} \|\widehat{a_m}\|_{L^1} \|\sigma\|_{L^\infty} \epsilon, \end{aligned}$$

which tends to 0 as ϵ approaches to 0.

Notice that φ is supported in the unit ball, therefore $\varphi_\epsilon(\xi_1 + \cdots + \xi_m)$ survives only if $|\xi_1 + \cdots + \xi_m| \le \varepsilon$. Identity (4.4) can be proved similarly by making use of the fact that for all $(\xi_1, \dots, \xi_m) \in \Delta_\epsilon^{m-1}$,

$$|\xi_m| \le |\xi_1 + \cdots + \xi_m| + |\xi_1 + \cdots + \xi_{m-1}| \le 3\epsilon,$$

and the vanishing moments of a_m.

Now we turn into II_ϵ and rewrite it in the following form

$$\begin{aligned} \mathrm{II}_\epsilon = \int_{\substack{|\xi_1|>\epsilon, \dots, |\xi_{m-1}|>\epsilon \\ |\xi_1 + \cdots + \xi_{m-1}| > 2\epsilon}} & \widehat{a_1}(\xi_1) \cdots \widehat{a_{m-1}}(\xi_{m-1}) \int_{\mathbb{R}^n} \widehat{a_m}(\xi - \xi_1 - \cdots - \xi_{m-1}) \times \\ & \times \sigma(\xi_1, \dots, \xi_{m-1}, \xi - \xi_1 - \cdots - \xi_{m-1}) \partial^\alpha \big[\varphi_\epsilon\big](\xi) \, d\xi \, d\xi_1 \cdots d\xi_{m-1}. \end{aligned}$$

Fix $\xi_1, \ldots, \xi_{m-1}$ so that $|\xi_1 + \cdots + \xi_{m-1}| > 2\epsilon$, and that $|\xi_i| > \epsilon$ for all $1 \le i \le m-1$. We easily see that the function $\xi \mapsto \sigma(\xi_1, \ldots, \xi_{m-1}, \xi - \xi_1 - \cdots - \xi_{m-1})$ is smooth on $B(0,\epsilon)$. Integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \widehat{a_m}(\xi - \xi_1 - \cdots - \xi_{m-1})\sigma(\xi_1, \ldots, \xi_{m-1}, \xi - \xi_1 - \cdots - \xi_{m-1})\partial^\alpha\big[\varphi_\epsilon\big](\xi)\, d\xi\\ &\quad = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^n} \partial^{\alpha-\beta}\widehat{a_m}(\xi - \xi_1 - \cdots - \xi_{m-1})\partial_m^\beta \sigma(\xi_1, \ldots, \xi_{m-1}, \xi - \xi_1 - \cdots - \xi_{m-1})\varphi_\epsilon(\xi)\, d\xi.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{II}_\epsilon = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \int_{\substack{|\xi_1|>\epsilon, \ldots, |\xi_{m-1}|>\epsilon\\ |\xi_1 + \cdots + \xi_{m-1}| > 2\epsilon}} &\widehat{a_1}(\xi_1) \cdots \widehat{a_{m-1}}(\xi_{m-1}) \Big\{ \int_{\mathbb{R}^n} \partial^{\alpha-\beta}\widehat{a_m}(\xi - \xi_1 - \cdots - \xi_{m-1})\\ &\partial_m^\beta \sigma(\xi_1, \ldots, \xi_{m-1}, \xi - \xi_1 - \cdots - \xi_{m-1})\varphi_\epsilon(\xi)\, d\xi \Big\}\, d\xi_1 \cdots d\xi_{m-1}.\end{aligned}$$

An argument similar to Lemma 3.2 allows us to use the Lebesgue dominated convergence theorem to pass the limit inside the above integral. Together with the use of the approximate identity we obtain

$$\begin{aligned}\lim_{\epsilon \to 0} \mathrm{II}_\epsilon = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \int_{\substack{|\xi_1|>0, \ldots, |\xi_{m-1}|>0\\ |\xi_1 + \cdots + \xi_{m-1}| > 0}} &\widehat{a_1}(\xi_1) \cdots \widehat{a_{m-1}}(\xi_{m-1}) \partial^{\alpha-\beta}\widehat{a_m}(-\xi_1 - \cdots - \xi_{m-1})\\ &\partial_m^\beta \sigma(\xi_1, \ldots, \xi_{m-1}, -\xi_1 - \cdots - \xi_{m-1})\, d\xi_1 \cdots d\xi_{m-1}.\end{aligned}$$

This identity completes the proof of the lemma. □

Proof of Theorem 1.4. By Lemma 3.3, it is clear that if (1.5) is valid then (1.4) holds automatically. For the reverse direction, we use an analogous extension of Lemma 3.5 and repeat the proof of Theorem 3.1. □

5. Proof of Theorem 1.1

Let $N \in \mathbb{N}$ be fixed and let σ be a bounded function in $\mathbb{R}^n$ that satisfies either (1.2) or (1.6), and let T_σ be the multilinear multiplier operator associated to σ. As showed in [14], T_σ is bounded from $H^{p_1}(\mathbb{R}^n) \times \cdots \times H^{p_m}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$, where $0 < p \le 1, 0 < p_j < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, if (1.4) holds, i.e.,

$$\int_{\mathbb{R}^n} x^\alpha T_\sigma(a_1, \ldots, a_m)(x)dx = 0,$$

for all $a_j \in \mathcal{O}_N(\mathbb{R}^n)$ and all $0 < |\alpha| \le \lfloor n(\frac{1}{p} - 1) \rfloor$. Therefore, the reverse direction from (2) to (1) of Theorem 1.1 follows from Theorem 1.4.

To obtain the other direction, since T_σ satisfies (1.8), $|x|^N T_\sigma(a_1, \ldots, a_m)$ is an integrable function. Therefore if $T_\sigma(a_1, \ldots, a_m) \in H^p(\mathbb{R}^n)$, then (1.4) is valid. This is a consequence of a result in [22, p. 128, 5.4 (c)]. Similarly, we can prove Theorem 1.2 by repeating the above argument.

6. Remarks, Examples, and Applications

It is noteworthy to mention that our results are also valid for symbols of intermediate or *mixed type*, i.e., of the form

$$\sigma(\xi_1, \ldots, \xi_m) = \sum_{\rho=1}^{T} \sum_{I_1^\rho, \ldots, I_{G(\rho)}^\rho} \prod_{g=1}^{G(\rho)} \sigma_{I_g^\rho}(\{\xi_l\}_{l \in I_g^\rho}), \tag{6.1}$$

where for each $\rho = 1, \dots, T$, $I_1^\rho, \dots, I_{G(\rho)}^\rho$ is a partition of $\{1, \dots, m\}$ and each $T_{\sigma_{I_g^\rho}}$ is an $|I_g^\rho|$-linear Coifman-Meyer multiplier operator. We write $I_1^\rho + \cdots + I_{G(\rho)}^\rho = \{1, \dots, m\}$ to denote such partitions. There is an analogous theorem for these general symbols.

Theorem 6.1. *Let σ be as in* (6.1). *Fix $0 < p_i < \infty$, $0 < p \le 1$ that satisfy* (1.3). *Then the following two statements are equivalent:*
(a) T_σ maps $H^{p_1}(\mathbb{R}^n) \times \cdots \times H^{p_m}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$.
(b) For all $|\alpha| \le \lfloor n(\frac{1}{p} - 1) \rfloor$ condition (1.5) *holds, i.e.*

$$\partial_m^\alpha \sigma(\xi_1, \dots, \xi_m) = 0$$

for all $(\xi_1, \dots, \xi_m)$ on the hyperplane Δ_n away from the points of singularity of σ.

For the sake of brevity we don't include the proof of Theorem 6.1 in this note, but we point out that similar techniques can be used to obtain it.

Next, we provide examples of functions that satisfy conditions (3.1); some of these examples are inspired by those given in [8]: On $\mathbb{R}^2 \times \mathbb{R}^2$ with coordinates $(\xi_1, \eta_2, \eta_1, \eta_2)$ consider the multipliers

$$\begin{aligned}\sigma_0(\xi_1, \xi_2, \eta_1, \eta_2) &= \frac{\xi_1\eta_2 - \xi_2\eta_1}{|\xi_1|^2 + |\xi_2|^2 + |\eta_1|^2 + |\eta_2|^2} \\ &= \frac{1}{|\xi_1|^2 + |\xi_2|^2 + |\eta_1|^2 + |\eta_2|^2} \det \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}.\end{aligned}$$

An alternative example is obtained by considering the multiplier

$$\begin{aligned}\sigma_1(\xi_1, \xi_2, \eta_1, \eta_2) &= \frac{\xi_1\eta_2 - \xi_2\eta_1}{\sqrt{|\xi_1|^2 + |\xi_2|^2}\sqrt{|\eta_1|^2 + |\eta_2|^2}} \\ &= \frac{1}{\sqrt{|\xi_1|^2 + |\xi_2|^2}\sqrt{|\eta_1|^2 + |\eta_2|^2}} \det \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}.\end{aligned}$$

It is easy to verify that for $(\xi_1, \xi_2) \neq (0, 0)$ we have

$$\sigma_0(\xi_1, \xi_2, -\xi_1, -\xi_2) = \sigma_1(\xi_1, \xi_2, -\xi_1, -\xi_2) = 0.$$

For higher order cancellation consider the examples

$$\sigma_2(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{\xi_1^2\eta_2^2 - 2\xi_1\xi_2\eta_1\eta_2 + \xi_2^2\eta_1^2}{(|\xi_1|^2 + |\xi_2|^2 + |\eta_1|^2 + |\eta_2|^2)^2}$$

and

$$\sigma_3(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{\xi_1^2\eta_2^2 - 2\xi_1\xi_2\eta_1\eta_2 + \xi_2^2\eta_1^2}{(|\xi_1|^2 + |\xi_2|^2)(|\eta_1|^2 + |\eta_2|^2)}$$

both of which satisfy:

$$\partial_{\xi_1}^k \partial_{\xi_2}^l \sigma_3(\xi_1, \xi_2, -\xi_1, -\xi_2) = \partial_{\xi_1}^k \partial_{\xi_2}^l \sigma_4(\xi_1, \xi_2, -\xi_1, -\xi_2) = 0, \quad |\xi_1|^2 + |\xi_2|^2 \neq 0,$$

for $(k, l) \in \{(0, 1), (1, 0), (0, 0)\}$. The symbols σ_1 and σ_3 are inspired by [8] and arise by expansions of the Hessian or by combinations of the Riesz transforms. Examples of σ_0 and σ_2 are of Coifman-Meyer type (case (i) in the introduction) while σ_1 and σ_3 are as in case (ii), i.e., sums of products of Calderón-Zygmund operators.

We generalize this example as follows:

$$\sigma_{2N-2}(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{1}{(|\xi_1|^2 + |\xi_2|^2 + |\eta_1|^2 + |\eta_2|^2)^{n_1 + n_2 + \cdots + n_N}} \prod_{j=1}^{N} \det \begin{pmatrix} {\xi_1}^{n_j} & {\xi_2}^{n_j} \\ {\eta_1}^{n_j} & {\eta_2}^{n_j}, \end{pmatrix}$$

where each n_j is positive integer. By the Leibniz rule we can check that

$$\partial_{\xi_1}^k \partial_{\xi_2}^l \sigma_3(\xi_1, \xi_2, -\xi_1, -\xi_2) = \partial_{\xi_1}^k \partial_{\xi_2}^l \sigma_4(\xi_1, \xi_2, -\xi_1, -\xi_2) = 0, \quad |\xi_1|^2 + |\xi_2|^2 \neq 0,$$

as long as $k + l \leq N - 1$.

Finally, we address the following question[1] and give a partial answer: Find a condition on a bilinear multiplier $B(f, g)$ such for any two sequences $f_k \to f$ weakly and $g_k \to g$ weakly, then $B(f_k, g_k) \to B(f, g)$ weakly. Suppose that B is given in multiplier form by

$$B(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{g}(\eta)\sigma(\xi, \eta)e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta$$

where f, g are defined on $\mathbb{R}^n$ and $\sigma(\xi, \eta)$ is a Coifman-Meyer multiplier, i.e., it satisfies:

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)\right| \leq C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}$$

for $|\alpha|, |\beta| \leq N$ with $N \gg 1$ We provide a condition on σ so that the associated operator preserves weak convergence. Obviously the product $B(f, g) = fg$ does not preserve weak convergence because the symbol $\sigma(\xi_1, \xi_2) = 1$ fails to satisfy condition (v) below.

Corollary 6.2. *Let $1 < p < \infty$ and let B be as above. Suppose that f_k, g_k, f, g, $k = 1, 2, \ldots$ are functions on $\mathbb{R}^n$ that satisfy:*

(i) $\sup_k \|f_k\|_{L^p(\mathbb{R}^n)} \leq C$.
(ii) $\sup_k \|g_k\|_{L^{p'}(\mathbb{R}^n)} \leq C$.
(iii) $f_k \to f$ *weakly in* $L^p(\mathbb{R}^n)$.
(iv) $g_k \to g$ *weakly in* $L^{p'}(\mathbb{R}^n)$.
(v) $\sigma(\xi, -\xi) = 0$ *for all* $\xi \neq 0$.
(vi) $B(f_k, g_k)$ *converges a.e. to* $B(f, g)$.

Then $B(f_k, g_k)$ converges to $B(f, g)$ weakly in $H^1(\mathbb{R}^n)$ in the sense that

$$\int_{\mathbb{R}^n} B(f_k, g_k)\varphi\, dx \to \int_{\mathbb{R}^n} B(f, g)\varphi\, dx \tag{6.2}$$

for all functions $\varphi \in \mathrm{VMO}(\mathbb{R}^n)$.

Proof. The boundedness of B from $L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$ can proved by combining condition (5) with Theorem 3.1 ($N = 1$) and the result in [14]; a version of this result was also proved by Dobyinski [6, Lemme 3.8]; see also [4]. It follows that

$$\sup_k \|B(f_k, g_k)\|_{H^1} \leq C_n \sup_k \|f_k\|_{L^p} \|g_k\|_{L^{p'}} \leq C_n C^2 .$$

Thus the sequence $B(f_k, g_k), k = 1, 2, \ldots$ is uniformly bounded in $H^1(\mathbb{R}^n)$ and converges a.e. to $B(f, g)$. Then we obtain (6.2) as a consequence of the result in [18]. $\square$

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